Three-element Mal'tsev algebras *

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Abstract

We prove that every three-element Mal'tsev algebra up to term equivalence is determined by its subalgebras, congruences, internal homomorphisms, labels of prime quotients (in the sense of tame congruence theory), and in certain special cases, by subalgebras of its cube. Thus there exist finitely many term inequivalent three-element Mal'tsev algebras.

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1 Introduction and results

Congruence permutability is the first considered and one of the most famous congruence conditions. Studied first by A.I.Mal'tsev it had initiated a long series of investigations of similar conditions that constitute one of the leading directions in universal algebraic research during last three decades. As well as many other congruence conditions, congruence permutability can be characterized by a term satisfying certain identities: a variety is congruence permutable if and only if it has a term $\mathbf{d}$ such that $\mathbf{d}(x, y, y) = \mathbf{d}(y, y, x) = x$. A term satisfying these identities is called a *Mal'tsev term*, and an algebra or a variety possessing a Mal'tsev term is said to be a *Mal'tsev algebra* or a *Mal'tsev variety*.

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In spite of extensive study, congruence permutability remains even less investigated in some aspects than other congruence conditions. One of such aspects is the number of different finite Mal'tsev algebras. To specify what "different algebras" means we need some definitions.

A *clone* on a finite set A is a set of operations on A closed under superposition and containing all projection operations. An operation is called a *term* [*polynomial*] *operation* of an algebra $\mathbb{A} = (A; F)$, if it belongs to the clone generated by the operations of F [F and constant operations]. The clone of term operations of $\mathbb{A}$ is denoted by $\mathsf{Clo}\ \mathbb{A}$. Two algebras are said to be *term* [*polynomial*] *equivalent* if they have the same clone of term [polynomial] operations. The question we deal with can be posed more precisely: *How many finite term* [*polynomial*] *inequivalent Mal'tsev algebras with the same universe are there?*

Surprisingly, the analogous problems for other well-known congruence conditions are much easier. If we set a strict condition like arithmeticity (congruence distributivity + congruence permutability), results of [2] imply that, for any finite set A, there are only finitely many term inequivalent arithmetical algebras with the universe A. If in the condition of arithmeticity we relax permutability to 3-permutability (recall that a variety is called *congruence 3-permutable* if $\alpha \circ \beta \circ \alpha = \beta \circ \alpha \circ \beta$ holds for any congruences α, β of any algebra $\mathbb{A}$ from the variety), we get continuum many polynomial inequivalent three-element algebras [12].

If the number of inequivalent algebras satisfying a condition is treated as a measure of the strictness of the condition, then congruence permutability is of intermediate strictness between strict conditions like arithmeticity, and weak conditions like 3-permutability. This makes the problem of finding the number of (term or polynomial) inequivalent finite Mal'tsev algebras highly fascinating and nontrivial. Up to now we have only particular results for small universes. The number of term (there exist 11 of them) and polynomial (there exist 2 of them) inequivalent two-element Mal'tsev algebras can be easily derived from Post's description of clones on a two-element set [14]. In [9], P.Idziak has found all 14 polynomial inequivalent three-element Mal'tsev algebras. In the same paper he proved that there are infinitely many Mal'tsev algebras having 4 or more elements (see also [10]). Finally, in [4], it was proved that there exist only countably infinite polynomial inequivalent 4-element Mal'tsev algebras. We believe that the following conjecture is true.

Conjecture 1.1 *Up to polynomial equivalence, there exist countably many finite Mal'tsev algebras.*

As for the number of term inequivalent Mal'tsev algebras, we need more evidences before posing any conjectures.

This is a remarkable fact that two- and three-element Mal'tsev algebras are determined up to polynomial equivalence by their congruences and types of prime quotients of the congruence lattice in the sense of tame congruence theory. As is noted above, there are 2 polynomial inequivalent Mal'tsev algebras: one of type **2**, which is polynomial equivalent to an Abelian group; and one of type **3**, which is primal. Moreover, it is easy to check that two-element Mal'tsev algebras are term equivalent whenever they have the same family of subalgebras, automorphism group, and the same type of the sole prime quotient of their congruence lattice. Recall that an algebra is said to be *functionally complete* if every operation is its polynomial operation, or, equivalently, if it is polynomial equivalent to a primal algebra.

Proposition 1.1 *Up to term equivalence there are 5 two-element Mal'tsev algebras polynomial equivalent to an Abelian group (that is of type* **2***): one having 2 one-element subalgebras and the non-identity automorphism, one having the automorphism and no subalgebras, two having a subalgebra and no non-identical automorphisms, one without subalgebras and automorphisms.*

There are 6 functionally complete two-element Mal'tsev algebras (of type **3***). Five of them are determined in the same way and the extra one has 2 one-element subalgebras and no non-identical automorphism.*

We prove that an analogous result holds for three-element Mal'tsev algebras; however, more objects related to an algebra must be involved. Let $\mathbb{B}$, $\mathbb{C}$ be subalgebras of $\mathbb{A}$. A homomorphism $\mathbb{B}$ into $\mathbb{C}$ is called an *internal homomorphism* of $\mathbb{A}$. Let $A = \{a, b, c\}$ be a three-element set. The equality relation and the total equivalence relation are denoted by $\underline{0}, \underline{1}$ respectively. The ternary relation $S_{a,b}$ is defined as follows

$$S_{a,b} = \begin{pmatrix} a & b & c & a & b & c \\ a & a & b & b & b & a \\ a & a & a & b & b & b \end{pmatrix}.$$

Notice also that a three-element Mal'tsev algebra may have at most one non-trivial congruence.

Theorem 1.1 *If a three-element Mal'tsev algebra* $\mathbb{A}$ *has a non-trivial congruence* μ, *the only non-trivial block* $\mathbb{B} = \{a, b\}$ *of* μ *is a subalgebra, and the type of both* $\mathbb{B}$ *and* $(\mu, \underline{1})$ *is* **2**, *then up to term equivalence it is determined by its subalgebras, internal homomorphisms, and the subuniverse* $S_{a,b}$ *of* $\mathbb{A}^3$. *Otherwise a three-element Mal'tsev algebra is determined by its subalgebras, internal homomorphisms, congruences and types of prime quotients of the congruence lattices of* $\mathbb{A}$ *and its subalgebras.*

Corollary 1.1 *Up to term equivalence there are finitely many three-element Mal'tsev algebras.*

2 Methods and tools

2.1 Relational clones

We shall constantly use Galois connection between clones and relational clones. Relational clones were introduced in [3, 7] (see also [13]) as sets of relations closed under some operations. We use the most general set of operations defining relational clones, namely, those that can be expressed through pp-formulas. We make use of a standard correspondence between relations and logical predicates: an n-ary relation ϱ consists of all tuples on which the corresponding n-ary predicate P_ϱ is true. An existential first-order formula is said to be *primitive positive (pp-)* if its quantifier-free part is a conjunction of atomic formulas. These formulas can be used as operations that allow one to express new predicates from given ones. A *relational clone* is a set of relations that contains the equality relation and is closed under all pp-formulas.

Recall that an operation f *preserves* an m-ary relation ϱ on a set A or ϱ is *invariant* with respect to f if ϱ is a subalgebra of $(A; f)^m$. All operations preserving a given set D of relations form a clone which is denoted by $\mathsf{Pol}\ D$. Similarly, all relations preserved by operations from a given set C form a relational clone denoted by $\mathsf{Inv}\ C$. Moreover, every clone [relational clone] can be represented in the form $\mathsf{Pol}\ D$ [respectively $\mathsf{Inv}\ C$] for certain D [respectively C] (for more details the reader is referred to [13]).

Every relation preserved by all term operations of an algebra can be treated as a subuniverse of its direct power, or as the universe of a subdirect product of its subalgebras. It will be convenient for us to use for relations the same notation as for algebras.

To obtain a description of algebras up to term equivalence it is enough to describe the clones of their term operations, that in its turn can be done by describing the corresponding relational clones. For this we are going to find a relatively small number of relations (there will be 46 of them) such that every relational clone on a three-element set invariant with respect to a Mal'tsev operation is generated by a certain set of these relations.

Notice, that for any internal homomorphism $\pi\colon \mathbb{B} \to \mathbb{C}$ of an algebra $\mathbb{A}$, the binary relation $G_\pi = \{(a, \pi(a)) \mid a \in \mathbb{B}\}$ is a subalgebra of $\mathbb{A}^2$. We shall call this relation the *graph* of π. As was shown in [4], the types of prime quotients of a Mal'tsev algebra are determined via certain 4-ary relations. For the special case of a three-element algebra we write out these relations explicitly in the next section. We shall prove that for every three-element Mal'tsev algebra $\mathbb{A}$ the set of all subalgebras of its direct powers, that is the relational clone of invariant relations, is generated by subalgebras of $\mathbb{A}$ (that is unary relations), congruences and internal homomorphisms (that is binary relations of specific kind), ternary relations of the form $S_{a,b}$, and 4-ary relations that determine the types of prime quotients.

In the case of two-element algebras the analogous result follows easily from Proposition 1.1.

Proposition 2.1 *Let* $\mathbf{C}$ *be a relational clone over the set* $\{0,1\}$ *such that* $\mathsf{Pol}\ \mathbf{C}$ *contains a Mal'tsev operation. Then* $\mathbf{C}$ *is generated by a subset of* $\{\{0\},\{1\},\varrho,\tau\}$ *where*

$$\varrho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

2.2 Minimal sets and types of prime quotients

In this subsection we recall some notions of tame congruence theory [8], which are intensively used throughout the paper. The definitions we reproduce here are enough for understanding all the statements, but proofs use more facts that can be found in [8]. Let $\mathbb{A}$ be a finite algebra, and α,β congruences of

$\mathbb{A}$ with $\alpha \prec \beta$, that is β covers α. The base set of $\mathbb{A}$ is denoted by A. A set $U \subseteq A$ is called an (α, β)-*minimal set* if it is minimal under inclusion amongst the subsets of A which can be represented in the form $f(A)$ where f is a unary idempotent polynomial with $f(\beta) \not\subseteq \alpha$. Further, let $T = B \cap U$ where B is a β-block such that $T^2 \not\subseteq \alpha$. Then T is called an (α, β)-*trace*, and the algebra $\mathbb{T} = \mathbb{A}|_T = (T; F)$ where $F = \{f|_T \mid f$ is a polynomial of $\mathbb{A}$ preserving $T\}$ is called an *induced algebra*. The algebra $\mathbb{T}/_{\alpha|_T}$ must be polynomial equivalent to an algebra of one of the five types:

1. a finite set with a group action on it,

2. a finite vector space over a finite field,

3. a two-element Boolean algebra,

4. a two-element lattice,

5. a two-element semilattice.

The type of $\mathbb{T}/_{\alpha|_T}$ depends only on the congruences α, β and therefore can be assigned to the quotient (α, β). It is denoted by $\mathsf{typ}(\alpha, \beta)$.

In the particular case of Mal'tsev algebras, for any prime quotient $\alpha \prec \beta$, $\mathsf{typ}(\alpha, \beta)$ equals either **2** or **3**, and a minimal set is a union of traces. Moreover, in the case of a three-element algebra, for any (α, β)-trace T, and (α, β)-minimal set U with $T \subseteq U$, we have $T = U$ and $|\mathbb{T}/_{\alpha|_T}| = 2, 3$. If $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$ then $\mathbb{T}/_{\alpha|_T}$ is polynomial equivalent to an Abelian group $\mathbb{Z}_2$ or $\mathbb{Z}_3$. In the first case the Mal'tsev operation $\mathbf{d}$ of $\mathbb{A}$ being restricted to $\mathbb{T}/_{\alpha|_T}$ is a minority operation $x + y + z$ of $\mathbb{Z}_2$, while in the second case $\mathbb{T}/_{\alpha|_T} = \mathbb{A}$ and $\mathbf{d} = x - y + z$ where $+, -$ are the operations of $\mathbb{Z}_3$.

2.3 Basic properties of Mal'tsev algebras

The following proposition contains some basic properties of Mal'tsev algebras, that will be constantly used. We denote tuples in boldface, say, $\mathbf{a} = (a_1, \ldots, a_n)$; in this case $a_1, \ldots, a_n$ are the components of the n-tuple $\mathbf{a}$. We use $\underline{n}$ to denote the set $\{1, \ldots, n\}$[1]. Let $\mathbb{D}$ be a subdirect product of alge-

[1] This notation will cause no confusion with the notation of the total equivalence relation, because we never denote the set $\{1\}$ by $\underline{1}$.

bras $\mathbb{D}_1, \ldots, \mathbb{D}_n$, and $I = \{i_1, \ldots, i_k\} \subseteq \underline{n}$. Then the tuple $\mathbf{a}_I = (a_{i_1}, \ldots, a_{i_s})$, $\mathbf{a} \in \mathbb{D}$, is said to be the *projection* of $\mathbf{a}$ onto I, and the relation $\mathbb{D}_I = \{\mathbf{a}_I \mid \mathbf{a} \in \mathbb{D}\}$ is said to be the *projection* of $\mathbb{D}$ onto I.

Similarly, polynomials of $\mathbb{D}$ are also denoted in boldface, say, $\mathbf{f}$. The polynomial $\mathbf{f}$ can be represented as $\mathbf{f}(x_1, \ldots, x_k) = g(x_1, \ldots, x_k, \mathbf{a}^1, \ldots, \mathbf{a}^l)$ where g is a term operation of $\mathbb{D}$ and $\mathbf{a}^1, \ldots, \mathbf{a}^l \in \mathbb{D}$. Then the polynomial $g(x_1, \ldots, x_k, a_i^1, \ldots, a_i^l)$ of $\mathbb{D}_i$ is denoted by f_i, and for $I = \{i_1, \ldots, i_s\} \subseteq \underline{n}$, $\mathbf{f}_I$ denotes the polynomial $g(x_1, \ldots, x_k, \mathbf{a}_I^1, \ldots, \mathbf{a}_I^l)$ of $\mathbb{D}_I$. For any i, and any polynomial f of $\mathbb{D}_i$, there is a polynomial $\mathbf{g}$ of $\mathbb{D}$ such that $g_i = f$. We shall call $\mathbf{g}$ an *extension* of f to a polynomial of $\mathbb{D}$. Finally, for $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_m)$, $(\mathbf{a}, \mathbf{b})$ denotes the tuple $(a_1, \ldots, a_n, b_1, \ldots, b_m)$, while $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the pair of tuples.

Proposition 2.2 *Let $\mathbb{D}$ be a subdirect product of Mal'tsev algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$ and $I \subseteq \underline{n}$. Then the following properties hold*
1. $\mathbb{D}$ is rectangular, *that is if* $\mathbf{a}, \mathbf{b} \in \mathbb{D}_I, \mathbf{c}, \mathbf{d} \in \mathbb{D}_{\underline{n}-I}$ *and* $(\mathbf{a}, \mathbf{c}), (\mathbf{a}, \mathbf{d}), (\mathbf{b}, \mathbf{c}) \in \mathbb{D}$*, then* $(\mathbf{b}, \mathbf{d}) \in \mathbb{D}$.
2. The relation $\theta_I = \{\langle \mathbf{a}, \mathbf{b} \rangle \in (\mathbb{D}_I)^2 \mid$ there is $\mathbf{c} \in \mathbb{D}_{\underline{n}-I}$ such that $(\mathbf{a}, \mathbf{c}), (\mathbf{b}, \mathbf{c}) \in \mathbb{D}\}$ is a congruence of $\mathbb{D}_I$.
3. $\mathbb{D}$ is a disjoint union of sets of the form $B \times C$ where B is a θ_I-block and C is a $\theta_{\underline{n}-I}$-block.

2.4 Coherent sets

In the last part of this section we set the main technical results.

Let $\mathbb{D}$ be a subdirect product of subdirectly irreducible Mal'tsev algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$, $\mathbf{d}$ the Mal'tsev term of $\mathbb{D}$, and let μ_i, $\underline{0}_i$ denote the monolith of $\mathbb{D}_i$ and the equality relation on D_i, $i \in \underline{n}$, respectively. We will say that an index $i \in \underline{n}$ *can be separated* from $j \in \underline{n}$ if there exists a unary polynomial $\mathbf{f}$ of $\mathbb{D}$ such that $f_i(\mu_i) \not\subseteq \underline{0}_i$ and $f_j(\mu_j) \subseteq \underline{0}_j$. If $\mathbf{f}$ satisfies this property we will also say that $\mathbf{f}$ *separates* i from j. As usual, by an idempotent unary polynomial we mean a polynomial $f(x)$ such that $f^2 = f$ or, equivalently, such that $f(x) = x$ for any x from its range.

Lemma 2.1 *If i can be separated from j then, for every $(\underline{0}_i, \mu_i)$-minimal set U, there is an idempotent unary polynomial $\mathbf{g}$ such that $g_i(\mathbb{D}_i) = U$, and $\mathbf{g}$ separates i from j.*

Proof. Let $\mathbf{f}$ separate i from j. Then $f_i(\mathbb{D}_i)$ contains a $(\underline{0}_i, \mu_i)$-minimal set V, and there is an idempotent polynomial h_i with $h_i(\mathbb{D}_i) = V$. The polynomial h_i can be extended to a polynomial $\mathbf{h}$ of $\mathbb{D}$. Then $\mathbf{f}' = \mathbf{hf}$ separates i from j and $f'_i(\mathbb{D}_i) = V$.

There is a $(\underline{0}_i, \mu_i)$-minimal set W with $f'_i(W) = V$ and an idempotent polynomial h'_i with $h'_i(V) = W$. As above the polynomial h'_i can be extended to a polynomial $\mathbf{h}'$ of $\mathbb{D}$. For a certain k, $(\mathbf{h}'\mathbf{f}')^k$ is idempotent, separates i from j, and $(h'_i f'_i)^k(\mathbb{D}_i) = W$. Now the lemma follows easily from the fact that any two $(\underline{0}_i, \mu_i)$-minimal sets are polynomial isomorphic. $\square$

Lemma 2.2 *If i can be separated from j then j can be separated from i.*

Proof. Let $U_1, \ldots, U_k$ be all the $(\underline{0}_i, \mu_i)$-minimal sets. By Lemma 2.1, for every U_l, there is an idempotent unary polynomial $\mathbf{g}^{(l)}$ separating i from j and such that $g_i^{(l)}(\mathbb{D}_i) = U_l$. Take a nontrivial μ_j-block B, a tuple $\mathbf{a} \in \mathbb{D}$ such that $a_j \in B$, and set $\mathbf{a}^{(l)} = \mathbf{g}^{(l)}(\mathbf{a})$. The operation $\mathbf{h}^{(l)}(x) = \mathbf{d}(x, \mathbf{g}^{(l)}(x), \mathbf{a}^{(l)})$ satisfies the conditions

- $h_i^{(l)}(x) = \mathbf{d}(x, g_i^{(l)}(x), a_i^{(l)}) = \mathbf{d}(x, x, a_i^{(l)}) = a_i^{(l)}$ for every $x \in U_l$;
- $h_j^{(l)}(x) = \mathbf{d}(x, g_j^{(l)}(x), a_j^{(l)}) = \mathbf{d}(x, a_j^{(l)}, a_j^{(l)}) = x$ for every $x \in B$.

We are going to compose the polynomials $\mathbf{h}^{(l)}$ such that the composition collapses μ_i. To this end take a sequence $1 = l_1, l_2, \ldots$ such that U_{l_2} is a subset of the range of $\overline{h}^{(1)} = h_i^{(l_1)}$, and, for $s > 2$, U_{l_s} is a subset of the range of $\overline{h}^{(s-1)} = h_i^{(l_{s-1})} \ldots h_i^{(l_1)}$. Since $|\mathsf{Im}\ (\overline{h}^{(s)})| < |\mathsf{Im}\ (\overline{h}^{(s-1)})|$, there is m such that $|\mathsf{Im}\ (\overline{h}^{(m)})|$ contains no $(\underline{0}_i, \mu_i)$-minimal sets. Therefore, setting $\mathbf{h}(x) = \mathbf{h}^{(l_m)}(\mathbf{h}^{(l_{m-1})}(\ldots \mathbf{h}^{(l_1)}(x) \ldots))$ we have h_i collapses all the $(\underline{0}_i, \mu_i)$-minimal sets, and h_j acts identically on B. Therefore, $\mathbf{h}$ separates j from i. $\square$

The relation η on $\underline{n}$ consisting of all pairs (i, j) such that i cannot be separated from j is obviously reflexive and transitive, and is symmetric by Lemma 2.2. So, η is an equivalence relation. We call its classes *coherent sets*.

We show that coherent sets possess some useful additional properties.

Lemma 2.3 *Let $I \subseteq \underline{n}$, and $I_1, \ldots, I_k$ the coherent sets of $\mathbb{D}$. Then the coherent sets of $\mathbb{D}_I$ are $I_1 \cap I, \ldots, I_k \cap I$.*

Lemma 2.3 follows from the observation that whether i can be separated from j or not depends entirely on the algebra $\mathbb{D}_{i,j}$.

Lemma 2.4 *If $I \subseteq \underline{n}$ is a coherent set then there is a unary polynomial* $\mathbf{f}$ *of* $\mathbb{D}$ *such that*

$$(*) \qquad \begin{array}{l} \mathbf{f} \textit{ is idempotent,} \\ f_i(\mu_i) \not\subseteq \underline{0}_i \textit{ if and only if } i \in I, \\ f_i(\mathbb{D}_i) \textit{ is a } (\underline{0}_i, \mu_i)\textit{-minimal set whenever } i \in I. \end{array}$$

Moreover, for any $l \in I$, and any $(\underline{0}_l, \mu_l)$-minimal set U_l, the polynomial $\mathbf{f}$ *can be chosen such that $f_l(\mathbb{D}_l) = U_l$.*

Proof. Take $i \in I$ and a $(\underline{0}_i, \mu_i)$-minimal set U. By Lemma 2.1, for every $j \notin I$, there is an idempotent polynomial $\mathbf{f}^{(j)}(x)$ such that $\mathbf{f}^{(j)}$ separates i from j and $f_i^{(j)}(\mathbb{D}_i) = U$. Composing all these polynomials we obtain an operation $\mathbf{f}$ such that $f_j(\mu_j) \subseteq \underline{0}_j$ for every $j \notin I$, and $f_i(x) = x$ for $x \in U$. Since for every $l \in I$ the polynomial $\mathbf{f}$ does not separate i from l, we have $f_l(\mu_l) \not\subseteq \underline{0}_l$.

Let now $\mathbf{f}$ be an operation separating each $i \in I$ from each $j \notin I$ and such that the sum of $|f_i(\mathbb{D}_i)|$, for $i \in I$, is minimal among the operations with this property. Then, for every $i \in I$, $f_i(\mathbb{D}_i)$ is a $(\underline{0}_i, \mu_i)$-minimal set. Indeed, suppose that there is $i \in I$ such that $f_i(\mathbb{D}_i)$ is not a minimal set. Then $f_i(\mathbb{D}_i)$ contains a minimal set V and there is an idempotent unary polynomial g_i with the range V. As is easily seen, for an extension $\mathbf{g}$ of g_i to a polynomial of $\mathbb{D}$, the operation $\mathbf{gf}$ still separates every $l \in I$ from every $j \notin I$, but the sum of $|f_j(\mathbb{D}_j)|$, $j \in I$, is less than that for $\mathbf{f}$; a contradiction.

Finally, arguing as in the proof of Lemma 2.1 we may derive an idempotent polynomial with the required properties. $\Box$

For a set $I \subseteq \underline{n}$, let us denote the equality relations on $\mathbb{D}$, $\mathbb{D}_I$ by $\underline{0}$, $\underline{0}_I$ respectively, and $\overline{\mu} = \prod_{i \in \underline{n}} \mu_i$, $\overline{\mu}_I = \prod_{i \in I} \mu_i$. For a congruence α we denote the α-block containing an element a by $a/_{\alpha}$. The following lemma shows a connection between minimal sets of $\mathbb{D}$ and its coherent sets.

Lemma 2.5 *Let $\underline{0} \leq \alpha \prec \beta \leq \overline{\mu}$, and $\mathbf{f}(\mathbb{D})$ be an (α, β)-minimal set for an idempotent polynomial* $\mathbf{f}$. *Then* $\mathbf{f}$ *satisfies $(*)$ for a certain coherent set.*

Proof. First we prove the claim of the lemma in a particular case. For $J \subseteq \underline{n}$ we denote by η_J the congruence

$$\eta_J = \{\langle \mathbf{a}, \mathbf{b}\rangle \in \mathbb{D}^2 \mid a_i = b_i \text{ if } i \in J, \text{ and } a_i \stackrel{\mu_i}{\equiv} b_i \text{ otherwise.}\}$$

For any $J \subseteq \underline{n}$ and $j \in J$, either $\eta_J = \eta_{J-\{j\}}$ or $\eta_J \prec \eta_{J-\{j\}}$. Notice that in the latter case the rectangularity of $\mathbb{D}$ implies that for any $(a, b) \in \mu_i$ there are $\mathbf{a}, \mathbf{b} \in \mathbb{D}$ with $a_i = a, b_i = b$, and such that $\langle \mathbf{a}, \mathbf{b}\rangle \in \eta_{J-\{j\}} - \eta_J$.

Claim. For any $J \subseteq \underline{n}$ and $j \in J$, such that $\eta_J \prec \eta_{J-\{j\}}$, every idempotent polynomial $\mathbf{f}$ of $\mathbb{D}$ such that $\mathbf{f}(\mathbb{D})$ is an $(\eta_J, \eta_{J-\{j\}})$-minimal set satisfies $(*)$ for the coherent set I containing j.

As is easily seen, $f_j(\mu_j) \not\subseteq \underline{0}_j$; therefore, there is a $(\underline{0}_j, \mu_j)$-minimal set $U \subseteq f_j(\mathbb{D}_j)$. Take a polynomial $\mathbf{g}$ satisfying $(*)$ for I, and such that $g_j(\mathbb{D}_j) = U$. Let k be such that $\mathbf{f}' = (\mathbf{fgf})^k$ is idempotent. Then $f'_j(U) = U$, $f'_l(\mu_l) \subseteq \underline{0}_l$ if and only if $l \notin I$, that is $\mathbf{f}'$ satisfies $(*)$ for I, and $\mathbf{f}'(\eta_{J-\{j\}}) \not\subseteq \eta_J$. Since $\mathsf{Im}\ (\mathbf{f}') \subseteq \mathsf{Im}\ (\mathbf{f})$, and $\mathsf{Im}\ (\mathbf{f})$ is an $(\eta_J, \eta_{J-\{j\}})$-minimal set, we have $\mathsf{Im}\ (\mathbf{f}') = \mathsf{Im}\ (\mathbf{f})$. Moreover, as $\mathsf{Ker}(\mathbf{f}) \subseteq \mathsf{Ker}(\mathbf{f}')$, we have $\mathbf{f}' = \mathbf{f}$, and the claim is proved.

Then we prove that, for any prime quotient $\underline{0} \leq \alpha \prec \beta \leq \overline{\mu}$, there is $J \subseteq \underline{n}$ and $j \in J$ such that (α, β) is projective to $(\eta_J, \eta_{J-\{j\}})$, and therefore, $\mathbf{f}(\mathbb{D})$ is an (α, β)-minimal set if and only if it is $(\eta_J, \eta_{J-\{j\}})$-minimal set.

Suppose the contrary, and let $\beta \leq \overline{\mu}$ be a maximal congruence such that, for some $\alpha \prec \beta$, (α, β) is projective to $(\eta_J, \eta_{J-\{j\}})$ for no $J \subseteq \underline{n}$, $j \in J$. Let J be a minimal set such that $\eta_J \leq \alpha$, and j any member of J. Then $\eta_J \prec \eta_{J-\{j\}}$. Since $\alpha \wedge \eta_{J-\{j\}} = \eta_J$, if $\eta_{J-\{j\}} \leq \beta$ the interval (α, β) is projective to $(\eta_J, \eta_{J-\{j\}})$, a contradiction with the assumption made. Otherwise, by the modularity of $\mathsf{Con}\ \mathbb{D}$, (α, β) is projective to $(\alpha \vee \eta_{J-\{j\}}, \beta \vee \eta_{J-\{j\}})$, a contradiction with the maximality of β. $\square$

Corollary 2.1 *(1) Let $I \subseteq \underline{n}$ be a coherent set, $\underline{0}_I \leq \alpha < \beta \leq \overline{\mu}_I$, $\alpha' = \{\langle \mathbf{a}, \mathbf{b}\rangle \in \overline{\mu} \mid \mathbf{a}_I \stackrel{\alpha}{\equiv} \mathbf{b}_I\}$, $\beta' = \{\langle \mathbf{a}, \mathbf{b}\rangle \in \overline{\mu} \mid \mathbf{a}_I \stackrel{\beta}{\equiv} \mathbf{b}_I\}$, and $\alpha' \prec \gamma \leq \beta'$. Then, any idempotent polynomial $\mathbf{f}$ of $\mathbb{D}$ such that $\mathbf{f}(\mathbb{D})$ is an (α', γ)-minimal set satisfies $(*)$ for I.*

(2) Let $\mathbf{a} \in \mathbb{D}$ be a tuple from a nontrivial $\overline{\mu}$-block, and $a_i/_{\mu_i}$ nontrivial. There is a polynomial $\mathbf{f}$ satisfying $(*)$ for I and such that $\mathbf{a} \in \mathbf{f}(\mathbb{D})$.

Proof. (1) By Lemma 2.5, $\mathbf{f}$ satisfies $(*)$ for a certain coherent set I'. If $I' \neq I$, then $f_i(\mu_i) \subseteq \underline{0}_i$ for any $i \in I$. Therefore, $\mathbf{f}_I(\beta) \subseteq \underline{0}_I \subseteq \alpha$, and $\mathbf{f}(\beta') \subseteq \alpha'$.

(2) Let α be the greatest congruence of $\mathbb{D}_I$ such that $\alpha \leq \overline{\mu}_I$ and $\mathbf{a}_I/_{\alpha}$ is a singleton, and $\alpha \prec \beta \leq \overline{\mu}_I$. Since $\mathbf{a}_I$ lies in a nontrivial $\overline{\mu}_I$-block, $\alpha \neq \overline{\mu}_I$ and $\mathbf{a}_I$ belongs to an (α, β)-minimal set. Let α', β', γ be congruences constructed as in part (1) of the corollary. Then there is an idempotent polynomial $\mathbf{f}$ satisfying $(*)$ for I, and such that $\mathbf{a}_I \in \mathbf{f}_I(\mathbb{D}_I)$.

Set $\mathbf{h}(x) = \mathbf{d}(\mathbf{f}(x), \mathbf{f}(\mathbf{a}), \mathbf{a})$. Since $f_i(\mu_i) \subseteq \underline{0}_i$ whenever $i \notin I$, we have $h_i(\mu_i) \subseteq \underline{0}_i$ in this case. If $i \in I$ and $x \in f_i(\mathbb{D}_i)$ then $f_i(x) = x$, and therefore

$$h_i(x) = \mathbf{d}(f_i(x), f_i(a_i), a_i) = \mathbf{d}(x, a_i, a_i) = x.$$

This means $h_i(\mu_i) \not\subseteq \underline{0}_i$. Moreover, as $f_i(\mathbb{D}_i)$ is a $(\underline{0}_i, \mu_i)$-minimal set, $h_i(\mathbb{D}_i)$ is also a $(\underline{0}_i, \mu_i)$-minimal set. Thus, for a certain k, $\mathbf{h}^k$ is idempotent and satisfies $(*)$ for I. Finally,

$$h_i(a_i) = \mathbf{d}(f_i(a_i), f_i(a_i), a_i) = \mathbf{d}(f_i(a_i), f_i(a_i), a_i) = a_i.$$

So, $\mathbf{a} \in \mathbf{h}^k(\mathbb{D})$; the corollary is proved. $\square$

Lemma 2.6 *Let B be a $\overline{\mu}$-block of $\mathbb{D}$. Then $B = B_{I_1} \times \ldots \times B_{I_k}$ where $I_1, \ldots, I_k$ are the coherent sets.*

Proof. Set $I = I_1$, $J = \underline{n} - I$. We prove that $B = B_I \times B_J$. To this end we consider two congruences on $\mathbb{D}_I$:

$$\theta = \theta_I = \{\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbb{D}_I^2 \mid \text{there is } \mathbf{c} \in \mathbb{D}_J \text{ such that } (\mathbf{a}, \mathbf{c}), (\mathbf{b}, \mathbf{c}) \in \mathbb{D}\}$$
$$\eta = \{\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbb{D}_I^2 \mid \text{there are } \mathbf{c}, \mathbf{e} \in \mathbb{D}_J \text{ such that } (\mathbf{a}, \mathbf{c}), (\mathbf{b}, \mathbf{e}) \in \mathbb{D},\ \langle \mathbf{c}, \mathbf{e} \rangle \in \overline{\mu}_J\}.$$

Notice that if for $\mathbf{a}^1, \mathbf{a}^2 \in B_I$ there is $\mathbf{b} \in \mathbb{D}_J$ with $(\mathbf{a}^1, \mathbf{b}), (\mathbf{a}^2, \mathbf{b}) \in \mathbb{D}$ then the rectangularity of $\mathbb{D}$ implies that, for any $\mathbf{c} \in B_J$ such that $(\mathbf{a}^1, \mathbf{c}) \in \mathbb{D}$, the tuple $(\mathbf{a}^2, \mathbf{c})$ also belongs to $\mathbb{D}$. Analogously, for any $\mathbf{a}^1, \mathbf{a}^2 \in B_I$ with $\mathbf{a}^1 \overset{\eta}{\equiv} \mathbf{a}^2$, there are $\mathbf{b}^1, \mathbf{b}^2 \in B_J$ such that $(\mathbf{a}^1, \mathbf{b}^1), (\mathbf{a}^2, \mathbf{b}^2) \in \mathbb{D}$. We are to prove that if $(\mathbf{a}^1, \mathbf{b}^1), (\mathbf{a}^2, \mathbf{b}^2) \in B$ then $(\mathbf{a}^1, \mathbf{b}^2), (\mathbf{a}^2, \mathbf{b}^1) \in \mathbb{D}$, or, equivalently, the equality $\theta \cap \overline{\mu}_I = \eta \cap \overline{\mu}_I$. Denote the congruences appearing in this equality by α and β respectively.

If $\alpha \neq \beta$ then, for congruences α', β', γ constructed as in Lemma 2.1(1), there is an idempotent unary polynomial $\mathbf{f}$ satisfying $(*)$ for I and such that $\mathbf{f}(\mathbb{D})$ is a (α', γ)-minimal set. However, this leads to a contradiction because on the one hand $\mathbf{f}(\gamma) \not\subseteq \alpha'$, hence $\mathbf{f}_I(\beta) \not\subseteq \alpha$, but on the other hand $\mathbf{f}_I(\eta) \subseteq \theta$ for any $\mathbf{f}$ satisfying $(*)$ for I.

Repeating the same argument for each coherent set we get what is required. □

A relation $\mathbb{D}$ is called *irreducible* if there is no pair i, j of indices such that $\mathbb{D}_{i,j}$ is a graph of a mapping.

Lemma 2.7 *If $\mathbb{D}$ is irreducible, then a coherent set containing an index i with $\mathsf{typ}(\underline{0}_i, \mu_i) = \mathbf{3}$ is a singleton.*

Proof. Without loss of generality we may assume that $I = \{1, \ldots, k\}$ is a coherent set, $k > 1$, and $\mathsf{typ}(\underline{0}_1, \mu_1) = \mathbf{3}$. Let $\mathbf{f}$ be a polynomial satisfying $(*)$ for I, and $f_i(\mathbb{D}_i) = U_i$, where U_i is a $(\underline{0}_i, \mu_i)$-minimal set, $i \in I$. By Theorem 8.5 of [8], the tail of U_i is empty. Therefore, $|U_1| = 2$. Since $\mathsf{typ}(\underline{0}_1, \mu_1) = \mathbf{3}$, there exists a polynomial operation $\mathbf{g}(x, y)$ of $\mathbb{D}$ such that $\mathsf{Im}\ (\mathbf{g}) = \mathsf{Im}\ (\mathbf{f})$; $\mathbf{g}(\mathbf{a}, \mathbf{a}) = \mathbf{a}$ for $\mathbf{a} \in \mathsf{Im}\ (\mathbf{f})$; and $g_1|_{U_1}$ is a semilattice operation. Let $a \in U_1$ be such that $g_1(a, x) = g_1(x, a) = a$ for any $x \in U_1$. Two cases are possible.

CASE 1. $\mathsf{typ}(\underline{0}_2, \mu_2) = \mathbf{2}$.
Take a $(\underline{0}_2, \mu_2)$-trace $N \subseteq U_2$ such that $(\{a\} \times N) \cap \mathbf{f}_{1,2}(\mathbb{D}_{1,2}) \neq \varnothing$. It is polynomial equivalent to a one-dimensional vector space over $\mathrm{GF}(q)$ where q is a prime power. Since $\mathbf{g}$ is idempotent, its restriction to N can be represented in the form $\alpha x + (1 - \alpha)y$, $\alpha \in \mathrm{GF}(q)$. We may assume that $\alpha = 1$. Indeed, if $\alpha = 0$ then consider $\mathbf{g}(y, x)$ instead of $\mathbf{g}(x, y)$. Otherwise, the operation

$$\underbrace{\mathbf{g}(\mathbf{g} \ldots \mathbf{g}}_{q-1 \text{ times}}(x, y), y \ldots, y)$$

satisfies the required conditions.

Let $\mathbf{a} \in \mathbf{f}(\mathbb{D})$ be such that $a_1 = a$ and $a_2 \in N$. Then for the operation $\mathbf{h}(x) = \mathbf{g}(\mathbf{f}(x), \mathbf{a})$ the following conditions hold: (a) $h_i(\mu_i) \subseteq \underline{0}_i$ whenever $i \notin I$; (b) $h_1(\mathbb{D}_1) = \{a\}$; (c) $h_2|_N(x) = x$. The last condition means that

$h_2(\mu_2) \not\subseteq \underline{0}_2$, while the second implies $h_1(\mu_1) \subseteq \underline{0}_1$. A contradiction with the assumption that I is a coherent set.

CASE 2. $\mathsf{typ}(\underline{0}_2, \mu_2) = \mathbf{3}$.
Since $\mathbb{D}$ is irreducible, the congruences

$$\begin{aligned}\eta_1 &= \{\langle a,b\rangle \in \mathbb{D}_1^2 \mid \text{ there is } c \in \mathbb{D}_2 \text{ such that } (a,c),(b,c) \in \mathbb{D}_{1,2}\}\\ \eta_2 &= \{\langle a,b\rangle \in \mathbb{D}_2^2 \mid \text{ there is } c \in \mathbb{D}_1 \text{ such that } (c,a),(c,b) \in \mathbb{D}_{1,2}\}\end{aligned}$$

are nontrivial, and therefore contain μ_1, μ_2 respectively. Hence, if $(a_1, a_2) \in \mathbb{D}_{1,2}$ then, for any $b_1 \stackrel{\mu_1}{\equiv} a_1$ $b_2 \stackrel{\mu_2}{\equiv} a_2$, we have $(b_1, b_2) \in \mathbb{D}_{1,2}$. In particular, as U_1, U_2 lie in a block of μ_1, μ_2 respectively, and $(U_1 \times U_2) \cap \mathbb{D}_{1,2} \neq \varnothing$, we get $U_1 \times U_2 \subseteq \mathbb{D}_{1,2}$, and moreover $U_1 \times U_2 \subseteq \mathbf{f}_{1,2}(\mathbb{D}_{1,2})$. Since $\mathbf{g}$ is idempotent, $g_2|_{U_2}$ is either a semilattice operation or a projection. If g_2 is a projection on U_2, then we may use the same argument as in Case 1. If g_2 is a semilattice operation, then let $U_2 = \{b, c\}$, and $g_2(b,c) = g_2(c,b) = c$. The operation $\mathbf{h} = \mathbf{g}(\mathbf{f}(x), \mathbf{b})$ where $b_1 = a$, $b_2 = b$ satisfies the conditions (a)–(c) from the Case 1, and we again get a contradiction. $\square$

3 Structure of relations preserved by a Mal'tsev operation on a three-element set

3.1 Standard relations

We are going to prove that the relational clone corresponding to a Mal'tsev algebra is generated by relations of a very restricted type. The Mal'tsev algebra we study will be denoted by $\mathbb{A}$, its universe by $A = \{0, 1, 2\}$. Below we give the complete list of these relations.

- one- and two-element subsets (subalgebras) $\{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{2,0\}$;
- equivalence relations (congruences)

$$\mu_1 = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 2 & 0 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 & 1 \end{pmatrix};$$

- internal homomorphisms of $\mathbb{A}$ including
  - automorphisms

$$\varphi_0 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix},$$
$$\psi_0 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix};$$

  - homomorphisms $\mathbb{A}$ onto a two-element subalgebra

$$\tau_1 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix},$$
$$\tau_3 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \tau_4 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix},$$
$$\tau_5 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 1 \end{pmatrix}, \quad \tau_6 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix};$$

  - internal isomorphisms

$$\psi'_0 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \psi'_1 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \psi'_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\varphi'_1 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \quad \varphi'_2 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \quad \varphi'_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix},$$
$$\varphi'_4 = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad \varphi'_5 = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \quad \varphi'_6 = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix};$$

- automorphisms of $\mathbb{A}/_{\mu}$ where μ is a proper congruence

$$\varrho_1 = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 2 & 2 & 0 & 1 \end{pmatrix}; \quad \varrho_2 = \begin{pmatrix} 0 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}, \quad \varrho_3 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 \end{pmatrix},$$

- linear dependences[2]

$$S_{0,1} = \begin{pmatrix} 0\,1\,2\,0\,1\,2 \\ 0\,0\,1\,1\,1\,0 \\ 0\,0\,0\,1\,1\,1 \end{pmatrix}, \quad S_{1,2} = \begin{pmatrix} 1\,2\,0\,1\,2\,0 \\ 1\,1\,2\,2\,2\,1 \\ 1\,1\,1\,2\,2\,2 \end{pmatrix}, \quad S_{2,0} = \begin{pmatrix} 2\,0\,1\,2\,0\,1 \\ 2\,2\,0\,0\,0\,2 \\ 2\,2\,2\,0\,0\,0 \end{pmatrix};$$

[2]Later we show that these relations are definable by linear equations.

- relations determining the type of a two-element subalgebra; the subalgebra $\{a,b\}$ has type $\mathbf{2}$ if and only if $\mathsf{Clo}\,\mathbb{A}$ preserve the relation $T_{a,b}$

$$\begin{aligned} T_{0,1} &= \{(a,b,c,d) \subseteq \{0,1\}^4 \mid a+b=c+d\} \\ T_{0,2} &= \{(a,b,c,d) \subseteq \{0,2\}^4 \mid a+b=c+d\} \\ T_{2,1} &= \{(a,b,c,d) \subseteq \{2,1\}^4 \mid a+b=c+d\} \end{aligned}$$

  where $+$ is the addition of the two-element group $\{a,b\}$;

- relations determining the type of $(\underline{0},\mu)$ where $\mu \in \{\mu_1,\mu_2,\mu_3\}$

$$\begin{aligned} T_1 &= \{(a,b,c,d) \subseteq \{0,1\}^4 \mid a+b=c+d\} \cup \{(2,2,2,2)\} \\ T_2 &= \{(a,b,c,d) \subseteq \{0,2\}^4 \mid a+b=c+d\} \cup \{(1,1,1,1)\} \\ T_3 &= \{(a,b,c,d) \subseteq \{2,1\}^4 \mid a+b=c+d\} \cup \{(0,0,0,0)\}; \end{aligned}$$

- relations determining the type of $(\mu,\underline{1})$ where $\mu \in \{\mu_1,\mu_2,\mu_3\}$

$$\begin{aligned} T_1' &= \{(a,b,c,d) \subseteq A^4 \mid a +_1 b = c +_1 d\} \\ T_2' &= \{(a,b,c,d) \subseteq A^4 \mid a +_2 b = c +_3 d\} \\ T_3' &= \{(a,b,c,d) \subseteq A^4 \mid a +_3 b = c +_3 d\} \end{aligned}$$

  where $+_1, +_2, +_3$ are defined as follows

$+_1$	0	1	2
0	0	0	2
1	0	0	2
2	2	2	0

$+_2$	0	1	2
0	0	1	0
1	1	0	1
2	0	1	0

$+_3$	0	1	2
0	2	0	0
1	0	2	2
2	0	2	2

- the relation determining the type of $(\underline{0},\underline{1})$ when the algebra is simple

$$T = \{(a,b,c,d) \mid a+b=c+d\}$$

  where $+$ is the addition of $\mathbb{Z}_3$.

3.2 Preliminary results

In this subsection we state several auxiliary results that allow us to restrict the type of relations to be studied.

Lemma 3.1 *A relational clone is generated by the set of all its irreducible relations and graphs of mappings.*

Proof. Let $\mathbf{C}$ be a relational clone, $\mathbb{D} \in \mathbf{C}$ an n-ary relation, and $I \subseteq \underline{n}$ a maximal set such that $\mathbb{D}_I$ is irreducible. Obviously, the set I can be chosen such that, for any $i \in \underline{n} - I$, there is $j_i \in I$ and $\mathbb{D}_{j_i,i}$ is the graph of a mapping $\pi_i : \mathbb{D}_{j_i} \to \mathbb{D}_i$. Then

$$P_{\mathbb{D}}(x_1, \dots, x_n) = P_{\mathbb{D}_I}(x_i)_{i \in I} \wedge \bigwedge_{i \in \underline{n} - I} P_{\pi_i}(x_{j_i}, x_i),$$

that is $\mathbb{D}$ belongs to the clone generated by the irreducible relation $\mathbb{D}_I \in \mathbf{C}$, and graphs $\mathbb{D}_{j_i,i} \in \mathbf{C}$ of mappings. $\square$

The lemma implies that we may restrict ourselves with the study of irreducible relations. Notice that any mapping whose graph is preserved by term operations of an algebra is an internal homomorphism of the algebra.

Lemma 3.2 *(1) Every relational clone is generated by one-element subalgebras and relations without trivial projections.*

(2) If $\mathbb{D}$ is a subdirect product of $\mathbb{D}_1, \dots, \mathbb{D}_n$, $\mathbb{D}_1$ is isomorphic to $\mathbb{B}$ and $\varphi : \mathbb{D}_1 \to \mathbb{B}$ is an isomorphism, then the relation $\mathbb{D}' = \{(\varphi(a_1), a_2, \dots, a_n) \mid (a_1, a_2, \dots, a_n) \in \mathbb{D}\}$ belongs to the relational clone generated by $\mathbb{D}$ and the graph G_φ of φ.

Proof. (1) If $\mathbb{D}$ is a subdirect product of $\mathbb{D}_1, \dots, \mathbb{D}_n$ and $|\mathbb{D}_1| = 1$ then $\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_{\{2,\dots,n\}}$.

(2) Indeed, $P_{\mathbb{D}'}(x_1, \dots, x_n) = \exists z P_{\mathbb{D}}(z, x_2, \dots, x_n) \wedge P_{G_\varphi}(z, x_1)$. $\square$

Then we consider interaction between two-element subalgebras of $\mathbb{A}$. Every proper subalgebra of $\mathbb{A}$ is two-element, and therefore simple. Finite Mal'tsev algebras all subalgebras of which are simple form the class of *paraprimal* algebras, see [5, 6, 11, 16]. Thus every proper subalgebra of $\mathbb{A}$ is paraprimal. Paraprimal algebras can also be characterized by the clear structure of subalgebras of their direct powers. The following statement is well known and easily follows from the rectangularity of subdirect products of paraprimal algebras.

Lemma 3.3 *Let $\mathbb{D}$ be a subdirect product of paraprimal algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$. Then, for any minimal set $I \subseteq \underline{n}$ such that the projection $\mathbb{D} \to \mathbb{D}_I$ is one-to-one, $\mathbb{D}_I = \prod_{i \in I} \mathbb{D}_i$.*

We need another aspect of the structure of subdirect product of paraprimal algebras.

Proposition 3.1 *Let $\mathbb{D}$ be a subdirect product of paraprimal algebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$ such that every $\mathbb{D}_i$ has a one-element subalgebra $\{a_i\}$, and the equivalence relation η on $\underline{n}$ defined by the rule: $(i,j) \in \eta$ if and only if $\mathbb{D}_i$ is isomorphic to $\mathbb{D}_j$. Then $\mathbb{D} = \mathbb{D}_{I_1} \times \ldots \times \mathbb{D}_{I_k}$ where $I_1, \ldots, I_k$ are the blocks of η.*

Proof. For $I \subseteq \underline{n}$ we denote by δ_I the kernel of the projection onto the set of indices I, that is $\langle \mathbf{b}, \mathbf{c} \rangle \in \delta_I$ if and only if $b_i = c_i$ for all $i \in I$. Clearly, δ_I is a congruence of $\mathbb{D}$. We shall write δ_i instead of $\delta_{\{i\}}$. By Lemma 3.3, if J is a minimal subset of $\underline{n}$ such that δ_J is the equality relation, then $\mathbb{D}_J = \prod_{i \in J} \mathbb{D}_i$ and the projection $\mathbb{D} \to \mathbb{D}_J$ is an isomorphism.

Claim. For any $l \notin \underline{n} - J$, there is $j \in J$ such that $\mathbb{D}_j$ is isomorphic to $\mathbb{D}_l$; $\delta_{J-\{i\}} \leq \delta_l$ whenever $\mathbb{D}_i$, $i \in J$, is not isomorphic to $\mathbb{D}_l$.

Notice that for any $I, I' \subseteq J$ we have $\delta_I \vee \delta_{I'} = \delta_{I \cap I'}$, $\delta_I \wedge \delta_{I'} = \delta_{I \cup I'}$. Since $\bigvee_{i \in J} \delta_{J-\{i\}} = \underline{1}$, for every index $l \in \underline{n} - J$, there is $j \in J$ such that $\delta_{J-\{j\}} \not\leq \delta_l$. Each δ_i, $i \in \underline{n}$ is a coatom in the congruence lattice $\mathsf{Con}\ \mathbb{D}$, hence, by the modularity of $\mathsf{Con}\ \mathbb{D}$, $\delta_{J-\{j\}}$ is an atom, and $\delta_{J-\{j\}} \wedge \delta_l = \underline{0}$, $\delta_{J-\{j\}} \vee \delta_l = \underline{1}$. Again by the modularity, for any $I \subset J - \{j\}$, we have $\delta_{I \cup \{l\}} \neq \underline{0}$, that is $J' = (J - \{j\}) \cup \{l\}$ is a minimal set such that $\delta_{J'} = \underline{0}$. Therefore $\mathbb{D}_{J'} = \prod_{i \in J'} \mathbb{D}_i$. If we assume, without loss of generality, that $J = \underline{r}$, $j = 1$, $l = r+1$, then this implies that the relation

$$\mathbb{D}' = \{(a,b) \in \mathbb{D}_1 \times \mathbb{D}_{r+1} \mid (a, a_2, \ldots, a_r, b) \in \mathbb{D}_{\underline{r+1}}\}$$

is a graph of a one-to-one mapping $\varphi \colon \mathbb{D}_1 \to \mathbb{D}_{r+1}$. Since $\mathbb{D}_{\underline{r+1}}, \{a_2\}, \ldots, \{a_r\} \in \mathsf{Inv\ Clo}\ \mathbb{A}$, φ is an internal isomorphism.

Set $I'_s = I_s \cap J$ for $s \in \underline{k}$. The Claim implies that $I'_s \neq \varnothing$ for any s, and for $l \in I_s - I'_s$ we have $\delta_{J-\{j\}} \leq \delta_l$ whenever $j \notin I'_s$. Therefore $\delta_{I'_s} = \bigvee_{j \in J - I'_s} \delta_{J-\{j\}} \leq \delta_l$, and $\delta_{I_s} = \delta_{I'_s}$. Since $\delta_{I'_s} \wedge \bigvee_{t \in \underline{k}-\{s\}} \delta_{I'_t} = \underline{0}$ and $\delta_{I'_s} \vee \bigwedge_{t \in \underline{k}-\{s\}} \delta_{I'_t} = \underline{1}$, the analogous equalities hold for I_s, I_t that implies $\mathbb{D} = \mathbb{D}_{I_1} \times \ldots \times \mathbb{D}_{I_k}$. $\square$

Corollary 3.1 *Let $\mathbb{D}$ be a subdirect product of two-element subalgebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$, and the equivalence relation η on $\underline{n}$ defined by the rule $(i, j) \in \eta$ if and only if $\mathbb{D}_i$ is isomorphic to $\mathbb{D}_j$. Then $\mathbb{D} = \mathbb{D}_{I_1} \times \ldots \times \mathbb{D}_{I_k}$ where $I_1, \ldots, I_k$ are the blocks of η.*

Indeed, if $\mathbb{A}$ has only one two-element subalgebra then the corollary is obvious. Otherwise, since the intersection of two subalgebras is a subalgebra, each two-element subalgebra of $\mathbb{A}$ has a one-element subalgebra.

Corollary 3.2 *The set of relations $\mathbb{D}$ from the relational clone $\mathbf{C} = \mathsf{Inv\,Clo}\,\mathbb{A}$ with $|\mathbb{D}_i| = 2$ for each index i is contained in the relational clone generated by standard relations which belong to $\mathbf{C}$.*

3.3 Simple three-element Mal'tsev algebras

Simple Mal'tsev algebra of type **2** is polynomial equivalent to a three-element module, that is to the group $\mathbb{Z}_3$. It is well known [15],[1] that there exist six such algebras determined, in the presence of the relation T, by one-element subalgebras and automorphisms.

Let $\mathbb{A}$ be a simple Mal'tsev algebra of type **3**. Let also $\mathbb{D}$ be an irreducible subdirect product of subalgebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$ of $\mathbb{A}$. The algebra $\mathbb{A}$ may have at most three two-element subalgebras belonging to at most three isomorphism classes. Set $I = \{i \in \underline{n} \mid \mathbb{D}_i = \mathbb{A}\}$ and $I_j = \{i \in \underline{n} \mid \mathbb{D}_i$ belongs to jth isomorphism class$\}$, $j = 1, 2, 3$. If $\mathbb{A}$ has less than three subalgebras or isomorphism classes, we just assume that some of I_1, I_2, I_3 are empty. The next proposition follows easily from Lemmas 2.7, 2.6, Proposition 3.1, and also can be derived from results of [6, 11].

Proposition 3.2 *Let $\mathbb{D}$ be an irreducible subdirect product of subalgebras $\mathbb{D}_1, \ldots, \mathbb{D}_n$ of $\mathbb{A}$, then $\mathbb{D} = \mathbb{D}_I \times \mathbb{D}_{I_1} \times \mathbb{D}_{I_2} \times \mathbb{D}_{I_3}$. Moreover, $\mathbb{D}_I = \mathbb{A}^{|I|}$.*

Proposition 3.2 reduces the structure of relations preserved by operations of $\mathbb{A}$ to the structure of relations over its subalgebras. Therefore it gives in view of Corollary 3.2 the statement of Theorem 1.1 for this case.

Corollary 3.3 *The relational clone $\mathsf{Inv\ Clo}\ \mathbb{A}$ for a simple three-element Mal'tsev algebra $\mathbb{A}$ is generated by the standard relations from this clone.*

3.4 Three-element Mal'tsev algebras with a proper congruence

Without loss of generality we may assume that the only nontrivial congruence μ has blocks $\{0,1\}$ and $\{2\}$, that is $\mu = \mu_1$. Let $\mathbb{D}$ be a subdirect product of subalgebras $\mathbb{D}_1, \dots, \mathbb{D}_n$. We have to study two aspects of the structure of $\mathbb{D}$: the structure inside a coherent set, and the interaction between coherent sets. We start from the structure of $\mathbb{D}$ inside a coherent set.

Lemma 3.4 *Let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \dots, \mathbb{D}_n$, and I coherent set. Then all the algebras $\mathbb{D}_i$, $i \in I$, have the same cardinality.*

Proof. Supposing the contrary, we may assume that $1,2 \in I$, $\mathbb{D}_1 = \mathbb{A}$, $\mathbb{D}_2 = \{a,b\}$. By Lemma 2.7, $\mathsf{typ}(\underline{0}_2, \underline{1}_2) = \mathbf{2}$. For some $c,d \in \mathbb{D}_2$, $(0,c),(2,d) \in \mathbb{D}_{1,2}$. Consider the operation $\mathbf{g}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \mathbf{d}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0 \\ c \end{pmatrix}, \begin{pmatrix} 2 \\ d \end{pmatrix}\right)$. Notice that $2 = \mathbf{d}(0,0,2) \stackrel{\mu}{\equiv} \mathbf{d}(1,0,2)$ implies $\mathbf{d}(1,0,2) = 2$. Since $\mathbf{d}$ is a minority operation on $\mathbb{D}_2$, the operation $\mathbf{g}$ maps $\{0,1\} \times \{a,b\}$ to $\{2\} \times \{a,b\}$. This contradicts the assumption that I is a coherent set. $\square$

Lemma 3.5 *Let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \dots, \mathbb{D}_n$, I a coherent set, and $\mathbb{D}_i = \mathbb{A}$ for $i \in I$. Then for any $\mathbf{a} \in \mathbb{D}$, either $\{a_i \mid i \in I\} \subseteq \{0,1\}$ or $\{a_i \mid i \in I\} = \{2\}$.*

Proof. If $\mathsf{typ}(\underline{0}, \mu) = \mathbf{3}$, then $|I| = 1$ and the lemma holds trivially. So, let us assume $\mathsf{typ}(\underline{0}, \mu) = \mathbf{2}$. In particular, $\mathbf{d}$ is the minority operation on $\{0,1\}$.

As always we may assume that $1,2 \in I$. Suppose also that $a_1 \in \{0,1\}$, $a_2 = 2$, for certain $\mathbf{a} \in \mathbb{D}$. Let $\mathbf{f}(x)$ be a polynomial satisfying $(*)$ for the coherent set I. Since $(\underline{0}_i, \mu_i)$-minimal sets, $i = 1,2$, have no tails, $f_1(\mathbb{D}_1) = f_2(\mathbb{D}_2) = \{0,1\}$. Then $\mathbf{f}_{1,2}\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) = \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} \in \{0,1\}^2$. The operation $\mathbf{d}$ is the minority operation on $\{0,1\}$, therefore, for $\mathbf{g}_{1,2}(x) = \mathbf{d}\left(\mathbf{f}_{1,2}(x), \begin{pmatrix} a_1' \\ a_2' \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right)$ we have $g_1(x) = \mathbf{d}(f_1(x), a_1', a_1)$ is a permutation of $\{0,1\}$, while $g_2(0) = \mathbf{d}(f_2(0), a_2', 2) = \mathbf{d}(f_2(1), a_2', 2) = 2$. A contradiction with the assumption that I is a coherent set. $\square$

By Lemmas 3.4,3.5, $\mathbb{D}_I$ where I is a coherent set is either a subalgebra of a direct product of two-element algebras, or is of the form $\mathbb{D}' \cup \{(2,\dots,2)\}$ where $\mathbb{D}' \subseteq \{0,1\}^I$. In the first case, Corollary 3.2 implies that $\mathbb{D}_I$ belongs to the relational clone generated by standard relations from $\mathsf{Inv\ Clo}\ \mathbb{A}$. Consider the second case. Let $\mathbf{Q}$ be the set of all relations $\mathbb{D}$ of the form $\mathbb{D} = \mathbb{D}' \cup \{(2,\dots,2)\}$ where $\mathbb{D}' \subseteq \{0,1\}^n$, $n \in \mathbb{N}$.

Lemma 3.6 $\mathbf{Q} \cap \mathsf{Inv\ Clo}\ \mathbb{A} = \mathbf{Q} \cap \mathbf{R}$ *where* $\mathbf{R}$ *is the relational clone generated by those relations* $\{0,2\}, \{1,2\}, \psi_2, \mu, T_1$ *which belong to* $\mathsf{Inv\ Clo}\ \mathbb{A}$.

Proof. The inclusion $\supseteq$ is obvious. Denote the set $\mathbf{Q} \cap \mathsf{Inv\ Clo}\ \mathbb{A}$ by $\mathbf{S}$, and set $\mathbf{S}' = \{\mathbb{D}' \subseteq \{0,1\}^n \mid \mathbb{D}' \cup \{(2,\dots,2)\} \in \mathbf{S}, n \in \mathbb{N}\}$. First we prove that $\mathbf{S}'$ is a relational clone on $\{0,1\}$.

The equality relation belongs to $\mathbf{S}'$, because the equality relation on A is the union of that on $\{0,1\}$ and the tuple $\{(2,2)\}$. Take relations $\mathbb{D}_1 = \mathbb{D}_1' \cup \{(2,\dots,2)\},\dots,\mathbb{D}_n = \mathbb{D}_n' \cup \{(2,\dots,2)\}$ from $\mathbf{S}$, and let $P_1,\dots,P_n$, $P_1',\dots,P_n'$ be the predicates corresponding to $\mathbb{D}_1,\dots,\mathbb{D}_n$, $\mathbb{D}_1',\dots,\mathbb{D}_n'$ respectively. Let also a relation $\mathbb{D}'$ be determined by a pp-formula

$$P_{\mathbb{D}'}(x_1,\dots,x_k) = \exists x_{k+1},\dots,x_l\ \Phi'(x_1,\dots,x_l)$$

where the predicates occurring in the elementary conjunction Φ' are from $\{P_1',\dots,P_n'\}$. It is easy to see that the relation $\mathbb{D} = \mathbb{D}' \cup \{(2,\dots,2)\}$ is determined by the pp-formula

$$P_{\mathbb{D}}(x_1,\dots,x_k) = \exists x_{k+1},\dots,x_l\ \Phi(x_1,\dots,x_l) \wedge \bigwedge_{i,j \in \underline{l}} P_\mu(x_i,x_j)$$

where P_μ is the predicate corresponding to μ, and Φ is obtained from Φ' by replacing $P_1',\dots,P_n'$ with $P_1,\dots,P_n$. Since $\mathbb{D} \in \mathbf{S}$, we have $\mathbb{D}' \in \mathbf{S}'$ which means that $\mathbf{S}'$ is closed under pp-formulas and is a relational clone.

Moreover, if $\mathbb{D}_1',\dots,\mathbb{D}_n'$ generate $\mathbf{S}'$ then $\mathbb{D}_1,\dots,\mathbb{D}_n$ generate $\mathbf{S}$. By Proposition 2.1, $\mathbf{S}'$ is generated by a subset of $\{\{0\},\{1\},\psi_2', T_{0,1}\}$, therefore, $\mathbf{S}$ is generated by μ and the corresponding subset of $\{\{0,2\},\{1,2\},\psi_2,T_1\}$. The lemma is proved. $\square$

Corollary 3.4 *If* $\mathbf{C} = \mathsf{Inv\ Clo}\ \mathbb{A}$*,* $\mathbb{D} \in \mathbf{C}$ *is a subdirect product of* $\mathbb{D}_1, \ldots, \mathbb{D}_n$*, and* $I \subseteq \underline{n}$ *is a coherent set such that* $\mathbb{D}_i = \mathbb{A}$ *for* $i \in I$*, then* $\mathbb{D}_I$ *belongs to the subclone of* $\mathbf{C}$ *generated by standard relations from* $\mathbf{C}$*.*

Now, we study how the coherent sets of a relation may interact. Recall that, for a subdirect product $\mathbb{D}$ of $\mathbb{D}_1, \ldots, \mathbb{D}_n$ and $I \subseteq \underline{n}$, by θ_I we denote the congruence of $\mathbb{D}_I$ such that $\mathbf{a} \stackrel{\theta_I}{\equiv} \mathbf{b}$ if and only if there is $\mathbf{c} \in \mathbb{D}_{\underline{n}-I}$ such that $(\mathbf{a}, \mathbf{c}), (\mathbf{b}, \mathbf{c}) \in \mathbb{D}$.

Lemma 3.7 *Let* $\mathbb{D}$ *be a subdirect product of* $\mathbb{D}_1, \ldots, \mathbb{D}_n$*,* $I \subseteq \underline{n}$ *a coherent set, and* $\mathbb{D}_i = \mathbb{A}$ *for* $i \in I$*. Let also* $\overline{\mu}_I = \prod_{i \in I} \mu_i$ *where* μ_i *is the monolith of* $\mathbb{D}_i$*. Then* $\overline{\mu}_I \subseteq \theta_I$*.*

Proof. Suppose that $\overline{\mu}_I \not\subseteq \theta_I$ and $\mathbb{D}$ is a relation of the smallest arity such that there is a coherent set with this property. Suppose also $I = \{1, \ldots, k\}$.

Take $\mathbf{a}, \mathbf{a}' \in \mathbb{D}_I$ such that $\langle \mathbf{a}, \mathbf{a}' \rangle \in \overline{\mu}_I - \theta_I$. Since by Lemma 2.3 $I - \{k\}$ is a coherent set of $\mathbb{D}_{\underline{n}-\{k\}}$, and the arity of $\mathbb{D}_{\underline{n}-\{k\}}$ is less than the arity of $\mathbb{D}$, the restriction of $\overline{\mu}_I$ to $I - \{k\}$ is less or equal than that of θ_I. Therefore, there exists a''_k such that $(a_1, \ldots, a_{k-1}, a''_k) \stackrel{\theta_I}{\equiv} \mathbf{a}'$. By Lemma 3.5, $a''_k \stackrel{\mu_k}{\equiv} a_k$. Without loss of generality we may assume $a_k = 0, a''_k = 1$. Denote the tuple $(a_1, \ldots, a_{k-1})$ by $\mathbf{a}''$.

Take $\mathbf{b} = (b_{k+1}, \ldots, b_n), \mathbf{b}' = (b'_{k+1}, \ldots, b'_n) \in \mathbb{D}_{\underline{n}-I}$ such that $(\mathbf{a}'', 0, \mathbf{b})$, $(\mathbf{a}'', 1, \mathbf{b}') \in \mathbb{D}$. If $b_i \stackrel{\mu_i}{\equiv} b'_i$ for each i from the range $k+1$ to n where μ_i is the monolith of $\mathbb{D}_i$, then $(\mathbf{a}'', 0, \mathbf{b}), (\mathbf{a}'', 1, \mathbf{b}')$ are in the same $\overline{\mu}$-block, $\overline{\mu} = \prod_{i \in \underline{n}} \mu_i$. Denoting this $\overline{\mu}$-block by B, by Lemma 2.6, we have $B = B_I \times B_{\underline{n}-I}$, and hence $\langle \mathbf{a}, \mathbf{a}' \rangle \in \theta_I$. So, there is i such that $\mathbb{D}_i = \mathbb{A}$ and $b_i \stackrel{\mu}{\not\equiv} b'_i$. Without loss of generality we may assume $i = k+1$.

Let $J = \underline{n} - \{k+1\}$. Once more by the minimality of n there is $\mathbf{e} = (e_{k+2}, \ldots, e_n) \in \mathbb{D}_{J-I}$ such that $(\mathbf{a}, 0, \mathbf{e}), (\mathbf{a}'', 1, \mathbf{e}) \in \mathbb{D}_J$. Now, as $(\mathbf{a}'', 0, b_{k+2}, \ldots, b_n) \in \mathbb{D}_J$, the rectangularity of $\mathbb{D}_J$ implies that $(\mathbf{a}'', 1, b_{k+2}, \ldots, b_n) \in \mathbb{D}_J$, hence, there is $b''_{k+1} \in \mathbb{D}_{k+1}$ with $(a_1, \ldots, a_{k-1}, 1, b''_{k+1}, b_{k+2}, \ldots, b_n) \in \mathbb{D}$. If $b_{k+1} \stackrel{\mu_{k+1}}{\equiv} b''_{k+1}$ we get $\langle \mathbf{a}, \mathbf{a}' \rangle \in \theta_I$ as above; so $b_{k+1} \stackrel{\mu_{k+1}}{\not\equiv} b''_{k+1}$ that is either b_{k+1} or b''_{k+1} equals 2. Suppose $b''_{k+1} = 2$.

Let $\mathbf{c} \in \mathbb{D}$ be such that $c_k = 2$ (by Lemma 3.5, $c_i = 2$ for all $i \in I$). Denote $\mathbf{b}'' = (b_{k+2}, \ldots, b_n)$, $\mathbf{c}' = (c_1, \ldots, c_{k-1})$, $\mathbf{c}'' = (c_{k+2}, \ldots, c_n)$. It is easy to see that $(\mathbf{c}', 2, 2, \mathbf{c}'') \in \mathbb{D}$. Indeed, we are done if $c_{k+1} = 2$, and if $c_{k+1} \in \{0, 1\}$ we have

$$\mathbf{d}\left(\begin{pmatrix} \mathbf{a}'' \\ 1 \\ 2 \\ \mathbf{b}'' \end{pmatrix}, \begin{pmatrix} \mathbf{a}'' \\ 0 \\ b_{k+1} \\ \mathbf{b}'' \end{pmatrix}, \begin{pmatrix} \mathbf{c}' \\ 2 \\ c_{k+1} \\ \mathbf{c}'' \end{pmatrix}\right) = \begin{pmatrix} \mathbf{c}' \\ 2 \\ 2 \\ \mathbf{c}'' \end{pmatrix} \in \mathbb{D}.$$

Furthermore,

$$\mathbf{d}\left(\begin{pmatrix} \mathbf{a}'' \\ 0 \\ b_{k+1} \\ \mathbf{b}'' \end{pmatrix}, \begin{pmatrix} \mathbf{a}'' \\ 1 \\ 2 \\ \mathbf{b}'' \end{pmatrix}, \begin{pmatrix} \mathbf{c}' \\ 2 \\ 2 \\ \mathbf{c}'' \end{pmatrix}\right) = \begin{pmatrix} \mathbf{c}' \\ 2 \\ b_{k+1} \\ \mathbf{c}'' \end{pmatrix} \in \mathbb{D}.$$

Finally,

$$\mathbf{d}\left(\begin{pmatrix} \mathbf{a}'' \\ 1 \\ 2 \\ \mathbf{b}'' \end{pmatrix}, \begin{pmatrix} \mathbf{c}' \\ 2 \\ 2 \\ \mathbf{c}'' \end{pmatrix}, \begin{pmatrix} \mathbf{c}' \\ 2 \\ b_{k+1} \\ \mathbf{c}'' \end{pmatrix}\right) = \begin{pmatrix} \mathbf{a}'' \\ 1 \\ b_{k+1} \\ \mathbf{b}'' \end{pmatrix} \in \mathbb{D}.$$

Thus $(\mathbf{a}'', 0) \overset{\theta_I}{\equiv} (\mathbf{a}'', 1)$, a contradiction. $\square$

Let $\mathbb{D}$ be a subdirect product of $\mathbb{D}_1, \ldots, \mathbb{D}_n$, and J_D the set of those indices $i \in \underline{n}$ for which $\mathbb{D}_i = \mathbb{A}$. We denote by ν_i the congruence μ if $i \in J_D$, and $\underline{0}_i$ if $i \notin J_D$; then set $\mathbb{D}'_i = \mathbb{D}_i/_{\nu_i}$, $\nu_D = \nu_1 \times \ldots \times \nu_n$, and let $\mathbb{D}'$ denote the factor-algebra $\mathbb{D}/_{\nu_D}$. Clearly, $\mathbb{D}'$ can be treated as a subdirect product of $\mathbb{D}'_1, \ldots, \mathbb{D}'_n$. We say that $\mathbb{D}$ is *μ-closed* if for any $\mathbf{a} \in \mathbb{D}$ every tuple $\mathbf{b}$ such that $\mathbf{b} \overset{\nu_D}{\equiv} \mathbf{a}$ belongs to $\mathbb{D}$. For every relation $\mathbb{D}$ there exists a μ-closed relation $\overline{\mathbb{D}}$ such that $\overline{\mathbb{D}}' = \mathbb{D}'$. The last equality means that $\mathbb{D}$ and $\overline{\mathbb{D}}$ have the same image with respect to the canonical mapping of $\prod_{i \in \underline{n}} \mathbb{D}_i$ onto $\prod_{i \in \underline{n}} \mathbb{D}'_i$. So, we shall also say that $\overline{\mathbb{D}}$ is the *full preimage* of $\mathbb{D}'$.

Denote by $\mathbf{C}$ the clone $\mathsf{Inv}\,\mathsf{Clo}\,\mathbb{A}$, by $\mathbf{C}'$ the set $\{\mathbb{D}' \mid \mathbb{D} \in \mathbf{C}\}$, and by $\mathbf{D}$ the set of all μ-closed relations from $\mathbf{C}$. Let U be the set consisting of subalgebras, internal homomorphisms, the congruence μ, the automorphism ϱ_1, linear dependence $S_{0,1}$, and the relations determining the type of $(\mu, \underline{1})$ and of two-element subalgebras. More precisely, $U = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{2, 0\}, \mu, \psi'_0, \psi'_1, \psi'_2, \varrho_1, \varphi'_1, \ldots, \varphi'_6, \tau_{02}, \tau_{12}, S_{0,1}, T'_1, T_{0,1}, T_{0,2}, T_{1,2}\}$.

Lemma 3.8 *(1) For any $\mathbb{D} \in \mathbf{C}$ the relation $\overline{\mathbb{D}}$ belongs to $\mathbf{C}$.*

(2) $\mathbf{D}$ is contained in the clone $\mathbf{D}'$ generated by $S = \mathbf{C} \cap U$.

Proof. (1) Let $\mathbb{D} \in \mathbf{C}$ be n-ary, and $J_D = \underline{k}$. Then $\overline{\mathbb{D}}$ is defined by the formula

$$P_{\overline{\mathbb{D}}}(x_1, \ldots, x_n) = \exists y_1, \ldots, y_k P_{\mathbb{D}}(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) \wedge \bigwedge_{i \in \underline{k}} P_\mu(x_i, y_i).$$

This implies also that the full preimage of any relation from $\mathbf{C}'$ belongs to $\mathbf{C}$.

(2) We prove first that every irreducible relation from $\mathbf{D}'$ is μ-closed. Every relation $\mathbb{D}$ from $\mathbf{D}'$ is representable by a pp-formula $\exists \overline{y} \Phi(\overline{x}, \overline{y})$ in relations from S and the equality relation. If $\mathbb{D}$ is irreducible then the equality relation may be imposed on no pair of variables of the form x_i, x_j. If the equality relation is imposed on a pair of the form x_i, y_j or y_i, y_j then we replace all occurrences of y_j with x_i or y_i respectively, and remove y_j as well as the equality relation. The obtained pp-formula still produces $\mathbb{D}$. Therefore any irreducible relation from $\mathbf{D}'$ is representable by a pp-formula in relations from S. Finally, notice that all relations from S are μ-closed and applying pp-formulas preserves this property.

To complete the prove we have to show that, for any relation $\mathbb{D}$, there is a relation $\mathbb{C} \in \mathbf{D}'$ such that $\mathbb{D}' = \mathbb{C}'$. Take an n-ary relation $\mathbb{D}$ from $\mathbf{C}$. Since $|\mathbb{A}/_\mu| = 2$, by Lemma 2.6, $\mathbb{D}' = \mathbb{D}'_{I_1} \times \ldots \times \mathbb{D}'_{I_k}$ where $I_1, \ldots, I_k$ are the coherent sets of $\mathbb{D}'$. We consider several cases started from the simplest one.

Case 1. $\mathsf{typ}(\mu, \underline{1}) = \mathbf{3}$.

In this case each $\mathbb{D}'_{I_j}$ is either $\mathbb{A}/_\mu$, or a subalgebra of a direct product of two-element subalgebras from $\mathbb{A}$. By Corollary 3.2, subdirect products of two-element subalgebras can be derived from standard relations. Therefore, to get $\mathbb{C}$ we just have to multiply the subdirect product to the appropriate power of $\mathbb{A}$.

Case 2. $\mathbb{A}/_\mu$ is isomorphic to a subalgebra $\mathbb{B}_0$ of $\mathbb{A}$.

In this case, by Lemma 3.2, $\mathbb{D}'$ can be derived from a subdirect product of two-element subalgebras, and the graph of an isomorphism of $\mathbb{A}/_\mu$ onto $\mathbb{B}_0$. In its turn, by Corollary 3.2, the subdirect product of two-element subalgebras belongs to the clone generated by the relations from S.

Notice that if $\{0,2\}$ or $\{1,2\}$ is a subalgebra then it is isomorphic to $\mathbb{A}/_{\mu}$. So, we may assume that $\mathbb{A}$ has no two-element subalgebras but, possibly, $\mathbb{B} = \{0,1\}$.

Case 3. $\mathbb{A}$ has no two-element subalgebras.

In this case $\mathbb{D}' \in \mathsf{Inv\ Clo}\ \mathbb{A}/_{\mu}$, and by Proposition 1.1, $\mathsf{Inv\ Clo}\ \mathbb{A}/_{\mu}$ is generated by subalgebras, automorphisms, and the relation determining the type of $\mathbb{A}/_{\mu}$. Let

$$P_{\mathbb{D}'} = \exists \overline{y} \Phi(\overline{x}, \overline{y}) \tag{1}$$

be a pp-formula expressing $\mathbb{D}'$ via standard relations for $\mathbb{A}/_{\mu}$. The full preimages of standard relations in this case are $\{0,1\}, \{2\}, \varrho_1, T_1'$, and each of them belongs to $\mathbf{C}$ whenever the corresponding standard relation belongs to $\mathbf{C}'$. By replacing standard relations in (1) with their full preimages we get a relation $\mathbb{C} \in \mathbf{D}'$ such that $\mathbb{C}' = \mathbb{D}'$, as required.

Case 4. $\mathsf{typ}\mathbb{B} = \mathbf{3}$ or $\mathbb{B}$ has a trivial subalgebra.

If $\mathsf{typ}\mathbb{B} = \mathbf{3}$ then each $\mathbb{D}'_{I_l}$ is either $\mathbb{B}$, or a subdirect power of $\mathbb{A}/_{\mu}$. If $\mathbb{B}$ has a trivial subalgebra then by Proposition 3.1, $\mathbb{D}'$ can be decomposed in the direct product $\mathbb{D}'_{J_D} \times \mathbb{D}'_{\underline{n}-J_D}$. In both cases we may prove separately for subdirect powers of $\mathbb{B}$ and those of $\mathbb{A}/_{\mu}$. For subdirect powers of $\mathbb{B}$ we apply Proposition 2.1 while for subdirect powers of $\mathbb{A}/_{\mu}$ we argue as in Case 3.

Case 5. $\mathsf{typ}\mathbb{B} = \mathbf{2}$, $\mathbb{B}$ has no trivial subalgebras.

The operation $\mathbf{d}$ can be represented as the minority operation $x+y+z$ on both $\mathbb{A}/_{\mu}$ and $\mathbb{B}$ where $+$ is the operation of a two-element Abelian group. Therefore, the algebras $\widetilde{\mathbb{B}} = (B, \mathbf{d})$ and $\widetilde{\mathbb{A}} = (A/_{\mu}, \mathbf{d})$ are isomorphic. Denote by g the operation taking $0, 1$ to 0 and 2 to 2, and by φ the isomorphism of $\widetilde{\mathbb{A}}$ onto $\widetilde{\mathbb{B}}$ taking $0/_{\mu}$ to 0 and $2/_{\mu}$ to 1. Notice that g is a homomorphism of $(A, \mathbf{d})$ onto $(\{0,1\}, \mathbf{d})$.

Claim 1. For any (n-ary) $\mathbb{D} \in \mathbf{D}$, $\mathbb{D}_i = \mathbb{A}$ whenever $i \in \underline{k}$ and $\mathbb{D}_i = \mathbb{B}$ otherwise, $\mathbb{D}$ can be determined by a system of linear equations of the form $\alpha_1 g(x_1) + \ldots + \alpha_k g(x_k) + \alpha_{k+1} x_{k+1} + \ldots + x_n = a$ with $\alpha_1, \ldots, \alpha_n \in \{0,1\}$, $a \in \mathbb{B}$.

Since $\mathbb{D} \in \mathsf{Inv\ Clo}\ \mathbb{A} \subseteq \mathsf{Inv\ Clo}\ (A, \mathbf{d})$, we may prove the claim for the algebra $\mathbb{A}'' = (A, \mathbf{d})$. By Lemma 3.2, the relation $\mathbb{D}''$ consisting of all tuples

$\mathbf{a}'$ such that

$$a'_i = \begin{cases} \varphi(a_i), & \text{if } i \in \underline{k}, \\ a_i, & \text{otherwise,} \end{cases}$$

for $\mathbf{a} \in \mathbb{D}'$, belongs to $\mathsf{Inv}\,\mathsf{Clo}\,\widetilde{\mathbb{B}}$. Every relation on a two-element set invariant with respect to the minority operation is a coset of a linear space over GF(2), and therefore can be represented as the solution space of a certain linear system. Replacing in each equation $\alpha_1 x_1 + \ldots + \alpha_n x_n = a$ of the system the unknowns $x_1, \ldots, x_k$ with $\varphi(x_1), \ldots, \varphi(x_k)$ we get a system determining $\mathbb{D}'$; furthermore, replacing $\varphi(x_1), \ldots, \varphi(x_k)$ with $g(x_1), \ldots, g(x_k)$ we get what is required.

Let $\mathbf{D}_1$ be the sets of those relations from $\mathbf{D}$ that are determined by a single equation.

Claim 2. $\mathbf{D} \subseteq \langle \mathbf{D}_1 \rangle$.

Take $\mathbb{D} \in \mathbf{D}$, say, n-ary. By Lemma 3.3, as $\mathbb{D}'$ is a subdirect product of paraprimal algebras, there is $J \subseteq \underline{n}$ such that $\mathbb{D}'_J$ is isomorphic to $\mathbb{D}'$ and $\mathbb{D}'_J = \prod_{j \in J} \mathbb{D}'_j$. Then

$$P_{\mathbb{D}} = \bigwedge_{j \in \underline{n} - J} P_{\mathbb{D}^j}$$

where $\mathbb{D}^j = \mathbb{D}_{J \cup \{j\}}$. Without loss of generality let $J = \underline{k}$, and $j = k+1$. Then, since $\mathbb{D}'_J = \prod_{j \in J} \mathbb{D}'_j$, $\alpha_{k+1} = 1$ in any equation that can be derived from a system determining $\mathbb{D}^j$. Therefore, such a system may contain at most one equation.

The variables corresponding to those factors of $\mathbb{D}$ which are equal to $\mathbb{A}$ we shall call $\mathbb{A}$-*variables*, and the others $\mathbb{B}$-*variables*. In other words, g is applied to $\mathbb{A}$-variables.

Claim 3. If the relation $\mathbb{D}$ determined by $g(x_1) + \ldots + g(x_k) + x_{k+1} + \ldots + x_n = a$ belongs to $\mathbf{D}_1$, then the relation $\widetilde{\mathbb{D}}$ determined by $g(x_1) + \ldots + g(x_{k-1}) + x_{k+1} + \ldots + x_n = a$ and $\widetilde{\widetilde{\mathbb{D}}}$ determined by $g(x_1) + \ldots + g(x_k) + x_{k+3} + \ldots + x_n = a$ also belong to $\mathbf{D}_1$.

Indeed,

$$\begin{aligned} P_{\widetilde{\mathbb{D}}} &= \exists x_k P_{\mathbb{D}}(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) \wedge P_{\{0,1\}}(x_k); \\ P_{\widetilde{\widetilde{\mathbb{D}}}} &= \exists x_{k+1} P_{\mathbb{D}}(x_1, \ldots, x_k, x_{k+1}, x_{k+1}, x_{k+3}, \ldots, x_n). \end{aligned}$$

Claim 3 implies, in particular, that if $k < n$, then the relation determined by $x_1 = a$, or $x_1+x_2 = a$ belongs to $\mathbf{D}_1$. However, since $\mathbb{B}$ has no one-element subalgebra, the first case is impossible. So, the number of $\mathbb{B}$-variables with non-zero coefficients which occur in an equation determining a relation from $\mathbf{D}_1$ must be even, while the number of $\mathbb{A}$-variables may be arbitrary.

If a relation $\mathbb{D}$ from $\mathbf{D}_1$ is determined by an equation that contains either only $\mathbb{A}$-variables or only $\mathbb{B}$-variables, then $\mathbb{D}' = \mathbb{D}'_{J_D} \times \mathbb{D}'_{\underline{n}-J_D}$. Therefore, arguing as in Case 4 $\mathbb{D}$ can be derived from standard relations and $\mathbb{D} \in \mathbf{D}'$. Consider the remaining possibility.

Claim 4. If $\mathbf{D}_1$ contains a relation determined by $g_1(x_1) + \ldots + g(x_k) + x_{k+1} + \ldots + x_n = a$ where $0 < k < n$, then $\mathbf{D}_1$ contains $S_{0,1}$, and if $a = 1$ then it also contains $S'_{0,1}$ defined by $g(x_1) + x_2 + x_3 = 1$.

By Claim 3, if $\mathbf{D}_1$ contains a relation determined by $g_1(x_1)+\ldots+g(x_k)+x_{k+1}+\ldots+x_n = a$ where $0 < k < n$, then $\mathbf{D}_1$ contains the relation $\mathbb{C}$ defined by $g_1(x_1) + x_2 + x_3 = a$. So, if $a = 1$ we get $S'_{0,1}$. Then the relation $\widetilde{\mathbb{C}}$ such that $P_{\widetilde{\mathbb{C}}} = \exists y, z\, P_{\mathbb{C}}(x_1, x_2, y) \wedge P_{\mathbb{C}}(z, x_3, y) \wedge P_{\{0,1\}}$ is defined by the equation $g(x_1) + x_2 + x_3 = 0$ that is $\widetilde{\mathbb{C}} = S_{0,1}$.

There are two subcases.

Subcase 5.1. All equations determining relations $\mathbb{D} \in \mathbf{D}_1$ and containing both $\mathbb{A}$- and $\mathbb{B}$-variables are homogeneous.

We prove by induction that each relation defined by a homogeneous equation containing at least 3 variables and even number of $\mathbb{B}$-variables belongs to $\mathbf{D}_1$. The relation $S_{0,1}$ gives the base case of induction. Further, if $\mathbb{D}$ is defined by $g(x_1) + \ldots + g(x_k) + x_{k+1} + \ldots + x_n = 0$, then

$$\begin{aligned}
\widetilde{\mathbb{D}} &= \exists y P_{\mathbb{D}}(x_1, \ldots, x_k, y, x_{k+3}, \ldots, x_{n+1}) \wedge P_{S_{0,1}}(x_{k+1}, x_{k+2}, y);\\
\widetilde{\widetilde{\mathbb{D}}} &= \exists y P_{\mathbb{D}}(x_1, \ldots, x_{k-1}, y, x_{k+2}, \ldots, x_{n+1}) \wedge P_{S_{0,1}}(y, x_k, x_{k+1})
\end{aligned}$$

are the relations defined by the equations $g(x_1)+\ldots+g(x_{k+1})+x_{k+2}+\ldots+x_{n+1} = 0$ and $g(x_1) + \ldots + g(x_{k-1}) + x_k + \ldots + x_{n+1} = 0$ respectively.

Subcase 5.2. $\mathbf{D}_1$ contains a relation defined by a non-homogeneous equation containing both $\mathbb{A}$- and $\mathbb{B}$-variables.

In this case starting with $S'_{0,1}$ one may also get all relations defined by non-homogeneous equations in a similar way.

Thus, $\mathbf{D}_1$ and, therefore, $\mathbf{D}$ is generated by the standard relations from $\mathbf{D}$. The lemma is proved. $\square$

Now we are in a position to complete the proof of Theorem 1.1. Set $\mathbf{C} = \mathsf{Inv\ Clo}\ \mathbb{A}$, and let $\mathbf{D}$ be the relational clone generated by those standard relations which belong to $\mathbf{C}$. We have to prove $\mathbf{C} \subseteq \mathbf{D}$. Take $\mathbb{D} \in \mathbf{C}$, and let $\mathbb{D}$ be n-ary, $I_1, \ldots, I_k$ its coherent sets. By Lemma 3.8, $\overline{\mathbb{D}} \in \mathbf{D}$, and by Corollary 3.4 $\mathbb{D}_{I_1}, \ldots, \mathbb{D}_{I_k} \in \mathbf{D}$. Finally, by Lemma 3.7

$$P_{\mathbb{D}} = P_{\overline{\mathbb{D}}} \wedge \bigwedge_{j=1}^{k} P_{\mathbb{D}_{I_j}}(x_i)_{i \in I_j},$$

and Theorem 1.1 is proved.

3.5 Enumeration of three-element Mal'tsev algebras

First we state some dependencies between subalgebras, automorphisms, and other standard relations. Each of these dependencies is either obvious or can be checked by a short calculation. We use the same notation as in the previous section.

Proposition 3.3 *1. The family of subalgebras is a lattice of subsets.*
2. The family of internal homomorphisms is a semigroup of mappings containing the identity mapping, and the inverse of each bijective member.
3. If $\mathbb{B}$ is a two-element algebra of type $\mathbf{2}$, and each its element form a subalgebra, then $\mathbb{B}$ has a non-trivial automorphism.
4. The restriction of an automorphism of an algebra to a subalgebra is an internal isomorphism.
5. A fixed point of a non-identical automorphism forms a subalgebra.

Now, let us assume that $\mathbb{A}$ has a proper congruence.

Proposition 3.4 *Let $\mu = \mu_1$ be a congruence of $\mathbb{A}$.*
1. If $\mathbb{B} \neq \{0,1\}$ is a two-element subalgebra of $\mathbb{A}$, then $\mathbb{A}/_{\mu}$ is isomorphic to $\mathbb{B}$.
2. 2 is a fixed point of any automorphism of $\mathbb{A}$.
3. If both $\{0,2\}, \{1,2\}$ are subalgebras, then φ_5' is an internal isomorphism.

4. If $\{0\}$ or $\{1\}$ is a subalgebra, then $\{0,1\}$ is also a subalgebra.
5. If $\mathbb{B} = \{0,1\}$ is a subalgebra, ψ_2' is its automorphism, and $\mathbb{A}/_{\mu} \cong \mathbb{B}$, then $\mathbb{A}/_{\mu}$ has a nontrivial automorphism.
6. If $S_{0,1}$ is a subalgebra of $\mathbb{A}^3$, then $\mathbb{B} = \{0,1\}$ is a subalgebra of $\mathbb{A}$, and $\mathsf{typ}(\mu, \underline{1}) = \mathsf{typ}\ \mathbb{B} = \mathbf{2}$.
7. If $\mathbb{B} = \{0,1\}$ is a subalgebra of $\mathbb{A}$, $\mathsf{typ}(\mu, \underline{1}) = \mathsf{typ}\ \mathbb{B} = \mathbf{2}$, and $\mathbb{A}/_{\mu} \cong \mathbb{B}$, then $S_{0,1}$ is a subalgebra of $\mathbb{A}^3$.
8. Suppose that $\mathbb{B} = \{0,1\}$ is a subalgebra of $\mathbb{A}$, and $\mathsf{typ}(\mu, \underline{1}) = \mathbf{2}$. Then the following conditions are equivalent: (a) $\mathbb{A}/_{\mu} \cong \mathbb{B}$; (b) $S_{0,1}$ is a subalgebra of $\mathbb{A}^3$ and $\mathbb{A}$ has a one-element subalgebra $\{0\}$ or $\{1\}$.
9. If $S_{0,1}$ is a subalgebra of $\mathbb{A}^3$, then the following conditions are equivalent: (a) $\{2\}$ is a subalgebra; (b) $\mathbb{B} = \{0,1\}$ has a non-identical automorphism.
10. If $\{0,2\}, \{1,2\}$ are subalgebras and $\mathsf{typ}(\underline{0}, \mu_1) = \mathbf{2}$, then ψ_2 is an automorphism.

It turns out that, for any combination of subalgebras, internal homomorphisms, equivalence relations, types of prime factors, and relations of the form $S_{a,b}$, which satisfy the restrictions stated, there exists a three-element Mal'tsev algebra, that has exactly this family of standard relations. The keen reader may check, that basic operations for such algebras can be chosen from the operations listed in Tables 1–3. Tables 4,5 indicates which standard relations are invariant under these operations. The operations from Tables 1–3 allow one to construct simple Mal'tsev algebras and algebras with the congruence μ_1. To construct Mal'tsev algebras with another non-trivial congruence one have to apply an appropriate permutation of $\{0,1,2\}$ to the listed operations.

Operations $h = x - y + z$, g, and d are Mal'tsev operations of a simple affine, a simple non-affine, and a non-simple 3-element algebras. The operations $h_O = 0, h_I = 1, h_{II} = 2, g_I, \ldots, g_X$, and $f_I, \ldots, f_{VII}$ control the family of subalgebras; $g_{322}, g_{332}, g_{333}, f_{232}, f_{223}, f_{332}, f_{333}$ determine the types of two-element subalgebras when the algebra is simple, and the types of $\mathbb{B} = \{0,1\}$ and of $(\underline{0}, \mu_1), (\mu_1, \underline{1})$ otherwise; the operations $h_\alpha = x + 1$ and $g_\alpha, \ldots, g_\lambda, f_\alpha, \ldots, f_\kappa$ control the internal homomorphisms of the algebra; finally, f_S destroys the relation $S_{0,1}$. Of course, the operations $g_I, \ldots, g_X$, $f_I, \ldots, f_{VII}$, $g_\alpha, \ldots, g_\lambda$, $f_\alpha, \ldots, f_\kappa$ do not allow us to represent all possible families of subalgebras and internal homomorphisms. Sometimes we need operations obtained from them by permuting elements $0, 1, 2$. However, using the listed operations we are able to represent all possible configurations of subalgebras and internal homomorphisms. For example, to get a simple algebra with 3 two-element subalgebras, two of which are isomorphic and of type **3** without the non-identical automorphism, and the third is of type **2**, we may consider the algebra $(\{0, 1, 2\}; g, g_\eta)$ or the algebra $(\{0, 1, 2\}; g, g_\theta)$ depending on whether the internal isomorphism has a fixed point or not. In Table 3 $f^0_{232}, f^1_{232}, f^2_{232}$ denote the operations $f_{232}(x, y, z, 0), f_{232}(x, y, z, 1)$, $f_{232}(x, y, z, 2)$ for the 4-ary operation f_{232}. When constructing Tables 1–5 the Universal Algebra Calculator by E. Kiss and R. Freese was used.

From such an enumeration some interesting facts can be derived.

Corollary 3.5 *There are 5 minimal Mal'tsev clones on a three-element set. One of them is generated by the operation h and is the clone of term operations of a simple affine algebra; another one is generated by g and is the clone of term operations of a simple non-affine algebra; the three remaining are generated by d or similar operations for the equivalence relations μ_2, μ_3, and are the clones of term operations of solvable algebras with a proper congruence.*

Corollary 3.6 *There are 1129 term inequivalent 3-element Mal'tsev algebras.*

References

[1] J. Bagyinszki, J. Demetrovics, *The lattice of linear classes in prime-valued logics,* Banach Center Publications (Warszawa) **7**(1982), 105–123.

[2] K. A. Baker and A. Pixley, *Polynomial interpolation and the Chinese remainder theorem for algebraic systems,* Math. Z., **143**(1975), 165–174.

[3] V. G. Bodnarchuk, L. A. Kalužnin, V. N. Kotov, B. A. Romov, *Galois Theory for Post algebras. I,II,* Kibernetika, **3,5**(1969), 1–10, 1–9 (Russian).

[4] A. A. Bulatov, P. M. Idziak *Counting Mal'tsev clones on small sets,* Discrete Math. (to appear).

[5] D. Clark, P. Krauss, *Para primal algebras,* Algebra Universalis, **6**(1976), no. 2, 165–192

[6] D. Clark, P. Krauss, *Varieties generated by para primal algebras,* Algebra Universalis, **7**(1977), no. 1, 93–114.

[7] D. Geiger, *Closed systems of functions and predicates,* Pacific J. Math. **27**(1968), 95–100.

[8] D. Hobby, R. McKenzie, *The structure of finite algebras,* Amer. Math. Soc. Contemporary Mathematics, vol. 76, Providence Rhode Island 1988.

[9] P. M. Idziak, *Clones with Malcev operations,* International Journal of Algebra and Computation, **9**(1999), 213–226.

[10] A. A. Krokhin, K. L. Safin, E. V. Sukhanov, *The structure of the lattice of closed classes of polynomials,* Diskret. Mat. **9**(1997), no. 2, 24–39 (Russian); [translation in Discrete Math. Appl. **7**(1997), no. 2, 131–146].

[11] R. McKenzie, *Para primal varieties: A study of finite axiomatizability and definable principal congruences in locally finite varieties,* Algebra Universalis, **8**(1978), no. 3, 336–348.

[12] M. Porębska, *Interpolation for fragments of intermediate logics,* Reports on Mathematical Logic, **21**(1987), 9–14.

[13] R. Pöschel, L. A. Kalužnin, *Funktionen- und Relationenalgebren,* Berlin 1979.

[14] E. L. Post, *The two-valued iterative systems of mathematical logic,* Annals of Math. Studies, no.5, Princeton University Press, N.J., 1941.

[15] A. A. Salomaa *On infinitely generated sets of operations in finite algebras.,* Ann. Univ. Turku, Ser. AI 74, 1964, 1–12.

[16] Á. Szendrei, *Clones in Universal Algebra,* University of Montreal, 1986, Seminaires de Mathematiques Superieures, vol. 99.

x	0	1	2
g_{II}	0	1	0
g_{III}	0	0	0
g_{VI}	1	0	2
g_{VII}	0	0	1
g_{VIII}	1	0	0
f_{III}	2	2	2
f_{IV}	2	2	0
f_{V}	1	1	1
f_{VI}	0	1	0
f_{VII}	0	0	2

Table 1

x	0	0	0	1	1	1	2	2	2
y	0	1	2	0	1	2	0	1	2
g_{α}	0	1	0	1	1	2	0	2	2
g_{β}	0	0	0	0	1	1	0	1	2
g_{γ}	0	0	0	1	1	2	2	1	2
g_{δ}	0	0	0	1	1	2	2	2	2
g_{η}	0	0	0	0	1	1	0	2	2
g_{θ}	0	0	2	0	1	1	2	2	2
g_{I}	0	0	0	1	1	0	2	0	2
g_{IV}	0	2	1	2	1	0	1	0	2
g_{V}	0	0	1	1	1	0	1	0	2
g_{IX}	0	2	1	2	0	0	1	0	0
g_{X}	1	2	1	2	2	0	1	0	0
f_{I}	0	0	0	1	1	0	2	2	2
f_{II}	0	0	1	1	1	0	2	2	2
f_{α}	0	1	0	1	1	1	2	2	2
f_{β}	0	0	1	0	0	1	2	2	2
f_{γ}	1	1	0	1	1	0	2	2	2
f_{δ}	0	0	2	1	1	2	2	2	2
f_{η}	0	0	2	1	1	2	0	1	2
f_{θ}	0	0	2	0	1	2	2	2	2
f_{S}	1	1	2	0	0	2	0	1	2

Table 2

x	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
y	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
z	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
g	0	1	2	1	0	0	2	0	0	1	0	1	0	1	2	1	2	1	2	2	0	2	2	1	0	1	2
g_{322}	0	0	2	0	1	0	2	0	0	0	1	1	1	1	2	1	2	1	2	2	0	2	2	1	0	1	2
g_{332}	0	0	0	0	1	0	0	0	2	0	1	1	1	1	2	1	2	1	0	2	2	2	2	1	2	1	2
g_{333}	0	0	0	0	1	0	0	0	2	0	1	1	1	1	1	1	1	2	0	2	2	2	1	2	2	2	2
g_κ	0	1	2	1	0	0	2	1	0	1	0	2	0	1	2	1	2	1	2	2	0	0	2	1	0	1	2
g_λ	0	1	2	1	0	0	2	1	0	1	0	1	0	1	2	0	2	1	2	0	0	1	2	1	0	1	2
d	0	1	2	1	0	2	2	2	0	1	0	2	0	1	2	2	2	1	2	2	0	2	2	1	0	1	2
f^0_{232}	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
f^1_{232}	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
f^2_{232}	0	1	0	1	0	0	0	0	0	1	0	1	0	1	1	1	1	1	2	2	2	2	2	2	2	2	2
f_{223}	0	0	0	0	0	0	0	0	2	1	1	1	1	1	1	1	1	2	0	1	2	0	1	2	2	2	2
f_{333}	0	0	0	0	1	0	0	0	2	0	1	1	1	1	1	1	1	2	0	0	2	1	1	2	2	2	2
f_{332}	0	0	2	0	1	2	0	0	0	0	1	2	1	1	2	1	1	1	2	2	0	2	2	1	0	1	2
f_κ	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	0	1	1	2	2	2	2	2	2	2	2	2

Table 3

	0	1	2	01	12	02	T_{01}	T_{02}	T_{12}	φ_1	ψ_0	ψ_1	ψ_2	ψ'_0	ψ'_1	ψ'_2	φ'_1	φ'_2	φ'_3	φ'_4	φ'_5	φ'_6
h	+	+	+	-	-	-	-	-	-	+	+	+	+	-	-	-	-	-	-	-	-	-
h_O	+	-	-	+	-	+	+	+	-	-	+	-	-	-	-	-	+	-	-	-	-	-
h_I	-	+	-	+	+	-	+	-	+	-	-	+	-	-	-	-	-	-	-	+	-	-
h_{II}	-	-	+	-	+	+	-	+	+	-	-	-	+	-	-	-	-	-	-	-	+	-
h_α	-	-	-	-	-	-	-	-	-	+	-	-	-	-	-	-	-	-	-	-	-	-
g	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
g_{322}	+	+	+	+	+	+	-	+	+	-	-	-	+	+	+	+	-	-	-	-	+	+
g_{332}	+	+	+	+	+	+	-	-	+	-	+	-	-	+	+	+	+	+	-	-	-	-
g_{333}	+	+	+	+	+	+	-	-	-	+	+	+	+	+	+	+	+	+	+	+	+	+
g_I	+	+	+	+	-	+	+	+	-	-	+	-	-	-	+	+	+	+	-	-	-	-
g_{II}	+	+	-	+	-	+	+	+	-	-	-	-	-	-	-	+	-	-	-	-	-	-
g_{III}	+	-	-	+	-	+	+	+	-	-	+	-	-	-	-	-	+	-	-	-	-	-
g_{IV}	+	+	+	-	-	-	-	-	-	+	+	+	+	-	-	-	-	-	-	-	-	-
g_V	+	+	+	+	-	-	+	-	-	-	-	-	+	-	-	+	-	-	-	-	-	-
g_{VI}	-	-	+	+	-	-	+	-	-	-	-	-	+	-	-	+	-	-	-	-	-	-
g_{VII}	+	-	-	+	-	-	+	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
g_{VIII}	-	-	-	+	-	-	+	-	-	-	-	-	-	-	-	+	-	-	-	-	-	-
g_{IX}	+	-	-	-	-	-	-	-	-	-	+	-	-	-	-	-	-	-	-	-	-	-
g_X	-	-	-	-	-	-	-	-	-	+	-	-	-	-	-	-	-	-	-	-	-	-
g_α	+	+	+	+	+	+	-	-	-	+	-	-	-	-	-	-	-	+	+	-	-	+
g_β	+	+	+	+	+	+	-	-	-	+	-	-	-	-	-	-	+	-	-	+	+	-
g_γ	+	+	+	+	+	+	+	+	+	-	+	-	-	+	+	+	+	+	-	-	-	-
g_δ	+	+	+	+	+	+	+	+	-	-	-	-	-	-	+	+	+	+	-	-	-	-
g_η	+	+	+	+	+	+	-	-	+	-	+	-	-	+	-	-	+	-	-	-	-	-
g_θ	+	+	+	+	+	+	-	-	+	-	+	-	-	+	-	-	-	+	-	-	-	-
g_κ	+	+	+	+	+	+	+	+	+	+	-	-	-	+	+	+	+	+	+	+	+	+
g_λ	+	+	+	+	+	+	+	+	+	-	-	-	+	+	+	+	+	+	+	+	+	+

Table 4

	0	1	2	01	12	02	ψ_2	τ_1	τ_2	ψ'_2	ϱ_1	S_{01}	T_{01}	T_1	T'_1
d	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
f_{232}	+	+	+	+	+	+	+	+	+	+	+	+	+	-	+
f_{223}	+	+	+	+	+	+	+	-	-	+	+	-	+	+	-
f_{333}	+	+	+	+	+	+	+	+	+	+	+	-	-	-	-
f_{332}	+	+	+	+	+	+	+	-	-	+	+	-	-	-	+
f_I	+	+	+	+	-	+	-	+	+	+	+	+	+	+	+
f_{II}	+	+	+	+	-	-	+	+	+	+	+	+	+	+	+
f_{III}	-	-	+	-	+	+	+	-	-	-	-	-	-	+	+
f_{IV}	-	-	-	-	-	+	-	-	-	-	+	-	-	+	+
f_V	-	+	-	+	+	-	-	-	+	-	-	+	+	+	+
g_{III}	+	-	-	+	-	+	-	+	-	-	-	+	+	+	+
f_{VI}	+	+	-	+	+	-	-	-	-	+	-	+	+	+	+
g_{VI}	-	-	+	+	-	-	+	-	-	+	+	+	+	+	+
f_{VII}	+	-	+	+	-	+	-	-	-	-	+	+	+	+	+
f_α	+	+	+	+	+	+	-	-	-	-	+	-	-	-	+
f_β	+	-	+	+	-	-	-	-	-	-	+	+	+	+	+
f_γ	-	+	+	+	-	-	-	-	-	-	+	+	+	+	+
f_δ	+	+	+	+	+	+	+	-	-	+	-	-	+	+	-
f_η	+	+	+	+	+	+	+	-	-	+	+	-	+	+	+
f_θ	+	+	+	+	+	+	-	-	+	-	-	-	-	-	-
f_S	-	-	+	+	-	-	+	-	-	+	+	-	+	+	+
f_κ	+	+	+	+	+	+	-	+	+	+	+	+	+	-	+

Table 5