The complexity of partition functions

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Abstract. We give a complexity theoretic classification of the counting versions of so-called H-colouring problems for graphs H that may have multiple edges between the same pair of vertices. More generally, we study the problem of computing a weighted sum of homomorphisms to a weighted graph H.
The problem has two interesting alternative formulations: First, it is equivalent to computing the partition function of a spin system as studied in statistical physics. And second, it is equivalent to counting the solutions to a constraint satisfaction problem whose constraint language consists of two equivalence relations.
In a nutshell, our result says that the problem is in polynomial time if the adjacency matrix of H has row rank 1, and #P-complete otherwise.

1 Introduction

This paper has two different motivations: The first is concerned with constraint satisfaction problems, the second with "spin-systems" as studied in statistical physics. A known link between the two are so-called H-colouring problems. Our main result is a complete complexity theoretic classification of the problem of counting the number of solutions of an H-colouring problem for an undirected graph H which may have multiple edges, and actually of a natural generalisation of this problem to weighted graphs H. Translated to the world of constraint satisfaction problems, this yields a classification of the problem of counting the solutions to constraint satisfaction problems for two equivalence relations. Translated to the world of statistical physics, it gives a classification of the problem of computing the partition function of a spin system.

Let us describe our result from each of the different perspectives: Let H be a graph, possibly with multiple edges between the same pair of vertices. An H*-colouring* of a graph G is a homomorphism from G to H. Both the decision problem, asking whether a given graph has an H-colouring, and the problem of counting the H-colourings of a given graph, have received considerable attention [5, 6, 9, 11, 12]. Here we are interested in the counting problem. Dyer and Greenhill [5] gave a complete complexity theoretic classification of the counting problem for undirected graphs H without multiple edges; they showed that the problem is in polynomial time if all connected components of H are complete bipartite without any loops or if H is complete with all loops present, and #P-complete otherwise. Here we are interested in counting H-colourings for graphs H that may have multiple edges. Note that, as opposed to the decision problem, multiple edges do make a difference for the counting problem. Let H be a graph with vertex set $\{1, \ldots, k\}$. H is best described in terms of its adjacency matrix $A = (A_{ij})$, where A_{ij} is the number of edges between vertices i and j. Given a graph $G = (V, E)$, we want to compute the number of homomorphisms from G to H. Observe that this number is

$$Z_A(G) = \sum_{\sigma : V \to \{1,\ldots,k\}} \prod_{e=\{u,v\} \in E} A_{\sigma(u)\sigma(v)}. \tag{1}$$

Borrowing from the physics terminology, we call Z_A the *partition function* of A (or H). We denote the problem of computing $Z_A(G)$ for a given graph G by EVAL(A). Of course if we define Z_A as in (1), the problem is not only meaningful for matrices A that are adjacency matrices of graphs, but for arbitrary square matrices A. We may view such matrices as adjacency matrices of weighted graphs. We call a symmetric matrix A *connected* (*bipartite*) if the corresponding graph is connected (*bipartite*, respectively).

We prove the following classification result:

Theorem 1. *Let A be a symmetric matrix with non-negative real entries.*

1. *If A is connected and not bipartite, then* EVAL(A) *is in polynomial time if the row rank of A is at most* 1*; otherwise* EVAL(A) *is #P-complete.*
2. *If A is connected and bipartite, then* EVAL(A) *is in polynomial time if the row rank of A is at most* 2*; otherwise* EVAL(A) *is #P-complete.*
3. *If A is not connected, then* EVAL(A) *is in polynomial time if each of its connected components satisfies the corresponding condition stated in (1) or (2); otherwise* EVAL(A) *is #P-complete.*

Note that this generalises Dyer and Greenhill's [5] classification result for graphs without multiple edges, whose adjacency matrices are symmetric 0-1 matrices.

Our proof builds on interpolation techniques similar to those used by Dyer and Greenhill, recent results on counting the number of solutions to constraint satisfaction problems due to Dalmau and the first author [3], and a considerable amount of polynomial arithmetic. Even though we present the proof in the language of constraint satisfaction problems here, in finding it it has been very useful to jump back and forth between the H-colouring and constraint satisfaction perspective; the reader may also find it useful to keep both perspectives in mind when studying the proof.

Let us now explain the result for constraint satisfaction problems. A *constraint language* Γ on a finite *domain* D is a set of relations on D. An instance of the problem CSP(Γ) is a triple $(V, D, \mathcal{C})$ consisting of a set V of *variables*, the domain D, and a set $\mathcal{C}$ of *constraints* (s, ρ), where, for some $r \geq 1$, $s \in V^r$ and ρ is an r-ary relation in Γ. A *solution* is a mapping $\sigma : V \to D$ such that for each constraint $((v_1, \ldots, v_r), \rho) \in \mathcal{C}$ we have $(\sigma(v_1), \ldots, \sigma(v_r)) \in \rho$. There has been considerable interest in the complexity of constraint satisfaction problems [16, 14, 7, 1, 2], which has mainly been driven by Feder and Vardi's [7] *dichotomy question*, asking whether for all languages Γ the problem CSP(Γ) is either solvable in polynomial time or NP-complete. A similar dichotomy question can be asked for the problem #CSP(Γ) of counting the solutions for a given instance [4, 3].

We consider constraint languages Γ consisting of two equivalence relations α, β. Suppose that α has k equivalence classes and β has ℓ equivalence classes. Then Γ can be described by a $(k \times \ell)$-matrix $B = (B_{ij})$, where B_{ij} is the number of elements in the intersection of the ith class of α and the jth class of β. We show that, provided that the matrix is "indecomposable" (in a sense made precise in Section 2.1), the problem #CSP(Γ) is in polynomial time if the row rank of B is 1 and #P-complete otherwise. There is also a straightforward extension to "decomposable" matrices. We can generalise the result to *weighted CSP*, where each domain element d carries a non-negative real weight $\omega(d)$. The weight of a solution $\sigma : V \to D$ is defined to be the product $\prod_{v \in V} \omega(\sigma(v))$, and the goal is to compute the weighted sum over all solutions. As an important intermediate step, we even prove our classification result for weights that are polynomials with integer coefficients.

Let us finally explain the connection with statistical physics. A *spin system* with k spins can be described by a $k \times k$-matrix $K = (K_{ij})$. We always assume K to be symmetric. A *configuration* of the system on a graph $G = (V, E)$ is a mapping $\sigma : V \to \{1, \dots, k\}$. The *energy* of σ is the sum $H(\sigma) = \sum_{e=\{u,v\} \in E} K_{\sigma(u)\sigma(v)}$. The *partition function* is $Z = \sum_\sigma \exp(-H(\sigma)/cT)$, where T is a parameter of the system (the *temperature*) and c is a constant. Observe that $Z = Z_A(G)$ for the matrix A with

$$A_{ij} = \exp(-K_{ij}/cT).$$

Thus EVAL(A) is just the problem of computing the partition function for the system described by A.

Preliminaries

$\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ denote the real numbers, rational numbers and integers, respectively, and $\mathbb{Q}[X]$ and $\mathbb{Z}[X]$ denote the polynomial rings over $\mathbb{Z}$ and $\mathbb{Q}$ in an indeterminate X. Throughout this paper, we let $\mathbb{S}$ denote one of these five rings.

For every set S, $S^{m \times n}$ denotes the set of all $m \times n$-matrices with entries from S. For a matrix A, A_{ij} denotes the entry in row i and column j. The row rank of a matrix $A \in \mathbb{S}^{m \times n}$ is denoted by $\mathsf{rank}(A)$. A matrix $A \in \mathbb{S}^{m \times n}$ is *non-negative* (*positive*), if, for $1 \le i \le m, 1 \le j \le n$, the leading coefficient of A_{ij} is non-negative (positive, respectively).

Graphs are always undirected, unless we explicitly call them *directed graphs.* Graphs and directed graphs may have loops and multiple edges. The *degree*, *in-degree*, and *out-degree* of a vertex in a (directed) graph are defined in the obvious way and denoted by $\mathsf{deg}(v)$, $\mathsf{indeg}(v)$, $\mathsf{outdeg}(v)$, respectively.

Our model of real number computation is a standard model, as it is, for example, underlying the complexity theoretic work on linear programming (cf. [10]). We can either assume that the numbers involved in our computations are polynomial time computable or that they are given by an oracle (see [15] for a detailed description of the model). However, our results do not seem to be very model dependent. All we really need is that the basic arithmetic operations are polynomial time computable. Our situation is fairly simple because all real numbers we encounter are the entries of some matrix A, which is always considered fixed, and numbers computed from the entries of A using a polynomial number of arithmetic operations. Instances of the problem EVAL(A) are just graphs, and we do not have to worry about real numbers as inputs of our computations.

2 The tractable cases

2.1 Block decompositions

Let $B \in \mathbb{S}^{k\times\ell}$. A *submatrix* of B is a matrix obtained from B by deleting some rows and columns. For non-empty sets $I \subseteq \{1,\dots,k\}$, $J \subseteq \{1,\dots,\ell\}$, where $I = \{i_1,\dots,i_p\}$ with $i_1 < \dots < i_p$ and $J = \{j_1,\dots,j_q\}$ with $j_1 < \dots < j_q$, B_{IJ} denotes the $(p \times q)$-submatrix with $(B_{IJ})_{rs} = B_{i_r j_s}$ for $1 \leq r \leq p, 1 \leq s \leq q$. A *proper submatrix* of B is a submatrix $B' \neq B$.

Definition 2. *Let $B \in \mathbb{S}^{k\times\ell}$.*

1. *A* decomposition *of B consists of two proper submatrices B_{IJ}, $B_{I'J'}$ such that*
   (a) $I = \{1,\dots,k\} \setminus I'$,
   (b) $J = \{1,\dots,\ell\} \setminus J'$,
   (c) $B_{ij} = 0$ for all $(i,j) \in (I \times J') \cup (I' \times J)$.
   B is indecomposable *if it has no decomposition.*
2. *A* block *of B is an indecomposable submatrix B_{IJ} with at least one non-zero entry such that $B_{IJ}, B_{I^{\complement}J^{\complement}}$ with $I^{\complement} = \{1,\dots,k\} \setminus I$ and $J^{\complement} = \{1,\dots,\ell\} \setminus J$ is a decomposition of B.*

Indecomposability may be viewed as a form of "connectedness" for arbitrary matrices. For square matrices there is also a natural graph based notion of connectedness.

Let $A \in \mathbb{S}^{k\times k}$ be a square matrix. A *principal submatrix* of A is a submatrix of the form A_{II} for some $I \subseteq \big\{1,\dots,k\big\}$. Instead of A_{II} we just write A_I. The *underlying graph* of A is the (undirected) graph $G(A)$ with vertex set $\{1,\dots,k\}$ and edge set $\big\{\{i,j\} \mid 1 \leq i,j \leq n \text{ such that } A_{ij} \neq 0\big\}$. Note that we define $G(A)$ to be an undirected graph even if A is not symmetric.

Definition 3. *Let $A \in \mathbb{S}^{k\times k}$.*

1. *A is* connected *if the graph $G(A)$ is connected.*
2. *A* connected component *of A is a principal submatrix A_C, where C is the vertex set of a connected component of $G(A)$.*

Lemma 4. *A connected symmetric matrix is either indecomposable or bipartite. In the latter case, the matrix has two blocks corresponding to the two parts of the bipartition.*

There is another useful connection between indecomposability and connectedness. For a matrix $B \in \mathbb{S}^{k\times\ell}$, let

$$\mathsf{bip}(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathbb{S}^{(k+\ell)\times(k+\ell)}.$$

Note that $\mathsf{bip}(B)$ is the adjacency matrix of a weighted bipartite directed graph. The following lemma is straightforward.

Lemma 5. *Let $B \in \mathbb{S}^{k\times\ell}$ and $A = \mathsf{bip}(B)$. Then for every block B_{IJ} of B there is a connected component A_C of A such that $A_C = \mathsf{bip}(B_{IJ})$, and conversely for every connected component A_C of A there is a block B_{IJ} of B such that $A_C = \mathsf{bip}(B_{IJ})$.*

In particular, B is indecomposable if, and only if, A is connected.

2.2 Partition functions of graphs

Even though our main result is about symmetric matrices and (undirected) graphs, it is useful generalise partition functions to directed graphs, which we do in the most straightforward way. Let $A \in \mathbb{S}^{k\times k}$ be a square matrix that is not necessarily symmetric and $G = (V, E)$ a directed graph. For every $\sigma : V \to \{1, \ldots, k\}$ we let

$$\omega_A(\sigma) = \prod_{(u,v)\in E} A_{\sigma(u)\sigma(v)},$$

and we let

$$Z_A(G) = \sum_{\sigma:V\to\{1,\ldots,k\}} \omega_A(\sigma).$$

Note that if A is symmetric, $G = (V, E)$ a directed graph, and G_U the underlying undirected graph, then

$$Z_A(G_U) = Z_A(G).$$

Thus by EVAL(A) we may denote the problem of computing $Z_A(G)$ for a given directed graph, with the understanding that for symmetric A we can always consider the input graph as undirected.

Theorem 6. *Let $A \in \mathbb{R}^{k\times k}$ be a matrix such that each connected component of A has row rank* 1*. Then* EVAL(A) *is in polynomial time.*

Proof: Let $A_1, \ldots, A_\ell$ be the connected components of A. Then for every graph G with connected components $G_1, \ldots, G_m$ we have

$$Z_A(G) = \prod_{i=1}^{m} \sum_{j=1}^{\ell} Z_{A_j}(G_i).$$

Thus without loss of generality we may assume that A is connected.

Then $\mathsf{rank}(A) \leq 1$, and thus there are numbers $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{R}$ such that for $1 \leq i, j \leq k$ we have:

$$A_{ij} = a_i \cdot b_j.$$

Let $G = (V, E)$ be a directed graph and $\sigma : V \to \{1, \ldots, k\}$. Then

$$\omega_A(\sigma) = \prod_{(v,w)\in E} A_{\sigma(v)\sigma(w)} = \prod_{(v,w)\in E} a_{\sigma(v)} b_{\sigma(w)} = \prod_{v\in V} a_{\sigma(v)}^{\mathsf{outdeg}(v)} b_{\sigma(v)}^{\mathsf{indeg}(v)}.$$

Thus

$$Z_A(G) = \sum_{\sigma:V\to\{1,\ldots,k\}} \omega_A(\sigma) = \sum_{\sigma} \prod_{v\in V} a_{\sigma(v)}^{\mathsf{outdeg}(v)} b_{\sigma(v)}^{\mathsf{indeg}(v)} = \prod_{v\in V} \sum_{i=1}^{k} a_i^{\mathsf{outdeg}(v)} b_i^{\mathsf{indeg}(v)}.$$

The last term can easily be evaluated in polynomial time. □

Corollary 7. *Let $A \in \mathbb{R}^{k\times k}$ be a symmetric matrix such that each connected component of A either has row rank at most* 1 *or is bipartite and has row rank at most* 2*. Then* EVAL(A) *is in polynomial time.*

Proof: We may assume that A is connected and bipartite with $\mathsf{rank}(A) = 2$. Then there are $k_1, k_2 \geq 1$ such that $k_1 + k_2 = k$ and a matrix $B \in \mathbb{R}^{k_1\times k_2}$ with $\mathsf{rank}(B) = 1$ and

$$A = \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix}.$$

Let $G = (V, E)$ be a graph. Without loss of generality we may assume that G is connected and bipartite, say, with bipartition V_1, V_2. Let G_{12} be the directed graph obtained from G by directing all edges from V_1 to V_2, and let G_{21} be the directed graph obtained from G by directing all edges from V_2 to V_1. Recall that

$$\mathsf{bip}(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{k\times k}.$$

We have

$$Z_A(G) = Z_{\mathsf{bip}(B)}(G_{12}) + Z_{\mathsf{bip}(B)}(G_{21}).$$

Since EVAL$(\mathsf{bip}(B))$ is in polynomial time by Theorem 6, this shows that $Z_A(G)$ can be computed in polynomial time. □

3 Weighted constraint satisfaction problems

It will be convenient for us to view constraint satisfaction problems as homomorphism problems (as first suggested by Feder and Vardi [7]). Recall that a *(relational) vocabulary* τ is a set of *relation symbols*, each with a prescribed arity. A *(relational) structure* $\mathcal{A}$ of vocabulary τ consists of a universe A and, for each r-ary relation symbol $\rho \in \tau$, a relation $\rho^{\mathcal{A}} \subseteq A^r$. Observe that a constraint language Γ on a domain D may be viewed as a relational structure $\mathcal{D}$ with universe D. Feder and Vardi call this structure the *template* of the problem $\mathrm{CSP}(\mathcal{D}) = \mathrm{CSP}(\Gamma)$. An instance $(V, D, \mathcal{C})$ of $\mathrm{CSP}(\mathcal{D})$ may be viewed as a structure $\mathcal{P}$ of the same vocabulary as $\mathcal{D}$. The universe of $\mathcal{P}$ is V, and for each relation symbol ρ in the vocabulary we let $\rho^{\mathcal{P}} = \{s \mid (s, \rho^{\mathcal{D}}) \in \mathcal{C}\}$. Then a *solution* is a homomorphism σ from $\mathcal{P}$ to $\mathcal{D}$. Note that with this notation the H-colouring problem simply becomes $\mathrm{CSP}(H)$.

The objective of the counting problem $\#\mathrm{CSP}(\mathcal{D})$ is to count the solutions for a given instance $\mathcal{P}$. We shall now define a weighted version of this problem. Let $\mathcal{D}$ be a template and $\omega : D \to \mathbb{S}$ be a *weight function*. Slightly abusing notation, we also use ω to denote the weight of a solution $\sigma : V \to D$ of $\mathrm{CSP}(\mathcal{D})$ for $\mathcal{V}$; we let

$$\omega(\sigma) = \prod_{v \in V} \omega(\sigma(v)).$$

As usually, V denotes the set of variables of the CSP, that is, the universe of the structure $\mathcal{P}$. We let

$$\mathcal{Z}_{\mathcal{D},\omega}(\mathcal{P}) := \sum_{\sigma} \omega(\sigma)$$

where the sum ranges over all solutions σ of $\mathrm{CSP}(\mathcal{D})$ for $\mathcal{P}$. We denote the problem of computing $\mathcal{Z}_{\mathcal{D},\omega}$ by $\mathrm{WCSP}(\mathcal{D}, \omega)$.

3.1 CSPs with two equivalence relations

For the rest of this paper, we let α and β be binary relation symbols. Let $\mathcal{D} = (D, \alpha^{\mathcal{D}}, \beta^{\mathcal{D}})$ be a structure in which $\alpha^{\mathcal{D}}$ and $\beta^{\mathcal{D}}$ are equivalence relations on D, and let $\omega : D \to \mathbb{S}$ be a weight function. Suppose that the equivalence classes of $\alpha^{\mathcal{D}}$ are $C_1, \ldots, C_k$ and those of $\beta^{\mathcal{D}}$ are $D_1, \ldots, D_\ell$. Let $B = B(\mathcal{D}, \omega) \in \mathbb{S}^{k \times \ell}$ be defined by

$$B_{ij} = \sum_{d \in C_i \cap D_j} \omega(d).$$

The following lemma shows that the function $\mathcal{Z}_{\mathcal{D},\omega}$ only depends on the matrix B.

Lemma 8. *Let $\mathcal{D}, \mathcal{D}'$ be templates with two equivalence relations and $\omega : D \to \mathbb{S}$, $\omega' : D' \to \mathbb{S}$ weight functions. Suppose that $B(\mathcal{D}, \omega) = B(\mathcal{D}', \omega')$. Then $\mathcal{Z}_{\mathcal{D},\omega} = \mathcal{Z}_{\mathcal{D}',\omega'}$.*

The proof is straightforward. □

Conversely, for every matrix $B \in \mathbb{S}^{k \times \ell}$ we define a *canonical template* $\mathcal{D}_B$ and a *canonical weight function* ω_B as follows: The universe of $\mathcal{D}_B$ is $D_B = \{1, \ldots, k\} \times \{1, \ldots, \ell\}$, the equivalence relation $\alpha^{\mathcal{D}(B)}$ is equality on the first component, and $\beta^{\mathcal{D}(B)}$ is equality on the second component. The weight function $\omega_B : D_B \to \mathbb{S}$ is defined by $\omega_B((i,j)) = B_{ij}$. Then clearly $B = B(\mathcal{D}, \omega)$.

In the following, we write $\mathcal{Z}_B$ instead of $Z_{\mathcal{D}_B,\omega_B}$ and WCSP(B) instead of WCSP($\mathcal{D}_B, \omega_B$).

The following useful lemma is an immediate consequence of the definitions.

Lemma 9. *Let $B, B' \in \mathbb{S}^{k\times\ell}$ such that B' is obtained from B by permuting rows and/or columns. Then $\mathcal{Z}_B = \mathcal{Z}_{B'}$.*

3.2 Back and forth between CSP and H-colouring

The next lemma shows that weighted CSP for two equivalence relations are equivalent to evaluation problems for weighted bipartite graphs.

Lemma 10. *Let $B \in \mathbb{S}^{k\times\ell}$. Then the problems* WCSP(B) *and* EVAL($\mathsf{bip}(B)$) *are polynomial time equivalent.*

Proof: Let

$$A = \mathsf{bip}(B) = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathbb{S}^{(k+\ell)\times(k+\ell)}.$$

Observe that for every directed graph $G = (V, E)$ we have $Z_A(G) = 0$ unless there is a bipartition V_1, V_2 of V such that $E \subseteq V_1 \times V_2$ (that is, all edges are directed from V_1 to V_2). Assuming that there is such a bipartition V_1, V_2, we let $\mathcal{P} = \mathcal{P}(G)$ be the $\{\alpha, \beta\}$-structure with universe E in which $\alpha^{\mathcal{P}}$ is the relation

$$\{(e, e') \in E^2 \mid e \text{ and } e' \text{ have the same endpoint in } V_1\}$$

and $\beta^{\mathcal{P}}$ is the relation

$$\{(e, e') \in E^2 \mid e \text{ and } e' \text{ have the same endpoint in } V_2\}.$$

Note that for every $\sigma : V \to \{1, \ldots, k+\ell\}$ we have $\omega_A(\sigma) = 0$ unless $\sigma(V_1) \subseteq \{1, \ldots, k\}$ and $\sigma(V_2) \subseteq \{k+1, \ldots, \ell\}$.

Recall the definition of the canonical template $\mathcal{D}_B$ and the canonical weight function ω_B. For a mapping $\sigma : V \to \{1, \ldots, k+\ell\}$ with $\omega_A(\sigma) \neq 0$, let $\sigma^* : E \to D_B$ be the mapping that maps $e = (u, v)$ with $u \in V_1$, $v \in V_2$ to $(\sigma(u), \sigma(v) - k)$. Observe that σ^* is a solution of CSP($\mathcal{D}_B$) for the instance $\mathcal{P}$. Conversely, every solution of CSP($\mathcal{D}_B$) for the instance $\mathcal{P}$ is of the form σ^* for some σ with $\omega_A(\sigma) \neq 0$. Furthermore, we have

$$\omega_A(\sigma) = \prod_{(v,w)\in E} B_{\sigma(v)\sigma(w)} = \omega_B(\sigma^*).$$

Thus

$$Z_A(G) = \sum_{\sigma} \omega_A(\sigma) = \sum_{\sigma^*} \omega_B(\sigma^*) = \mathcal{Z}_B(\mathcal{P}).$$

This yields a reduction from EVAL(A) to WCSP(B).

Let $\mathcal{P} = (P, \alpha^{\mathcal{P}}, \beta^{\mathcal{P}})$ be an instance of WCSP(B). Without loss of generality we may assume that $\alpha^{\mathcal{P}}$ and $\beta^{\mathcal{P}}$ are equivalence relations. To see this, just note that every solution of WCSP(B) for $\mathcal{P}$ is also a solution for the instance $(P, \overline{\alpha}^{\mathcal{P}}, \overline{\beta}^{\mathcal{P}})$, where $\overline{\alpha}^{\mathcal{P}}$ and $\overline{\beta}^{\mathcal{P}}$ are the reflexive symmetric transitive closures of $\alpha^{\mathcal{P}}$ and $\beta^{\mathcal{P}}$, respectively. Let $C_1, \ldots, C_k$ be the equivalence classes of $\alpha^{\mathcal{P}}$ and $D_1, \ldots, D_\ell$ the equivalence classes of $\beta^{\mathcal{P}}$. Let $G = (V, E)$ be the directed graph defined as follows: The vertex set is $V = \{1, \ldots, k + \ell\}$, and for $1 \leq i \leq k$, $1 \leq j \leq \ell$ there are $|C_i \cap D_j|$ edges from i to j. It is easy to see that

$$\mathcal{Z}_B(\mathcal{P}) = Z_A(G).$$

This yields a reduction from WCSP(B) to EVAL(A). $\square$

The following corollary is an immediate consequence of the preceding lemma and Lemma 5:

Corollary 11. *Let $B \in \mathbb{R}^{k\times\ell}$ such that every block of B has row rank at most 1. Then* WCSP(B) *is in polynomial time.*

The following lemma is needed to derive the hardness part of Theorem 1 from the hardness results on weighted CSP.

Lemma 12. *Let $A \in \mathbb{S}^{k\times k}$. Then* WCSP$(A)$ *is polynomial time reducible to* EVAL(A).

Proof: Let $A' = \mathsf{bip}(A)$. By Lemma 10, it suffices to prove that EVAL(A') is reducible to EVAL(A).

Let $G = (V, E)$ be a directed graph. If G is not bipartite with all edges directed from one part to the other, then $Z_{A'}(G) = 0$. Therefore, we assume that there is a partition V_1, V_2 of V such that $E \subseteq V_1 \times V_2$. We claim that

$$Z_{A'}(G) = Z_A(G). \tag{2}$$

Note that for every $\sigma' : V \to \{1, \ldots, 2k\}$ with $\omega_{A'}(\sigma) \neq 0$ we have $\sigma(V_1) \subseteq \{1, \ldots, k\}$ and $\sigma(V_2) \subseteq \{k + 1, \ldots, 2k\}$.

For $\sigma : V \to \{1, \ldots, k\}$, let $f(\sigma) : V \to \{1, \ldots, 2k\}$ be defined by $f(\sigma)(v_1) = \sigma(v_1)$ for all $v_1 \in V_1$ and $f(\sigma)(v_2) = \sigma(v_2) + k$ for all $v_2 \in V_2$. Then $\omega_A(\sigma) = \omega_{A'}(\sigma')$. Moreover, f is one-to-one, and for every $\sigma' : V \to \{1, \ldots, 2k\}$ with $\omega_{A'}(\sigma') \neq 0$ there exists $\sigma : V \to \{1, \ldots, k\}$ such that $\sigma' = f(\sigma)$. This proves (2). $\square$

4 The main hardness theorem

Theorem 13. *Let $B \in \mathbb{S}^{k\times\ell}$ be non-negative such that at least one block of B has row rank at least 2. Then* WCSP(B) *is #P-complete.*

Due to space limitations we have to defer the proof of Theorem 13 to Appendix A. A brief outline of the proof will be given in the next subsection.

Note that, combined with Corollary 11, Theorem 13 yields a complete complexity theoretic classification of problems WCSP(B) for non-negative matrices $B \in \mathbb{R}^{k\times\ell}$.

Furthermore Theorem 1 follows easily from Theorem 6, Corollary 7 (for the tractability results) and Lemma 12, Theorem 13 (for the hardness results). Note that there is no contradiction between Theorem 1(2) and Theorem 13, because if the graph $G(A)$ of a symmetric matrix A is bipartite then A is not indecomposable.

4.1 Outline of the proof

In this subsection we sketch the proof of Theorem 13 (details are given in Appendix A). Let $B \in \mathbb{S}^{k\times\ell}$ be a non-negative matrix such that at least one block of B has row rank at least 2.

Step 1: From numbers to polynomials. In this first step of the proof we show that we can assume that all positive (i.e. non-zero) entries of B are powers of some indeterminate X. More precisely, we prove that there is a matrix B^* whose positive entries are powers of X such that B^* also has a block of row rank at least 2 and WCSP(B^*) is polynomial time reducible to WCSP(B). The construction is based on a lemma, which essentially goes back to [5], stating that the problem WCSP(B) is equivalent to the problem counting all solutions of a given weight. For simplicity, let us assume here that all entries of B are non-negative integers; additional tricks are required for real matrices. We can use the lemma to filter out powers of a particular prime p from all entries of B. This way we obtain a matrix B' whose positive entries are powers of a prime p. Using a technique which corresponds to "thickening" in the graph context (cf. [13, 5]), we can replace the entries of this matrix by arbitrary powers, and by interpolation we can then replace p by the indeterminate X. This gives us the desired matrix B^*.

From now on, we assume that all positive entries of B are powers of X.

Step 2: Further preparations. Now we employ two results due to [3]. A consequence of the first is a lemma stating that if there are rows i, i' and columns j, j' such that the entries $B_{ij}, B_{ij'}, B_{i'j}$ are non-zero and $B_{i'j'}$ is zero, then WCSP(B) is #P-complete. This implies that we may assume that in every block of B all entries are positive.

The second result due to [3] is that the more general problem where we only count solutions in which the values of some variables are fixed (i.e., solutions extending some fixed partial solutions) is reducible to WCSP(B). This implies that we may assume that B is indecomposable.

Together, the two assumptions imply that all entries of B are positive.

Another simple reduction shows that either (a) each row of B and each column of B contains a 1, or (b) B has principal submatrix in which all entries are 1, and no 1s appear outside of this principal submatrix. From here we branch into cases (a) and (b).

Step 3(a): Separate 1s. We assume that all entries of B are positive and each row of B and each column of B contains a 1. Since we may permute rows and columns of B, we may assume that all diagonal entries of B are 1. It is not hard to see that then we can reduce the problem EVAL(A) for a symmetric non-singular 2×2-matrices to WCSP(B). For such matrices A the problem EVAL(A) is #P-hard. We believe that this is known (implicitly it is underlying [8]), but in absence of a clear reference we give a proof which boils down to a reduction from the problem of counting MAXCUTs of a graph.

Step 3(b): All 1s together. This part of the proof is the hardest, and it is difficult to describe on a high level. We assume that all entries of B are positive and that a principal submatrix in the upper left corner of B contains all 1s. We define a sequence $B^{[k]}$, for $k \geq 1$, of matrices that are obtained from B by some construction on the instances that is remotely similar to "stretching" and "thickening" (cf. [13, 5]), but more complicated. We show that WCSP($B^{[k]}$) is reducible to WCSP(B) for all k.

The entries of the $B^{[k]}$ are polynomials with integer coefficients (no longer just powers of X as the entries of B). Employing a little bit of complex analysis, we prove that for some k, $B^{[k]}_{11}$ has an irreducible factor $p(X)$ such that the multiplicity of $p(X)$ in $B^{[k]}_{11}$ is higher than in all other entries in the first row and column, and the multiplicity in the corresponding diagonal entries is also sufficiently high. Using similar tricks as in Step 1, we can filter out the powers of this irreducible polynomial $p(X)$. We obtain a matrix whose weighted CSP is #P-complete by the results of Steps 2 and 3(a).

5 Conclusions

We give a complete complexity theoretic classification for the problem of evaluating the partition function of a symmetric non-negative matrix A, which may be viewed as the adjacency matrix of an undirected weighted graph H. Our proofs explore a correspondence between this evaluation problem and weighted constraint satisfaction problems for constraint languages with two equivalence relations.

Peculiarly, our proof does not go through for matrices with negative entries. Indeed, we do not know whether the evaluation problem for the matrix

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

is #P-complete. (Observe that the evaluation problem for this matrix is equivalent to the problem of counting induced subgraphs with an even number of edges.)

The more important open problem is to obtain a classification result for the evaluation problem for non-symmetric matrices, corresponding to directed graphs. We believe that with our results such a classification may now be within reach, in particular because our main hardness result goes through for directed graphs. The ultimate goal of this line of research is a classification of counting and weighted CSP for arbitrary constraint languages. Towards a solution of this problem, one may try to reduce the weighted CSP to evaluation problems for directed graphs. It is interesting to note that the known reduction between the corresponding decision problems does not give a reduction between the counting problems we are interested in here.

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A Proof of Theorem 13

We recall some definitions and notation. Along with the homomorphism problem we will use the original form of the constraint satisfaction problem.
WEIGHTED CSP (WCSP(Γ,ω)). Let D be a finite set, $\omega : D \to \mathbb{S}$ a weight function, where $\mathbb{S}$ is a semiring, and Γ a constraint language on D.

Instance: A CSP instance $\mathcal{P} = (V; D; \mathcal{C})$, where all the constraint relations belong to Γ;
Objective: The number

$$\mathcal{Z}_{\Gamma,\omega}(\mathcal{P}) = \sum_{\varphi} \prod_{v \in V} \omega(\varphi(v))$$

where φ runs over the solutions to $\mathcal{P}$.

For the weighted constraint satisfaction problem WCSP(B), D, α, β will denote the domain and the relations of the canonical template. In several cases we distinguish integer (rational) matrices and polynomial matrices. Then in the latter case we use the notation $B(X)$. For the counting constraint satisfaction problem #CSP(α, β), where α, β are equivalence relations $C_1, \ldots, C_m$, $D_1, \ldots, D_n$ will denote the classes of α and β respectively, $B(\alpha, \beta)$ will denote the $m \times n$-matrix B with entries $B_{ij} = |C_i \cap D_j|$. Conversely, given a non-negative $m \times n$-matrix B, we fix a domain D'_B with $|D'_B| = \sum_{i=1,j=1}^{m,n} B_{ij}$ and equivalence relations α_B, β_B with $B = B(\alpha, \beta)$.

Lemma 14. *Let $B \in \mathbb{Z}^{m \times n}$ be a non-negative matrix. Then* WCSP(B) *is polynomial time equivalent to* #CSP($\{\alpha_B, \beta_B\}$).

Proof is straightforward.

In this section notation $\mathsf{deg}(f)$ denotes the *degree* of a polynomial $f(X)$.

A.1 From numbers to polynomials

We need more general versions of some results of [5]. First of all we show that a transformation similar to 'thickening' exists in the case of weighted CSP.

Let $B^{(\ell)}(X)$ denote the matrix whose entries are $(B_{ij}(X))^{\ell}$.

Lemma 15. *For any $\ell \geq 1$,* WCSP($B^{(\ell)}(X)$) *is polynomial time reducible to* WCSP($B(X)$).

Proof: Let α, β be equivalence relations on D corresponding to the problems WCSP(B), WCSP($B^{(\ell)}$). Take an instance $\mathcal{P} = (V; D; \mathcal{C})$ of WCSP($B^{(\ell)}$). Then

- every $v \in V$ replace with $v_1, \ldots, v_l$ and denote the obtained set by V';
- every constraint $\langle (u, v), \alpha \rangle$ [$\langle (u, v), \beta \rangle$] replace with $\langle (u_1, v_1), \alpha \rangle$ [$\langle (u_1, v_1), \beta \rangle$];
- for any $v \in V$ and any $1 \leq i, j \leq l$, include the constraints $\langle (v_i, v_j); \alpha \rangle$, $\langle (v_i, v_j); \beta \rangle$;
- denote the obtained set of constraints by $\mathcal{C}'$ and the problem $(V'; D; \mathcal{C}')$ by $\mathcal{P}'$.

Clearly, $\mathcal{P}' \in \mathrm{WCSP}(B)$ and $\mathcal{Z}_B(\mathcal{P}) = \mathcal{Z}_{B^{(\ell)}}(\mathcal{P}')$. $\square$

A proof of the following lemma is straightforward.

Lemma 16. *Let $B(X) \in \mathbb{Q}[X]^{m\times n}$ be a matrix and a an integer. Then* $\mathrm{WCSP}(B(a))$*, where $B(a)$ denotes the numerical matrix obtained from $B(X)$ by substituting a instead of X, is polynomial time reducible to* $\mathrm{WCSP}(B(X))$*. Moreover, a can be chosen such that* $\mathsf{rank}(B(a)) = \mathsf{rank}(B(X))$.

For a matrix $B \in \mathbb{S}^{m\times n}$ and an instance $\mathcal{P} = (V; D; \mathcal{C})$ of $\mathrm{WCSP}(B)$, we let

$$P_B(\mathcal{P}) = \Big\{ \prod_{\substack{1\le i\le m\\ 1\le j\le n}} (B)_{ij}^{m_{ij}} \mid 0 \le m_{ij} \le |V| \text{ for } 1\le i\le m, 1\le j\le n \text{ such that } \textstyle\sum_{\substack{1\le i\le m\\ 1\le j\le n}} m_{ij} = |V| \Big\}.$$

Then

$$\{\omega_B(\sigma) \mid \ \sigma \text{ is a solution of } \mathcal{P}\} \subseteq P_B(\mathcal{P}).$$

Note that for fixed B the size of $P_B(\mathcal{P})$ is polynomial in size of $\mathcal{P}$ and that $P_B(\mathcal{P})$ can be computed in polynomial time. Also note that

$$\mathcal{Z}_B(\mathcal{P}) = \sum_{w\in P_B(\mathcal{P})} w \cdot N_B(\mathcal{P}, w), \tag{3}$$

where $N_B(\mathcal{P}, w)$ denotes the number of solutions σ of $\mathcal{P}$ such that $\omega_B(\sigma) = w$. $\mathrm{COUNT}(B)$ denotes the following problem:

Input: $\mathrm{WCSP}(B)$-instance $\mathcal{P}$, $w \in \mathbb{S}$.
Objective: Compute $N_B(\mathcal{P}, w)$.

The next proposition is an analogue of Lemma 3.3 from [5].

Lemma 17. *Let $B \in \mathbb{S}^{m\times n}$. Then the problems* $\mathrm{WCSP}(B)$ *and* $\mathrm{COUNT}(B)$ *are polynomial time equivalent.*

Proof: The following proof mainly follows the proof of Lemma 3.3 from [5]. As is noticed above the set $P_B(\mathcal{P})$ can be constructed in polynomial time. Suppose that $w_1, \dots, w_t$ are the non-zero elements of $P_B(\mathcal{P})$. If $w \notin P_B(\mathcal{P})$ then $N_B(\mathcal{P}, w) = 0$. Assume now that $w = w_j \in P_B(\mathcal{P})$. For $1 \le \ell \le t$, consider the number $\mathcal{Z}_{B^{(\ell)}}(\mathcal{P})$. We have

$$\mathcal{Z}_{B^{(\ell)}}(\mathcal{P}) = \sum_{\sigma \text{ a solution to } \mathcal{P}} \prod_{v\in V} \omega_B^{\ell}(v) = \sum_{w\in P_B(\mathcal{P})} w^{\ell} N_B(\mathcal{P}, w).$$

If $\mathbb{S}$ is a numerical ring then we complete the proof as in [5]. If $\mathbb{S}$ is a polynomial ring then denote the value $\mathcal{Z}_{B^{(\ell)}}(\mathcal{P})$ by $f_\ell(X)$ and notice the equations above can be represented in the matrix form

$$\begin{pmatrix} f_1(X)\\ f_2(X)\\ \vdots\\ f_t(X)\end{pmatrix} = \begin{pmatrix} w_1(X) & w_2(X) & \cdots & w_t(X)\\ w_1^2(X) & w_2^2(X) & \cdots & w_t^2(X)\\ \vdots & \vdots & & \vdots\\ w_1^t(X) & w_2^t(X) & \cdots & w_t^t(X)\end{pmatrix} \cdot \begin{pmatrix} N_B(\mathcal{P}, w_1)\\ N_B(\mathcal{P}, w_1)\\ \vdots\\ N_B(\mathcal{P}, w_1)\end{pmatrix}. \tag{4}$$

On the one hand, the determinant of the square matrix is Vandermonde. Since all $w_1, \dots, w_t$ are distinct and non-zero, it is also non-zero. On the other hand, this determinant is a non-zero polynomial; let us denote it by $d(X)$. Therefore, there is an integer

$$a \leq t \cdot \max_{1 \leq j \leq t} \mathsf{deg}(w_j) + 1$$

such that $d(a) \neq 0$. Substituting a instead of X in (4), we obtain a numerical matrix equation. $\square$

Lemma 18. *(1) Let $B \in \mathbb{Q}^{k \times \ell}$ be a non-negative matrix with at least one block of rank at least 2. Then there is a non-negative matrix $C \in \mathbb{Z}^{k \times \ell}$ satisfying the same condition and such that* WCSP(C) *is polynomial time reducible to* WCSP(B).

(2) Let $B(X) \in \mathbb{Q}[X]^{k \times \ell}$ be a non-negative matrix with at least one block of rank at least 2. Then there is a non-negative matrix $C(X) \in \mathbb{Z}[X]^{k \times \ell}$ satisfying the same condition and such that WCSP$(C(X))$ *is polynomial time reducible to* WCSP$(B(X))$.

Proof: Let N be the least common denominator of entries of B [of coefficients of entries of $B(X)$]. We set C to be the matrix with entries $C_{ij} = N \cdot B_{ij}$. Since for any WCSP(B)-instance $\mathcal{P}$

$$\mathcal{Z}_C(\mathcal{P}) = N^{|V|} \mathcal{Z}_B(\mathcal{P}),$$

the problems WCSP(B) and WCSP(C) are polynomial time equivalent. $\square$

Lemma 19. *(1) Let $B \in \mathbb{Z}^{m \times n}$ be a non-negative matrix, and p a prime number. Let C be the matrix obtained from B by replacing all entries divisible by p with* 0. *Then there is a polynomial time reduction from* WCSP(C) *to* WCSP(B).

(2) Let $B(X) \in \mathbb{Z}[X]^{m \times n}$ be a matrix, and $p(X)$ an irreducible polynomial. Let $C(X)$ be the matrix obtained from $B(X)$ by replacing all entries divisible by p with 0. *Then there is a polynomial time reduction from* WCSP$(C(X))$ *to* WCSP$(B(X))$.

Proof: We first prove (2) by reducing WCSP$(C(X))$ to COUNT$(B(X))$; this is sufficient by Lemma 17. Given an instance $\mathcal{P} = (V; D; \mathcal{C})$ of WCSP$(C(X))$, we first compute the set $P_C(\mathcal{P}) = (P_B(\mathcal{P}) - \{w \mid w \text{ divisible by } p\}) \cup \{0\}$. Then for each $w \in P_C(\mathcal{P}) - \{0\}$, we compute then number $N_B(\mathcal{P}, w)$ using an oracle COUNT(B). Then we compute

$$\mathcal{Z}_C(\mathcal{P}) = \sum_{w \in P_C(\mathcal{P}) - \{0\}} N_B(\mathcal{P}, w) \cdot w.$$

For (1) the proof is quite similar. $\square$

Let p be a prime number [an irreducible polynomial]. For an integer $a \in \mathbb{Z}$ [a polynomial $a \in \mathbb{Z}[X]$] we let

$$a\big|_p = \begin{cases} p^{\max\{k \mid k \geq 0,\ p^k \text{ divides } a\}}, & \text{if } a \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For a matrix $B \in \mathbb{Z}^{m\times n}$ [a matrix $B \in \mathbb{Z}[X]^{m\times n}$] we let $B\big|_p$ be the matrix entries $(B\big|_p)_{ij} = (B_{ij})\big|_p$.

Lemma 20. *(1) Let $B \in \mathbb{Z}^{m\times n}$ be a non-negative matrix, and p a prime number. Then* $\mathrm{WCSP}(B\big|_p)$ *is polynomial time reducible to* $\mathrm{WCSP}(B)$.

(2) Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a matrix, and $p(X)$ an irreducible polynomial. Then $\mathrm{WCSP}(B(X)\big|_p)$ *is polynomial time reducible to* $\mathrm{WCSP}(B(X))$.

Proof: We prove part (2) of the lemma. For (1) the proof is quite similar. Let ω be the weight function corresponding B and ω' the one corresponding to $B\big|_p$. Let $\mathcal{P}$ be an instance of $\mathrm{WCSP}(B(X)|_p)$. Note that for every solution σ of $\mathcal{P}$ we have $\omega'(\sigma) = (\omega(\sigma))\big|_p$. Then by (3),

$$\mathcal{Z}_{B\big|_p}(\mathcal{P}) = \sum_{w\in P_B(\mathcal{P})} w\big|_p \cdot N_B(\mathcal{P}, w).$$

Thus $\mathrm{WCSP}(B\big|_p)$ is polynomial time reducible to $\mathrm{COUNT}(B)$ and hence, by Lemma 17, to $\mathrm{WCSP}(B)$. $\Box$

Lemma 21. *(1) Let $B \in \mathbb{Z}^{m\times n}$ be a non-negative matrix which has a block of rank at least 2. Then there is some prime number p such that there is a block of $B|_p$ of rank at least 2.*

(2) Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix which has a block of rank at least 2. Then there is some irreducible polynomial $p(X)$ such that there is a block of $B|_p$ of rank at least 2.

Proof: As usual, we prove only (2), because for (1) the proof is quite similar. Suppose that for all irreducible p every block of the matrix $B\big|_p$ has rank at most 1. We shall prove that any two rows from the same block of B are linearly dependent, which is impossible.

Let $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ be two rows from a block of rank 2 of B. Then we have

$$a_i = \prod_p a_i\big|_p \quad \text{and} \quad b_i = \prod_p b_i\big|_p$$

for all i, where the (finite) products are taken over all irreducible p dividing an entry of B. Since for every irreducible p the rank of $B\big|_p$ is 1, there is λ_p, μ_p such that $\lambda_p a_i\big|_p = \mu_p b_i\big|_p$ for $1 \le i \le n$. Let

$$\Lambda = \prod_p \lambda_p \quad \text{and} \quad M = \prod_p \mu_p.$$

Then

$$\Lambda a_i = \Lambda \cdot \prod_p a_i\big|_p = \prod_p \lambda_p a_i\big|_p = \prod_p \mu_p b_i\big|_p = M \cdot \prod_p b_i\big|_p = M b_i$$

for $1 \le i \le n$. This shows that indeed $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent. $\Box$

Lemma 22. *Let $p(X)$ an irreducible polynomial, $B \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix whose entries are either 0 or powers of $p(X)$ and $a \in \mathbb{Z}$ such that $B(a)$ is non-negative and $\mathsf{rank}(B(a)) = \mathsf{rank}(B(X))$. Then the problem* WCSP$(B(X))$ *is polynomial time reducible to* WCSP$(B(a))$.

Proof: Let us denote $Y = p(X)$, let $B'(Y)$ be such that $B'(Y) = B(X)$ and $b = p(a)$. Then $B'(Y)$ is a matrix whose entries are either 0 or powers of Y. Let $\ell_{\max}$ be maximum such that $Y^{\ell_{\max}}$ is an entry of $B'(Y)$. For every instance $\mathcal{P}$ of WCSP$(B(X))$ with, say, m variables, $Z_{B'(Y)}(\mathcal{P})$ is a polynomial in Y of degree at most $m \cdot \ell_{\max}$. By Lemma 15, we can evaluate this polynomial for $Y = b^i$ for all $i \geq 0$. Thus we can interpolate and compute its coefficients. $\square$

Corollary 23. *Let a be an integer, and B be an integer $m \times n$-matrix whose entries are either 0 or powers of a. Let $B(X)$ be the matrix obtained from B by replacing powers of a with corresponding powers of indeterminate X, that is, by letting*

$$(B(X))_{ij} = \begin{cases} X^l, \text{ if } B_{ij} = a^l \text{ for some } l \geq 0, \\ 0, \quad \text{if } B_{ij} = 0. \end{cases}$$

Then the problem WCSP$(B(X))$ *is polynomial time reducible to* WCSP(B).

The following corollary summarizes the results of this subsection:

Corollary 24. *Let $\mathbb{S} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{Z}[X], \mathbb{Q}[X]\}$, and let $B \in \mathbb{S}^{k\times\ell}$ be a non-negative matrix that has a block of row rank at least 2.*

Then there exists a matrix $B^* \in \mathbb{Z}[X]^{k\times\ell}$ whose non-zero entries are powers of X such that B^* has a block of row rank at least 2, and WCSP(B^*) *is reducible to* WCSP(B).

We now prove the analogue of Corollary 24 for real matrices. The proof is very similar to the proof for the integers, the only difference being that we have to replace the primes involved there by suitable real numbers.

Lemma 25. *Let $B \in \mathbb{R}^{k\times\ell}$ be a non-negative matrix that has a block of row rank at least 2.*

Then there exists a matrix $B^* \in \mathbb{Z}[X]^{k\times\ell}$ whose non-zero entries are powers of X such that B^* has a block of row rank at least 2, and WCSP(B^*) *is reducible to* WCSP(B).

Proof: Let $B \in \mathbb{R}^{k\times\ell}$ be a non-negative matrix. Recall the definition of the set $P_B(\mathcal{P})$ for an instance $\mathcal{P}$, the problem COUNT(B), and Lemma 17 stating that COUNT(B) and WCSP(B) are polynomial time equivalent. (Lemma 17 also holds for real matrices without any changes.)

Let $a_1, \ldots, a_m$ be the positive entries of B. For $1 \leq i \leq m$, let $r_i = \ln a_i$. We view $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and are interested in the subspace L_B generated by $r_1, \ldots, r_m$. Let $1 \leq i_1 < \ldots < i_n \leq m$ such that $r_{i_1}, \ldots, r_{i_n}$ form a basis of L_B. For $1 \leq j \leq n$, let $b_j = a_{i_j}$. We call $\{b_1, \ldots, b_n\}$ an *l-basis* for B. Note that every positive entry of B has a unique representation

$b_1^{q_1}\dots b_n^{q_n}$, where $q_1,\dots,q_n \in \mathbb{Q}$. Since for every positive $s \in \mathbb{Z}$ the problem WCSP($B^{(s)}$) is reducible to WCSP(B), we may actually assume without loss of generality that every positive entry of B has a unique representation $b_1^{\ell_1}\dots b_n^{\ell_n}$, where $\ell_1,\dots,\ell_n \in \mathbb{Z}$. Even though then the elements of our l-basis are no longer entries of (the new) matrix B, we may still assume that we have an oracle to the elements of the l-basis and that we know the representation of all positive entries of B in terms of this l-basis. We can make this last assumption because B is fixed and not part of the input. Observe that for every instance $\mathcal{P}$ and $w \in P_B(\mathcal{P})$, $w \neq 0$, we have $\ln w \in L_B$. Thus w has a unique representation $b_1^{m_1}\dots b_n^{m_n}$ with $m_1,\dots,m_n \in \mathbb{Z}$.

For $1 \le s \le n$ and $a = b_1^{\ell_1}\dots b_n^{\ell_n}$ we let $a\|_{b_s} = b_s^{\ell_s}$, and we let $A\|_{b_s}$ be the matrix with $(A\|_{b_s})_{ij} = (A_{ij})\|_{b_s}$ if $A_{ij} > 0$ and $(A\|_{b_s})_{ij} = 0$ if $A_{ij} = 0$.

Analogously to Lemma 20 we can prove that for $1 \le s \le n$ the problem WCSP($B\|_{b_s}$) is polynomial time reducible to WCSP(B). Here we use the fact that for every instance $\mathcal{P}$ every element of the set $P_B(\mathcal{P})$ has a unique representation in terms of our l-basis.

Analogously to Lemma 21 we can prove that if B has a block of row rank at least 2 then for some $b \in \{b_1,\dots,b_n\}$ the matrix $B\|_b$ has a block of row rank at least 2.

To complete the proof of Lemma 25, assume that B has a block of row rank at least 2 and let $b \in \{b_1,\dots,b_n\}$ such that $B\|_b$ also satisfies this condition. Let X be an indeterminate and B^* the matrix obtained from $B\|_b$ by replacing each entry b^ℓ by X^ℓ. Let $\mathcal{P}$ be an instance with m variables. We want to compute the polynomial $q(X) = \mathcal{Z}_{B^*}(\mathcal{P})$, which is a polynomial of degree at most m in X. Observing that for $0 \le r \le m$ we have

$$\mathcal{Z}_{(B\|_b)^{(r)}}(\mathcal{P}) = q(b^r),$$

we can compute the coefficients of q by Lemma 15 and interpolation. This completes the proof of Lemma 25. □

A.2 Basic reductions

Recall that equivalence relations α, β are said to be *permutable* if $\alpha \circ \beta = \beta \circ \alpha$.

Lemma 26 ([3]). *If equivalence relations α, β are not permutable, then the problem #CSP(α, β) is #P-complete.*

As is easily seen, α, β are not permutable if and only if there are $1 \le i, j \le m$, $1 \le k, l \le n$ such that $B(\alpha,\beta)_{ik}, B(\alpha,\beta)_{il}, B(\alpha,\beta)_{jk} \neq 0$, but $B(\alpha,\beta)_{jl} = 0$. This property is also sufficient for #P-completeness of the weighted problem.

Lemma 27. *If, for some i, j, k, l, $B_{ik}(X), B_{il}(X), B_{jk}(X) \neq 0$ while $B_{jl}(X) = 0$ then WCSP($B(X)$) is #P-complete.*

Proof: Take a integer a such that $B(a)$ is non-negative, $B_{ik}(a), B_{il}(a), B_{jk}(a) \neq 0$, and let $B' = B(a)$. By Lemmas 16,14, the problems WCSP$(B(a))$, WCSP(B') and #CSP$(\alpha_{B'}, \beta_{B'})$ are polynomial time reducible to WCSP$(B(X))$. Since $B'_{ik}, B'_{il}, B'_{jk} \neq 0$ and $B'_{jl} = 0$, by Lemma 26, #CSP$(\alpha_{B'}, \beta_{B'})$ is #P-complete. $\Box$

We shall frequently use weighted constraint problems with larger constraint languages than that corresponding to the canonical template. Let $\mathcal{D}_B$ be the canonical template for a matrix $B \in \mathbb{Z}[X]^{m\times n}$. For $d \in D$, we use κ_d to denote the unary one-element relation $\{(d)\}$ and CON(D) to denote the set $\{\kappa_d \mid d \in D\}$. If $m = n$ then we also use the relation $\theta = \bigcup_{1\le i\le m} C_i \cap D_i$ consisting of all 'diagonal' elements of the canonical template. We shall use WCSP$^+(B)$ to denote the problem WCSP$(\{\alpha, \beta\} \cup \text{CON}(D) \cup \{\theta\})$ and WCSP$^*(B)$ to denote the problem WCSP$(\{\alpha, \beta\} \cup \text{CON}(D))$.

As is easily seen, for an instance $\mathcal{P} = (V; D; \mathcal{C})$ of the problem WCSP$^+(B)$ or the problem #CSP$(\{\alpha, \beta\} \cup \text{CON}(D) \cup \{\theta\})$, specifying a set of constraints of the form $\langle (v), \kappa_d\rangle$ is equivalent to specifying a partial solution $\varphi_{\mathcal{P}}$ of $\mathcal{P}$, where $\varphi_{\mathcal{P}}(v)$ is defined to be d on every variable occurring in a constraint $\langle (v), \kappa_d\rangle$. Analogously, the set of constraints of the form $\langle (v), \theta\rangle$ can be represented by just a subset $V_{\mathcal{P}} \subseteq V$ of all variables to which such a constraint applies. To simplify the notation we will use $\mathcal{Z}_B^{\varphi_{\mathcal{P}}, V_{\mathcal{P}}}(\mathcal{P})$ to denote the objective value for an instance of WCSP$^+(B)$, or $\mathcal{Z}_B^{\varphi_{\mathcal{P}}}(\mathcal{P})$ or $\mathcal{Z}_B^{V_{\mathcal{P}}}(\mathcal{P})$ if $\mathcal{P}$ is an instance of WCSP$^*(B)$ or WCSP$(\{\alpha, \beta\} \cup \{\theta\})$ respectively.

Lemma 28. *Let $p(X)$ be an irreducible polynomial and $B(X) \in \mathbb{Z}[X]^{m\times n}$ a non-negative matrix. Then $WCSP^*(B)$ is polynomial time reducible to* WCSP(B).

Proof: By Lemma 16, there is an integer non-negative $m \times n$-matrix C such that WCSP$(B(X))$ and WCSP(C) are polynomial time equivalent. Furthermore, by Lemma 14, WCSP(C) is reducible to #CSP$(\{\alpha_C, \beta_C\})$. By Theorem 12 of [3], the problem #CSP$(\{\alpha_C, \beta_C\} \cup \text{CON}(D'_C))$ is polynomial time reducible to #CSP(α_B, β_B).

We show that WCSP$^*(C)$ is polynomial time reducible to #CSP$(\{\alpha_B, \beta_B\} \cup \text{CON}(D'_C)$. Let $\mathcal{P} = (V; D; \mathcal{C})$ be an instance of WCSP$^*(C)$. For every element $d \in D$, say, from α-class C_i and β-class D_j, we denote the $\alpha_B \cap \beta_B$-class $C_i^B \cap D_j^B$ by $d^{\alpha_B\cap\beta_B}$. If κ_d occurs amongst the constraint relations of $\mathcal{P}$, then we may assume that this class is non-empty, because otherwise $Z_C^{\varphi_{\mathcal{P}}}(\mathcal{P}) = 0$. Fix an element d^* from this class. Then we define an instance $\mathcal{P}' = (V \cup V'; D'_C; \mathcal{C}')$ from #CSP$(\{\alpha_B, \alpha_B\} \cup \text{CON}(D'_B))$ as follows.

- $V' = \{v' \mid v \in V_{\mathcal{P}}\}$;
- for every constraint of the form $\langle (u_1, u_2), \alpha\rangle, \langle (v_1, v_2), \beta\rangle \in \mathcal{C}$, include the constraints $\langle (u_1, u_2), \alpha_B\rangle$, $\langle (v_1, v_2), \beta_B\rangle$ into $\mathcal{C}'$;
- for every $\langle (v), \kappa_d\rangle \in \mathcal{C}$, include $\langle (v, v'), \alpha_B\rangle$, $\langle (v, v'), \beta_B\rangle$ and $\langle (v'), d^*\rangle$ into $\mathcal{C}'$.

As is easily seen, for any solution $\sigma : V \to D'_C$ to $\mathcal{P}'$, the following conditions hold:

– $\sigma(v') = (\varphi(v))^*$ for every $v \in V_{\mathcal{P}}$;
– $(\sigma(v), (\varphi(v))^*) \in \alpha_B \cap \beta_B$ for every $v \in V_{\mathcal{P}}$;
– the restriction of σ onto V is a solution to $\mathcal{P}$.

The last two conditions imply, as in Lemma 14, that $Z_C^{\varphi_{\mathcal{P}}}(\mathcal{P})$ equals the number of solutions to $\mathcal{P}'$.

To finish the proof we just note that $Z_{B(X)}^{\varphi_{\mathcal{P}}}$ can be evaluated using $C^{(i)}$ in the same way as in Lemma 22. □

Lemma 29. *Let $B \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix. Then $C = BB^\top$ is a symmetric non-negative matrix,* $\mathsf{rank}(C) = \mathsf{rank}(B)$*, if B has a block of rank at least 2 then C also has such a block, and* $\mathrm{WCSP}^+(C)$ *is polynomial time reducible to* $\mathrm{WCSP}^*(B)$.

Proof: As usual, we denote the classes of α by $C_1, \dots, C_m$ and the classes of β by $D_1, \dots, D_n$. Let $\mathcal{D}_C = (D'; \alpha', \beta')$ be the canonical template for C, where D' is m^2-element set. Notice that elements of D' correspond to pairs of rows of B; we therefore denote them by pairs (i, j). Since C is Gram matrix of the collection of rows of B, its rank equals $\mathsf{rank}(B)$. Notice that the same holds for every block of B. Therefore, for any block of B, the matrix C has a block of the same rank.

Take an instance $\mathcal{P}' = (V'; D'; \mathcal{C}') \in \mathrm{WCSP}^+(C)$. We consider the instance $\mathcal{P} = (V; D; \mathcal{C})$ from $\mathrm{WCSP}(\{\alpha, \beta\} \cup \mathrm{CON}(D))$ defined as follows:

– $V = \{v^1, v^2 \mid v \in V'\} \cup \{v^3, v^4 \mid \langle v, \kappa_{(i,j)}\rangle \in \mathcal{C}'\}$;
– for every constraint $\langle (v, w), \alpha'\rangle \in \mathcal{C}'$, we include $\langle (v^1, w^1), \alpha\rangle$ into $\mathcal{C}$, and, for every constraint $\langle (v, w), \beta'\rangle \in \mathcal{C}'$, we include $\langle (v^2, w^2), \alpha\rangle$ into $\mathcal{C}$;
– for every $v \in V'$ we include $\langle (v^1, v^2), \beta\rangle \in \mathcal{C}$;
– for every constraint $\langle (v), \kappa_{(i,j)}\rangle \in \mathcal{C}'$, we include into $\mathcal{C}$ the constraints $\langle (v^3), \kappa_{d_i}\}\rangle$, $\langle (v^4), \kappa_{d_j}\}\rangle$, $\langle (v^1, v^3), \alpha\rangle$, $\langle (v^2, v^4), \alpha\rangle$, where $d_i \in C_i$, $d_j \in C_j$ arbitrary;
– for every constraint $\langle (v), \theta\rangle \in \mathcal{C}'$, we include $\langle (v^1, v^2), \alpha\rangle$ into $\mathcal{C}$.

The domain of $\varphi_{\mathcal{P}}$ is the set of all variables of the form v^3, v^4.

For every solution σ to $\mathcal{P}'$, there are $m^{|V|}$ solutions ψ to $\mathcal{P}$ such that, for any $v \in V$, if $\sigma(v) = (i, j)$ then $\psi(v^1) \in C_i$, $\psi(v^2) \in C_j$. (Notice that $\psi(v^3), \psi(v^4)$ are the same for any solution ψ.) Let us denote the set of such solutions by Ψ_σ. Vice versa, if ψ is a solution to $\mathcal{P}$, let σ be the mapping from V' to D' defined by $\sigma(v) = (i, j)$, where $\psi(v^1) \in C_i, \psi(v^2) \in C_j$. Then the following conditions hold.

– For any $\langle (u, v), \alpha'\rangle \in \mathcal{C}$, the constraint $\langle (u^1, v^1), \alpha\rangle$ implies that $\psi(u^1), \psi(v^1)$ are in the same α-class, say, C_i. Hence, $(\sigma(u), \sigma(v)) = ((i, j), (i, k)) \in \alpha'$ for certain $1 \le j, k \le m$.
– For any $\langle (u, v), \beta'\rangle \in \mathcal{C}$, the constraint $\langle (u^2, v^2), \alpha\rangle$ implies that $\psi(u^2), \psi(v^2)$ are in the same α-class, say, C_j. Therefore, $(\sigma(u), \sigma(v)) = ((i, j), (k, j)) \in \beta'$ for certain $1 \le i, k \le m$.

– For any $v \in V''$, the constraint $\langle (v^1, v^2), \alpha \rangle$ implies that $\psi(v^1) = \psi(v^2) \in C_i$ for a certain $1 \leq i \leq m$. Hence $\sigma(v) = (i,i) \in \theta$.
– For any v such that $\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'$, $\psi(v^3) \in C_i$, $\psi(v^4) \in C_j$; therefore, $\psi(v^1) \in C_i$, $\psi(v^2) \in C_j$; which means $\sigma(v) = (i,j) \in \kappa_{(i,j)}$.

Thus σ is a solution to $\mathcal{P}'$.

Finally, we have

$$\begin{aligned}
\mathcal{Z}_B^{\varphi_{\mathcal{P}}}(\mathcal{P}) &= \prod_{\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'} \omega_B(d_i)\omega_B(d_j) \left(\sum_{\substack{\sigma \text{ solution} \\ \text{to } \mathcal{P}'}} \sum_{\psi \in \Psi_\sigma} \prod_{\substack{v \in V \\ \sigma(v)=(i,j) \\ \psi(v^1), \psi(v^2) \in D_k}} B_{ik}B_{jk} \right) \\
&= \prod_{\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'} \omega_B(d_i)\omega_B(d_j) \left(\sum_{\substack{\sigma \text{ solution} \\ \text{to } \mathcal{P}'}} \prod_{\substack{v \in V \\ \sigma(v)=(i,j)}} (B_{i1}B_{j1} + \ldots + B_{jn}B_{jn}) \right) \\
&= \prod_{\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'} \omega_B(d_i)\omega_B(d_j) \left(\sum_{\substack{\sigma \text{ solution} \\ \text{to } \mathcal{P}'}} \prod_{\substack{v \in V \\ \sigma(v)=(i,j)}} C_{ij} \right) \\
&= \mathcal{Z}_C^{\varphi_{\mathcal{P}'}, V_{\mathcal{P}'}}(\mathcal{P}') \prod_{\langle (v), \kappa_{(i,j)} \rangle \in \mathcal{C}'} \omega_B(d_i)\omega_B(d_j)
\end{aligned}$$

□

Next we show that the reductions established in Lemma 17 and Lemmas 15, 19, 20 work for the more general problem $\mathrm{WCSP}^+(B)$.

Lemma 30. *Let $B \in \mathbb{Z}[X]^{m \times n}$ be a non-negative matrix. For any $\ell \geq 1$, $\mathrm{WCSP}^+(B^{(\ell)})$ is polynomial time reducible to $\mathrm{WCSP}^+(B(X))$.*

Proof: To obtain the required reduction we use the reduction from Lemma 15 and extend it by replacing unary constraints $\langle (v), \kappa_d \rangle$ and $\langle (v), \theta \rangle$ with $\langle (v_1), \kappa_d \rangle$ and $\langle (v_1), \theta \rangle$. □

By $\mathrm{COUNT}^+(B(X))$ we denote the problem of finding the number $N_B^+(\mathcal{P}, w)$ of solutions σ to a $\mathrm{WCSP}^+(B(X))$ instance $\mathcal{P}$ such that $\omega_B(\sigma) = w$. The following statement can be proved using the same argument as in Lemma 17 and Lemma 30 instead of Lemma 15.

Lemma 31. *Let $B(X) \in \mathbb{Z}[X]^{m \times n}$ be a non-negative matrix. Then the problems $\mathrm{WCSP}^+(B(X))$ and $\mathrm{COUNT}^+(B(X))$ are polynomial time equivalent.*

Lemmas 19,20 can also be proved in a stronger form using Lemma 31 instead of Lemma 17.

Lemma 32. *Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix, and $p(X)$ an irreducible polynomial. Let $C(X)$ be the matrix obtained from $B(X)$ by replacing all entries divisible by p with 0. Then there is a polynomial time reduction from* $\mathrm{WCSP}^+(C(X))$ *to* $\mathrm{WCSP}^+(B(X))$.

Lemma 33. *Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix, and $p(X)$ an irreducible polynomial. Then* $\mathrm{WCSP}^+(B(X)|_p)$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B(X))$.

Note that if a symmetric matrix B is not connected then permuting rows and columns it can be made *quasi-diagonal*, that is of the form

$$\left(\begin{array}{c|c|c|c} * & 0 & \cdots & 0 \\ \hline 0 & * & \cdots & 0 \\ \hline \vdots & \vdots & & \vdots \\ \hline 0 & 0 & \cdots & * \end{array}\right)$$

where $*$ denotes non-zero blocks (of different size). Let us denote these submatrices by $B_1,\ldots,B_k$.

Lemma 34. *Let $B(X) \in \mathbb{Z}[X]^{m\times m}$ be non-negative symmetric matrix with blocks $B_1,\ldots,B_k$.*

(1) For any $r \in \{1,\ldots,k\}$ and any $s,t \in \{1,\ldots,k\}$ such that sth row and tth column cross B_r, $\mathrm{WCSP}^+(C)$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B)$*, where $C_{ij} = (B_r)_{ij}\cdot(B_r)_{it}\cdot(B_r)_{sj}$.*

(2) If B_r is positive and $\mathsf{rank}(B_r) \ge 2$ *then* $\mathsf{rank}(C) \ge 2$.

Proof: Without loss of generality we assume that $s = t = 1$.

(1) Let $\mathcal{D}_r = (D_r;\alpha_r,\beta_r)$ be the canonical template for B_r with $D_r \subseteq D$, $\alpha_r = \alpha\cap D_r^2, \beta_r = \beta\cap D_r^2$ and $\theta_r = \theta\cap D_r$. Pick up an element $d_0 \in C_1\cap D_1$ and take an instance $\mathcal{P} = (V;D_r;\mathcal{C})$ of $\mathrm{WCSP}^+(C)$. We transform it to an instance $\mathcal{P}' = (V\cup V'\cup\{v_0\};D;\mathcal{C}')$ of $\mathrm{WCSP}^+(B(X))$ as follows.

- $V' = \{v^1,v^2 \mid v\in V\}$;
- for every constraint from $\mathcal{C}$ we include into $\mathcal{C}'$ the analogous constraint replacing $\alpha_r,\beta_r,\theta_r$ with α,β,θ respectively and κ_d with κ_d;
- for every $v\in V$ we include in $\mathcal{C}'$ the constraints $\langle(v,v^1),\alpha\rangle$, $\langle(v,v^2),\beta\rangle$, $\langle(v^1,v_0),\beta\rangle$, $\langle(v^2,v_0),\alpha\rangle$;
- we include the constraint $\langle(v_0),\kappa_{d_0}\rangle$.

As is easily seen, for any solution σ of $\mathcal{P}'$ such that $\omega_B(\sigma)\ne 0$ and any $v\in V$, $\sigma(v^1),\sigma(v^2)\in D_r$. Hence, $\sigma(v)\in D_s$. Thus the restriction σ' of σ onto V is a solution of $\mathcal{P}$. Conversely, every solution σ' of $\mathcal{P}$ can be extended to a solution of $\mathcal{P}'$ in a unique way. Finally,

$$\begin{aligned}
\omega_B(\sigma') &= B_{11} \cdot \prod_{v\in V\cup V'} B_{\sigma_1(v)\sigma_2(v)} \\
&= B_{11} \cdot \prod_{v\in V} B_{\sigma_1(v)\sigma_2(v)} \cdot \prod_{v\in V} B_{\sigma_1(v)1} \cdot \prod_{v\in V} B_{1\sigma_2(v)} \\
&= B_{11} \cdot \prod_{v\in V\cup V'} B_{\sigma_1(v)\sigma_2(v)} B_{\sigma_1(v)1} B_{1\sigma_2(v)} \\
&= B_{11} \cdot \prod_{v\in V} C_{\sigma_1(v)\sigma_2(v)} \\
&= B_{11} \cdot \omega_C(\sigma).
\end{aligned}$$

where we use $\sigma(v)$ to denote the pair $(\sigma_1(v), \sigma_2(v))$ of the numbers of the α- and β-classes containing $\sigma(v)$.

(2) Suppose that the following determinant is non-zero:

$$\begin{vmatrix} B_{ij} & B_{ij'} \\ B_{i'j} & B_{i'j'} \end{vmatrix} \neq 0.$$

Then we have

$$\begin{vmatrix} C_{ij} & C_{ij'} \\ C_{i'j} & C_{i'j'} \end{vmatrix} = \begin{vmatrix} B_{ij}B_{i1}B_{1j} & B_{ij'}B_{i1}B_{1j'} \\ B_{i'j}B_{i'1}B_{1j} & B_{i'j'}B_{i'1}B_{1j'} \end{vmatrix} = B_{i1}B_{i'1}B_{1j}B_{1j'} \cdot \begin{vmatrix} B_{ij} & B_{ij'} \\ B_{i'j} & B_{i'j'} \end{vmatrix} \neq 0.$$

□

Making use of Lemmas 29 and 33 we may assume that our matrix B is symmetric and every entry of B is a power of an indeterminate X. Then Lemma 27 allows us to assume that all blocks of B are positive, finally by Lemma 34 B can be assumed connected. We show that in some cases the degrees of powers of X in B can be decreased.

Lemma 35. *Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix, and $p(X)$ an irreducible polynomial such that every entry of $B(X)$ is divisible by $p(X)$. Then* $\mathrm{WCSP}^+(\frac{1}{p}B(X))$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B(X))$.

Proof: Take $\mathcal{P} = (V; D; \mathcal{C}) \in \mathrm{WCSP}^+(\frac{1}{p}B(X))$. Then

$$\mathcal{Z}^{\varphi_{\mathcal{P}}, V_{\mathcal{P}}}_{\frac{1}{p}B(X)}(\mathcal{P}) = \frac{1}{p^{|V|}} \mathcal{Z}^{\varphi_{\mathcal{P}}, V_{\mathcal{P}}}_{B}(\mathcal{P}).$$

□

A row of the matrix $B(X)$ is called *1-row* if 1 occurs in it. If a polynomial $g(X)$ is a sum of different powers of an irreducible polynomial $p(X)$, then we use $\mathsf{deg}_p(g)$ to denote the maximal degree of $p(X)$ occurring in $g(X)$.

Lemma 36. *Let $B(X) \in \mathbb{Z}[X]^{m\times n}$ be a non-negative matrix, whose entries are powers of an irreducible polynomial $p(X)$, and let $C(X)$ be the matrix obtained from $B(X)$ by removing all non-1-rows. Then* $\mathrm{WCSP}^*(C(X))$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B(X))$.

Proof: Let D' be set of elements from D belonging to α-classes that contains an element of weight 1 (that is elements corresponding to the 1-rows of $B(X)$). Let also α', β' denote the restrictions of α, β onto D'.

We use Δ to denote the maximal degree of $p(X)$ in $C(X)$, ℓ_i the minimal degree of $p(X)$ in the ith row, $\mathcal{P} = (V; D'; \mathcal{C}')$ a problem instance from WCSP$^*(C(X))$, and $\mathcal{P}' = (V; D; \mathcal{C})$ its associated problem from WCSP$^*(B(X))$, i.e. every constraint $\langle (u,v), \alpha' \rangle \in \mathcal{C}'$ $[\langle (u,v), \beta' \rangle, \langle (v), \kappa_d \rangle \in \mathcal{C}']$ is replaced with $\langle (u,v), \alpha \rangle \in \mathcal{C}$ $[\langle (u,v), \beta \rangle, \langle (v), \kappa_d \rangle \in \mathcal{C}]$. Let also $k = |V| \cdot \Delta + 1$. Then we make the following transformation:

- replace every variable $v \in V$ with $v_1, \ldots, v_k$;
- replace every $\langle (u,v), \alpha' \rangle \in \mathcal{C}'$ with $\langle (u_1, v_1), \alpha \rangle$ and every $\langle (u,v), \beta' \rangle \in \mathcal{C}'$ with $\langle (u_1, v_1), \beta \rangle$;
- for every $v \in V$ and every $1 \leq i, j \leq k$, add the constraint $\langle (v_i, v_j), \alpha \rangle$;
- denote the obtained problem $\mathcal{P}'' = (V''; D; \mathcal{C}'')$.

For every solution σ to $\mathcal{P}''$ we have

$$\omega_B(\sigma) = \prod_{v \in V} (B_{\sigma_1(v)1} + \ldots + B_{\sigma_1(v)n}) B_{\sigma(v)},$$

where we use $\sigma(v)$ to denote the pair $(\sigma_1(v), \sigma_2(v))$ of the numbers of the α- and β-classes containing $\sigma(v)$. Since the minimal degree of $p(X)$ in $B_{i1}(X) + \ldots + B_{in}(X)$ is ℓ_i, $\mathsf{deg}_p(\omega(\sigma)|_p) < k$ if and only if the range of σ belongs to the 1-rows. $\square$

A.3 Matrices with at least two 1-cells

Recall, that by the results of Section A.1, it is enough to prove the hardness of WCSP(B) for matrices $B(X) \in \mathbb{Z}[X]^{m \times n}$ whose entries are powers of the indeterminate X. Moreover, using Lemma 35 we may assume that B has at least one 1-row.

By Lemmas 19,27, if $B_{ik}(X) = B_{il}(X) = B_{jk}(X) = 1$ and $B_{jl}(X) \neq 1$, then WCSP($B(X)$) is #P-complete. Otherwise permuting rows and columns if required, it can be represented in the form shown in Figure 1. So, the matrix B has a cellular structure. We shall call B a *cell* matrix if, for every cell, its entries are all equal. A cell consisting of 1s is said to be a *1-cell.* Notice that 1-cells in the figure above are not necessary on the diagonal or square.

We first assume that $B(X)$ has at least two 1-cells.

Lemma 37. *Let $B(X) \in \mathbb{Z}[X]^{m \times n}$ be a non-negative matrix whose entries are powers of an irreducible polynomial $p(X)$, $B(X)$ contains at least two 1-cells. Then $\mathsf{rank}((BB^\top)|_p) \geq 2$ and this matrix contains at least two 1-cells.*

Proof: We assume that B has the structure shown above and that the upper left 1-cell is spanned by the first $\ell - 1$ rows. Then in the matrix $C = BB^\top$ we have $C_{11}|_p = C_{\ell\ell}|_p = 1$ while $C_{1\ell}|_p \neq 1$ that proves both claims. $\square$

$$\left(\begin{array}{c|c|c|c|c}
\begin{matrix}1&\cdots&1\\ \cdots&&\cdots\\ 1&\cdots&1\end{matrix} & * & & * & * \\ \hline
* & \begin{matrix}1&\cdots&1\\ \cdots&&\cdots\\ 1&\cdots&1\end{matrix} & & * & * \\ \hline
 & & \ddots & & \\ \hline
* & * & & \begin{matrix}1&\cdots&1\\ \cdots&&\cdots\\ 1&\cdots&1\end{matrix} & * \\ \hline
* & * & & * & *
\end{array}\right).$$

Fig. 1. The cellular structure of the matrix B

By Lemmas 36,29, we may replace $B(X)$ with $C(X)$, where $C(X)$ is obtained from $B(X)$ by removing all rows which are not 1-row, and then $C(X)$ with $C(X)C^\top(X)$ reducing $\mathrm{WCSP}^+(C(X)C^\top(X))$ to $\mathrm{WCSP}^+(B(X))$. If all entries of $B(X)$ are powers of an irreducible polynomial $p(X)$, then, as is easily seen $D(X) = (C(X)C^\top(X))|_p$ is square, symmetric, all its diagonal entries equal 1 and $\mathsf{rank}(D) \geq 2$. Therefore, we may assume that $B(X)$ satisfies all these conditions. Again, permuting rows and columns $B(X)$ can be represented in the form shown above, but all its 1-cells are square and diagonal, and all its diagonal blocks are 1-cells.

Lemma 38. *There is a cell matrix $C(X)$ of the same size as $B(X)$ and rank at least 2, such that* $\mathrm{WCSP}^+(C(X))$ *is polynomial time reducible to* $\mathrm{WCSP}^+(B(X))$.

Proof: Consider the matrix $B^1(X) = (B(X)B^\top(X))|_p$. As is easily seen, we have $B^1_{ij}(X) = p^{m_{ij}}(X)$ where $m_{ij} = \min_{k\leq n}\{\mathsf{deg}_p B_{ik}(X) + \mathsf{deg}_p B_{jk}(X)\}$. Since $B_{jj}(X) = 1$, $\mathsf{deg}_p B^1_{ij}(X) \leq \mathsf{deg}_p B_{ij}(X)$. On the other hand, if $B_{ij}(X) \neq 1$ then $B^1_{ij}(X) \neq 1$. Therefore, there is k such that in the sequence $B^1(X), B^2(X), \ldots$ where $B^{\ell+1}(X) = (B^\ell B^{\ell\top})(X)|_p$, we have $B^{k+1}(X) = B^k(X)$. We show that $B^k(X)$ is a cell matrix.

Take $B^k_{ij}(X) \neq 1$. To simplify the notation let us assume that this entry lies in the cell consisting of elements from $1, \ldots, \ell$ rows and $\ell+1, \ldots, m$ columns. Assume also that $B^k_{ij}(X)$ is minimal amongst the entries from this cell. Then $\mathsf{deg}_p B^k_{ij'}(X) = \mathsf{deg}_p B^{k+1}_{ij'}(X) = \min_{q\leq n}\{\mathsf{deg}_p B^k_{iq}(X) + \mathsf{deg}_p B^k_{j'q}(X)\} \leq \mathsf{deg}_p B^k_{ij}(X)$ for any $j' \in \{l+1, \ldots, m\}$ because $B^k_{j'j}(X) = 1$. Analogously, $B^k_{i'j}(X) = B^{k+1}_{i'j}(X) = \min_{q\leq n}\{\mathsf{deg}_p B^k_{i'q}(X) + \mathsf{deg}_p B^k_{jq}(X)\} \leq \mathsf{deg}_p B^k_{ji}(X) = \mathsf{deg}_p B^k_{ij}(X)$ for any $i' \in \{1, \ldots, \ell\}$ because $B^k_{i'i}(X) = 1$.

Finally, by Lemma 20,29, $\mathrm{WCSP}^+(B^k(X))$ is polynomial time reducible to $\mathrm{WCSP}(B(X))$. $\Box$

Thus $B(X)$ can be assumed to be cell matrix and its entries to be powers of the indeterminate X.

Lemma 39. *Let*

$$C = \left(\begin{array}{ccc|ccc} 1 & \cdots & 1 & q & \cdots & q \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots & 1 & q & \cdots & q \\ \hline q & \cdots & q & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ q & \cdots & q & 1 & \cdots & 1 \end{array}\right).$$

be a matrix with the diagonal 1-cells of size ℓ and k respectively and $q > \ell + k$ a prime number. Then the problem $\mathrm{WCSP}(C)$ *is #P-complete.*

Proof: By Lemma 10, $\mathrm{EVAL}(\mathsf{bip}(C))$ is polynomial time reducible to $\mathrm{WCSP}(C)$. Then, as is easily seen, $\mathrm{EVAL}(\mathsf{bip}(C)\cdot(\mathsf{bip}(C))^\top)$ is polynomial time reducible to $\mathrm{EVAL}(\mathsf{bip}(C))$. Notice that for every graph $G = (V;E)$ and every mapping $\sigma : V \to \{1,\dots,2(\ell+k)\}$, $\omega(\sigma) = 0$ unless the range of σ is a subset of $\{1,\dots,\ell+k\}$. Therefore, $\mathrm{EVAL}(\mathsf{bip}(C)\cdot(\mathsf{bip}(C))^\top)$ is polynomial time equivalent to $\mathrm{EVAL}(D)$, where

$$D = \left(\begin{array}{ccc|ccc} 1+\ell q^2 & \cdots & 1+\ell q^2 & (\ell+k)q & \cdots & (\ell+k)q \\ \vdots & & \vdots & \vdots & & \vdots \\ 1+\ell q^2 & \cdots & 1+\ell q^2 & (\ell+k)q & \cdots & (\ell+k)q \\ \hline (\ell+k)q & \cdots & (\ell+k)q & 1+kp^2 & \cdots & 1+kq^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ (\ell+k)q & \cdots & (\ell+k)q & 1+kp^2 & \cdots & 1+kq^2 \end{array}\right).$$

In [5], a generalised version of $\mathrm{EVAL}(A)$ was considered. In this generalised version vertex-weights are also allowed. So, if A is a symmetric $m \times m$-matrix and F is a diagonal matrix of the same size, then in $\mathrm{EVAL}(A,F)$ the aim is, given a graph G, compute the number

$$Z_{A,F} = \sum_{\sigma:V\to\{1,\dots,m\}} \prod_{\{u,v\}\in E} A_{\sigma(u)\sigma(v)} \prod_{v\in V} F_{\sigma(v)\sigma(v)}.$$

In that paper it has been also proved (Theorem 3.2) that $\mathrm{EVAL}(A)$ is polynomial time reducible to $\mathrm{EVAL}(A,F)$.

Now we define

$$D' = \begin{pmatrix} 1+\ell q^2 & (\ell+k)q \\ (\ell+k)q & 1+kq^2 \end{pmatrix}, \qquad F = \begin{pmatrix} \ell & 0 \\ 0 & k \end{pmatrix},$$

and show that $\mathrm{EVAL}(D)$ is equivalent to $\mathrm{EVAL}(D',F)$.

Let $G = (V; E)$ be a graph. To every mapping $\sigma : V \to \{1, \ldots, \ell + k\}$ we associate the mapping $\overline{\sigma} : V \to \{1, 2\}$ such that $\overline{\sigma}(v) = 1$ iff $\sigma(v) \leq \ell$ and $\overline{\sigma}(v) = 2$ otherwise. We have

$$\begin{aligned}
Z_D(G) &= \sum_{\sigma : V \to \{1,\ldots,m\}} \prod_{\{u,v\} \in E} D_{\sigma(u)\sigma(v)} \\
&= \sum_{\sigma : V \to \{1,\ldots,m\}} \prod_{\substack{\{u,v\} \in E \\ \overline{\sigma}(u) \neq \overline{\sigma}(v)}} ((\ell + k)q) \prod_{\substack{\{u,v\} \in E \\ \overline{\sigma}(u) = \overline{\sigma}(v) = 1}} (1 + \ell q^2) \prod_{\substack{\{u,v\} \in E \\ \overline{\sigma}(u) = \overline{\sigma}(v) = 2}} (1 + k q^2) \\
&= \sum_{\overline{\sigma} : V \to \{1,2\}} \left(\prod_{\substack{\{u,v\} \in E \\ \overline{\sigma}(u) \neq \overline{\sigma}(v)}} ((\ell + k)q) \prod_{\substack{\{u,v\} \in E \\ \overline{\sigma}(u) = \overline{\sigma}(v) = 1}} (1 + \ell q^2) \prod_{\substack{\{u,v\} \in E \\ \overline{\sigma}(u) = \overline{\sigma}(v) = 2}} (1 + k q^2) \right) \cdot \ell^{\overline{\sigma}^{-1}(1)} \cdot k^{\overline{\sigma}^{-1}(2)} \\
&= \sum_{\overline{\sigma} : V \to \{1,2\}} \prod_{\{u,v\} \in E} D'_{\overline{\sigma}(u)\overline{\sigma}(v)} \prod_{v \in V} F_{\overline{\sigma}(v)} \\
&= Z_{D',F}(G).
\end{aligned}$$

Thus, it is enough to prove #P-completeness of WCSP(D'). We show that #MAX-CUT is reducible to EVAL(D'). Indeed, if $G = (V; E)$ is a graph and V_1, V_2 is its partition, then the number of edges connecting V_1 and V_2 equals K if and only if, for the mapping $\sigma : V \to \{1, 2\}$ with $\sigma(v) = 1$ for $v \in V_1$ and $\sigma(v) = 2$ for $v \in V_2$ we have $\omega(\sigma)\rfloor_q = q^K$. □

Lemma 40. *If $B(X)$ is a cellular with at least two 1-cells, then* WCSP$^+(B(X))$ *is #P-complete.*

Proof: We show that WCSP$^*(B(X))$ is #P-complete. Let $\delta \geq 1$ be the minimum such that $p^\delta(X)$ is an entry of A, and Δ the maximum such that $p^\Delta(X)$ is an entry of A.

We shall reduce WCSP(C) for a matrix C of the form specified in Lemma 39, where $z = p^{2\delta}(a)$ and a is an integer such that the value of $p^\delta(a)$ is bigger than 1.

Take a cell with entries p^δ. Without loss of generality, we may assume that this cell is the submatrix spanned by rows from 1 to ℓ and columns from $\ell + 1$ to $\ell + k$. Since $B(X)$ is symmetric, entries of the cell spanned by rows from $\ell + 1$ to $\ell + k$ and columns from 1 to ℓ are also p^δ. So, we choose C to be of size $\ell + k$ with diagonal cells of size ℓ and k, respectively. Let D' be a $(\ell + k)$-element set, α', β', and w' the equivalence relations and the weight function on D' corresponding to C.

Let $\mathcal{P} = (V; D'; \mathcal{C}')$ be an instance from WCSP(C), and $\mathcal{P}' = (V; D; \mathcal{C})$ the corresponding instance from WCSP($B(X)$), i.e. every constraint $\langle (u, v), \alpha' \rangle \in \mathcal{C}'$ $[\langle (u, v), \beta' \rangle \in \mathcal{C}']$ is replaced with $\langle (u, v), \alpha \rangle \in \mathcal{C}$ $[\langle (u, v), \beta \rangle, \in \mathcal{C}]$. Choose $1 \leq s \leq \ell$, $\ell + 1 \leq t \leq \ell + k$ and $a, b \in D$, $a \in C_s \cap D_t$, $b \in C_t \cap D_s$. If $|V| = n$ then set $q = n \cdot \Delta + 1$. Let $\mathcal{P}'' = (V''; D; \mathcal{C}'')$ be the problem instance obtained by the following transformation.

- add 2 new variables x, z;

– for every variable $v \in V$, add $4q$ new variables $v^1, \ldots, v^1_q, v^2, \ldots, v^2_q$, $v^3, \ldots, v^3_q, v^4, \ldots, v^4_q$;
– for every $v \in V$, $1 \le i < q$, $j = 1, 2, 3, 4$, add the constraint $\langle (v^j_i, v^j_{i+1}), \alpha \cap \beta \rangle$;
– add constraints $\langle (x), \{(a)\} \rangle, \langle (z), \{(b)\} \rangle$;
– for every $v \in V$, add constraints $\langle (v, v^1_1), \alpha \rangle$, $\langle (v, v^2_1), \beta \rangle$, $\langle (v, v^3_1), \alpha \rangle$, $\langle (v, v^4_1), \beta \rangle$, $\langle (x, v^1_1), \beta \rangle$, $\langle (x, v^2_1), \alpha \rangle$, $\langle (z, v^3_1), \beta \rangle$, $\langle (z, v^4_1), \alpha \rangle$.

If σ is a solution to $\mathcal{P}''$ then $\sigma(v^j_i) = \sigma(v^j_{i'})$, for any $1 \le i, i' \le q$, $j = 1, 2, 3, 4$, $v \in V$. There is also a solution σ' to $\mathcal{P}'$ with $\sigma'(v) = \sigma(v)$ for $v \in V$, and $0 \le Q \le 4n \cdot \Delta$, such that $w(\varphi)$ equals

$$\omega(\sigma) = \omega(\sigma') \cdot p^{Qq}(X) \cdot p^{2\delta}(X).$$

Claim. $\omega(\sigma) = \sigma(\sigma') \cdot p^{2n\delta q}(X) \cdot p^{2\delta}(X)$ if and only if the following conditions hold:

– for every $v \in V$, one of $\omega(\sigma(v^1_1)), \omega(\sigma(v^3_1))$ equals 1, the other equals $p^{\delta}(X)$;
– for every $v \in V$, one of $\omega(\sigma(v^2_1)), \omega(\sigma(v^4_1))$ equals 1, the other equals $p^{\delta}(X)$;
– for every $v \in V$, $\sigma(v) \in C_i \cap D_j$, where $1 \le i, j \le \ell + k$.

The last condition listed in the Claim implies that WCSP($B'(X)$), where $B'(X)$ is the submatrix of $B(X)$ spanned by the first $\ell + k$ rows and columns, is polynomial time reducible to WCSP$^*(B(X))$. Since $C = B'(a)$, WCSP(C) is reducible to WCSP($B'(X)$) by Lemma 16. □

A.4 Matrices with a single 1-cell

In this section we consider the case when $B(X) \in \mathbb{Z}[X]^{m \times m}$ is a symmetric matrix with entries $B_{ij} = X^{\ell_{ij}}$ and one 1-cell. We therefore assume that

(1) $B_{ij}(X) = 1$ if and only if $1 \le i, j \le r - 1$ and $B_{ij}(X) \ne 1$ otherwise;
(2) the submatrix spanned by the first and rth rows and columns has rank 2.

Lemma 41. *Let $C(X)$ be the matrix with*

$$C_{ij}(X) = B_{ij}(X) \cdot (B_{ii}(X))^s \cdot (B_{jj}(X))^t.$$

Then WCSP$^+(C(X))$ *is polynomial time reducible to* WCSP$^+(B(X))$.

Proof: Let $\mathcal{P} = (V; D; \mathcal{C})$ be an instance of WCSP$^+(C)$. We construct an instance $\mathcal{P}' = (W; D; \mathcal{C}')$ of WCSP$^+(B)$ as follows

– $W = V \cup U$ where $U = \{v^1_1, \ldots, v^1_s, v^2_1, \ldots, v^2_t \mid v \in V\}$;
– $\mathcal{C}' = \mathcal{C} \cup \mathcal{C}''$ where $\mathcal{C}'' = \mathcal{C}_1 \cup \mathcal{C}_2$ consists of 2 parts:
  - $\mathcal{C}_1 = \{\langle (v^1_1), \theta \rangle, \ldots, \langle (v^1_s), \theta \rangle, \langle (v^2_1), \theta \rangle, \ldots, \langle (v^2_t), \theta \rangle \mid v \in V\}$,
  - $\mathcal{C}_2 = \{\langle (v, v^1_1), \alpha \rangle, \ldots, \langle (v, v^1_s), \alpha \rangle, \langle (v, v^2_1), \beta \rangle, \ldots, \langle (v, v^2_t), \beta \rangle \mid v \in V\}$.

Every solution σ of $\mathcal{P}$ can be extended to a solution of $\mathcal{P}'$ in a unique way, because, for any element $d \in D$, there is only one 'diagonal' element from the α-class and only one from β-class containing d. Conversely, the restriction of any solution of $\mathcal{P}'$ onto V is a solution of $\mathcal{P}$. Finally, for a solution of $\mathcal{P}$ and the corresponding solution σ' of $\mathcal{P}'$, we have

$$\begin{aligned}\omega(\sigma') &= \prod_{v\in V} B_{\sigma_1(v)\sigma_2(v)} \cdot B^s_{\sigma_1(v)\sigma_1(v)} \cdot B^t_{\sigma_2(v)\sigma_2(v)} \\ &= \prod_{v\in V} C_{\sigma_1(v)\sigma_2(v)} \\ &= \omega'(\sigma).\end{aligned}$$

Here $\sigma(v) \in C_{\sigma_1(v)} \cap D_{\sigma_2(v)}$. □

Lemma 41 implies that B can be assumed satisfying the following conditions.

(3) $B_{r1}(X)$ is an entry with the smallest degree in the rth row.
To obtain this we first permuting rows and columns assume that ℓ_{r1} is minimal amongst $\ell_{r1}, \dots, \ell_{rr-1}$. Then we replace B with C from Lemma 41 for $s = t$ such that $\ell_{r1} < s(u-1)$ where $u = \min\{\ell_{rr}, \dots, \ell_{mm}\}$.
(4) $\mathsf{deg} B_{1i}(X) < \mathsf{deg} B_{ri}(X)$.
This inequality holds trivially for $i \in \{1, \dots, r-1\}$. For other entries we replace B with C from Lemma 41 for $s = t$ such that $t\ell_{r1} > v$ where $v = \max\{\ell_{1r} - \ell_{rr}, \dots, \ell_{1m} - \ell_{rm}\}$.

Lemma 42. *Let k be a natural number and $B^{[k]}(X)$ the matrix with $B^{[k]}_{ij}(X) = B_{ij}(X) \cdot (B_{1j}(X))^{k-1}$. Then* WCSP$^+(B^{[k]}(X))$ *is polynomial time reducible to* WCSP$^+(B(X))$.

Proof: Let $\mathcal{P} = (V; D; \mathcal{C})$ be an instance of WCSP$^+(B^{[k]})$. We construct an instance $\mathcal{P}' = (W; D; \mathcal{C}')$ of WCSP$^+(B)$ as follows

- $W = V \cup U$ where $U = \{v_1, \dots, v_{k-1} \mid v \in V\} \cup \{u_0\}$;
- $\mathcal{C}' = \mathcal{C} \cup \mathcal{C}'' \cup \{\langle (u_0), \kappa_{d_0} \rangle\}$ where $d_0 \in C_1$ and $\mathcal{C}'' = \mathcal{C}_1 \cup \mathcal{C}_2$ consists of 2 parts:
  - $\mathcal{C}_1 = \{\langle (v_1, u_0), \alpha \rangle, \dots, \langle (v_{k-1}, u_0), \alpha \rangle, \mid v \in V\}$,
  - $\mathcal{C}_2 = \{\langle (v, v_1), \beta \rangle, \dots, \langle (v, v_{k-1}), \beta \rangle \mid v \in V\}$.

Every solution σ of $\mathcal{P}$ can be extended to a solution of $\mathcal{P}'$ in a unique way, because, for any element $d \in D$, there is only one element from the β-class containing d and C_1. Conversely, the restriction of any solution of $\mathcal{P}'$ onto V is a solution of $\mathcal{P}$. Finally, for a solution of $\mathcal{P}$ and the corresponding solution σ' of $\mathcal{P}'$, we have

$$\begin{aligned}\omega(\sigma') &= \omega(d_0) \cdot \prod_{v\in V} B_{\sigma_1(v)\sigma_2(v)} \cdot B^{k-1}_{1\sigma_2(v)} \\ &= \omega(d_0) \cdot \prod_{v\in V} C_{\sigma_1(v)\sigma_2(v)} \\ &= \omega(d_0) \cdot \omega'(\sigma).\end{aligned}$$

Here $\sigma(v) \in C_{\sigma_1(v)} \cap D_{\sigma_2(v)}$. □

Lemma 43. *Let $C(X)$ be a symmetric polynomial matrix. If there is a root of $C_{11}(X)$ which is not a root of $C_{1r}(X)$ then $\mathrm{WCSP}^+(C(X))$ is NP-complete.*

Proof: If $C_{11}(X)$ has a root that is not a root of $C_{1r}(X)$, then there is an irreducible polynomial $p(X)$ such that $p \mid C_{11}$, but $p \not| C_{1r}$. Let C' denote the matrix with $C'_{ij}(X) = p^{m_{ij}}(X)$ where m_{ij} is the maximal degree such that $p^{m_{ij}} \mid C_{ij}$. By Lemma 20, $\mathrm{WCSP}^+(C'(X))$ is polynomial time reducible to $\mathrm{WCSP}^+(C'(X))$. Clearly, $C'_{11}(X) \neq 1$, $C'_{12}(X) = C'_{21}(X) = 1$, which means, in particular, that the first and rth rows are linearly independent; and that $C'(X)$ has at least two 1-cells. By Lemma 40, $\mathrm{WCSP}^+(C'(X))$ is #P-complete. □

We use $C^{[k]}(X)$ to denote the matrix $B^{[k]}(X) \cdot (B^{[k]})^\top(X)$. The entries of $C^{[k]}(X)$ are polynomials which are not powers of X anymore. Certain properties of the roots of these polynomials are quite useful.

Let f be a polynomial and λ a complex number. Then $\mathsf{mult}(\lambda, f)$ denotes the *multiplicity* of λ in f if λ is a root of f and $\mathsf{mult}(\lambda, f) = 0$ otherwise. We shall use the following basic facts on polynomials, roots and their multiplicity.

Lemma 44. *Let f be a polynomial and λ a complex number.*

- *There exists a unique (up to a scalar factor) irreducible polynomial p_λ such that λ is a root of p_λ. If λ is a root of f then $p_\lambda | f$.*
- *If $\mathsf{mult}(\lambda, f) = s$ then $f = p_\lambda^s \overline{f}$ and $\overline{f}(\lambda) \neq 0$.*
- *Every root of $f(X^k)$ is of the form $\lambda^{1/k}$ where λ is a root of $f(X)$. Moreover, $\mathsf{mult}(\lambda^{1/k}, f(X^k)) = \mathsf{mult}(\lambda, f)$.*

Lemma 45. *(1) For any $j = \{r, \dots, m\}$, there exist a natural k and a root λ of $B^{[k]}_{11}$ such that $\mathsf{mult}(\lambda, B^{[k]}_{1j}) < \mathsf{mult}(\lambda, B^{[k]}_{11})$.*

(2) For any $j = \{r, \dots, m\}$, a root λ of $B^{[k]}_{11}$ and a natural k, $\mathsf{mult}(\lambda, B^{[k]}_{1j}) \leq \mathsf{mult}(\lambda, B^{[k]}_{jj})$.

Proof: To simplify the notation, set $n = m-1$, $b = \min\{\ell_{j1}, \dots, \ell_{jr-1}\}$. Without loss of generality let us assume $b = \ell_{j1}$. Then set $a_1 = \ell_{12} = 0, \dots, a_{r-2} = \ell_{1r-1} = 0$, $a_{r-1} = \ell_{1r}, \dots, a_n = l_{1m}$, $b_1 = \ell_{j2} - b, \dots, b_m = \ell_{jn} - b$ and $c_i = b_i - a_i$ for $1 \leq i \leq n$. Then

$$B = \begin{pmatrix} 1 & X^{a_1} & X^{a_2} & \cdots & X^{a_n} \\ \vdots & \vdots & \vdots & & \vdots \\ X^b & X^{b+b_1} & X^{b+b_2} & \cdots & X^{b+b_n} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix},$$

$$B^{[k]} = \begin{pmatrix} 1 & X^{ka_1} & X^{ka_2} & \cdots & X^{ka_n} \\ \vdots & \vdots & \vdots & & \vdots \\ X^b & X^{b+b_1-a_1+ka_1} & X^{b+b_2-a_2+ka_2} & \cdots & X^{b+b_n-a_n+ka_n} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}.$$

For the matrix $C^{[k]}$ we have

$$C_{11}^{[k]} = 1 + X^{2ka_1} + X^{2ka_2} + \ldots + X^{2ka_n} = r + X^{2ka_{r-1}} + X^{2ka_r} + \ldots + X^{2ka_n}$$
$$C_{1j}^{[k]} = X^b(1 + X^{c_1+2ka_1} + X^{c_2+2ka_2} + \ldots + X^{c_n+2ka_n}).$$

Let k be a natural number, and $\mu = e^\alpha$ a complex number such that $\mu^{1/k}$ is a root of $C_{11}^{[k]}$. If μ is not a root of $C_{1j}^{[k]}$ then (1) is proved. Otherwise, we get

$$C_{1j}^{[k]}(\mu^{1/k}) = e^{b/k}\left(1 + e^{\alpha(2a_1+\frac{c_1}{k})} + e^{\alpha(2a_2+\frac{c_2}{k})} + \ldots + e^{\alpha(2a_n+\frac{c_n}{k})}\right) = 0.$$

We denote $z = \frac{1}{k}$ and consider the function

$$f_\mu(z) = 1 + e^{\alpha(2a_1+c_1z)} + e^{\alpha(2a_2+c_2z)} + \ldots + e^{\alpha(2a_n+c_nz)}.$$

Claim 1. Suppose that $g(z) = u(z) + iv(z)$ is an analytic function at 0 and there are sequences $\{r_n\}$, $\{s_n\}$ from the real segment $[0,1]$ such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = 0$ and $u(r_n) = v(s_n) = 0$. Then

(a) $g(0) = 0$;
(b) there are sequences $\{r'_n\}$, $\{s'_n\}$ from the real segment $[0,1]$ such that

$$\lim_{n\to\infty} r'_n = \lim_{n\to\infty} s'_n = 0$$

and $u'(r'_n) = v'(s'_n) = 0$ where u', v' denote the derivatives of the respective functions.

Without loss of generality we may assume that $\{r_n\}$, $\{s_n\}$ are monotone. Then, since g is continuous,

$$g(0) = \lim_{z\to 0} g(z) = \lim_{z\to 0} u(z) + i \lim_{z\to 0} v(z) = \lim_{n\to\infty} u(r_n) + i \lim_{n\to\infty} v(s_n) = 0.$$

Furthermore, let u_0, v_0 denote the restrictions of u, v onto the real interval $[0,1]$. Then u_0, v_0 are continuous and differentiable real functions. Therefore, for any n, there are $r'_n \in [r_{n+1}, r_n]$ and $s'_n \in [s_{n+1}, s_n]$ such that $u'_0(r'_n) = v'_0(s'_n) = 0$. Clearly, $\lim_{n\to\infty} r'_n = \lim_{n\to\infty} s'_n = 0$ and $u'(r'_n) = u'_0(r'_n) = 0$, $v'(s'_n) = v'_0(s'_n) = 0$.

The function $f_\mu(z)$ is analytic everywhere including 0. Moreover,

$$f_\mu(1/k) = 1 + e^{\alpha(2a_1+\frac{c_1}{k})} + e^{\alpha(2a_2+\frac{c_2}{k})} + \ldots + e^{\alpha(2a_n+\frac{c_n}{k})} = \frac{C_{1j}^{[k]}(\mu^{1/k})}{e^{b/k}} = 0.$$

Therefore, by Claim 1, for any $l \geq 1$, the lth derivative $f_\mu^{(l)}(0) = 0$.

Computing the derivatives at 0 we get

$$f_\mu^{(l)} = (\alpha c_1)^l e^{\alpha a_1} + (\alpha c_2)^l e^{\alpha a_2} + \ldots + (\alpha c_n)^l e^{\alpha a_n} = 0.$$

Without loss of generality we may assume that $c_1 = \ldots = c_{s_1}$, $c_{s_1+1} = \ldots = c_{s_2}, \ldots, c_{s_{t-1}+1} = \ldots = c_{s_t} = 0$, $s_0 = 0, s_t = n$, and that $c_1, c_{s_1}, \ldots, c_{s_{t-1}}, c_n$ are all different.

Denoting $Y_i = e^{\alpha 2a_{s_{i-1}+1}} + \ldots + e^{\alpha 2a_{s_i}}$, $1 \leq i \leq t-1$, we get a system of linear equations

$$\begin{cases} c_{s_1}Y_1 + c_{s_2}Y_2 + \ldots + c_{s_{t-1}}Y_{t-1} = 0 \\ (c_{s_1})^2 Y_1 + (c_{s_2})^2 Y_2 + \ldots + (c_{s_{t-1}})^2 Y_{t-1} = 0 \\ \vdots \\ (c_{s_1})^{t-1} Y_1 + (c_{s_2})^{t-1} Y_2 + \ldots + (c_{s_{t-1}})^{t-1} Y_{t-1} = 0 \end{cases}$$

The determinant of the system is Vandermonde. Therefore, $Y_1 = \ldots = Y_{t-1} = 0$. Denoting

$$\begin{aligned} g_1(X) &= X^{2a_1} + \ldots + X^{2a_{s_1}} \\ &\vdots \\ g_{t-1}(X) &= X^{2a_{s_{t-2}+1}} + \ldots + X^{2a_{s_{t-1}}} \\ g_t(X) &= X^{2a_{s_{t-1}+1}} + \ldots + X^{2a_{s_t}}, \end{aligned}$$

we have $g_1(\mu) = \ldots = g_{t-1}(\mu) = 0$ and, since $C_{11}^{[k]}(X) = g_1(X) + \ldots + g_{t_1}(X) + g_t(X)$ and μ is a root of $C_{11}^{[k]}$, $g_t(\mu) = 0$ as well.

Thus, for every irreducible polynomial g with $g|C_{11}^{[k]}$, we have $g|g_1, \ldots, g_t$. Let $h_1, \ldots, h_q$ be the different irreducible divisors of $C_{11}^{[k]}$. Then

$$\begin{aligned} C_{11}^{[k]}(X) &= \overline{g}_1(X)h_1^{r_{11}}(X)\ldots h_q^{r_{1q}}(X) + \ldots \overline{g}_t(X)h_1^{r_{t1}}(X)\ldots h_q^{r_{tq}}(X) \\ &= h_1^{m_1}(X)\ldots h_q^{m_q}(X) \\ &\qquad \big(\overline{g}_1(X)h_1^{r'_{11}}(X)\ldots h_q^{r'_{1q}}(X) + \ldots + \overline{g}_t(X)h_1^{r'_{t1}}(X)\ldots h_q^{r'_{tq}}(X)\big), \end{aligned}$$

where $m_i = \min(r_{1i}, \ldots, r_{ti})$. Since g_1 and g_t are clearly different, the degree of the polynomial in the brackets is non-zero.

Claim 2. For any root λ of $C_{11}^{[k]}(X)$, say λ is a root of $h_i(X)$, and any natural k, $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}(X)) = m_i$.

Let $\lambda = e^{\alpha}$. Regroup the summands in $f_\lambda(z)$:

$$\begin{aligned} f_\lambda(z) &= e^{\alpha c_{s_1} z}(e^{\alpha a_1} + \ldots + e^{\alpha a_{s_1}}) + \ldots + e^{\alpha c_{s_{t-1}} z}(e^{\alpha a_{s_{t-2}+1}} + \ldots + e^{\alpha a_{s_{t-1}}}) \\ &\qquad + (1 + e^{\alpha a_{s_{t-1}+1}} + \ldots + e^{\alpha a_{s_t}}) \\ &= h_1^{m_1}(\lambda)\ldots h_q^{m_q}(\lambda)(\lambda^{c_{s_1} z}\overline{g}_1(\lambda)h_1^{r'_{11}}(\lambda)\ldots h_q^{r'_{1q}}(\lambda) + \ldots \\ &\qquad + \lambda^{c_{s_{t-1}} z}\overline{g}_{t-1}(\lambda)h_1^{r'_{t-11}}(\lambda)\ldots h_q^{r'_{t-1q}}(\lambda) + \overline{g}_t(\lambda)h_1^{r'_{t1}}(\lambda)\ldots h_q^{r'_{tq}}(\lambda)). \end{aligned}$$

To simplify the notation, we set

$$h(X) = h_1^{m_1}(X)\ldots h_q^{m_q}(X), \qquad f_i(X) = \overline{g}_i(X)h_1^{r'_{i1}}(X)\ldots h_q^{r'_{iq}}(X),$$

and $\overline{f}(X) = X^{c_{s_1}z} f_1(X) + \ldots + X^{c_{s_{t-1}}z} f_{t-1}(X) + f_t(X)$.

Let $\beta_1 = \lambda^{c_{s_1}/t!}, \ldots, \beta_{t-1} = \lambda^{c_{s_{t-1}}/t!}, \beta_t = 1$. Then, for any $l \leq t$,

$$f_\lambda\left(\frac{1}{t!/l}\right) = h(\lambda)(\beta_1^l f_1(\lambda) + \ldots + \beta_{t-1}^l f_{t-1}(\lambda) + \beta_t^l f_t(\lambda)).$$

If $\overline{f}\left(\frac{1}{t!/l}\right) = 0$ for $l = 1, \ldots, t$, then

$$\begin{cases} \beta_1 f_1(\lambda) + \ldots + \beta_t f_t(\lambda) = 0 \\ (\beta_1)^2 f_1(\lambda) + \ldots + (\beta_t)^2 f_t(\lambda) = 0 \\ \vdots \\ (\beta_1)^t f_1(\lambda) + \ldots + (\beta_t)^t f_t(\lambda) = 0 \end{cases}$$

Since $\beta_i \neq \beta_{i'}$ whenever $i \neq i'$, we get $f_1(\lambda) = \ldots = f_t(\lambda) = 0$ which is impossible by the choice of $m_1, \ldots, m_q$. Therefore, for a certain k, $\overline{f}_\lambda(1/k) \neq 0$. Finally,

$$\begin{aligned} C^{[k]}(X) &= X^b(X^{c_{s_1}} g_1(X^k) + \ldots + X^{c_{s_{t-1}}} g_{t-1}(X^k) + g_t(X^k)) \\ &= X^b h(X^k)(X^{c_{s_1}} f_1(X^k) + \ldots + X^{c_{s_{t-1}}} f_{t-1}(X^k) + f_t(X^k)), \end{aligned}$$

$\mathsf{mult}(\lambda^{1/k}, h(X^k)) = \mathsf{mult}(\lambda, h(X)) = m_1$ and, since $\overline{f}(k) \neq 0$, $\lambda^{1/k}$ is not a root of the polynomial in the brackets. Thus, $\mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}) = m_i$.

Every root of $\overline{g}_1(X) h_1^{r'_{11}}(X) \ldots h_q^{r'_{1q}}(X) + \ldots + \overline{g}_t(X) h_1^{r'_{t1}}(X) \ldots h_q^{r'_{tq}}(X)$, λ, is also a root of $B_{11}(X)$ and, therefore, it is a root of one of $h_1, \ldots, h_q$. Without loss of generality let λ be a root of h_1. Then $\mathsf{mult}(\lambda, C_{11}^{[k]}) > m_1$ and, for any k, $\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]}) = \mathsf{mult}(\lambda, C_{11}^{[k]}) > m_1 = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}$.

To prove (2), consider $C_{jj}^{[k]}$:

$$\begin{aligned} C_{jj}^{[k]}(X) &= X^{2b}(1 + X^{2c_1+2ka_1} + \ldots + X^{2c_m+2ka_m}) \\ &= X^{2b}(1 + X^{2c_1}(X^k)^{2a_1} + \ldots + X^{2c_m}(X^k)^{2a_m}) \\ &= X^{2b} h(X^k)(X^{2c_{s_1}} f_1(X^k) + \ldots + X^{2c_{s_{t-1}}} f_{t-1}(X^k) + f_t(X^k)). \end{aligned}$$

For any root of $C_{11}^{[k]}$, λ, say λ is root of $m_i(X)$, we have

$$\mathsf{mult}(\lambda^{1/k}, C_{jj}^{[k]}) \geq \mathsf{mult}(\lambda^{1/k}, h(X^k)) = \mathsf{mult}(\lambda, h(X)) = m_i = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]}).$$

□

Now, take $j \in \{r, \ldots, m\}$, a root λ of $C_{11}^{[k]}$ and a natural k such that $t = \mathsf{mult}(\lambda^{1/k}, C_{1j}^{[k]})$, is minimal. We also may assume that $\mathsf{mult}(\lambda^{1/k}, C_{11}^{[k]})$ is minimal amongst $\mathsf{mult}(\lambda^{1/k}, C_{ii'}^{[k]})$, $1 \leq i, i' \leq r$. In particular, for any $i \in \{1, \ldots, n\}$, $\mathsf{mult}(\lambda^{1/k}, C_{1i}^{[k]}) \geq t$ and $\mathsf{mult}(\lambda^{1/k}, C_{ii}^{[k]}) \geq t$. Let also $p(X)$ be the irreducible polynomial with the root $\lambda^{1/k}$.

Case 1. There is an entry $C^{[k]}_{rq}$ such that $s = \mathsf{mult}(\lambda^{1/k}, C^{[k]}_{rq}) < t$.

Let s be the least multiplicity of $\lambda^{1/k}$ in entries of $C^{[k]}$. Denote by C the matrix $C^{[k]}|_p$ divided by p^s. Then $C_{rq} = C_{qr} = 1$, $C_{rr}, C_{qq} \neq 1$. Therefore rth and qth rows of C are linearly independent and C has at least two 1-cell. Thus, by results of Section A.3, #CSP(C) is #P-complete.

Case 2. For any r, q, $\mathsf{mult}(\lambda^{1/k}, C^{[k]}_{rq}) \geq t$.

In this case, the matrix C obtained from $C^{[k]}|_p$ by dividing by p^t has at least two 1-cells, since $C_{1j} = C_{j1} = 1$, but $C_{11} \neq 1$. $\square$