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Conditions satisfied by clone lattices

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Dedicated to the memory of Viktor Aleksandrovich Gorbunov

Abstract. For any $k \geq 4$ the clone lattice $\mathcal{L}_k$ on a k-element set contains any finite lattice as a sublattice. This means in particular that $\mathcal{L}_k$ does not satisfy any nontrivial lattice identity or quasi-identity. We exhibit two conditions looking like "infinite quasi-identities" that are satisfied by every clone lattice on a finite set.

1. Introduction and results

Let A be a finite set. By a clone on A we mean a set of operations on A which is closed under superposition and contains all projection operations. The lattice of all clones on A ordered by $\subseteq$ is denoted by $\mathcal{L}_A$. Clearly $\mathcal{L}_A$ depends only upon $|A|$, and so we use the notation $\mathcal{L}_k$ where $k = |A|$ rather than $\mathcal{L}_A$.

It is well known that $\mathcal{L}_2$ is countably infinite [9] while $\mathcal{L}_k$ for $k \geq 3$ is of continuum cardinality [11, 6]. In the former case the lattice was completely described and in the latter case a complete description of the lattice is believed to be unreachable. The complexity of clone lattices in the latter case was confirmed by results of [1, 2], see also [5].

In [1, 3] it was proved that the subsemigroup lattice of the additive semigroup of natural numbers and even the free lattice of countable rank are embeddable in $\mathcal{L}_k$, $k \geq 3$, and therefore $\mathcal{L}_k$ does not satisfy any nontrivial lattice identity. Actually, many known identities, such as distributivity, modularity etc., fail already in $\mathcal{L}_2$, as is easily derived from the description of $\mathcal{L}_2$ [9]. Moreover, for clone lattices on sets having 4 or more elements we have a stronger result [2]:

THEOREM 1.1. ([2], see also [5]) *Let $\{L_i \mid i \in \mathbb{N}\}$ be finite lattices and let $k \geq 4$. Then $\prod_{i\in\mathbb{N}} L_i$ is isomorphic to a sublattice of $\mathcal{L}_k$.*

Theorem 1.1 implies that no nontrivial lattice quasi-identity holds in $\mathcal{L}_k$ whenever $k \geq 4$. So, if a condition holds in some clone lattice, it have to be of much more complicated form than an identity or a quasi-identity.

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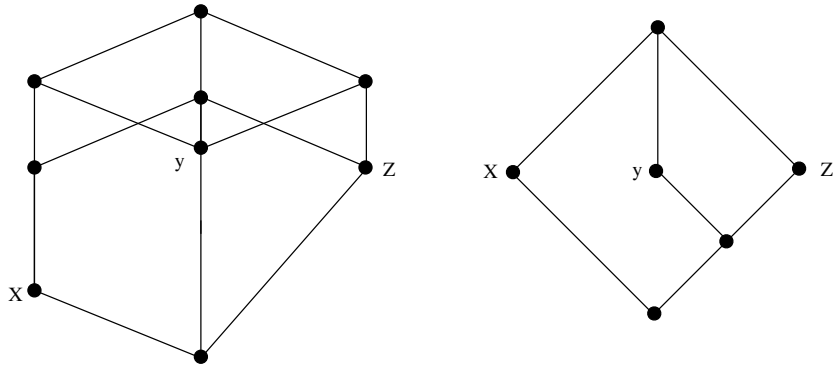

Figure 1

Here we exhibit two mutually dual conditions that look like quasi-identities depending on infinitely many variables.

We say that a complete lattice L satisfies the condition (ω-SD$\wedge$) if whenever elements $a_0, a_1, \ldots$ of L satisfy $a_i \wedge a_j = a_s \wedge a_t$ for any $i, j, s, t \in \{0, 1, 2, \ldots\}$, $i \neq j$, $s \neq t$, they also satisfy $a_0 = \bigwedge_{0<i<\omega}(a_0 \vee a_i)$. Dually, a complete lattice L satisfies the condition (ω-SD$\vee$) if whenever elements $a_0, a_1, \ldots$ of L satisfy $a_i \vee a_j = a_s \vee a_t$ for any $i, j, s, t \in \{0, 1, 2, \ldots\}$, $i \neq j$, $s \neq t$, they also satisfy $a_0 = \bigvee_{0<i<\omega}(a_0 \wedge a_i)$.

In order to compare these conditions with known ones let us consider a condition that obtained from one of them, say (ω-SD$\wedge$), by reducing the number of variables to three:

$$x \wedge y = x \wedge z = y \wedge z \quad \text{implies} \quad x = (x \vee y) \wedge (x \vee z) \tag{$*$}$$

It looks very similar to the condition of meet-semidistributivity

$$x \wedge y = x \wedge z \quad \text{implies} \quad x \wedge y = x \wedge (y \vee z). \tag{SD$\wedge$}$$

However, these two conditions are incomparable, as the examples on Figure 1 show.

The first lattice in the picture is meet-semidistributive, while it does not satisfy the condition ($*$). Conversely, the second lattice satisfies ($*$), but is not meet-semidistributive.

A natural infinite analogue of meet-semidistributivity was proved for subvariety lattices in [7]. However, the 'zipper-condition' obtained in [7] is too strong, since in particular, it implies meet-semidistributivity. The conditions (ω-SD$\wedge$) and (ω-SD$\vee$) seem to be more appropriate infinite 'semidistributive-like' properties that hold in every finite lattice. As the following theorem shows (ω-SD$\wedge$) and (ω-SD$\vee$) hold in every clone lattice.

THEOREM 1.2. *For any $k \geq 2$ the lattice $\mathcal{L}_k$ satisfies both conditions (ω-SD$\wedge$) and (ω-SD$\vee$).*

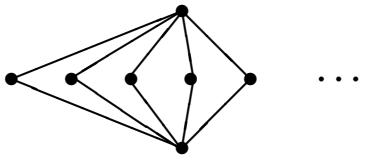

Figure 2

The simplest lattice that satisfies neither (ω-SD$\wedge$) nor (ω-SD$\vee$) is the lattice M_ω:

COROLLARY 1.3. *The lattice M_ω is isomorphic to no sublattice of $\mathcal{L}_k$, $k \geq 2$.*

2. Proof of the theorem

To prove the theorem we need more definitions and notation. An operation f on A *preserves* an m-ary relation ϱ on A (or ϱ is invariant under f) if ϱ is a subuniverse of $(A, f)^m$. For a clone $\mathbf{C}$, the set of all m-ary relations invariant under each operation from $\mathbf{C}$ is denoted by $\mathrm{Inv}^{(m)}\mathbf{C}$, moreover $\mathrm{Inv}\mathbf{C} = \bigcup_{m\in N} \mathrm{Inv}^{(m)}\mathbf{C}$. The following equality is well known and easy to prove

$$\mathrm{Inv}(\mathbf{C}_1 \vee \mathbf{C}_2) = \mathrm{Inv}\mathbf{C}_1 \cap \mathrm{Inv}\mathbf{C}_2$$

for any clones $\mathbf{C}_1, \mathbf{C}_2$.

Further, the set of all n-ary operations from $\mathbf{C}$ is denoted by $\mathbf{C}^{(n)}$. Each operation from $\mathbf{C}^{(n)}$ is a mapping $A^n \to A$, and therefore is an element of A^{A^n}. Thus $\mathbf{C}^{(n)}$ corresponds to a subset of A^{A^n} that we will denote by $\mathrm{Gr}_n\mathbf{C}$. The first part of the following proposition can be found in [10], the second part is obvious.

PROPOSITION 2.1. *Let $\mathbf{C}$ be a clone.*

(1) *Every operation from $\mathbf{C}$ preserves $\mathrm{Gr}_n\mathbf{C}'$ for each clone $\mathbf{C}' \supseteq \mathbf{C}$. Moreover, an operation f belongs to $\mathbf{C}^{(n)}$ if and only if f preserves $\mathrm{Gr}_n\mathbf{C}$.*
(2) *An n-ary relation ϱ belongs to $\mathrm{Inv}\mathbf{C}$ if and only if it is invariant under all operations from $\mathbf{C}^{(|A|^n)}$.*

Now we are in a position to prove the main result of the paper.

Let us prove first the condition (ω-SD$\wedge$). Let $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots$ be clones such that $\mathbf{C}_i \wedge \mathbf{C}_j = \mathbf{C}_s \wedge \mathbf{C}_t$ whenever $i \neq j, s \neq t$, and let $\mathbf{C} = \mathbf{C}_0 \wedge \mathbf{C}_1$. For any $n \in \mathbb{N}$, set $m = |A|^{|A|^n}$. The set of all m-ary operations is finite, therefore, there are only finitely many possibilities for $\mathbf{C}^{(m)}$. So, there exist distinct $i, j \neq 0$ such that $\mathbf{C}_i^{(m)} = \mathbf{C}_j^{(m)}$. Since $(\mathbf{C}_i \wedge \mathbf{C}_j)^{(m)} = \mathbf{C}_i^{(m)} \cap \mathbf{C}_j^{(m)}$, we have $\mathbf{C}_i^{(m)} = \mathbf{C}_j^{(m)} = \mathbf{C}^{(m)}$. By Proposition 2.1(2), this implies $\mathrm{Inv}^{(|A|^n)}\mathbf{C} = \mathrm{Inv}^{(|A|^n)}\mathbf{C}_i$. Further, by Proposition 2.1(1), $\mathrm{Gr}_n\mathbf{C}_0$ is in $\mathrm{Inv}^{(|A|^n)}\mathbf{C}_i$,

and therefore, in $\mathrm{Inv}^{(|A|^n)}(\mathbf{C}_i \vee \mathbf{C}_0)$. Hence, $(\mathbf{C}_0 \vee \mathbf{C}_i)^{(n)} \subseteq \mathbf{C}_0^{(n)}$. As for every $n \in \mathbb{N}$ there is i such that the latter inclusion holds, we obtain

$$\bigwedge_{0<i<\omega} (\mathbf{C}_0 \vee \mathbf{C}_i) = \mathbf{C}_0.$$

Then let us check the condition (ω-SD$\vee$). Let clones $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2, \ldots$ be such that $\mathbf{C}_i \vee \mathbf{C}_j = \mathbf{C}_s \vee \mathbf{C}_t$ whenever $i \neq j$, $s \neq t$, and set $\mathbf{C} = \mathbf{C}_0 \vee \mathbf{C}_1$. Let $n \in \mathbb{N}$ be arbitrary and $m = |A|^n$. The set $\{\mathrm{Inv}^{(m)}\mathbf{C}_i \mid i \in \mathbb{N}\}$ is finite, therefore $\mathrm{Inv}^{(m)}\mathbf{C}_i = \mathrm{Inv}^{(m)}\mathbf{C}_j$ for certain $i \neq j$. As $\mathrm{Inv}^{(m)}(\mathbf{C}_i \vee \mathbf{C}_j) = \mathrm{Inv}^{(m)}\mathbf{C}_i \cap \mathrm{Inv}^{(m)}\mathbf{C}_j$, this means that $\mathrm{Inv}^{(m)}\mathbf{C}_i = \mathrm{Inv}^{(m)}\mathbf{C}$. In particular, $\mathrm{Gr}_n\mathbf{C}_i \in \mathrm{Inv}^{(m)}\mathbf{C}$ that implies $\mathbf{C}_i^{(n)} = \mathbf{C}^{(n)} \supseteq \mathbf{C}_0^{(n)}$. Therefore we have $(\mathbf{C}_0 \wedge \mathbf{C}_i)^{(n)} = \mathbf{C}_0^{(n)}$ and the required equality

$$\bigvee_{0<i<\omega} (\mathbf{C}_0 \wedge \mathbf{C}_i) = \mathbf{C}_0$$

holds. The theorem is proved.

3. Concluding remarks

In this section we consider some properties of the class of all lattices which are isomorphic to sublattices of clone lattices. For $k \geq 2$, we denote by $\mathcal{S}_k$ the class of all lattices embeddable into $\mathcal{L}_k$. Let also $\mathcal{S} = \bigcup_{k\geq 2} \mathcal{S}_k$.

PROPOSITION 3.1. ([8]) 1) *For any $k \geq 2$, the lattice $\mathcal{L}_k$ is embeddable into $\mathcal{L}_{k+1}$.*
2) *For any $k, l \geq 2$ the direct product $\mathcal{L}_k \times \mathcal{L}_l$ is embeddable into $\mathcal{L}_{k\cdot l}$.*

From this proposition one easily derives the following properties of classes $\mathcal{S}_k$.

COROLLARY 3.2. 1) $\mathcal{S}_2 \subseteq \mathcal{S}_3 \subseteq \mathcal{S}_4 \subseteq \ldots$;
2) *$\mathcal{S}$ is closed under finite direct products.*

It follows from the results of [9] and [11, 6] that were mentioned above that $\mathcal{S}_2 \neq \mathcal{S}_3$. However, up to now it is unknown whether $\mathcal{S}_i = \mathcal{S}_j$ for some i, j.

Another property of $\mathcal{S}$ is given by the result proved in [4].

PROPOSITION 3.3. ([4]) *For any $k \geq 2$, the countable power $\mathcal{L}_k^\omega$ is embeddable into $\mathcal{L}_{k+3}$.*

This means that the class $\mathcal{S}$ is closed under countable direct powers. However, $\mathcal{S}$ behaves worse with respect to homomorphic images.

PROPOSITION 3.4. *The class $\mathcal{S}$ is not closed under homomorphic images.*

Proof. As was proved in [3], the free lattice $FL(\omega)$ with countably many generators is embeddable into $\mathcal{L}_3$, and therefore belongs to $\mathcal{S}$. However, M_ω is obviously a homomorphic image of $FL(\omega)$, but does not belong to $\mathcal{S}$. □

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