

Constraint Satisfaction Problems over semilattice block Mal'tsev algebras

Andrei A. Bulatov

Abstract—There are two well known types of algorithms for solving CSPs: local propagation and generating a basis of the solution space. For several years the focus of the CSP research has been on ‘hybrid’ algorithms that somehow combine the two approaches. In this paper we present a new method of such hybridization that allows us to solve certain CSPs that has been out of reach for a quite a while. We consider these method on a fairly restricted class of CSPs given by algebras we will call semilattice block Mal'tsev. An algebra \mathbb{A} is called semilattice block Mal'tsev if it has a binary operation f , a ternary operation m , and a congruence σ such that the quotient \mathbb{A}/σ with operation f is a semilattice, f is a projection on every block of σ , and every block of σ is a Mal'tsev algebra with Mal'tsev operation m . We show that the constraint satisfaction problem over a semilattice block Mal'tsev algebra is solvable in polynomial time.

I. INTRODUCTION

The study of the complexity of the Constraint Satisfaction Problem (CSP, for short) has been initiated by Schaefer [32]. In that paper Schaefer studied the complexity of $\text{CSP}(\Gamma)$, the CSP parametrized by a set Γ of allowed constraints over a certain set, often called a constraint language. More precisely he determined the complexity of $\text{CSP}(\Gamma)$ for constraint languages on a 2-element set. The complexity of problems $\text{CSP}(\Gamma)$ for constraint languages over finite sets has been attracting much attention since then. This research is guided by the Dichotomy Conjecture proposed by Feder and Vardi [19], [20] that states that every CSP of the form $\text{CSP}(\Gamma)$ for a constraint language Γ on a finite set is either solvable in polynomial time or is NP-complete. The Dichotomy Conjecture has been restated and made more precise in different languages, see, e.g. [12], [30]. Also, several powerful approaches to the problem have been developed, through algebra, logic, and graph theory. So far the most successful method of studying the complexity of the CSP has been the algebraic approach introduced by Jeavons et al. [11], [12], [14], [24]. This approach relates the complexity of $\text{CSP}(\Gamma)$ to the properties of a certain universal algebra \mathbb{A}_Γ associated with Γ . In particular it allows one to expand $\text{CSP}(\Gamma)$ to the problem $\text{CSP}(\mathbb{A}_\Gamma)$ depending only on the associated algebra without changing its complexity. It therefore suffices to restrict ourselves to the study of the complexity of problems of the form $\text{CSP}(\mathbb{A})$, where \mathbb{A} is a finite universal algebra.

Although the dichotomy conjecture remains open in general, it has been confirmed in a number of cases: for constraint languages on 2- and 3-element sets [7], [32] (a dichotomy result was also announced for languages over 4-, 5-, and 7-element sets [26], [33], [34]), for constraint languages containing all unary relations [1], [8], [9], and several others, see, e.g. [2],

[3], [23]. One of the most remarkable phenomena discovered is that, generally, there are only two types of algorithms applicable to CSPs solvable in polynomial time. The first one has long been known to researchers in Artificial Intelligence as constraint propagation [18]. Algorithms of the other type resemble Gaussian elimination in the sense that they construct a small generating set of the set of all solutions [10], [23]. The scope of both types of algorithms is precisely known [2], [23].

General dichotomy results, however, cannot be proved using only algorithms of a single ‘pure’ type. In all such results, see, e.g. [1], [7], [8], [9] a certain mix of the two types of algorithms is needed. In some cases, for instance, [7] such a hybrid algorithm is somewhat ad hoc; in other cases, [1], [8], [9] it is based on intricate decompositions of the problem instance. It is clear however that ad hoc hybridization and the decomposition techniques developed in the mentioned works are not sufficient. Therefore trying to identify new polynomial time solvable cases of the CSP through combining the two types of algorithms is the key to approaching the Dichotomy Conjecture. There have been several further attempts to design hybrid algorithms; however, most of them were not quite successful. In more successful cases such as [27], [28], [29], [31] the researchers tried to tackle somewhat limited cases, in which a combination of local consistency properties and Gaussian elimination type fragments is very explicit. To provide the context for our results we explain those cases in details.

Suppose an idempotent algebra \mathbb{A} is such that it has a congruence σ with the property that the CSP of its quotient \mathbb{A}/σ can be solved by, say, a local propagation algorithm, while for every σ -block \mathbb{B} (a subalgebra of \mathbb{A}) the CSP over \mathbb{B} can be solved by the small generating set algorithm; or the other way round, see Figure 1. How can one solve the CSP over \mathbb{A} itself? Maroti in [28] considered the second case, when \mathbb{A}/σ can be solved by the small generating set algorithm, say, it is Mal'tsev. This case turns out to be easier because of the property of the σ -blocks we can exploit. Suppose for simplicity that every σ -block \mathbb{B} is a semilattice, as shown in Figure 1. Then every CSP instance on \mathbb{B} has some sort of a canonical solution that assigns the maximal element of the semilattice (that is element $a \in \mathbb{B}$ such that $ab = a$ for all $b \in \mathbb{B}$) to every variable. It then can be shown that if we find a solution $\varphi : V \rightarrow \mathbb{A}/\sigma$ where V is the set of variables of the instance on \mathbb{A}/σ , and then assigning the maximal elements of the σ -block $\varphi(v)$ to v , we obtain a solution of the original instance.

The case when \mathbb{A}/σ is a semilattice, while every σ -block is Mal'tsev is much more difficult. We will call such algebras *semilattice block Mal'tsev* algebras (SBM algebras, for short). More precisely, we consider idempotent algebras \mathbb{A} with the

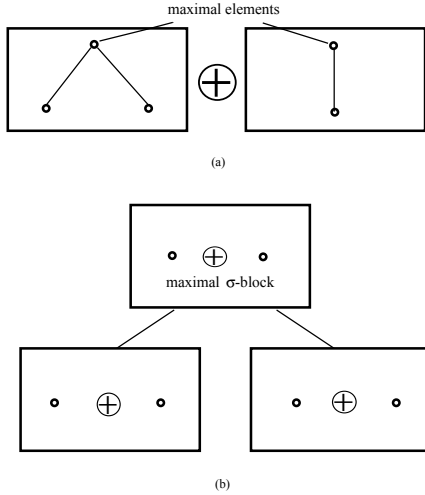


Fig. 1. (a) Algebra \mathbb{A} such that \mathbb{A}/σ is Mal'tsev; (b) an SBM algebra. Rectangles represent σ -blocks, dots represent elements, lines show the semilattice structure, and \oplus represents a Mal'tsev operation acting on elements or σ -blocks.

following property: There are a binary operation f and a ternary operation m , and a congruence σ of \mathbb{A} such that \mathbb{A}/σ is a semilattice with a semilattice operation f , and every σ -block B is a Mal'tsev algebra with Mal'tsev operation m , and $f|_B$ is a projection. The main difficulty with this kind of algebras is that the only solution of a CSP over a semilattice we can reliably find is the canonical one assigning the maximal available element to each variable. Finding a second solution is already hard. On the other hand, if we restrict our instance only to the maximal σ -block \mathbb{B} , it may have no solution there, even though the original instance has a solution, which simply does not belong to the maximal block. If this is the case, it has been unclear for nearly 10 years how the domain can be reduced so that the maximal block is eliminated.

The problem has been resolved in some special cases. Firstly, Maroti in [29] showed that it suffices to consider SBM algebras of a certain restricted type. We will use this result in this paper. Marcovic and McKenzie suggested an algorithm that solves the CSP over SBM algebras \mathbb{A} when \mathbb{A}/σ is a chain, that is, $ab \in \{a, b\}$ for any $a, b \in \mathbb{A}/\sigma$. In this case their algorithm is capable of eliminating the maximal block using the fact that if a semilattice is a chain, any of its subsets is a subalgebra. Finally, very recently Payne in [31] suggested an algorithm that works for a more general class of algebras than SBM, but algebras in this class have to satisfy an extra condition that in SBM algebras manifests itself as the existence of certain well behaving mappings between σ -blocks. In particular, this condition guarantees that the instance restricted to the maximal σ -block has a solution whenever the original problem has a solution.

In this paper we continue the effort started in [27], [29], [31] and present an algorithm that solves the CSP over an arbitrary SBM algebra.

Theorem 1: If \mathbb{A} is a SBM algebra then $\text{CSP}(\mathbb{A})$ is solvable in polynomial time.

The algorithm is based upon a new local consistency notion that we call *block-minimality* (although in our case it is

necessarily not quite local, since it has to deal with Mal'tsev algebras). More specifically, our algorithm first separates the set V of variables of a CSP instance into overlapping subsets, so-called coherent sets, and considers subproblems on these sets of variables. For block-minimality these subproblems have to be minimal, that is, every tuple from every constraint relation has to be a part of a solution. This can be achieved by solving the problem many times with additional constraints. However, this is not very straightforward, because coherent sets may contain all the variables from V . To overcome this problem we show that the subproblems restricted to coherent sets are either over a Mal'tsev domain and therefore can be solved efficiently, or they split up into a collection of disjoint instances, each of which has a strictly smaller domain. In the latter case we can recurse on these smaller instances. Finally, we prove that any block-minimal instance has a solution.

The results of this paper can easily be made more general by removing some of the restrictions on the basic functions of SBM algebras. However, we hope that these results can be generalized well beyond SBM-like algebras and so we stop short of giving more general but also more technically involved proofs just restricting ourselves to demonstrating the general idea.

In Section II we recall the basic definitions and study certain properties of SBM algebras. In Section V we strengthen the results of [5] about the structure of relations over Mal'tsev algebras and extend them to SBM algebras¹. In Section VI we extend these notions to CSP instances. Finally, in Section VII we prove the main results and present a solution algorithm.

Omitted proofs can be found in the full version of the paper [16].

II. PRELIMINARIES

For a detailed introduction to CSP and the algebraic approach to its structure the reader is referred to a very recent and very nice survey by Barto et al. [4]. Basics of universal algebra can be learned from the textbook [17] and monograph [22].

A. Multisorted Constraint Satisfaction Problem

By $[n]$ we denote the set $\{1, \dots, n\}$. Let A_1, \dots, A_n be finite sets. Tuples from $A_1 \times \dots \times A_n$ are denoted in boldface, say, \mathbf{a} , and their entries by $\mathbf{a}[1], \dots, \mathbf{a}[n]$. A relation R over A_1, \dots, A_n is a subset of $A_1 \times \dots \times A_n$. We refer to n as the *arity* of the tuple \mathbf{a} and the relation R . Let $I = (i_1, \dots, i_k)$ be an (ordered) multiset, a subset of $[n]$. Then let $\text{pr}_I \mathbf{a} = (\mathbf{a}[i_1], \dots, \mathbf{a}[i_k])$ and $\text{pr}_I R = \{\text{pr}_I \mathbf{a} \mid \mathbf{a} \in R\}$. Relation R is said to be a *subdirect product* of A_1, \dots, A_n if $\text{pr}_i R = A_i$ for $i \in [n]$. In some cases it will be convenient to consider tuples and relations whose entries are indexed by sets other than subsets of $[n]$, most often those will be sets of variables. Then we either assume the index set is somehow ordered, or

¹Kearnes and Szendrei in [25] developed a technique based on so-called critical relations that resembles in certain aspects what can be achieved through coherent sets. However, [25] only concerns congruence modular algebras, and so cannot be used for SBM algebras.

consider tuples as functions from the index set to the domain and relations as sets of such functions.

Let \mathcal{A} be a set of sets, in this paper \mathcal{A} is usually the set of universes of finite algebras derived from an SBM algebra; we clarify ‘derived’ later. An instance of a (*Multisorted*) *Constraint Satisfaction Problem* (CSP) over \mathcal{A} is given by $\mathcal{P} = (V, \mathcal{D}, \mathcal{C})$, where V is a set of *variables*, \mathcal{D} is a collection of *domains* $D_v \in \mathcal{A}$, and \mathcal{C} is a set of *constraints*; every constraint $\langle \mathbf{s}, R \rangle$ is a pair consisting of an ordered multiset $\mathbf{s} = (v_1, \dots, v_k)$, a subset of V , called the *constraint scope* and R , a relation over D_{v_1}, \dots, D_{v_k} , called the *constraint relation*.

Let \mathfrak{A} be a class of finite algebras of the same type and \mathcal{A} the set of universes of algebras from \mathfrak{A} . Then $\text{CSP}(\mathfrak{A})$ is the class of instances $(V, \mathcal{D}, \mathcal{C})$ of CSPs over \mathcal{A} such that every constraint relation R from $\langle \mathbf{s}, R \rangle \in \mathcal{C}$, $\mathbf{s} = (v_1, \dots, v_k)$, is a subalgebra of $D_{v_1} \times \dots \times D_{v_k}$, where $D_v, v \in V$, are viewed as algebras from \mathfrak{A}

Let $W \subseteq V$. By \mathcal{P}_W we denote the instance $(W, \mathcal{D}^W, \mathcal{C}^W)$ defined as follows: $D_v^W = D_v$ for each $v \in W$; for every constraint $C = \langle \mathbf{s}, R \rangle$, $C \in \mathcal{C}$, the set \mathcal{C}^W includes the constraint $C^W = \langle \mathbf{s}', R' \rangle$, where $\mathbf{s}' = \mathbf{s} \cap W$ and $R' = \text{pr}_{\mathbf{s}'} R$. A solution of \mathcal{P}_W is called a *partial solution* of \mathcal{P} on W . The set of all such solutions is denoted by \mathcal{S}_W . If $W = \{v\}$ or $W = \{u, v\}$, we simplify notation to $\mathcal{P}_v, \mathcal{S}_v$ and $\mathcal{P}_{uv}, \mathcal{S}_{uv}$, respectively.

Instance \mathcal{P} is called *minimal* if every tuple $\mathbf{a} \in R$ for any constraint $\langle \mathbf{s}, R \rangle \in \mathcal{C}$ can be extended to a solution of \mathcal{P} ; that is, there is $\varphi \in \mathcal{S}$ such that $\varphi(v) = \mathbf{a}[v]$ for $v \in \mathbf{s}$. Instance \mathcal{P} is called *k-minimal* if \mathcal{P}_W is minimal for all k -element $W \subseteq V$. For any fixed k every instance can be reduced to a k -minimal instance in polynomial time by a standard algorithm [13]: cycle over all k element subsets $W \subseteq V$, solve the problem \mathcal{P}_W , and for every constraint $\langle \mathbf{s}, R \rangle$ exclude from R all tuples inconsistent with \mathcal{S}_W . If $\mathcal{P} \in \text{CSP}(\mathfrak{A})$ for some class of finite algebras \mathfrak{A} closed under subalgebras, the resulting problem also belongs to $\text{CSP}(\mathfrak{A})$. In particular, from now on we will assume that all the instances we deal with are 1-minimal. For such problems we can also *tighten* the instance reducing the domains $D_v, v \in V$, to the sets \mathcal{S}_v . Every constraint relation will therefore be assumed to be a subdirect product of the respective domains. If \mathfrak{A} consists of idempotent algebras, then any problem from $\text{CSP}(\mathfrak{A})$ can be reduced to a minimal one by solving polynomially many instances of $\text{CSP}(\mathfrak{A})$. First of all, *constant relations*, $R_a = \{(a)\}$, $a \in \mathbb{A} \in \mathfrak{A}$, are subalgebras of \mathbb{A} and therefore can be used in constraints. Then the algorithm proceeds as follows: cycle over all constraints $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ and all $\mathbf{a} \in R$; replace C with the collection of unary constraints $\langle (\mathbf{s}[i]), R_{\mathbf{a}[\mathbf{s}[i]i]} \rangle$; solve the resulting instance $\mathcal{P}_{C, \mathbf{a}}$; remove \mathbf{a} from R if $\mathcal{P}_{C, \mathbf{a}}$ has no solutions. However, this procedure obviously amounts to solving instances from $\text{CSP}(\mathfrak{A})$, and therefore there is no guarantee this can be done in polynomial time.

B. Congruences and polynomials

The set (lattice) of congruences of an algebra \mathbb{A} will be denoted by $\text{Con}(\mathbb{A})$. The smallest congruence of \mathbb{A} , the equality relation, is denoted by $\underline{0}_{\mathbb{A}}$, and the greatest congruence,

the total relation, is denoted by $\underline{1}_{\mathbb{A}}$. If a, b are related by a congruence α , we write $a \stackrel{\alpha}{\equiv} b$; the α -block containing a is denoted a^α . Let R be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_k$, and $\alpha_i \in \text{Con}(\mathbb{A}_i), i \in [k]$. Then by $\bar{\alpha}_R$, or simply $\bar{\alpha}$ if R is clear from the context, we denote the congruence $\alpha_1 \times \dots \times \alpha_k$ of R given by $\mathbf{a} \stackrel{\bar{\alpha}}{\equiv} \mathbf{b}$ if and only if $\mathbf{a}[i] \stackrel{\alpha_i}{\equiv} \mathbf{b}[i]$ for all $i \in [k]$. Also, if $I = \{i_1, \dots, i_\ell\} \subseteq [k]$ then by $\bar{\alpha}_I$ we denote the congruence $\alpha_{i_1} \times \dots \times \alpha_{i_\ell}$ of $\text{pr}_I R$.

Let $\mathcal{P} = (V, \mathcal{S}, \mathcal{C})$ be an instance and α_v a congruence of \mathcal{S}_v for each $v \in V$. By $\mathcal{P}_{\bar{\alpha}}$ we denote the instance $(V, \mathcal{S}^{\bar{\alpha}}, \mathcal{C}^{\bar{\alpha}})$, in which $\mathcal{S}_v^{\bar{\alpha}} = \mathcal{S}_v / \alpha_v$, and a constraint $\langle \mathbf{s}, R' \rangle$, $\mathbf{s} = (v_1, \dots, v_k)$, belongs to $\mathcal{C}^{\bar{\alpha}}$ if and only if a constraint $\langle \mathbf{s}, R \rangle$, where

$$R' = R / \bar{\alpha} = \{ \mathbf{a}^{\bar{\alpha}} = (\mathbf{a}[1]^{\alpha_{v_1}}, \dots, \mathbf{a}[k]^{\alpha_{v_k}}) : \mathbf{a} \in R \},$$

belongs to \mathcal{C} .

A pair of congruences $\alpha, \beta \in \text{Con}(\mathbb{A})$ is said to be a *prime interval*, denoted $\alpha \prec \beta$, if $\alpha \leq \beta$ and $\alpha < \gamma < \beta$ for no congruence $\gamma \in \text{Con}(\mathbb{A})$. Then $\alpha \preceq \beta$ means that $\alpha \prec \beta$ or $\alpha = \beta$. For an operation f on \mathbb{A} we write $f(\beta) \subseteq \alpha$ if, for any $a, b \in \mathbb{A}$ with $a \stackrel{\beta}{\equiv} b$, $f(a) \stackrel{\alpha}{\equiv} f(b)$.

Polynomials of \mathbb{A} are formed from term operations as follows. Let $f(x_1, \dots, x_k, y_1, \dots, y_\ell)$ be a term operation of \mathbb{A} and $a_1, \dots, a_\ell \in \mathbb{A}$. Then the operation $g(x_1, \dots, x_k) = f(x_1, \dots, x_k, a_1, \dots, a_\ell)$ is said to be a *polynomial* of \mathbb{A} . Note that although a polynomial does not have to be a polymorphism of invariant relations of \mathbb{A} , unary polynomials and congruences of \mathbb{A} are in a special relationship: an equivalence relation α is a congruence of \mathbb{A} if and only if it is preserved by every unary polynomial f , that is, $f(\alpha) \subseteq \alpha$. As usual, by an *idempotent unary polynomial* we mean a polynomial $f(x)$ such that $f \circ f = f$ or, equivalently, such that $f(x) = x$ for any x from its range.

Recall that algebra \mathbb{A} is called *Mal'tsev* if it has a ternary term operation $m(x, y, z)$ satisfying the equations $m(x, y, y) = m(y, y, x) = x$. Every algebra from the variety generated by a Mal'tsev algebra is *congruence permutable*, that is, any two of its congruences α, β satisfy the condition $\alpha \circ \beta = \beta \circ \alpha$. In particular, the congruence lattice of a Mal'tsev algebra is *modular*.

Let R be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_k$. Similar to tuples from R , polynomials of R are also denoted in boldface, say, \mathbf{f} . The polynomial \mathbf{f} can be represented as $\mathbf{f}(x_1, \dots, x_k) = g(x_1, \dots, x_k, \mathbf{a}^1, \dots, \mathbf{a}^\ell)$ where g is a term operation of R and $\mathbf{a}^1, \dots, \mathbf{a}^\ell \in R$. Then the polynomial $g(x_1, \dots, x_k, \mathbf{a}^1[i], \dots, \mathbf{a}^\ell[i])$ of \mathbb{A}_i is denoted by f_i , and for $I = \{i_1, \dots, i_s\} \subseteq [n]$, \mathbf{f}_I denotes the polynomial $g(x_1, \dots, x_k, \text{pr}_I \mathbf{a}^1, \dots, \text{pr}_I \mathbf{a}^\ell)$ of $\text{pr}_I R$. For any i , and any polynomial f of \mathbb{A}_i , there is a polynomial \mathbf{g} of R such that $g_i = f$. We shall call \mathbf{g} an *extension* of f to a polynomial of R . Finally, for $I \subseteq [k]$, and $\mathbf{a} \in \prod_{i \in I} \mathbb{A}_i$ and $\mathbf{b} \in \prod_{i \in [k] - I} \mathbb{A}_i$, (\mathbf{a}, \mathbf{b}) denotes the tuple \mathbf{c} such that $\mathbf{c}[i] = \mathbf{a}[i]$ for $i \in I$ and $\mathbf{c}[i] = \mathbf{b}[i]$ if $i \in [k] - I$. To distinguish such concatenation of tuples from pairs of tuples, we will denote pairs of tuples by $\langle \mathbf{a}, \mathbf{b} \rangle$.

The proposition below lists the main basic properties of relations over Mal'tsev algebras.

Proposition 1 (Folklore): Let R be a subdirect product of Mal'tsev algebras $\mathbb{A}_1, \dots, \mathbb{A}_k$ and $I \subseteq [k]$. Then the following properties hold

- (1) R is *rectangular*, that is if $\mathbf{a}, \mathbf{b} \in \text{pr}_I R, \mathbf{c}, \mathbf{d} \in \text{pr}_{[k]-I} R$ and $(\mathbf{a}, \mathbf{c}), (\mathbf{a}, \mathbf{d}), (\mathbf{b}, \mathbf{c}) \in R$, then $(\mathbf{b}, \mathbf{d}) \in R$.
- (2) The relation $\nu_I = \{(\mathbf{a}, \mathbf{b}) \in (\text{pr}_I R)^2 \mid \text{there is } \mathbf{c} \in \text{pr}_{[k]-I} R \text{ such that } (\mathbf{a}, \mathbf{c}), (\mathbf{b}, \mathbf{c}) \in R\}$ is a congruence of $\text{pr}_I R$.
- (3) R is a disjoint union of sets of the form $B \times C$ where B is a ν_I -block and C is a $\nu_{[k]-I}$ -block.

III. OUTLINE OF THE ALGORITHM

Our solution algorithm works by establishing some sort of minimality condition and repeatedly alternates two phases. The first phase is based on the results of Maroti [29] that allow us to reduce an instance over SBM algebras to a one over SBM algebras with a *minimal* element. If \mathbb{A} is an SBM algebra then there is a congruence σ such that \mathbb{A}/σ is a semilattice. This means that \mathbb{A}/σ has a maximal or absorbing element a such that $ax = xa = a$ for any $x \in \mathbb{A}/\sigma$. This element will be in the focus of our argument. We will also show with help of [29], Corollary 1, that it can always be assumed that \mathbb{A}/σ has a minimal or *neutral* element b such that $bx = xb = x$ for any $x \in \mathbb{A}/\sigma$. In fact, one can assume an even stronger condition: that b is a 1-element σ -block.

For the second phase we introduce the *block-minimality* condition defined with the help of congruences and polynomials of an algebra. Let R be a subdirect product of $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$ and $\alpha, \beta \in \text{Con}(\mathbb{A}_i), \gamma, \delta \in \text{Con}(\mathbb{A}_j)$ such that $\alpha \prec \beta, \gamma \prec \delta$ for some $i, j \in [n]$. Interval (α, β) can be separated from (γ, δ) if there is a unary polynomial f of R such that $f_i(\beta) \not\subseteq \alpha$ while $f_j(\delta) \subseteq \gamma$. We are mostly interested in the situation when prime intervals cannot be separated.

Suppose that $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ is a 3-minimal instance and the domain \mathbb{A}_v of $v \in V$ is a SBM algebra and σ_v is such that \mathbb{A}_v/σ_v is a semilattice. For each $v \in V$ select a chain of congruences $\underline{0}_{\mathbb{A}_v} = \alpha_{v1} \prec \alpha_{v2} \prec \dots \prec \alpha_{vk_v} = \theta_{\mathbb{A}_v}$, where $\theta_{\mathbb{A}_v}$ denotes the congruence of \mathbb{A}_v such that the maximal element of $\mathbb{A}_v/\theta_{\mathbb{A}_v}$ is one block of $\theta_{\mathbb{A}_v}$, and all other $\theta_{\mathbb{A}_v}$ -blocks are singletons. We show, Lemma 6, that this is indeed a congruence. For every $v \in V$ and $i \in [k_v - 1]$ let $W_{vi} \subseteq V$ denote the set of variables w such that $(\alpha_{vi}, \alpha_{vi+1})$ and $(\alpha_{wj}, \alpha_{wj+1})$ for some $j \in [k_w - 1]$ cannot be separated from each other in the binary relation \mathcal{S}_{vw} . Instance \mathcal{P} is said to be *block-minimal* if for every $v \in V, i \in [k_v - 1]$ the problem $\mathcal{P}_{W_{vi}}$ is minimal.

The result now follows from the following two statements. First, Proposition 2 claims that any instance \mathcal{P} over SBM algebras can be efficiently reduced to an equivalent block-minimal instance by solving polynomially many SBM instances over domains of smaller size. The second statement, Theorem 2, claims that any block-minimal SBM instance has a solution.

The key to the proof of Proposition 2 is that every problem $\mathcal{P}_{W_{vi}}$ is a disjoint union of problems over smaller domains. More precisely, there is k such that for every $w \in W_{vi}$ the domain \mathbb{A}_w can be partitioned into a disjoint union $\mathbb{A}_w^{(1)} \cup \dots \cup \mathbb{A}_w^{(k)}$ in such a way that for any constraint $\langle (v_1, \dots, v_\ell), R \rangle$

of $\mathcal{P}_{W_{vi}}$, every tuple $\mathbf{a} \in R$ belongs to $\mathbb{A}_{v_1}^{(j)} \times \dots \times \mathbb{A}_{v_k}^{(j)}$ for some $j \in [k]$. This property follows from the existence of a minimal element in every domain and the fact that certain prime intervals in congruence lattices of the domains of $\mathcal{P}_{W_{vi}}$ cannot be separated from each other, Lemma 23. It means, of course, that it suffices to solve k problems $\mathcal{P}_{W_{vi}}^{(j)}$ whose domains are $\mathbb{A}_w^{(j)}$.

We prove Theorem 2 by induction, showing that for every $\bar{\beta} = (\beta_v)_{v \in V}$ with $\beta_v \in \{\alpha_{v1}, \dots, \alpha_{vk_v}\}$ there is a collection of solutions $\varphi_{v,i}$ of $\mathcal{P}_{W_{vi}}$ such that whenever $u \in W_{v,i} \cap W_{w,j}$ we have $\varphi_{v,i}(u) \stackrel{\beta_u}{=} \varphi_{w,j}(u)$. If every β_w equals $\theta_{\mathbb{A}_w}$ then such a collection exists because the maximal element of \mathbb{A}_w/β_w is a singleton, and we always can choose mappings $\varphi_{v,i}$ such that $\varphi_{v,i}(w)/\beta_v$ is the maximal element. On the other hand, if β_w is the equality relation for every $w \in V$ then solutions $\varphi_{v,i}$ agree with each other and provide a solution of \mathcal{P} . Thus, showing that the existence of solutions $\varphi_{v,i}$ for some $\bar{\beta}$ implies the existence of such solutions for smaller congruences $\bar{\beta}'$ is the crux of our argument.

IV. SEMILATTICE BLOCK MAL'TSEV ALGEBRAS AND MINIMAL ELEMENTS

A. Minimal sets and polynomials

We will use several basic concepts of the tame congruence theory, [22].

An (α, β) -*minimal set* is a minimal (under inclusion) set U such that $U = \text{Im}(f)$ for a unary polynomial of \mathbb{A} satisfying $f(\beta) \not\subseteq \alpha$. Sets B, C are said to be *polynomially isomorphic* in \mathbb{A} if there are unary polynomials f, g such that $f(B) = C, g(C) = B$, and $f \circ g, g \circ f$ are identity mappings on C and B , respectively.

Lemma 1 (Theorem 2.8, [22]): Let $\alpha, \beta \in \text{Con}(\mathbb{A}), \alpha \prec \beta$. Then the following hold.

- (1) Any (α, β) -minimal sets U, V are polynomially isomorphic.
- (2) For any (α, β) -minimal set U and any unary polynomial f , if $f(\beta_U) \not\subseteq \alpha$ then $f(U)$ is an (α, β) -minimal set, U and $f(U)$ are polynomially isomorphic, and f witnesses this fact.
- (3) For any (α, β) -minimal set U there is a unary polynomial f such that $f(\mathbb{A}) = U, f(\beta) \not\subseteq \alpha$, and f is idempotent, in particular, f is the identity mapping on U .
- (4) For any $(a, b) \in \beta - \alpha$ and an (α, β) -minimal set U there is a unary polynomial f such that $f(\mathbb{A}) = U$ and $(f(a), f(b)) \in \beta_U - \alpha_U$. Moreover, f can be chosen to satisfy the conditions of item (3).
- (5) For any a unary polynomial f such that $f(\beta) \not\subseteq \alpha$ there is an (α, β) -minimal set U such that f witnesses that U and $f(U)$ are polynomially isomorphic.

Recall that prime intervals $\alpha \prec \beta$ and $\gamma \prec \delta$ in $\text{Con}(\mathbb{A})$ are said to be *perspective* if $\alpha \vee \delta = \beta, \alpha \wedge \delta = \gamma$ or $\beta \vee \gamma = \delta, \beta \wedge \gamma = \alpha$.

Lemma 2 (Lemma 6.2, [22]): Let $\alpha, \beta, \gamma, \delta \in \text{Con}(\mathbb{A})$ be such that $\alpha \prec \beta, \gamma \prec \delta$ and intervals $(\alpha, \beta), (\gamma, \delta)$ are perspective. Then $\text{typ}(\alpha, \beta) = \text{typ}(\gamma, \delta)$ and a set U is (α, β) -minimal if and only if it is (γ, δ) -minimal.

We will also use the following property of modular lattices.

Lemma 3 (Theorem 348,350 [21]): Let a, b, c be elements of a modular lattice L such that $a \leq b, c$. For any $d, e \in L$ with $a \leq d \prec e \leq b \vee c$ the interval $[d, e]$ is perspective to $[d \wedge b, e \wedge b]$ or to $[d \wedge c, e \wedge c]$.

Minimal sets of a Mal'tsev algebra form a particularly dense collection.

Lemma 4 (Folklore): Let \mathbb{A} be a finite Mal'tsev algebra and $\alpha \prec \beta$ for $\alpha, \beta \in \text{Con}(\mathbb{A})$. Then for any $a, b \in \mathbb{A}$ with $\langle a, b \rangle \in \beta - \alpha$, there is an (α, β) -minimal set containing both a and b .

B. Semilattice block Mal'tsev algebras

Since the fewer operations an algebra has, the richer the corresponding constraint language, we assume that the algebras we are dealing with have only two basic operations, just enough to guarantee the required properties. An algebra \mathbb{A} is called a *semilattice block Mal'tsev* (SBM) algebra if it has two basic operations: a binary operation \cdot that we will often omit, and a ternary operation m that satisfy the following conditions. There is a congruence $\sigma_{\mathbb{A}}$ of \mathbb{A} such that $\mathbb{A}/\sigma_{\mathbb{A}}$ is term equivalent to a semilattice with semilattice operation \cdot , and every $\sigma_{\mathbb{A}}$ -block B is a Mal'tsev algebra, where m is a Mal'tsev operation and \cdot is the first projection. For elements $a, b \in \mathbb{A}$ such that $ab = ba = b$ we write $a \leq b$.

Lemma 5: Let \mathbb{A} be an SBM algebra. By choosing a reduct of \mathbb{A} we may assume that

- (1) Operation \cdot satisfies the equation $x(xy) = xy$; in particular, for any $a, b \in \mathbb{A}$ $a \leq ab$.
- (2) Operation m can be chosen such that for any $a, b, c \in \mathbb{A}$, $m(a, b, c)^{\sigma_{\mathbb{A}}} = (abc)^{\sigma_{\mathbb{A}}}$.

Proof: (1) Follows from Proposition 10 of [15].

(2) Consider the operation $m'(x, y, z) = m(x, y, z)xyz$. If B is a $\sigma_{\mathbb{A}}$ -block, then, since $ab = a$ for any $a, b \in B$, operation m' is Mal'tsev on B . Also, as $\mathbb{A}/\sigma_{\mathbb{A}}$ is term equivalent to a semilattice, $d = m(a, b, c)^{\sigma_{\mathbb{A}}}$ belongs to the subsemilattice of $\mathbb{A}/\sigma_{\mathbb{A}}$ generated by $a^{\sigma_{\mathbb{A}}}, b^{\sigma_{\mathbb{A}}}, c^{\sigma_{\mathbb{A}}}$. Therefore $m'(a, b, c)^{\sigma_{\mathbb{A}}} = d(abc)^{\sigma_{\mathbb{A}}} = (abc)^{\sigma_{\mathbb{A}}}$, and we can choose m' for m . \square

Next we show several useful properties of SBM algebras. Let \mathbb{A} be an SBM algebra and $\max(\mathbb{A})$ the maximal block of σ , that is, $\max(\mathbb{A}) \cdot a \subseteq \max(\mathbb{A})$ for all $a \in \mathbb{A}$.

Lemma 6: (1) The equivalence relation $\theta_{\mathbb{A}}$ whose blocks are $\max(\mathbb{A})$ and all the remaining elements form singleton blocks, is a congruence.

(2) Let R be a subdirect product of SBM algebras $\mathbb{A}_1, \dots, \mathbb{A}_n$ and the equivalence relation θ_R is such that its blocks are $R \cap (\max(\mathbb{A}_1) \times \dots \times \max(\mathbb{A}_n))$, and all the remaining elements form singleton blocks. Then θ_R is a congruence.

Proof: (1) It suffices to observe that for any $a \in \max(\mathbb{A})$ we have $ax, xa, m(a, x, y), m(x, a, y), m(x, y, a) \in \max(\mathbb{A})$ for any x, y , and therefore all non-constant polynomials of \mathbb{A} preserve $\max(\mathbb{A})$.

(2) is similar to (1). \square

Lemma 7: Every (α, β) -minimal set, for $\alpha \prec \beta \leq \theta_{\mathbb{A}}$, is a subset of $\max(\mathbb{A})$.

Proof: Let U be a (α, β) -minimal set and f an idempotent polynomial with $f(\mathbb{A}) = U$ and $f(\beta) \not\subseteq \alpha$. Since $\beta \leq \theta_{\mathbb{A}}$, $|U \cap$

$\max(\mathbb{A})| \geq 2$, as otherwise we would have $f(\beta) \subseteq \alpha$. Take $a \in \max(\mathbb{A})$ and set $g(x) = f(x)a$. For any $b \in U \cap \max(\mathbb{A})$ we have $g(b) = f(b)a = ba = b$. Therefore $g(\beta) \not\subseteq \alpha$ and $g(\mathbb{A}) \subseteq \max(\mathbb{A})$. Finally, $f(\max(\mathbb{A})) \subseteq \max(\mathbb{A})$, therefore $f g(\mathbb{A}) \subseteq U \cap \max(\mathbb{A})$ and $f g(x) = x$ for $x \in U \cap \max(\mathbb{A})$. As U is minimal, $U = U \cap \max(\mathbb{A})$. \square

Lemma 8: Let R be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$. The interval $(\underline{0}, \theta_R)$ in $\text{Con}(R)$ is modular.

Proof: Let $\underline{0}'_i, \theta'_i$ denote the restriction of $\underline{0}_i, \theta_i$ on $\max(\mathbb{A}_i)$ and $R' = R \cap (\max(\mathbb{A}_1) \times \dots \times \max(\mathbb{A}_n))$ with added unary operations that are restrictions of unary polynomials of R . This definition is proper, as every polynomial of R except for constant ones preserves R' . Since the only non-trivial block of θ_R is contained in R' , interval $(\underline{0}, \theta_R)$ in $\text{Con}(R)$ is isomorphic to $(\underline{0}', \bar{\theta}')$. As R' is a Mal'tsev algebra, the claim follows. \square

C. Maroti's reduction

In this section we describe a reduction introduced by Maroti in [29] that allows us to reduce CSPs over SBM algebras to CSPs over SBM algebras of a certain restricted type. More precisely, it allows us to assume that every domain \mathbb{A} is either a Mal'tsev algebra with m as a Mal'tsev operation, or it contains a *minimal element* a , that is, an element such that $ab = ba = b$ for all $b \in \mathbb{A}$. Moreover, as is easily seen, such element is unique and forms a $\sigma_{\mathbb{A}}$ -block, which is also the smallest element of the semilattice $\mathbb{A}/\sigma_{\mathbb{A}}$.

Let f be an idempotent unary polynomial of algebra \mathbb{A} , A is the universe of \mathbb{A} . The *retract* $f(\mathbb{A})$ of \mathbb{A} is the algebra with universe $f(A)$, and whose basic operations are of the form $f g$, given by $f g(x_1, \dots, x_n) = f(g(x_1, \dots, x_n))$ for $x_1, \dots, x_n \in f(A)$, where g is a basic operation of \mathbb{A} .

Lemma 9: A retract of an SBM algebra through an idempotent polynomial is an SBM algebra.

Proof: Let f be an idempotent polynomial. Let $g_1(x, y) = f(xy)$, $m_1(x, y, z) = f(m(x, y, z))$ be the basic operations of the reduct, and $\mathbb{A}_1 = f(\mathbb{A})$, and $\sigma_1 = \sigma_{\mathbb{A}|_{\mathbb{A}_1}}$. Firstly, note that σ_1 is a congruence of \mathbb{A}_1 and \mathbb{A}_1 is an idempotent algebra. Since $\mathbb{A}/\sigma_{\mathbb{A}}$ is term equivalent to a semilattice and any retract of a semilattice by a semilattice polynomial is a semilattice, so is \mathbb{A}_1/σ_1 . Finally,

$$m_1(x, y, y) = f(m(x, y, y)) = f(x) = x$$

$$m_1(y, y, x) = f(m(y, y, x)) = f(x) = x,$$

for any $x, y \in \mathbb{A}_1$ with $x \equiv y^{\sigma_1}$. \square

The results of [29] imply the following. Let \mathfrak{A} be a class of finite algebras of similar type closed under subalgebras, and retracts via idempotent unary polynomials. Suppose that \mathfrak{A} has a term operation f satisfying the following conditions for some $\mathbb{B} \in \mathfrak{A}$:

- (1) $f(x, f(x, y)) = f(x, y)$ for any $x, y \in \mathbb{B}$;
- (2) for each $a \in \mathbb{B}$ the mapping $x \mapsto f(a, x)$ is not surjective;
- (3) the set C of $a \in \mathbb{B}$ such that $x \mapsto f(x, a)$ is surjective generates a proper subalgebra of \mathbb{B} .

Then $\text{CSP}(\mathfrak{A})$ is polynomial time reducible to $\text{CSP}(\mathfrak{A} - \{\mathbb{B}\})$.

By Lemma 5 the operation \cdot of the class of SBM algebras from \mathfrak{A} satisfies condition (1). If the operation $a \cdot x$ is surjective

for some a , then $a \leq x$ for all $x \in \mathbb{B}$. Therefore the only case when condition (2) is not satisfied is when \mathbb{B} has a minimal element. Finally, condition (3) is satisfied whenever \mathbb{B} is not a Mal'tsev algebra. Therefore, choosing \mathbb{B} to be a maximal (in terms of cardinality) algebra from \mathfrak{A} satisfying conditions (1)–(3) we may only consider instances of $\text{CSP}(\mathfrak{A})$, in which every domain has a minimal element or is a Mal'tsev algebra.

Corollary 1: Every instance $\mathcal{P} \in \text{CSP}(\mathfrak{A})$ can be reduced in polynomial time to polynomially many instances over algebras each of which either is Mal'tsev or has a minimal element.

Throughout the rest of the paper \mathfrak{A} is a finite class of finite SBM algebras closed under taking subalgebras, homomorphic images, and retracts through unary idempotent polynomials.

V. COHERENT SETS

In this section we introduce and study a method of decomposition of subdirect products of SBM algebras. First, we introduce and study the notion of separation of prime intervals. Let R be a subdirect product of $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$ and $\alpha, \beta \in \text{Con}(\mathbb{A}_i)$, $\gamma, \delta \in \text{Con}(\mathbb{A}_j)$ such that $\alpha \prec \beta$, $\gamma \prec \delta$ for some $i, j \in [n]$. Recall that interval (α, β) can be separated from (γ, δ) if there is a unary polynomial \mathbf{f} of R such that $f_i(\beta) \not\subseteq \alpha$ while $f_j(\delta) \subseteq \gamma$. If \mathbf{f} satisfies this property we will also say that \mathbf{f} *separates* (α, β) from (γ, δ) . We consider two cases of separation of intervals: when they are fixed for each factor of R , say, $\alpha_i, \beta_i \in \text{Con}(\mathbb{A}_i)$ with $\alpha_i \preceq \beta_i$ for $i \in [n]$, and when they are from the congruence lattice of the same algebra. In both cases we study the sets of intervals that cannot be separated from each other. In the second case we arrive to the notion of e-minimal congruences that to some extent generalize prime intervals of the affine type. In the first case this gives rise to the notion of coherent sets. It turns out that coherent sets are closely related to (γ, δ) -minimal sets for $\bar{\alpha} \leq \gamma \prec \delta \leq \bar{\beta}$, Lemmas 14, 17, and that $R/\bar{\alpha}$ admits some sort of direct decomposition, Lemma 18. This result provides a strong structural property of subdirect products of SBM algebras, which is one of the two main ingredients of our argument. Collapsing polynomials introduced in the penultimate subsection provide the second ingredient.

A. Separation: sequences of congruences

Let again $R \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ be a subdirect product of SBM algebras and $\alpha_i \preceq \beta_i \leq \theta_{\mathbb{A}_i}$. We will say that an index $i \in [n]$ can be separated from $j \in [n]$ with respect to $\bar{\alpha}, \bar{\beta}$, if (α_i, β_i) can be separated from (α_j, β_j) . A polynomial that separates the intervals will be said to separate i from j with respect to $\bar{\alpha}, \bar{\beta}$.

First, we show that separating polynomials can be chosen to satisfy certain simple conditions.

Lemma 10: If i can be separated from j then there is a polynomial \mathbf{f} that separates i from j and such that $f_\ell(\mathbb{A}_\ell) \subseteq \max(\mathbb{A}_\ell)$ for every $\ell \in [n]$.

Proof: Let \mathbf{g} separate i from j . Choose a tuple $\mathbf{a} \in R \cap (\max(\mathbb{A}_1) \times \dots \times \max(\mathbb{A}_n))$ and consider the polynomial $\mathbf{f}(x) = \mathbf{g}(x) \cdot \mathbf{a}$. As is easily seen, $f_\ell(\mathbb{A}_\ell) \subseteq \max(\mathbb{A}_\ell)$ for $\ell \in [n]$. Since $g_j(\beta_j) \subseteq \alpha_j$, we have $f_j(\beta_j) \subseteq \alpha_j$. Finally, take $a, b \in \max(\mathbb{A}_i) \cap \text{Im}(g_i)$ with $(a, b) \in \beta_i = \alpha_i$ and

$a', b' \in \max(\mathbb{A}_i)$ such that $g_i(a') = a$, $g_i(b') = b$. Such elements exist, because $g_i(\beta_i) \not\subseteq \alpha_i$ and all the nontrivial (that is, different from an α_i -block) β_i -blocks are inside $\max(\mathbb{A}_i)$. Then

$$\begin{aligned} f_i(a') &= g_i(a')\mathbf{a}[i] = a\mathbf{a}[i] = a \\ &\neq b = b\mathbf{a}[i] = g_i(b')\mathbf{a}[i] = f_i(b'). \end{aligned}$$

□

From now on we assume that all polynomials separating coordinate positions satisfy the conditions of Lemma 10.

Lemma 11: If i can be separated from j then, for any (α_i, β_i) -minimal set U , there is an idempotent unary polynomial \mathbf{g} such that $g_i(\mathbb{A}_i) = U$, and \mathbf{g} separates i from j .

Proof: Let \mathbf{f} separate i from j . Then by Lemma 1 $f_i(\mathbb{A}_i)$ contains an (α_i, β_i) -minimal set U , and there is an idempotent polynomial h_i with $h_i(\mathbb{A}_i) = U$. The polynomial h_i can be extended to a polynomial \mathbf{h} of R . Then $\mathbf{f}' = \mathbf{h} \circ \mathbf{f}$ separates i from j and $f'_i(\mathbb{A}_i) = U$.

Again by Lemma 1 there is a (α_i, β_i) -minimal set U' with $f'_i(U') = U$ and an idempotent polynomial h'_i with $h'_i(U) = U'$. As above the polynomial h'_i can be extended to a polynomial \mathbf{h}' of R . For a certain k , $(\mathbf{h}' \circ \mathbf{f}')^k$ is idempotent, separates i from j , and $(h'_i \circ f'_i)^k(\mathbb{A}_i) = U'$. Now the lemma follows easily from the fact that any two (α_i, β_i) -minimal sets are polynomially isomorphic. □

If $\alpha_i \prec \beta_i$ while $\alpha_j = \beta_j$, then (α_i, β_i) can obviously be separated from (α_j, β_j) . Let $J \subseteq [n]$ be the set of all $i \in [n]$ with $\alpha_i \prec \beta_i$. The relation $\eta_{\bar{\alpha}, \bar{\beta}}$ on J consisting of all pairs (i, j) such that i cannot be separated from j with respect to $\bar{\alpha}, \bar{\beta}$ is obviously reflexive and transitive. Now, we also prove it is symmetric

Lemma 12: If i can be separated from j and $\alpha_j \prec \beta_j$ then j can be separated from i .

Proof: Let U_1, \dots, U_k be all the (α_i, β_i) -minimal sets. By Lemma 11, for every U_ℓ , there is an idempotent unary polynomial $\mathbf{g}^{(\ell)}$ separating i from j and such that $g_i^{(\ell)}(\mathbb{A}_i) = U_\ell$. Take a β_j -block B that contains more than one α_j -block, a tuple $\mathbf{a} \in R$ such that $a_j \in B$, and set $\mathbf{a}^{(\ell)} = \mathbf{g}^{(\ell)}(\mathbf{a})$. By Lemmas 7 and 10 $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(k)} \in R \cap (\max(\mathbb{A}_1)) \times \dots \times \max(\mathbb{A}_n)$ and $U_1, \dots, U_k \subseteq \max(\mathbb{A}_i)$, and $B \subseteq \max(\mathbb{A}_j)$. The operation $\mathbf{h}^{(\ell)}(x) = m(x, \mathbf{g}^{(\ell)}(x), \mathbf{a}^{(\ell)})$ satisfies the conditions

- $h_i^{(\ell)}(x) = m(x, g_i^{(\ell)}(x), \mathbf{a}^{(\ell)}[i]) = m(x, x, \mathbf{a}^{(\ell)}[i]) = a^{(\ell)}[i]$ for every $x \in U_\ell$;
- $h_j^{(\ell)}(x) = m(x, g_j^{(\ell)}(x), \mathbf{a}^{(\ell)}[j]) \stackrel{\alpha_j}{=} m(x, \mathbf{a}^{(\ell)}[j], \mathbf{a}^{(\ell)}[j]) = x$ for every $x \in B$;
- $\mathbf{h}^{(\ell)}(R) \subseteq R \cap (\max(\mathbb{A}_1) \times \dots \times \max(\mathbb{A}_n))$.

We are going to compose the polynomials $\mathbf{h}^{(\ell)}$ such that the composition collapses β_i . To this end take a sequence $1 = \ell_1, \ell_2, \dots$ such that U_{ℓ_2} is a subset of the range of $\bar{h}^{(1)} = h_i^{(\ell_1)}$, and, for $s > 2$, U_{ℓ_s} is a subset of the range of $\bar{h}^{(s-1)} = h_i^{(\ell_{s-1})} \dots h_i^{(\ell_1)}$. Since $|\text{Im}(\bar{h}^{(s)})| < |\text{Im}(\bar{h}^{(s-1)})|$, there is r such that $|\text{Im}(\bar{h}^{(r)})|$ contains no (α_i, β_i) -minimal sets. Therefore, setting $\mathbf{h}(x) = \mathbf{h}^{(\ell_r)}(\mathbf{h}^{(\ell_{r-1})}(\dots \mathbf{h}^{(\ell_1)}(x) \dots))$ we have that h_i collapses all the (α_i, β_i) -minimal sets, and h_j acts identically on B/α_j . Thus, \mathbf{h} separates j from i . □

Lemma 12 together with the observation before it shows that $\eta_{\bar{\alpha}, \bar{\beta}}$ is an equivalence relation on J . We call its classes *coherent sets* with respect to $\bar{\alpha}, \bar{\beta}$.

We show that coherent sets possess some additional useful properties.

Lemma 13: Let $I \subseteq [n]$, and I_1, \dots, I_k the coherent sets of R with respect to $\bar{\alpha}, \bar{\beta}$. Then the coherent sets of $\text{pr}_I R$ with respect to $\bar{\alpha}_I, \bar{\beta}_I$ are $I_1 \cap I, \dots, I_k \cap I$.

Lemma 13 follows from the observation that whether i can be separated from j or not depends entirely on the algebra $\text{pr}_{i,j} R$.

Using coherent sets we can impose further conditions on separating polynomials.

Lemma 14: If $I \subseteq [n]$ is a coherent set then there is a unary polynomial \mathbf{f} of R such that

\mathbf{f} is idempotent,

(*) $\mathbf{f}(R) \subseteq R \cap (\max(\mathbb{A}_1) \times \dots \times \max(\mathbb{A}_n))$,

$f_i(\beta_i) \not\subseteq \alpha_i$ if and only if $i \in I$,

$f_i(\mathbb{A}_i)$ is an (α_i, β_i) -minimal set whenever $i \in I$.

Moreover, for any $\ell \in I$, and any $(\alpha_\ell, \beta_\ell)$ -minimal set U_ℓ , the polynomial \mathbf{f} can be chosen such that $f_\ell(\mathbb{A}_\ell) = U_\ell$.

Proof: Take $i \in I$ and an (α_i, β_i) -minimal set U . By Lemma 11, for every $j \notin I$, there is an idempotent polynomial $\mathbf{f}^{(j)}(x)$ such that $\mathbf{f}^{(j)}$ separates i from j and $f_i^{(j)}(\mathbb{A}_i) = U$. Composing all these polynomials we obtain an operation \mathbf{f} such that $f_j(\beta_j) \subseteq \alpha_j$ for every $j \notin I$, and $f_i(x) = x$ for $x \in U$. Since for every $\ell \in I$ the polynomial \mathbf{f} does not separate i from ℓ , we have $f_\ell(\beta_\ell) \not\subseteq \alpha_\ell$.

Let now \mathbf{f} be an operation separating each $i \in I$ from each $j \notin I$ and such that the sum of $|f_i(\mathbb{A}_i)|$, for $i \in I$, is minimal among the operations with this property. Then, for every $i \in I$, $f_i(\mathbb{A}_i)$ is an (α_i, β_i) -minimal set. Indeed, suppose that there is $i \in I$ such that $f_i(\mathbb{A}_i)$ is not a minimal set. Then by Lemma 1 $f_i(\mathbb{A}_i)$ contains an (α_i, β_i) -minimal set U and there is an idempotent unary polynomial g_i with the range U . As is easily seen, for an extension \mathbf{g} of g_i to a polynomial of R , the operation $\mathbf{g} \circ \mathbf{f}$ still separates every $\ell \in I$ from every $j \notin I$, but the sum of $|f_j(\mathbb{A}_j)|$, $j \in I$, is less than that for \mathbf{f} ; a contradiction.

Finally, arguing as in the proof of Lemma 11 we may derive an idempotent polynomial with the required properties. \square

B. Separation in a single algebra

Let $\alpha \prec \beta$ be prime interval in $\text{Con}(\mathbb{A})$. We say that a congruence $\xi \geq \alpha$ is *e-minimal* for the interval α, β if for every $\alpha \leq \gamma \prec \delta \leq \xi$, the interval $\gamma \prec \delta$ cannot be separated from $\alpha \prec \beta$.

The following lemma allows us to use some of the results of Section V-A for intervals of the same algebra.

Lemma 15: Let Q be the binary equality relation on \mathbb{A} . Prime interval $\alpha \prec \beta \leq \theta_{\mathbb{A}}$ can be separated from $\gamma \prec \delta \leq \theta_{\mathbb{A}}$ as intervals in $\text{Con}(\mathbb{A})$ if and only if 1 can be separated from 2 in Q (as coordinate positions of a binary relation) with respect to (α, γ) and (β, δ) .

Proof: Note that for any polynomial \mathbf{f} of Q its action on the first and second projections of Q is the same polynomial

of \mathbb{A} . By definition $\alpha \prec \beta$ can be separated from $\gamma \prec \delta$ in $\text{Con}(\mathbb{A})$ if and only if there is a unary polynomial f of \mathbb{A} , $f(\beta) \not\subseteq \alpha$ while $f(\delta) \subseteq \gamma$. This condition can be expressed as follows: there is a unary polynomial \mathbf{f} of Q , $f_1(\beta) \not\subseteq \alpha$ while $f_2(\delta) \subseteq \gamma$, which precisely means that 1 can be separated from 2 in Q . \square

Now the results of Section V-A imply the following

Corollary 2: Let \mathbb{A} be any SBM algebra.

(1) If $\alpha \prec \beta \leq \theta_{\mathbb{A}}$ can be separated from $\gamma \prec \delta \leq \theta_{\mathbb{A}}$, then $\gamma \prec \delta$ can be separated from $\alpha \prec \beta$.

(2) If prime intervals $\alpha \prec \beta$ and $\gamma \prec \delta$ are perspective, then they cannot be separated from each other.

(3) If $\alpha \prec \beta \leq \theta_{\mathbb{A}}$ and $\gamma \prec \delta \leq \theta_{\mathbb{A}}$ cannot be separated, then a set U is a (α, β) -minimal set if and only if it is a (γ, δ) -minimal set.

Proof: (1) follows from Lemmas 12 and 15.

(2) follows from Lemma 2, as (α, β) - and (γ, δ) -minimal sets are the same.

(3) Consider the binary equality relation Q on \mathbb{A} . By Lemma 15 coordinate position 1 cannot be separated from 2 with respect to (α, γ) and (β, δ) . By Lemma 14 for any (α, β) -minimal set U there is a polynomial \mathbf{f} of Q such that $f_1(\mathbb{A}) = U$ and $f_2(\mathbb{A})$ is a (γ, δ) -minimal set. However, $f_2(\mathbb{A}) = U$, the result follows. \square

The next lemma amounts to saying that for any prime interval $\alpha \prec \beta \leq \theta_{\mathbb{A}}$ there is the greatest e-minimal congruence.

Lemma 16: (1) If γ is e-minimal for $\alpha \prec \beta \leq \theta_{\mathbb{A}}$, then $\gamma \leq \theta_{\mathbb{A}}$.

(2) If γ_1, γ_2 are e-minimal for $\alpha \prec \beta \leq \theta_{\mathbb{A}}$, then so is $\gamma_1 \vee \gamma_2$.

Proof: (1) Take $a \in \max(\mathbb{A})$ and consider the polynomial $f(x) = x \cdot a$. This polynomial does not collapse β to α but collapses \mathbb{A} onto $\max(\mathbb{A})$, and so it collapses every interval $\eta \prec \zeta$ with $\eta \leq \theta_{\mathbb{A}}$ and $\zeta \not\leq \theta_{\mathbb{A}}$.

(2) By Lemma 8 the interval $[\theta_{\mathbb{A}}, \theta_{\mathbb{A}}]$ in $\text{Con}(\mathbb{A})$ is modular. Therefore by Lemma 3 every prime interval in $[\alpha, \gamma]$, $\gamma = \gamma_1 \vee \gamma_2$, is perspective to an interval in $[\alpha, \gamma_1]$ or to an interval in $[\alpha, \gamma_2]$. Therefore, for every prime interval $\eta \prec \zeta \leq \gamma$, every (η, ζ) -minimal set U is a minimal set of some prime interval $\delta \prec \pi \leq \gamma_i$, $i = 0$ or $i = 1$. Thus if some f collapses $[\delta, \pi]$, we have $U \not\subseteq \text{Im}(f)$ for any (η, ζ) -minimal set U . Therefore, f collapses $[\eta, \zeta]$. \square

By Lemma 16(2) there is the greatest congruence of \mathbb{A} e-minimal for $\alpha \prec \beta$; it will be denoted by $\text{emin}(\alpha, \beta)$.

C. Minimal sets and direct decompositions

The main result of this subsection is Lemma 18 that provides some sort of direct decomposition of a subdirect product R of SBM algebras $\mathbb{A}_1, \dots, \mathbb{A}_n$ according to its coherent sets. However, first of all we need to relate minimal sets of R with its coherent sets.

For a set $I \subseteq [n]$, let us denote the equality relations on R , R_I by $\underline{0}, \underline{0}_I$ respectively, and for any $\gamma_i \in \text{Con}(\mathbb{A}_i)$, $\bar{\gamma} = \prod_{i \in [n]} \gamma_i$, $\bar{\gamma}_I = \prod_{i \in I} \gamma_i$.

Lemma 17: Let $\xi_i = \text{emin}(\alpha_i, \beta_i)$ if $\alpha_i \prec \beta_i$ and $\xi_i = \alpha_i$ otherwise. Let $\bar{\alpha} \leq \gamma \prec \delta \leq \bar{\xi}$, and let $\mathbf{f}(R)$ be a (γ, δ) -minimal set for an idempotent polynomial \mathbf{f} . Then \mathbf{f} satisfies condition (*) for a certain coherent set with respect to $\bar{\alpha}, \bar{\beta}$.

Aproof of Lemma 17 can be found in the full version of the paper [16]

Corollary 3: Let $I \subseteq [n]$ be a coherent set.

(1) Let $\bar{\alpha}_I \leq \gamma < \delta \leq \bar{\xi}_I$, $\gamma' = \{\langle \mathbf{a}, \mathbf{b} \rangle \in \bar{\xi} \mid \text{pr}_I \mathbf{a} \stackrel{\gamma}{\equiv} \text{pr}_I \mathbf{b}\}$, $\delta' = \{\langle \mathbf{a}, \mathbf{b} \rangle \in \bar{\xi} \mid \text{pr}_I \mathbf{a} \stackrel{\delta}{\equiv} \text{pr}_I \mathbf{b}\}$, and $\gamma' \prec \eta \leq \delta'$ in $\text{Con}(R)$. Then, any idempotent polynomial \mathbf{f} of R such that $\mathbf{f}(R)$ is an (γ', η) -minimal set satisfies $(*)$ for I .

(2) Let $\mathbf{a} \in \max(R)$ be a tuple from a $\bar{\xi}$ -block that is not equal to an $\bar{\alpha}$ -block and such that $\text{pr}_I \mathbf{a}$ is in a $\bar{\xi}_I$ -block that is not equal to an $\bar{\alpha}_I$ -block. There is a polynomial \mathbf{f} satisfying $(*)$ for I and such that $\mathbf{a} \in \mathbf{f}(R)$.

Proof: (1) By Lemma 17, \mathbf{f} satisfies $(*)$ for a certain coherent set I' . If $I' \neq I$, then $f_i(\xi_i) \subseteq \alpha_i$ for any $i \in I$. Therefore, $\mathbf{f}_I(\delta) \subseteq \bar{\alpha}_I \subseteq \gamma$, and $\mathbf{f}(\delta') \subseteq \gamma'$, a contradiction.

(2) Let $\underline{\gamma}$ be the greatest congruence of $\text{pr}_I R$ such that $\bar{\alpha}_I \leq \underline{\gamma} \leq \bar{\xi}_I$ and $\mathbf{a}_I^{\underline{\gamma}}$ contains only one $\bar{\alpha}_I$ -block, and $\bar{\alpha}_I \leq \underline{\gamma} \prec \delta \leq \bar{\xi}_I$. Since \mathbf{a}_I lies in a nontrivial $\bar{\xi}_I$ -block, $\underline{\gamma} \neq \bar{\xi}_I$ and by Lemma 4, \mathbf{a}_I belongs to a (γ, δ) -minimal set. Let γ', δ', η be congruences constructed as in part (1) of the corollary. Then there is an idempotent polynomial \mathbf{f} satisfying $(*)$ for I , and such that $\mathbf{a}_I \in \mathbf{f}_I(\text{pr}_I R)$.

Set $\mathbf{h}(x) = m(\mathbf{f}(x), \mathbf{f}(\mathbf{a}), \mathbf{a})$. Since $f_i(\xi_i) \subseteq \alpha_i$ whenever $i \notin I$, we have $h_i(\xi_i) \subseteq \alpha_i$ and $h_i(\mathbf{a}[i]) = \mathbf{a}[i]$. If $i \in I$ and $x \in f_i(\mathbb{A}_i)$ then $f_i(x) = x$, and therefore

$$h_i(x) = m(f_i(x), f_i(\mathbf{a}[i]), \mathbf{a}[i]) = m(x, \mathbf{a}[i], \mathbf{a}[i]) = x.$$

This means $h_i(\xi_i) \not\subseteq \alpha_i$. Moreover, as $f_i(\mathbb{A}_i)$ is a (α_i, β_i) -minimal set, $h_i(\mathbb{A}_i)$ is also a (α_i, β_i) -minimal set. Thus, for a certain k , \mathbf{h}^k is idempotent and satisfies $(*)$ for I . Finally,

$$h_i(\mathbf{a}[i]) = m(f_i(\mathbf{a}[i]), f_i(\mathbf{a}[i]), \mathbf{a}[i]) = \mathbf{a}[i].$$

So, $\mathbf{a} \in \mathbf{h}^k(R)$; the corollary is proved. \square

We are now in a position to prove our direct decomposability result.

Lemma 18: Let B be a $\bar{\xi}$ -block of R , such that $\text{pr}_i B \subseteq \max(\mathbb{A}_i)$ for $i \in [n]$. Then $B/\bar{\alpha} = \text{pr}_{I_1} B/\bar{\alpha}_{I_1} \times \dots \times \text{pr}_{I_k} B/\bar{\alpha}_{I_k} \times \text{pr}_K B/\bar{\alpha}_K$ where I_1, \dots, I_k are the coherent sets with respect $\bar{\alpha}, \bar{\beta}$ and $K = \{i \in [n] \mid \alpha_i = \beta_i\}$.

Proof: Set $I = I_1$, $J = [n] - I$. We prove that $B = B_I/\bar{\alpha}_I \times B_J/\bar{\alpha}_J$. By taking factors modulo α_i we may assume $\alpha_i = \underline{0}_i$, $i \in [n]$. Consider two congruences on $\text{pr}_I R$:

$$\begin{aligned} \nu &= \{\langle \mathbf{a}, \mathbf{b} \rangle \in (\text{pr}_I R)^2 \mid \text{there is } \mathbf{c} \in \text{pr}_I R \text{ such} \\ &\quad \text{that } \langle \mathbf{a}, \mathbf{c} \rangle, \langle \mathbf{b}, \mathbf{c} \rangle \in R\} \\ \zeta &= \{\langle \mathbf{a}, \mathbf{b} \rangle \in (\text{pr}_I R)^2 \mid \text{there are } \mathbf{c}, \mathbf{e} \in \text{pr}_I R \\ &\quad \text{such that } \langle \mathbf{a}, \mathbf{c} \rangle, \langle \mathbf{b}, \mathbf{e} \rangle \in R, \langle \mathbf{c}, \mathbf{e} \rangle \in \bar{\xi}_I\}, \end{aligned}$$

and $\nu' = \nu \cap \theta_{\text{pr}_I R}$, $\zeta' = \zeta \cap \theta_{\text{pr}_I R}$. Notice that if for $\mathbf{a}^1, \mathbf{a}^2 \in \text{pr}_I B$ there is $\mathbf{b} \in \text{pr}_I R$ with $\langle \mathbf{a}^1, \mathbf{b} \rangle, \langle \mathbf{a}^2, \mathbf{b} \rangle \in R$ then the rectangularity of $R' = R \cap (\max(\mathbb{A}_1) \times \dots \times \max(\mathbb{A}_n))$ implies that, for any $\mathbf{c} \in \text{pr}_I B$ such that $\langle \mathbf{a}^1, \mathbf{c} \rangle \in R$, the tuple $(\mathbf{a}^2, \mathbf{c})$ also belongs to R . Also, for any $\mathbf{a}^1, \mathbf{a}^2 \in \text{pr}_I B$ with $\mathbf{a}^1 \stackrel{\zeta}{\equiv} \mathbf{a}^2$, there are $\mathbf{b}^1, \mathbf{b}^2 \in \text{pr}_I B$ such that $\langle \mathbf{a}^1, \mathbf{b}^1 \rangle, \langle \mathbf{a}^2, \mathbf{b}^2 \rangle \in R$. We are to prove that if $\langle \mathbf{a}^1, \mathbf{b}^1 \rangle, \langle \mathbf{a}^2, \mathbf{b}^2 \rangle \in B$ then $\langle \mathbf{a}^1, \mathbf{b}^2 \rangle, \langle \mathbf{a}^2, \mathbf{b}^1 \rangle \in R$, or, equivalently, the equality $\nu' \cap \bar{\xi}'_I = \zeta' \cap \bar{\xi}'_I$. Denote the congruences appearing in this equality by γ and δ , respectively.

If $\gamma \neq \delta$ then, for congruences γ', δ', η constructed as in Corollary 3(1), there is an idempotent unary polynomial \mathbf{f} satisfying $(*)$ for I and such that $\mathbf{f}(R)$ is a (γ', η) -minimal set. However, this leads to a contradiction because on the one hand $\mathbf{f}(\eta) \not\subseteq \gamma'$, hence $\mathbf{f}_I(\delta) \not\subseteq \gamma$, but on the other hand $\mathbf{f}_I(\zeta) \subseteq \nu$ for any \mathbf{f} satisfying $(*)$ for I .

Repeating the same argument for each coherent set we get what is required. \square

D. Collapsing polynomials

Intuitively, a collapsing polynomial for some prime interval $\alpha \prec \beta$ in an algebra or a subdirect product of algebras is a polynomial that collapses all prime intervals that cannot be separated from $\alpha \prec \beta$ and only such prime intervals. We will use collapsing polynomials in the proof of the correctness of our algorithm to obtain solutions of quotient instances for smaller and smaller congruences.

Let $\alpha \prec \beta \leq \theta_{\mathbb{A}} \in \text{Con}(\mathbb{A})$, and let α' be the smallest congruence such that $\alpha' \prec \beta'$ and $\alpha \prec \beta$ cannot be separated for some β' . Observe that since the interval $[\underline{0}_{\mathbb{A}}, \theta_{\mathbb{A}}]$ in $\text{Con}(\mathbb{A})$ is modular, by Lemma 3 the intersection of all $\alpha'' \in [\underline{0}_{\mathbb{A}}, \theta_{\mathbb{A}}]$ such that $\alpha'' \prec \beta''$ cannot be separated from $\alpha \prec \beta$ for some β'' , can be chosen for such an α' . For $\gamma \in \text{Con}(\mathbb{A})$, $\gamma \leq \theta_{\mathbb{A}}$, if $\alpha' \leq \gamma$, let $\zeta_{\alpha\beta}(\gamma)$ denote the smallest congruence such that there is an irreducible chain $\zeta_{\alpha\beta}(\gamma) = \gamma_0 \prec \gamma_1 \prec \dots \prec \gamma_k = \gamma$ and every prime interval $\gamma_i \prec \gamma_{i+1}$ can be separated from $\alpha \prec \beta$. By $\nu_{\alpha\beta}(\gamma)$ we denote the smallest congruence such that there is an irreducible chain $\nu_{\alpha\beta}(\gamma) = \gamma_0 \prec \gamma_1 \prec \dots \prec \gamma_k = \gamma$ and every prime interval $\gamma_i \prec \gamma_{i+1}$ cannot be separated from $\alpha \prec \beta$. Again by Lemma 3 the modularity of $[\underline{0}_{\mathbb{A}}, \theta_{\mathbb{A}}]$ implies that such smallest congruences exist.

A unary idempotent polynomial f is called (α, β) -collapsing with respect to $a \in \max(\mathbb{A})$ if the following conditions hold:

- (C1) for any $\gamma \in \text{Con}(\mathbb{A})$, $\alpha' \leq \gamma \leq \theta_{\mathbb{A}}$, it holds $f(a^\gamma) \subseteq a^{\nu_{\alpha\beta}(\gamma)}$;
- (C2) for any $\gamma \in \text{Con}(\mathbb{A})$, $\gamma \leq \theta_{\mathbb{A}}$ and any $x \in \mathbb{A}$ with $x \stackrel{\gamma}{\equiv} a$, $f(x) \stackrel{\zeta_{\alpha\beta}(\gamma)}{\equiv} x$.

First, we show that (α, β) -collapsing polynomials exist for a single algebra.

Lemma 19: For every SBM algebra \mathbb{A} , any $\alpha, \beta \in \text{Con}\mathbb{A}$, $\alpha \prec \beta \leq \theta_{\mathbb{A}}$, and any $a \in \mathbb{A}$ from a β -block containing more than one α -block, there is an (α, β) -collapsing polynomial f with respect to a .

Proof: Let α' be the smallest congruence such that $\alpha' \prec \beta'$ and $\alpha \prec \beta$ cannot be separated for some β' . First, we show that \mathbb{A} has a polynomial f satisfying the following conditions.

- (U1) $f(\beta) \not\subseteq \alpha$;
- (U2) $f(\gamma) \subseteq \zeta_{\alpha\beta}(\gamma)$ for any $\gamma \in \text{Con}(\mathbb{A})$, $\gamma \leq \theta_{\mathbb{A}}$.

For any prime interval $\gamma \prec \delta$ that can be separated from $\alpha \prec \beta$, there is a unary polynomial $f_{\gamma\delta}$ such that $f_{\gamma\delta}(\beta) \not\subseteq \alpha$, but $f_{\gamma\delta}(\delta) \subseteq \gamma$. We may assume that all the $f_{\gamma\delta}$ are idempotent and have the same (α, β) -minimal set as image. As is easily seen, composing all such polynomials we obtain the result. Let us denote the resulting polynomial by f .

Now we convert f into an (α, β) -collapsing polynomial. We may assume f is idempotent and $U = \text{Im}(f)$ is a (α, β) -minimal set such that $a \in U$. Set $g_U(x) = m(x, f(x), a)$; then $g_U(U) = a$. Take $\gamma \in \text{Con}(\mathbb{A})$, $\gamma \leq \theta_{\mathbb{A}}$, and $x \in a^\gamma$. Since $f(\gamma) \subseteq \zeta_{\alpha\beta}(\gamma)$, we obtain $f(x) \stackrel{\zeta_{\alpha\beta}(\gamma)}{\equiv} a$. Therefore $g_U(x) \stackrel{\zeta_{\alpha\beta}(\gamma)}{\equiv} x$. Now let U_1, \dots, U_k be a list of all (α, β) -minimal sets containing a , and g_{U_i} a polynomial with $|g_{U_i}(U_i)| = 1$ and $g_{U_i}(x) \stackrel{\zeta_{\alpha\beta}(\gamma)}{\equiv} x$ for all $\gamma \in \text{Con}(\mathbb{A})$, $\gamma \leq \theta_{\mathbb{A}}$, and $x \in a^\gamma$. Composing these polynomials as in the proof of Lemma 12 we obtain

$$h(x) = g_{U_{\ell_1}}(\dots g_{U_{\ell_m}}(x)\dots)$$

that satisfies the following conditions: $|h(U)| = 1$ for any (α, β) -minimal set U containing a , and $h(x) \stackrel{\zeta_{\alpha\beta}(\gamma)}{\equiv} x$ for all $\gamma \in \text{Con}(\mathbb{A})$ and $x \in a^\gamma$. The first condition implies $h(a^\gamma) \subseteq a^{\nu_{\alpha\beta}(\gamma)}$. \square

Next, we expand Lemma 19 to subdirect products. The following statement can be proved along the same lines as Lemma 19.

Lemma 20: Let $R \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ be a subdirect product of SBM algebras, $\bar{\alpha} = \alpha_1 \times \dots \times \alpha_n$ for $\alpha_i \in \text{Con}(\mathbb{A}_i)$, and $\bar{\beta} = \beta_1 \times \dots \times \beta_n$ for $\alpha_i \preceq \beta_i$. Let also I be a coherent set with respect to $\bar{\alpha}, \bar{\beta}$ and let $\mathbf{a} \in \text{pr}_I R$ be such that $\mathbf{a}[i]$ belongs to a β_i -block containing more than one α_i -block for some $i \in I$. There is an idempotent unary polynomial \mathbf{g} of R such that g_j is (α_j, β_j) -collapsing with respect to $\mathbf{a}[j]$ for all $j \in I$.

We will call a polynomial \mathbf{g} constructed in Lemma 20 $(\bar{\alpha}_I, \bar{\beta}_I)$ -collapsing.

E. Splits and alignments

In this section we present a sufficient condition for two coordinate positions to belong to different coherent sets. As we shall see using this condition a projection of a relation onto its coherent set can be partitioned into a small number of subdirect products of smaller algebras.

An element $a \in \mathbb{A}_i$ is called $\alpha_i \beta_i$ -split if there is a β_i -block B and $b, c \in B$ such that $ab \not\stackrel{\alpha_i}{\equiv} ac$. Note that no element from $\max(\mathbb{A}_i)$ is an $\alpha_i \beta_i$ -split, while the minimal element is an $\alpha_i \beta_i$ -split. We say that $i, j \in [n]$ are not $\bar{\alpha} \bar{\beta}$ -aligned if there is $a \in R$ such that $\mathbf{a}[i]$ is not $\alpha_i \beta_i$ -split and $\mathbf{a}[j]$ is $\alpha_j \beta_j$ -split, or the other way round.

Lemma 21: If i, j are not $\bar{\alpha} \bar{\beta}$ -aligned then they are in different coherent sets with respect to $\bar{\alpha}, \bar{\beta}$.

Proof: It suffices to consider the case $n = 2$, $i = 1$, $j = 2$. Let $(a, b) \in R$ be such that a is $\alpha_1 \beta_1$ -split, while b is not $\alpha_2 \beta_2$ -split. Let also $(c, d) \in R' = R \cap (\max(\mathbb{A}_1) \times \max(\mathbb{A}_2))$. Consider operation $\mathbf{f}((x_1, x_2)) = (a, b) \cdot ((x_1, x_2) \cdot (c, d))$. We claim that $f_1(\beta_1) \not\subseteq \alpha_1$ while $f_2(\beta_2) \subseteq \alpha_2$.

First, observe that all the values of the operation $g((x_1, x_2)) = (x_1, x_2) \cdot (c, d)$ belong to R' , and $g((x_1, x_2)) = (x_1, x_2)$ for any $(x_1, x_2) \in R'$. Then, for any β_2 -block B_2 and any $a', b' \in B_2$ we have $b(a'd) \stackrel{\alpha_2}{\equiv} b(b'd)$, as b is not $\alpha_2 \beta_2$ -split. Thus $f_2(\beta_2) \subseteq \alpha_2$. On the other hand, since a is $\alpha_1 \beta_1$ -split, there is a β_1 -block B_1 and $a'', b'' \in B_1$ such that $a(a''c) = aa'' \not\stackrel{\alpha_1}{\equiv} ab'' = a(b''c)$. Therefore $f_1(\beta_1) \not\subseteq \alpha_1$. \square

VI. FROM RELATIONS TO INSTANCES

Here we apply the results of the previous section to CSP instances. In particular, we introduce coherent sets of an instance and show that if an instance has solutions on every coherent set, which are consistent in some weak sense, then the entire instance has a solution.

Let $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ be a 3-minimal instance of $\text{CSP}(\mathfrak{A})$. We assume that the domain \mathbb{A}_v of each variable $v \in V$ is the set of solutions \mathcal{S}_v , and so the constraint relations are subdirect products of the domains.

Since coherent sets depend only on binary projections of a relation, coherent sets can be defined for 3-minimal instances as well. More precisely, let $\alpha_v, \beta_v \in \text{Con}(\mathbb{A}_v)$, $\alpha_v \preceq \beta_v \leq \theta_{\mathbb{A}_v}$ for $v \in V$; we say that v is not separated from w , $v, w \in V$, with respect to $\bar{\alpha}, \bar{\beta}$, if this is the case for \mathcal{S}_{vw} . Due to 3-minimality — we can consider ternary sets of solutions — this relation is transitive. It is also reflexive and symmetric. The equivalence classes will be called coherent sets of \mathcal{P} with respect to $\bar{\alpha}, \bar{\beta}$.

Recall that $\mathcal{P}_{\bar{\alpha}}$ denotes the quotient instance with domains \mathbb{A}_v / α_v defined in Section II-B.

Lemma 22: Let $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ be a 3-minimal instance of $\text{CSP}(\mathfrak{A})$ and V_1, \dots, V_k the coherent sets of \mathcal{P} with respect to $\bar{\alpha}, \bar{\beta}$, $\bar{\alpha} = (\alpha_v)_{v \in V}$, $\bar{\beta} = (\beta_v)_{v \in V}$, $\alpha_v \preceq \beta_v \leq \theta_v \in \text{Con}(\mathbb{A}_v)$, $v \in V$. If $\mathcal{P}_{\bar{\beta}}$ has a solution φ such that $\varphi(v) \in \max(\mathbb{A}_v / \beta_v)$ and for every $i \in [k]$ the problem $(\mathcal{P}_{\bar{\alpha}})_{V_i}$ has a solution φ_i such that $\varphi_i(v) \subseteq \varphi(v)$ for every $v \in V_i$, then $\mathcal{P}_{\bar{\alpha}}$ has a solution ψ , where $\psi(v) = \varphi_i(v)$ whenever $v \in V_i$, $i \in [k]$.

Proof: Replacing \mathbb{A}_v with the quotient \mathbb{A}_v / α_v we can assume $\alpha_v = \underline{0}_v$. Let $\langle \mathbf{s}, R \rangle \in \mathcal{C}$ and $V'_i = \mathbf{s} \cap V_i$. By Lemma 18

$$\left(\text{pr}_{V'_1} R \cap \prod_{v \in V'_1} \varphi(v) \right) \times \dots \times \left(\text{pr}_{V'_k} R \cap \prod_{v \in V'_k} \varphi(v) \right) \subseteq R.$$

Therefore $\psi(\mathbf{s}) \in R$, and ψ is a solution. \square

Next we define two partitions of a CSP instance \mathcal{P} . The first one, link partition allows us to reduce solving subinstances of \mathcal{P} to instances over smaller domains. The second one provides a sufficient condition to have a link partition and is defined through alignment properties.

Let again $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ be a 3-minimal instance of $\text{CSP}(\mathfrak{A})$. A partition $V_1 \cup \dots \cup V_k = V$ of the set of variables is called a *link partition* if the following condition holds:

- For every $v \in V$ there is a partition $A_{v_1} \cup \dots \cup A_{v_{k_v}} = \mathbb{A}_v$ such that whenever $v, w \in V_i$ for some i , we have $k_v = k_w$, and there is a bijection $\varphi_{vw} : [k_v] \rightarrow [k_w]$ such that for any $(a, b) \in \mathcal{S}_{vw}$ and any $j \in [k_v]$, $a \in A_{v_j}$ if and only if $b \in A_{w_{\varphi_{vw}(j)}}$.

Observe that, since \mathcal{P} is 3-minimal, the mappings φ_{vw} are consistent, that is, for any u, v, w from the same class V_i it holds that $\varphi_{vw} \circ \varphi_{uv} = \varphi_{uw}$. Without loss of generality we will assume that φ_{vw} is an identity mapping whenever v, w belong to the same class of the partition.

As is easily seen the partition $A_{v_1} \cup \dots \cup A_{v_{k_v}} = \mathbb{A}_v$ defines a congruence of \mathbb{A}_v . In particular, each of A_{v_i} is a subalgebra of \mathbb{A}_v .

In a similar way we define another partition of V based on $\overline{\alpha\beta}$ -alignment properties. Variables $v, w \in V$ are $\overline{\alpha\beta}$ -aligned if they are $\overline{\alpha\beta}$ -aligned in \mathcal{S}_{vw} . As is easily seen, this property defines an equivalence relation; let V^1, \dots, V^ℓ be the classes of this relation. By Lemma 21 the $\overline{\alpha\beta}$ -alignment partition is coarser than the partition into coherent sets with respect to $\overline{\alpha}, \overline{\beta}$.

In the following lemma we assume that every domain \mathbb{A}_v of \mathcal{P} either has a minimal element, or $\sigma_{\mathbb{A}_v}$ is the full congruence, and so \mathbb{A}_v is a Mal'tsev algebra.

Lemma 23: (1) If variables $v, w \in V$ of an instance $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ are $\overline{\alpha\beta}$ -aligned and \mathbb{A}_v has a minimal element then \mathbb{A}_w also has a minimal element.

(2) If every domain of an instance $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ has a minimal element the partition of V into aligned sets is a link partition.

Proof: For every $v \in V$ let L_v denote the set of $\alpha_v \beta_v$ -split elements of \mathbb{A}_v and let N_v denote the set of $\alpha_v \beta_v$ -non-split elements. As we observed before Lemma 21 both sets are nonempty if \mathbb{A}_v has a minimal element, and $L_v = \emptyset$ if \mathbb{A}_v is a Mal'tsev algebra.

(1) If \mathbb{A}_w is a Mal'tsev algebra then v, w cannot be $\overline{\alpha\beta}$ -aligned since $L_w = \emptyset$, while $L_v, N_v \neq \emptyset$.

(2) For any $v, w \in V^i$ and any pair $(a, b) \in \mathcal{S}_{vw}$, $a \in L_v$ if and only if $b \in L_w$. Therefore \mathcal{S}_{vw} is link-partitioned, as well as $\text{pr}_{\mathcal{S} \cap V^i} R$ for any constraint $C = \langle \mathcal{S}, R \rangle \in \mathcal{C}$. \square

VII. THE ALGORITHM

In the first part of this section we introduce the property of block-minimality, the key property of CSP instances for our algorithm. We also prove that block-minimality can be efficiently established. Then in the second part we show that block-minimality is sufficient for the existence of a solution, Theorem 2, which is the main result of this section.

A. Block-minimality

Let $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ be a 3-minimal instance such that for every its domain \mathbb{A}_v either $\sigma_{\mathbb{A}_v}$ is the full congruence, and so \mathbb{A}_v is a Mal'tsev algebra with respect to m , or \mathbb{A}_v has a minimal element. Choose an irreducible chain of congruences $\underline{0}_{\mathbb{A}_v} = \alpha_{v1} \prec \alpha_{v2} \prec \dots \prec \alpha_{vk_v} = \theta_{\mathbb{A}_v}$ in $\text{Con}(\mathbb{A}_v)$. Let $k = \max\{k_v \mid v \in V\}$. If $k_v < k$ we set $\alpha_{vk_v+1} = \dots = \alpha_{vk} = \theta_{\mathbb{A}_v}$. We use the following notation. For $i \in [k]$:

- $\xi'_v(i)$ is the greatest $j \in [k_v]$ such that $j \geq i$ and every prime interval in $\alpha_{vi} \prec \alpha_{vi+1} \prec \dots \prec \alpha_{vj}$ cannot be separated from $(\alpha_{vi}, \alpha_{vi+1})$ if $i < k_v$, and $\xi'_v(i) = i$ for $i \geq k_v$;
- $\text{next}_v(i)$ is the minimal $j \in [k_v]$ such that $j > \xi'_v(i)$ and $(\alpha_{vj}, \alpha_{vj+1})$ cannot be separated from $(\alpha_{vi}, \alpha_{vi+1})$, if one exists and if $i < k_v$; $\text{next}_v(i) = k$ otherwise.

For $\beta_v \in \{\alpha_{v1}, \dots, \alpha_{vk}\}$, say, $\beta_v = \alpha_{vi_v}$, let β_v^+ denote α_{vi_v+1} and $\beta_v^+ = \alpha_{vk}$ if $\beta_v = \alpha_{vk}$.

Lemma 24: Let $\overline{\beta} = (\beta_v)_{v \in V}$ be such that $\beta_v \in \{\alpha_{v1}, \dots, \alpha_{vk-1}\}$, say, $\beta_v = \alpha_{vi_v}$. Let also $\eta_v = \alpha_{v \text{next}_v(i_v)}$. (1) the coherent sets with respect to $\overline{\beta}, \overline{\beta}^+$, and $\overline{\eta}, \overline{\eta}^+$ are the same.

(2) for any such coherent set W the $(\overline{\beta}_W, \overline{\beta}_W^+)$ - and $(\overline{\eta}_W, \overline{\eta}_W^+)$ -collapsing polynomials are the same.

Proof: The lemma straightforwardly follows from the definitions. \square

For every $v \in V$ and $i \in [k_v - 1]$ let W_{vi} be the set of $w \in V$ such that for some $j \in [k_w - 1]$ v cannot be separated from w in \mathcal{S}_{vw} with respect to $(\alpha_{vi}, \alpha_{wj}), (\alpha_{vi+1}, \alpha_{wj+1})$. For $w \in W_{vi}$ let $\ell_w = j$, where j is as above. Let $\beta_w = \alpha_{w\ell_w}$ for $w \in W_{vi}$ and $\beta_w = \alpha_{w1}$ for $w \in V - W_{vi}$. As is easily seen, W_{vi} is a coherent set of \mathcal{P} with respect to $\overline{\beta}, \overline{\beta}^+$ and does not depend on the choice of j .

Instance \mathcal{P} is said to be *block-minimal* if for any $v \in V$ and $i \in [k_v - 1]$ the instance $\mathcal{P}_{W_{vi}}$ is minimal.

To show that Theorem 2 below gives rise to a polynomial-time algorithm for $\text{CSP}(\mathfrak{A})$ we need to show how block-minimality can be established. We prove that establishing block-minimality can be reduced to solving polynomially many smaller instances of $\text{CSP}(\mathfrak{A})$.

Proposition 2: Transforming an instance $\mathcal{P} = (V, \mathcal{A}, \mathcal{C}) \in \text{CSP}(\mathfrak{A})$ to a block-minimal instance can be reduced to solving polynomially many instances $\mathcal{P}^i = (V^i, \mathcal{A}^i, \mathcal{C}^i) \in \text{CSP}(\mathfrak{A})$ such that $V^i \subseteq V$ and for all $v \in V^i$ either \mathbb{A}_v^i is a Mal'tsev algebra, or $|\mathbb{A}_v^i| < |\mathbb{A}_v|$.

Since the cardinalities of algebras in \mathfrak{A} are bounded, together with Theorem 2 this proposition gives a polynomial time algorithm for $\text{CSP}(\mathfrak{A})$.

Proof: Using the standard propagation algorithm and Maroti's reduction (Section IV-C) we may assume that \mathcal{P} is 3-minimal and every \mathbb{A}_v is either Mal'tsev or has a minimal element. Let W_{vi} be the coherent sets as in the definition of block-minimality. We need to show how to make problems $\mathcal{P}_{W_{vi}}$ minimal. If every \mathbb{A}_w for $w \in W_{vi}$ is Mal'tsev, $\mathcal{P}_{W_{vi}}$ can be made minimal using the algorithm from [10]. If \mathbb{A}_w has a minimal element for some $w \in W_{vi}$ then by Lemma 23 and 21 $\mathcal{P}_{W_{vi}}$ is link partitioned, that is, it is a disjoint union of instances $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_m$, where $\mathcal{P}_i = (W_{vi}, \mathcal{A}^i, \mathcal{C}^i)$ are such that $\mathbb{A}_w = \mathbb{A}_w^1 \cup \dots \cup \mathbb{A}_w^m$ is a disjoint union. We then transform them to minimal instances separately.

If at any stage there is a tuple from a constraint relation that does not extend to a solution of a certain subinstance, we tighten the original problem \mathcal{P} and start all over again. Observing that the set tuples from a constraint relation that can be extended to a solution of the subinstance is a subalgebra, the resulting instance belongs to $\text{CSP}(\mathfrak{A})$ as well. \square

B. Block-minimality and solutions of the CSP

We now prove that block-minimality is a sufficient condition to have a solution.

Theorem 2: Every block-minimal instance $\mathcal{P} \in \text{CSP}(\mathfrak{A})$ with nonempty constraint relations has a solution.

We start with an auxiliary lemma. As is easily seen, for any $\overline{\gamma}$ such that $\gamma_v \in \{\alpha_{v1}, \dots, \alpha_{vk}\}$ for each $v \in V$, any coherent set W of \mathcal{P} with respect to $\overline{\gamma}, \overline{\gamma}^+$ is a subset of W_{wj} for any $w \in W$ and j such that $\gamma_w = \alpha_{wj}$.

Lemma 25: Let $\overline{\gamma}$ be such that $\gamma_v \in \{\alpha_{v1}, \dots, \alpha_{vk}\}$ for each $v \in V$ and W a coherent set of \mathcal{P} with respect to $\overline{\gamma}, \overline{\gamma}^+$; and let w, j be such that $W \subseteq W_{wj}$. For $v \in W$ if $\gamma_v = \alpha_{vi_v}$

let $j_v = \xi'_v(i_v)$ and $\delta_v = \alpha_{v j_v}$. Suppose φ is a solution of $(\mathcal{P}_{\bar{\delta}})_W$ such that if $\text{next}_v(i_v) \neq k$ for some $v \in W$, then there is a $(\bar{\gamma}_W, \bar{\gamma}_W^+)$ -collapsing polynomial \mathbf{f} of $\mathcal{S}'_W = \text{pr}_W \mathcal{S}_{W_{w_j}}$ with $\varphi \in \mathbf{f}(\mathcal{S}'_W)/\bar{\delta}$. Then there is a solution ψ to $(\mathcal{P}_{\bar{\gamma}_W})_W$ such that $\psi \in \varphi$.

Now we are in a position to prove Theorem 2.

Proof:[of Theorem 2] We prove two claims by induction for any $\bar{\gamma} = (\gamma_v)_{v \in V}$ such that $\gamma_v \in \{\alpha_{v1}, \dots, \alpha_{vk}\}$ for $v \in V$:

- (1) the instance $\mathcal{P}_{\bar{\gamma}}$ has a solution $\varphi_{\bar{\gamma}}$ such that if $\bar{\gamma} \leq \bar{\delta}$ then $\varphi_{\bar{\gamma}} \in \varphi_{\bar{\delta}}$;
- (2) solution $\varphi_{\bar{\gamma}}$ can be chosen such that for any coherent set W with respect to $\bar{\gamma}, \bar{\gamma}^+$, there is a $(\bar{\gamma}_W, \bar{\gamma}_W^+)$ -collapsing polynomial \mathbf{f} of $\text{pr}_W \mathcal{S}_{W_{w_j}}$ for w, j with $W \subseteq W_{w_j}$ such that $\varphi(W) \in \mathbf{f}(\mathcal{S}_W)/\bar{\gamma}_W$.

If $\gamma_v = \alpha_{v1} = \underline{0}_{\mathcal{S}_v}$ then $\mathcal{P}_{\bar{\gamma}} = \mathcal{P}$ and the result follows. The base case of induction is given by $\gamma_v = \alpha_{vk} = \theta_{\mathcal{S}_v}$; then the mapping $\varphi(v) = \max(\mathcal{S}_v)$ is a solution since $\mathcal{S}_v/\sigma_{\mathcal{S}_v}$ is a semilattice.

Suppose $\bar{\gamma}$ is such that for any $\bar{\delta}$ with $\gamma_v \leq \delta_v$ for $v \in V$, where at least one inequality is strict, claims (1) and (2) are true. By the induction hypothesis there is a solution ψ of $\mathcal{P}_{\bar{\gamma}^+}$. Let W be a coherent set with respect to $\bar{\gamma}, \bar{\gamma}^+$. For $v \in W$ let $\gamma_v = \alpha_{v i_v}$, $\delta_v = \alpha_{v \xi'_v(i_v)}$, and $\eta_v = \alpha_{v \text{next}_v(i_v)}$. Let $W' \subseteq W$ be the set of those v for which $\text{next}_v(i_v) \neq k$. Let also $\eta_v = \gamma_v$ for $v \notin W$.

As is easily seen, W' is a coherent set with respect to $\bar{\eta}, \bar{\eta}^+$. Again by the induction hypothesis $\psi(W')/\bar{\eta}_{W'}$ is a solution of $(\mathcal{P}_{\bar{\eta}_{W'}})_{W'}$ and belongs to the range of some $(\bar{\eta}_{W'}, \bar{\eta}_{W'}^+)$ -collapsing polynomial \mathbf{f}' of $\text{pr}_{W'} \mathcal{S}_{W_{w_j}}$. Note that \mathbf{f}' is also a $(\bar{\gamma}_{W'}, \bar{\gamma}_{W'}^+)$ -collapsing. Therefore, there is an extension of \mathbf{f}' to a $(\bar{\gamma}_W, \bar{\gamma}_W^+)$ -collapsing polynomial \mathbf{f} of $\text{pr}_W \mathcal{S}_{W_{w_j}}$. Then $\psi(v) \in \mathbf{f}'_v(\mathcal{S}_v)/\eta_v = \mathbf{f}_v(\mathcal{S}_v)/\eta_v$ for $v \in W'$ and $\psi(v) \in \mathcal{S}_v = \mathbf{f}_v(\mathcal{S}_v)/\eta_v$. Also, since $\zeta_{\gamma_v} \gamma_v^+(\eta_v) \leq \delta_v$. Therefore, as \mathbf{f} is $(\bar{\gamma}_W, \bar{\gamma}_W^+)$ -collapsing, $\mathbf{f}(\psi) \stackrel{\bar{\delta}_W}{\equiv} \psi$. By Lemma 25 there is a solution φ of $(\mathcal{P}_{\bar{\gamma}_W})_W$ such that $\varphi \in \psi(W)$. Since as in the proof of Lemma 25 φ is an image under polynomial \mathbf{f} , this also proves (2) for W .

Now (1) follows by Lemma 22. \square

REFERENCES

- [1] Libor Barto. The dichotomy for conservative constraint satisfaction problems revisited. In *LICS*, pages 301–310, 2011.
- [2] Libor Barto and Marcin Kozik. Constraint satisfaction problems solvable by local consistency methods. *J. ACM*, 61(1):3, 2014.
- [3] Libor Barto, Marcin Kozik, and Todd Niven. The CSP dichotomy holds for digraphs with no sources and no sinks (A positive answer to a conjecture of Bang-Jensen and Hell). *SIAM J. Comput.*, 38(5):1782–1802, 2009.
- [4] Libor Barto, Andrei A. Krokhin., and Ross Willard. Polymorphisms, and How to Use Them. In *The Constraint Satisfaction Problem: Complexity and Approximability [Result of a Dagstuhl Seminar]*, pages 1–44, 2017.
- [5] Andrei A. Bulatov. Three-element Mal'tsev algebras. *Acta Sci. Math (Szeged)*, 71(3-4):469–500, 2002.
- [6] Andrei A. Bulatov. A graph of a relational structure and constraint satisfaction problems. In *LICS*, pages 448–457, 2004.
- [7] Andrei A. Bulatov. A dichotomy theorem for constraint satisfaction problems on a 3-element set. *J. ACM*, 53(1):66–120, 2006.
- [8] Andrei A. Bulatov. Complexity of conservative constraint satisfaction problems. *ACM Trans. Comput. Log.*, 12(4):24, 2011.
- [9] Andrei A. Bulatov. Conservative constraint satisfaction re-revisited. *CoRR*, abs/1408.3690, 2014.

- [10] Andrei A. Bulatov and Víctor Dalmau. A simple algorithm for mal'tsev constraints. *SIAM J. Comput.*, 36(1):16–27, 2006.
- [11] Andrei A. Bulatov and Peter Jeavons. An algebraic approach to multi-sorted constraints. In *CP*, pages 183–198, 2003.
- [12] Andrei A. Bulatov, Peter Jeavons, and Andrei A. Krokhin. Classifying the complexity of constraints using finite algebras. *SIAM J. Comput.*, 34(3):720–742, 2005.
- [13] Andrei A. Bulatov, Andrei A. Krokhin, and Benoit Larose. Dualities for constraint satisfaction problems. In *Complexity of Constraints - An Overview of Current Research Themes [Result of a Dagstuhl Seminar]*, pages 93–124, 2008.
- [14] Andrei A. Bulatov and Matthew Valeriote. Recent results on the algebraic approach to the CSP. In *Complexity of Constraints - An Overview of Current Research Themes [Result of a Dagstuhl Seminar]*, pages 68–92, 2008.
- [15] Andrei A. Bulatov. Graphs of finite algebras, edges, and connectivity. *CoRR*, abs/1601.07403, 2016.
- [16] Andrei A. Bulatov. Constraint satisfaction problems over semilattice block Mal'tsev algebras. *CoRR*, abs/1701.02623, 2017.
- [17] Stanley Burris and H.P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981.
- [18] Rina Dechter. *Constraint processing*. Elsevier Morgan Kaufmann, 2003.
- [19] Tomas Feder and Moshe Y. Vardi. Monotone monadic SNP and constraint satisfaction. In *Proceedings of 25th ACM Symposium on the Theory of Computing (STOC)*, pages 612–622, 1993.
- [20] Tomas Feder and Moshe Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: A study through Datalog and group theory. *SIAM Journal of Computing*, 28:57–104, 1998.
- [21] Georg Grätzer. *General Lattice Theory*, Birkhäuser Verlag, Basel, 1998.
- [22] David Hobby and Ralph N. McKenzie. *The Structure of Finite Algebras*, volume 76 of *Contemporary Mathematics*. American Mathematical Society, Providence, R.I., 1988.
- [23] Pawel M. Idziak, Petar Marković, Ralph McKenzie, Matthew Valeriote, and Ross Willard. Tractability and learnability arising from algebras with few subpowers. *SIAM J. Comput.*, 39(7):3023–3037, 2010.
- [24] Peter Jeavons, David A. Cohen, and Marc Gyssens. Closure properties of constraints. *J. ACM*, 44(4):527–548, 1997.
- [25] Keith A. Kearnes and Ágnes Szendrei. Clones of algebras with parallelogram terms. *Internat. J. Algebra Comput.*, 22(1), 2012
- [26] Petar Marković. The complexity of cps on a 4-element set. Personal communication, 2011.
- [27] Petar Marković and Ralph McKenzie. Block algorithm. Personal communication, 2009.
- [28] Miklós Maróti. Malcev on top. Manuscript, available at <http://www.math.u-szeged.hu/~mmaroti/pdf/200x%20Maltsev%20on%20top.pdf>, 2011.
- [29] Miklós Maróti. Tree on top of Malcev. Manuscript, available at <http://www.math.u-szeged.hu/~mmaroti/pdf/200x%20Tree%20on%20top%20of%20Maltsev.pdf>, 2011.
- [30] Jaroslav Nesetril, Mark H. Siggers, and László Zádori. A combinatorial constraint satisfaction problem dichotomy classification conjecture. *Eur. J. Comb.*, 31(1):280–296, 2010.
- [31] Ian Payne. A CSP Algorithm for some subvarieties of Maltsev products. *A workshop on constraint satisfaction, structure theory and computation in algebra, Boulder CO*, 2016. (oral communication)
- [32] T.J. Schaefer. The complexity of satisfiability problems. In *Proceedings of the 10th ACM Symposium on Theory of Computing (STOC'78)*, pages 216–226, 1978.
- [33] Dmitriy Zhuk. The proof of CSP Dichotomy Conjecture for 5-element domain, In *Arbeitstagung Allgemeine Algebra AAA'91 Brno, Czech Republic*, 2016
- [34] Dmitriy Zhuk. On CSP Dichotomy Conjecture, In *Arbeitstagung Allgemeine Algebra AAA'92 Prague, Czech Republic*, pages 32, 2016,