Graphs of relational structures: restricted types

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Abstract

In our LICS 2004 paper we introduced an approach to the study of the local structure of finite algebras and relational structures that aims at applications in the Constraint Satisfaction Problem (CSP). This approach involves a graph associated with an algebra $\mathbb{A}$ or a relational structure $\mathbf{A}$, whose vertices are the elements of $\mathbb{A}$ (or $\mathbf{A}$), the edges represent subsets of $\mathbb{A}$ such that the restriction of some term operation of $\mathbb{A}$ is 'good' on the subset, that is, act as an operation of one of the 3 types: semilattice, majority, or affine. In this paper we significantly refine and advance this approach. In particular, we prove certain connectivity and rectangularity properties of relations over algebras related to components of the graph connected by semilattice and affine edges. We also prove a result similar to 2-decomposition of relations invariant under a majority operation, only here we do not impose any restrictions on the relation. These results allow us to give a new, somewhat more intuitive proof of the bounded width theorem: the CSP over algebra $\mathbb{A}$ has bounded width if and only if $\mathbb{A}$ does not contain affine edges. Actually, this result shows that bounded width implies width (2,3). We also consider algebras with edges from a restricted set of types. In particular, it can be proved that type restrictions are preserved under the standard algebraic constructions. Finally, we prove that algebras without semilattice edges have few subalgebras of powers, that is, the CSP over such algebras is also polynomial time.

Categories and Subject Descriptors F.2.2 [*Nonnumerical Algorithms and Problems*]: Computations on discrete structures; G.2.1 [*Combinatorics*]: Combinatorial algorithms

Keywords constraint satisfaction problem, algebraic approach, bounded width, few subalgebras

1. Introduction

The Constraint Satisfaction Problem (CSP) has received a great deal of attention over the last several decades from various areas including logic, artificial intelligence, computer science, discrete mathematics, and algebra. Different facets of the CSP play an important role in all these disciplines. In this paper we focus on the complexity of and algorithms for the CSP. This direction in the study of the CSP revolves around the Dichotomy conjecture by (Feder and Vardi 1993, 1998) for the decision version of the problem, and the Unique Games conjecture by (Khot 2002) for the optimization version.

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One of the several possible forms of the CSP asks whether there exists a homomorphism between two given relational structures. The Dichotomy conjecture deals with the so called *nonuniform* CSP parametrized by the target structure $\mathbf{B}$; that is, for a given relational structure $\mathbf{A}$ the goal is to decide the existence of a homomorphism from $\mathbf{A}$ to the fixed target structure $\mathbf{B}$. Such a problem is usually denoted by $\mathrm{CSP}(\mathbf{B})$. The conjecture claims that every problem $\mathrm{CSP}(\mathbf{B})$ is either NP-complete, or is solvable in polynomial time; so no intermediate complexity class is attained by problems $\mathrm{CSP}(\mathbf{B})$. This conjecture has been attacked using different approaches, see, e.g. (Kolaitis and Vardi 2000; Kolaitis 2003; Hell and Nešetřil 1990; Hell and Nešetřil 2004), however, the algebraic approach using invariance properties of relational structures seems the most promising at the moment. This approach is based on exploiting the properties of *polymorphisms* of relational structures, which can be thought of as homomorphisms from a power $\mathbf{A}^n$ of a structure $\mathbf{A}$ to the structure itself, but are usually viewed as multi-ary operations on $\mathbf{A}$ 'preserving' the relations of $\mathbf{A}$. The use of polymorphisms was first proposed by (Jeavons et al. 1997; Jeavons 1998; Jeavons et al. 1998), who showed that the complexity of $\mathrm{CSP}(\mathbf{B})$ is completely determined by the polymorphisms of $\mathbf{B}$, and identified several types of polymorphisms whose presence guarantees the solvability of $\mathrm{CSP}(\mathbf{B})$ in polynomial time. We will be using these types of operations all the time in this paper, so we name them here: semilattice, majority, and affine operations, for exact definitions see Section 2.1. The algebraic approach was later developed further in (Bulatov et al. 2005; Bulatov and Jeavons 2003) to use universal algebras associated with relational structures rather than polymorphisms; which allowed for applications of structural results from universal algebra. This connection has been used first to state the Dichotomy conjecture in a precise form, that basically boils down to the presence of 'nontrivial' polymorphisms (in which case $\mathrm{CSP}(\mathbf{B})$ is polynomial time solvable) (Bulatov et al. 2005), and to obtain a number of strong tractability and dichotomy results (Bulatov 2002, 2006; Bulatov and Dalmau 2006; Bulatov 2011b, 2016b; Barto 2011; Barto et al. 2012; Barto and Kozik 2014; Idziak et al. 2010). However, in spite of all these achievements the Dichotomy conjecture remains open.

The main obstacle in proving (or refuting) the Dichotomy conjecture seems to be that the existing structural theories of universal algebras are not designed for the CSP. Therefore, the study of the CSP has triggered substantial research in algebra aiming to obtain more advanced results on the structure of finite algebras. Probably the most well developed approach at the moment is based on the absorbing properties of algebras, see, e.g. (Barto and Kozik 2012; Barto 2015). Within this approach the bounded width conjecture has been proved (Barto and Kozik 2014) (see more about this conjecture in subsequent sections), along with many algebraic results and generalizations of the known CSP complexity results (Barto 2011; Barto et al. 2012; Barto and Kozik 2014; Barto 2016). Another potential approach is to use so called *key relations*, i.e. relations that cannot be represented through a combination of simpler ones, see, e.g. (Zhuk 2014); although this method requires further development. The third approach has been originally introduced in (Bulatov 2004; Bulatov and Valeriote 2008; Bulatov 2011a) and uses the local structure of universal algebras. More precisely, it identifies small sets of elements of a relational structure or an algebra — in most cases 2-element sets — such that there is a polymorphism of the structure or a term operation of the algebra that behaves well on this subset, where 'well' means that the operation is close to a semilattice, majority, or an affine one. These subsets are then considered edges of a graph that can have one of the three types, corresponding to the three types of good operations: semilattice, majority, or affine. For a relational structure $\mathbf{A}$ or an algebra $\mathbb{A}$ the resulting graph will be denoted by $\mathcal{G}(\mathbf{A})$ and $\mathcal{G}(\mathbb{A})$, respectively. It turns out that for every algebra $\mathbb{A}$ that gives rise to a tractable CSP according to the Dichotomy conjecture, the graph $\mathcal{G}(\mathbb{A})$ is connected, moreover, the types of edges present in the graph are related to other properties of the CSP. In particular, the absence of affine edges corresponds to the bounded width of the CSP.

In this paper we refine and advance the approach from (Bulatov 2004; Bulatov and Valeriote 2008). The main motif of this work is to consider algebras $\mathbb{A}$ for which the graph $\mathcal{G}(\mathbb{A})$ contains edges from a restricted set of types. We first show that the property to have edges from a certain set of types is preserved under the standard algebraic constructions.

THEOREM 1. *Let $S \subseteq \{semilattice, majority, affine\}$ and $\mathbb{A}$ be a finite idempotent algebra such that every edge of $\mathcal{G}(\mathbb{A})$ has a type from S. Then every edge of any finite algebra from the variety generated by $\mathbb{A}$ belongs to S.*

Then we prove that if we restrict the type of edges to semilattice and majority, or majority and affine then the algebra belongs to one of the two major known classes of tractable algebras. In the semilattice-majority case we also give a somewhat more intuitive proof for the characterization of CSPs of bounded width than that in (Barto and Kozik 2014) and (Bulatov 2009).

THEOREM 2. *Let $\mathbb{A}$ be an idempotent algebra every edge of which is semilattice or majority. Then $\mathrm{CSP}(\mathbb{A})$ has bounded width. Moreover, every algebra that gives rise to a CSP of bounded width satisfies this condition.*

An algebra $\mathbb{A}$ is said to have *few subpowers* if the number of subalgebras of direct products of several copies of $\mathbb{A}$ is exponentially smaller than it generally can be. (Idziak et al. 2010) proved that if $\mathbb{A}$ has few subpowers then $\mathrm{CSP}(\mathbb{A})$ can be solved in polynomial time. Moreover, for such CSPs it is possible to construct a small (polynomial size) set of generators of the set of all solutions to the problem. We show that every algebra whose edges are majority or affine has few subpowers, although it is not true that every algebra with few subpowers satisfies this condition.

THEOREM 3. *Let $\mathbb{A}$ be an idempotent algebra every edge of which is majority or affine. Then $\mathbb{A}$ has few subpowers. In particular, $\mathrm{CSP}(\mathbb{A})$ can be solved in polynomial time.*

Finally, we consider the components of the graph $\mathcal{G}(\mathbb{A})$ that are connected by edges of the semilattice and affine types. Such components are called *as-components*. We prove that every relation over $\mathbb{A}$ is *rectangular* for every as-component. (For a binary relation $R \subseteq \mathbb{A} \times \mathbb{A}$ rectangularity means that whenever $(a,c),(b,c),(b,d) \in R$ for some elements $a,b,c,d \in \mathbb{A}$, it also holds that $(a,d) \in R$; for a general definition see Section 5). It is well known (Baker and Pixley 1975) that if a relational structure has a majority polymorphism then every relation of this structure is *2-decomposable*, that is, it is completely determined by its binary projections. This property allows for a simple local algorithm for the corresponding CSP. Here we show that a similar property of *quasi-2-decomposability* holds for arbitrary structures (algebras) satisfying the conditions of the Dichotomy conjecture. More precisely, this condition means that for every relation $R \subseteq \mathbb{A}^n$ and any collection $B_1, \dots, B_n$ of as-components of $\mathcal{G}(\mathbb{A})$ whenever there is a tuple $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{A}^n$ (not necessarily from R), $a_\ell \in B_\ell$ for $\ell \in \{1, \dots, n\}$, such that for every $i, j \in \{1, \dots, n\}$ there is a tuple $\mathbf{b}_{ij} \in R$ that agrees with $\mathbf{a}$ in positions i, j, then there is a $\mathbf{b} = (b_1, \dots, b_n) \in R$ with $b_\ell \in B_\ell$ for $\ell \in \{1, \dots, n\}$.

THEOREM 4. *Let R be a subalgebra of $\mathbb{A}^n$, where $\mathbb{A}$ satisfies the conditions of the Dichotomy conjecture. Then R is quasi-2-decomposable.*

Theorem 4 allows us to construct a reduction that reduces any CSP to a CSP which is essentially over a single as-component.

2. Preliminaries

2.1 Relational structures, algebras, and the CSP

By $[n]$ we denote the set $\{1, \dots, n\}$. For sets $A_1, \dots, A_n$ tuples from $A_1 \times \cdots \times A_n$ are denoted in boldface, say, $\mathbf{a}$; the ith component of $\mathbf{a}$ is referred to as $\mathbf{a}[i]$. An n-ary relation R over sets $A_1, \dots, A_n$ is any subset of $A_1 \times \cdots \times A_n$. For $I = \{i_1, \dots, i_k\} \subseteq [n]$ by $\mathrm{pr}_I \mathbf{a}, \mathrm{pr}_I R$ we denote the *projections* $\mathrm{pr}_I \mathbf{a} = (\mathbf{a}[i_1], \dots, \mathbf{a}[i_k])$, $\mathrm{pr}_I R = \{\mathrm{pr}_I \mathbf{a} \mid \mathbf{a} \in R\}$ of tuple $\mathbf{a}$ and relation R. If $\mathrm{pr}_i R = A_i$ for each $i \in [n]$, relation R is said to be a *subdirect product* of $A_1 \times \cdots \times A_n$. As usual, a *relational structure* $\mathbf{A}$ with a *(relational) alphabet* $(R_1, \dots, R_m)$ is a set A equipped with interpretations $R_i^{\mathbf{A}}$ of predicate symbols with relations over A of matching arity.

We assume familiarity with basic concepts of universal algebra, for references see (Burris and Sankappanavar 1981). A (universal) algebra $\mathbb{A}$ with a *functional alphabet* $f_1, \dots, f_\ell$ is a set A, called the *universe* equipped with interpretations $f_i^{\mathbb{A}}$ of functional symbols with (multi-ary) operations on A of matching arity. In this paper all structures and algebras are assumed finite. Algebras with the same functional alphabet are said to be *similar*. Operations that can be derived from $f_1, \dots, f_\ell$ by means of composition are called *term operations*. Let $\mathbb{A}, \mathbb{B}$ be similar algebras with universes A and B, respectively. A mapping $\varphi : A \to B$ is a *homomorphism* of algebras, if it preserves all the operations, that is $\varphi(f_i^{\mathbb{A}}(a_1, \dots, a_k)) = f_i^{\mathbb{B}}(\varphi(a_1), \dots, \varphi(a_k))$ for any $i \in [\ell]$ and any $a_1, \dots, a_k \in A$. A bijective homomorphism is an *isomorphism*. A set $B \subseteq A$ is a *subuniverse* of $\mathbb{A}$ if, for every $i \in [\ell]$, the operation $f_i^{\mathbb{A}}$ restricted to B takes values from B only. For a nonempty subuniverse B of algebra $\mathbb{A}$ the algebra $\mathbb{B}$ with universe B and operations $f_1^{\mathbb{B}}, \dots, f_\ell^{\mathbb{B}}$ (where $f_i^{\mathbb{B}}$ is a restriction of $f_i^{\mathbb{A}}$ to B) is a *subalgebra* of $\mathbb{A}$. Given similar algebras $\mathbb{A}, \mathbb{B}$, a *product* $\mathbb{A} \times \mathbb{B}$ of $\mathbb{A}$ and $\mathbb{B}$ is the algebra similar to $\mathbb{A}$ and $\mathbb{B}$ with universe $A \times B$ and operations computed coordinate-wise. An algebra $\mathbb{C}$ is a *subdirect product* of $\mathbb{A}$ and $\mathbb{B}$ if it is a subalgebra of $\mathbb{A} \times \mathbb{B}$ whose universe is a subdirect product of A and B. An equivalence relation θ on A is called a *congruence* of algebra $\mathbb{A}$ if θ is a subalgebra of $\mathbb{A} \times \mathbb{A}$. Given a congruence θ on $\mathbb{A}$ we can form the *factor algebra* $\mathbb{A}/_\theta$ similar to $\mathbb{A}$, whose elements are the equivalence classes of θ and the operations are defined so that the natural projection mapping is a homomorphism $\mathbb{A} \to \mathbb{A}/_\theta$. The θ-block containing element $a \in \mathbb{A}$ is denoted by a^θ. We often abuse the notation and use the same operation symbol for all similar algebras including factor algebras. In particular, to make notation lighter we use f rather than $f/_\theta$ for operations on a factor algebra. Algebra $\mathbb{A}$ is *simple*, if it has the trivial congruences only (i.e. the equality relation and the full congruence). If θ is a maximal congruence of $\mathbb{A}$, then $\mathbb{A}/_\theta$ is simple. A *variety* is a class of algebras closed under direct products (including infinite products), subalgebras, and homomorphic images (or factor algebras). Algebra $\mathbb{A}$ is said to be idempotent if $f_i(x,\dots,x) = x$ for all $x \in A$ and any $i \in [\ell]$. If θ is a congruence of an idempotent algebra $\mathbb{A}$, then θ-blocks are subuniverses of $\mathbb{A}$. The subalgebra of $\mathbb{A}$ generated by a set $B \subseteq \mathbb{A}$ is denoted $\mathsf{Sg}_{\mathbb{A}}(B)$. In most cases $\mathbb{A}$ is clear from the context and is omitted.

The connection between algebras and relational structures is given by the invariance relation. Let $A_1,\dots,A_n$ be sets, operation f is defined on each of the A_i, and R is a relation over $A_1,\dots,A_n$. An operation $f(x_1,\dots,x_k)$ is said to *preserve* relation R, or f is a *polymorphism* of R, or R is *invariant* with respect to f, if for any $\mathbf{a}_1,\dots,\mathbf{a}_k \in R$ the tuple $f(\mathbf{a}_1,\dots,\mathbf{a}_k) \in R$. Operation f on a set A is a polymorphism of relational structure $\mathbf{A} = (A; R_1,\dots,R_m)$ if it is a polymorphism of every relation of $\mathbf{A}$. This definition can be generalized to multi-sorted relational structures, but we do not need it here. For a (finite) class of finite algebras $\mathcal{A}$ with basic operations $f_1,\dots,f_\ell$ by $\mathsf{Inv}(\mathcal{A})$ we denote the class of all finitary relations over the universes of algebras from $\mathcal{A}$ invariant under every f_i, $i \in [\ell]$. Alternatively, $\mathsf{Inv}(\mathcal{A})$ is the class of subalgebras of direct products of algebras from $\mathcal{A}$.

The (*nonuniform*) *Constraint Satisfaction Problem* (*CSP*) associated with a relational structure $\mathbf{B}$ is the problem $\mathrm{CSP}(\mathbf{B})$, in which, given a structure $\mathbf{A}$ of the same signature as $\mathbf{B}$, the goal is to decide whether or not there is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. For a class of algebras $\mathcal{A} = \{\mathbb{A}_i \mid i \in I\}$ for some set I an instance of $\mathrm{CSP}(\mathcal{A})$ is a triple $(V,\delta,\mathcal{C})$, where V is a set variables; $\delta : V \to \mathcal{A}$ is a type function that associates every variable with a domain in $\mathcal{A}$. Finally, $\mathcal{C}$ is a set of constraints, i.e. pairs $\langle \mathbf{s}, R\rangle$, where $\mathbf{s} = (v_1,\dots,v_k)$ is a tuple of variables from V, and $R \in \mathsf{Inv}(\mathcal{A})$, a subset of $A_{\delta(v_1)} \times \dots A_{\delta(v_k)}$. The goal is to find a solution, that is a mapping $\varphi : V \to \bigcup \mathcal{A}$ such that $\varphi(v) \in \delta(v)$ and for every constraint $\langle \mathbf{s}, R\rangle$, $\varphi(\mathbf{s}) \in R$. It is easy to see that if $\mathcal{A}$ is a class containing just one algebra $\mathbb{A}$, then $\mathrm{CSP}(\mathcal{A})$ can be viewed as the union of $\mathrm{CSP}(\mathbf{A})$ for all relational structures $\mathbf{A}$ invariant under the operations of $\mathbb{A}$.

The CSP dichotomy conjecture (Feder and Vardi 1993) states that for every relational structure $\mathbf{B}$ $\mathrm{CSP}(\mathbf{B})$ is either solvable in polynomial time or is NP-complete. In its algebraic form (Bulatov et al. 2005) it claims that for any finite algebra $\mathbb{A}$ the problem $\mathrm{CSP}(\mathbb{A})$ is either solvable in polynomial time or NP-complete; the single algebra $\mathbb{A}$ can also be replaced here with a finite class of finite similar algebras. The algebraic approach also helps to make the conjecture more precise: for a class $\mathcal{A}$ of idempotent algebras the problem $\mathrm{CSP}(\mathcal{A})$ is solvable in polynomial time if and only if the variety generated by $\mathcal{A}$ does not contain 'trivial' algebras, or, equivalently, when it omits type **1** in the sense of tame congruence theory (Hobby and McKenzie 1988). Otherwise $\mathrm{CSP}(\mathcal{A})$ is NP-complete. Note that all CSPs for non-idempotent algebras or relational structures are equivalent to some CSPs over idempotent algebras under log-space reductions (Bulatov et al. 2005). In the next section we give an alternative characterization of algebras omitting type **1** that will be used in the paper. In particular, all algebras we deal with will be assumed finite, idempotent, and omitting type **1**.

2.2 Coloured graphs

In (Bulatov 2004; Bulatov and Valeriote 2008) we introduced a local approach to the structure of finite algebras. As we use this approach throughout the paper, we present it here in some details, see also (Bulatov 2016a).

Let $\mathbb{A}$ be an algebra with universe A. Recall that a binary operation f on A is said to be *semilattice* if it satisfies the equations $f(x,x) = x$, $f(x,y) = f(y,x)$, and $f(x,f(y,z)) = f(f(x,y),z)$ for any $x,y,z \in A$. A ternary operation g is said to be *majority* if it satisfies the equations $g(x,x,y) = g(x,y,x) = g(y,x,x) = x$ for all $x,y \in A$. It is called *Mal'tsev* if it satisfies $g(x,y,y) = g(y,y,x) = x$. An operation is said to be semilatiice (majority, Mal'tsev) on a set $B \subseteq A$ or $B \subseteq A/_\theta$ for an equivalence relation θ, if the above equalities hold for all $x,y,z \in B$. We will often use a special type of algebras. A *module* over a ring $\mathbb{R}$ with unity 1 is an algebra $\mathbb{M}$ with universe M, whose basic operations are a constant 0, binary addition $+$, unary $-$, and also unary multiplication by a scalar r for each $r \in \mathbb{R}$. The operations have to satisfy the following conditions: M with 0, $+$, and $-$ is an Abelian group, $r(x+y) = rx + ry$, $1x = x$, and $r_1(r_2x) = (r_1r_2)x$ for any $r, r_1, r_2 \in \mathbb{R}$ and any $x,y \in M$. Every term operation of $\mathbb{M}$ has the form $r_1x_1 + \dots + r_kx_k$, it is idempotent if $r_1 + \dots + r_k = 1$. Every module has a Mal'tsev operation $x - y + z$; we call this operation of a module *affine*. Modules are not idempotent, and so in this paper they are replaced with their *full idempotent reducts*, in which we remove all the non-idempotent operations from the module. For every submodule $\mathbb{M}'$ the set of its cosets (sets of the form $\mathbb{M}' + a$) form a partition that gives rise to a congruence of $\mathbb{M}$. Therefore simple modules do not have proper submodules. This is also true for the full idempotent reduct of a module (in what follows we abuse the terminology and call such reducts just modules).

Graph $\mathcal{G}(\mathbb{A})$ is introduced as follows. The vertex set is the set A. A pair ab of vertices is an *edge* iff there exists a congruence θ of $\mathsf{Sg}(a,b)$, other than the full congruence and a term operation f of $\mathbb{A}$ such that either $\mathsf{Sg}(a,b)/_\theta$ is a module and f is an affine operation on it, or f is a semilattice operation on $\{a^\theta, b^\theta\}$, or f is a majority operation on $\{a^\theta, b^\theta\}$. (Note that we use the same operation symbol in this case.) In most cases θ can be chosen to be a maximal congruence of $\mathsf{Sg}(a,b)$, however, sometimes we want it as small as possible.

If there are a congruence and a term operation of $\mathbb{A}$ such that f is a semilattice operation on $\{a^\theta, b^\theta\}$ then ab is said to have the *semilattice type*. An edge ab is of *majority type* if there are a congruence θ and a term operation f such that f is a majority operation on $\{a^\theta, b^\theta\}$ and there is no semilattice term operation on $\{a^\theta, b^\theta\}$. Finally, ab has the *affine type* if there are θ and f such that f is an affine operation on $\mathsf{Sg}(a,b)/_\theta$ and $\mathsf{Sg}(a,b)/_\theta$ is a module; in particular it implies that there is no semilattice or majority operation on $\{a^\theta, b^\theta\}$. In all cases we say that congruence θ *witnesses* the type of edge ab. Observe that a pair ab can still be an edge of more than one type as witnessed by different congruences.

Omitting type **1** can be characterized as follows.

THEOREM 5 ((Bulatov 2004, 2016a)). *An idempotent algebra $\mathbb{A}$ omits type* **1** *iff $\mathcal{G}(\mathbb{B})$ is connected for every subalgebra $\mathbb{B}$ of $\mathbb{A}$.*

For the sake of the dichotomy conjecture, it suffices to consider *reducts* of an algebra $\mathbb{A}$ omitting type **1**, that is, algebras with same universe but reduced set of term operations, as long as reducts also omit type **1**. In particular, we are interested in reducts of $\mathbb{A}$, in which semilattice and majority edges are subalgebras.

THEOREM 6 ((Bulatov 2004, 2016a)). *Let $\mathbb{A}$ be an idempotent algebra omitting type* **1**, *ab an edge of $\mathcal{G}(\mathbb{A})$ of the semilattice or majority type witnessed by congruence θ, and $R_{ab} = a^\theta \cup b^\theta$. Let also F_{ab} denote set of term operations of $\mathbb{A}$ preserving R_{ab}*

(1) $\mathbb{A}' = (A, F_{ab})$ omits type **1**.

(2) If ab is majority and $\mathcal{G}(\mathbb{A})$ has no affine edges, then $\mathcal{G}(\mathbb{A}')$ also contains no affine edges.

An algebra $\mathbb{A}$ such that $a^\theta \cup b^\theta$ is a subuniverse of $\mathbb{A}$ for every semilattice or majority edge ab of $\mathbb{A}$ is called *sm-smooth*.

Operations witnessing the type of edges can be significantly uniformized.

THEOREM 7 ((Bulatov 2004, 2016a)). *Let $\mathbb{A}$ be an idempotent algebra. There are term operations f, g, h of $\mathbb{A}$ such that $f|_{\{a^\theta, b^\theta\}}$ is a semilattice operation if ab is a semilattice edge; $g|_{\{a^\theta, b^\theta\}}$ is a majority operation if ab is a majority edge; $h|_{\mathsf{Sg}(ab)/_\theta}$ is an affine operation operation if ab is an affine edge, where θ witnesses the type of the edge. Moreover, f, g, h can be chosen such that*

(1) $f(x, f(x, y)) = f(x, y)$ for all $x, y \in \mathbb{A}$;
(2) $g(x, g(x, y, y), g(x, y, y)) = g(x, y, y)$ for all $x, y \in \mathbb{A}$;
(3) $h(h(x, y, y), y, y) = h(x, y, y)$ for all $x, y \in \mathbb{A}$.

Unlike majority and affine operations, for a semilattice edge ab and a congruence θ of $\mathsf{Sg}(a, b)$ witnessing that, there can be semilattice operations acting differently on $\{a^\theta, b^\theta\}$,which corresponds to the two possible orientations of ab. In every such case by fixing operation f from Theorem 7 we effectively choose one of the two orientations. In this paper we do not really care about what orientation is preferable.

Edges as defined above do not help too much. In (Bulatov 2016a) we therefore refine these notions. A pair ab of elements of algebra $\mathbb{A}$ is called a *thin semilattice edge* if ab is a semilattice edge, and the congruence witnessing that is the equality relation. In other words, $f(a, a) = a$ and $f(a, b) = f(b, a) = f(b, b) = b$. We denote the fact that ab is a thin semilattice edge by $a \le b$. Thin semilattice edges allow us to introduce a directed graph $\mathcal{G}_s(\mathbb{A})$, whose vertices are the elements of $\mathbb{A}$, and the arcs are the thin semilattice edges. We then can define *semilattice-connected* and *strongly semilattice-connected* components of $\mathcal{G}_s(\mathbb{A})$. We will also use the natural order on the set of strongly semilattice-connected components of $\mathcal{G}_s(\mathbb{A})$: for components A, B, we write $A \le B$ if there is a directed path in $\mathcal{G}_s(\mathbb{A})$ connecting a vertex from A with a vertex from B. Elements from the maximal strongly connected components (or simply *maximal components*) of $\mathcal{G}_s(\mathbb{A})$ are called *maximal* elements of $\mathbb{A}$ and the set of all such elements is denoted by $\mathsf{max}(\mathbb{A})$. A directed path in $\mathcal{G}_s(\mathbb{A})$ is called a *semilattice path* or *s-path*. If there is an s-path from a to b we write $a \sqsubseteq b$.

PROPOSITION 8 ((Bulatov 2004, 2016a)). *Let $\mathbb{A}$ be a finite algebra omitting type $\mathbf{1}$. There is a binary term operation f of $\mathbb{A}$ such that f is a semilattice operation on $\{a^\theta, b^\theta\}$ for every semilattice edge ab of $\mathbb{A}$, where congruence θ witnesses that, and, for any $a, b \in \mathbb{A}$, either $a = f(a, b)$ or the pair $(a, f(a, b))$ is a thin semilattice edge of $\mathbb{A}$. Operation f with this property will be denoted by a dot (think multiplication).*

Let operatons f, g, h be as in Theorem 7. A pair ab from $\mathbb{A}$ is called a *thin majority edge* if (a) it is a majority edge, let congruence θ witness this, (b) for any $c \in b^\theta$, $b \in \mathsf{Sg}(a, c)$, (c) $g(a, b, b) = b$, and (d) there exists a ternary term operation g' such that $g'(a, b, b) = g'(b, a, b) = g'(b, b, a) = b$. Finally, a pair ab is called a *thin affine edge* if (a) it is an affine edge, let congruence θ witness this, (b) for any $c \in b^\theta$, $b \in \mathsf{Sg}(a, c)$, (c) $h(b, a, a) = b$, (d) there exists a ternary term operation h' such that $h'(b, a, a) = h'(a, a, b) = b$, and (e) a is maximal in $\mathsf{Sg}(a, b)$. Note that the operations h, g from Theorem 7 do not have to satisfy conditions (2), (3) of that theorem on thin edges; thin edges even do not have to be closed under g, h. Thin edges of all types are oriented. We therefore can define yet another directed graph, $\mathcal{G}'(\mathbb{A})$, in which the arcs are the thin edges of all types.

LEMMA 9 ((Bulatov 2016a)). *Let $\mathbb{A}$ be an algebra.*

(1) Let ab be a semilattice or majority edge in $\mathbb{A}$, and θ the congruence of $\mathsf{Sg}(a, b)$ witnessing that. Then there is $b' \in b^\theta$ such that ab' is a thin semilattice or majority edge, respectively.

(2) Let ab be an affine edge, and θ the congruence of $\mathsf{Sg}(a, b)$ witnessing that. Then there are $a' \in a^\theta$ and $b' \in b^\theta$ such that $a \sqsubseteq a'$ and $a'b'$ is a thin affine edge.

A directed path in $\mathcal{G}'(\mathbb{A})$ is called a *path*. If all edges of this path are semilattice or affine, it is called an *affine-semilattice path* or an *as-path*, if there is an as-path from a to b we write $a \sqsubseteq_{as} b$. Similar to maximal components, we consider the strongly connected components of $\mathcal{G}'(\mathbb{A})$ with majority edges removed, and the natural partial order on such components. The maximal components will be called *as-components*, and the elements from as-components are called *as-maximal*; the set of all as-maximal elements of $\mathbb{A}$ is denoted by $\mathsf{amax}(\mathbb{A})$ If a is an as-maximal element, the as-component containing a is denoted $\mathsf{as}(a)$.

PROPOSITION 10 ((Bulatov 2016a)). *Let $\mathbb{A}$ be an algebra omitting type $\mathbf{1}$. Then*

(1) any $a, b \in \mathbb{A}$ are connected in $\mathcal{G}'(\mathbb{A})$ with an undirected path;

(2) any $a, b \in \mathsf{max}(\mathbb{A})$ (or $a, b \in \mathsf{amax}(\mathbb{A})$) are connected in $\mathcal{G}'(\mathbb{A})$ with a directed path.

The following simple properties of thin edges will be useful. Note that a subdirect product of algebras (a relation) is also an algebras, and so edges and thin edges can be defined for relations as well.

LEMMA 11 ((Bulatov 2016a)). *(1) Let $\mathbb{A}$ be an algebra omitting type $\mathbf{1}$ and ab a thin edge. Then ab is a thin edge in any subalgebra of $\mathbb{A}$ containing a, b, and $a^\theta b^\theta$ is a thin edge in $\mathbb{A}/_\theta$ for any congruence θ.*

(2) Let R be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$, $I \subseteq [n]$, and $\mathbf{ab}$ a thin edge in R. Then $\mathrm{pr}_I \mathbf{a} \mathrm{pr}_I \mathbf{b}$ is a thin edge in $\mathrm{pr}_I R$ of the same type as $\mathbf{ab}$.

The graphs $\mathcal{G}(\mathbb{A})$ and $\mathcal{G}'(\mathbb{A})$ retain substantial amount of crucial information required for solving CSPs. They witness that the omitting type $\mathbf{1}$ condition and the bounded width condition hold, and also can certify some other useful properties. However, in general they also erase much information about the algebra. As an extreme example, if $\mathbb{A}$ is a prime algebra, that is, every possible operation on its universe is a term operation of $\mathbb{A}$, then it satisfies all the CSP related conditions on an algebra. It has few subpowers (see the Section 6), $\mathrm{CSP}(\mathbb{A})$ has bounded width, etc. But according to the definitions graphs $\mathcal{G}(\mathbb{A})$ and $\mathcal{G}'(\mathbb{A})$ have only semilattice edges that are oriented in an arbitrary way. In particular, there is no way to know from these graphs that $\mathbb{A}$ has few subpowers, unless one picks a very special orientation of semilattice edges.

3. Algebras with graphs of restricted types

Let $T \subseteq$ {semilattice, majority, affine}. An algebra $\mathbb{A}$ is said to be *T-restricted* if every edge of $\mathbb{A}$ has a type from T.

THEOREM 12. *Let $T \subseteq$ {semilattice, majority, affine} and $\mathcal{A}$ a finite collection of similar T-restricted algebras omitting type $\mathbf{1}$. Then every finite algebra from the variety generated by $\mathcal{A}$ is T-restricted.*

Proof: Every subalgebra of a T-restricted algebra is T-restricted, as it follows from the definition of types of edges. Let $\mathbb{A} = \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ where all $\mathbb{A}_1, \dots, \mathbb{A}_n$ are T-restricted. Suppose there is an edge $\mathbf{ab}$ in $\mathbb{A}$ of type $z \in$ {semilattice, majority, affine} $- T$, and θ is the congruence of $\mathbb{B} = \mathsf{Sg}(\mathbf{a}, \mathbf{b})$ witnessing that. Let $I(\mathbf{a}', \mathbf{b}') = \{i \in [n] \mid \mathbf{a}'[i] = \mathbf{b}'[i]\}$ for $\mathbf{a}', \mathbf{b}' \in \mathbb{A}$. Tuples $\mathbf{a}, \mathbf{b}$ can be assumed to be such that $I = I(\mathbf{a}, \mathbf{b})$ is maximal among pairs $\mathbf{a}', \mathbf{b}'$ with $\mathbf{a}' \in \mathbf{a}^\theta$, $\mathbf{b}' \in \mathbf{b}^\theta$. Then for any $\mathbf{c} \in \mathbb{B}$ and any $i \in I$, $\mathbf{c}[i] = \mathbf{a}[i]$. Take $i \in [n] - I$ and set $A' = \{\mathbf{a}'[i] \mid \mathbf{a}' \in \mathbf{a}^\theta\}$ and $B' = \{\mathbf{b}'[i] \mid \mathbf{b}' \in \mathbf{b}^\theta\}$. By the choice of $\mathbf{a}, \mathbf{b}$, $A' \cap B' = \varnothing$.

This means that the projection of θ on the ith coordinate, that is, the congruence of $\mathbb{C} = \mathrm{pr}_i\mathbb{B}$ given by

$$\eta = \{(a,b) \mid \text{ for some } \mathbf{c},\mathbf{d} \in \mathbb{B}, a = \mathbf{c}[i], b = \mathbf{d}[i], (\mathbf{c},\mathbf{d}) \in \theta\}$$

is nontrivial. Moreover, every term operation that is semilattice, majority or affine on $\mathbb{B}/_\theta$ is semilattice, majority, or affine on $\mathbb{C}/_\eta$, as well. Since $\mathbb{C}$ is generated by $\mathbf{a}[i], \mathbf{b}[i]$, this pair is an edge of $\mathbb{A}_i$ of type z, a contradiction.

Now suppose that $\mathbb{A}$ is T-restricted and $\mathbb{B} = \mathbb{A}/_\alpha$ for some congruence α. Let ab, $a,b \in \mathbb{B}$, be an edge of type $z \in$ {semilattice, majority, affine} and θ a maximal congruence of $\mathbb{C} = \mathsf{Sg}_{\mathbb{B}}(a,b)$ witnessing that. We will find $a',b' \in \mathbb{A}$ such that $a'b'$ is an edge of $\mathbb{A}$ of type z. Let $\mathbb{C}' = \bigcup_{c\in\mathbb{C}} c$ (elements of $\mathbb{C}$ are subsets of $\mathbb{A}$), $\alpha' = \alpha \cap \mathbb{C}'^2$, and $\theta' = \alpha' \vee \theta$, a congruence of $\mathbb{C}'$. Choose $a',b' \in \mathbb{C}'$ such that $a' \in a$ and $b' \in b$. Let θ'' be the restriction of θ' on $\mathbb{C}'' = \mathsf{Sg}_{\mathbb{A}}(a',b')$. Then clearly, $\mathbb{C}''/_{\theta''}$ is isomorphic to $\mathbb{C}/_\theta$, and therefore θ'' witnesses that $a'b'$ is an edge of type z in $\mathbb{A}$. □

4. Path extension

In this section we state some technical results. The main result claims that an as-path in a projection of a relation can always be extended to an as-path in the relation.

Let R be a subdirect product of $\mathbb{A}_1,\mathbb{A}_2$. Binary relations Q_1,Q_2 on $\mathbb{A}_1,\mathbb{A}_2$ given by $Q_1 = \{(a,b) \mid \exists\, c \in \mathbb{A}_2 \text{ with } (a,c),(b,c) \in R\}$ and $Q_2 = \{(a,b) \mid \exists\, c \in \mathbb{A}_1 \text{ with } (c,a),(c,b) \in R\}$, respectively, are called *link tolerances* of R. They are tolerances of $\mathbb{A}_1$, $\mathbb{A}_2$, respectively, that is invariant reflexive and symmetric relations. The transitive closures of Q_1,Q_2 are called *link congruences*, and they are, indeed, congruences. Relation R is said to be *linked* if the link congruences are full congruences.

Let $\mathbb{A}$ be an algebra and $a \in \mathbb{A}$. By $\mathrm{Ft}_{\mathbb{A}}(a)$ we denote the set of elements $b \in \mathbb{A}$ such that $a \sqsubseteq b$; similarly, by $\mathrm{Ft}^{as}_{\mathbb{A}}(a)$ we denote the set of elements $b \in \mathbb{A}$ such that $a \sqsubseteq_{as}$. Note that if a is an as-maximal element then $a \in \mathrm{Ft}^{as}_{\mathbb{A}}(b)$ for any $b \in \mathbb{A}$.

LEMMA 13 ((Bulatov 2016a)). *Let $\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_3$ be similar idempotent algebras omitting type* **1**.

(1) Let ab and cd be thin edges of different types in $\mathbb{A}_1,\mathbb{A}_2$, resp. Then there is a term operation r with $r(b,a) = b$, $r(c,d) = d$.

(2) Let ab and cd be thin affine edges in $\mathbb{A}_1,\mathbb{A}_2$. Then there is a term operation h' such that $h'(b,a,a) = b$ and $h'(c,c,d) = d$.

(3) Let a_1b_1, a_2b_2, and a_3b_3 be thin majority edges in $\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_3$, respectively. Then there is a term operation g' such that $g'(a_1,b_1,b_1) = b_1$, $g'(b_2,a_2,b_2) = b_2$, $g'(b_3,b_3,a_3) = b_3$.

We formulate the following statements without proofs.

LEMMA 14. *Let R be a subdirect product of $\mathbb{A}_1,\dots,\mathbb{A}_n$, $I \subseteq [n]$.*

(1) For any $\mathbf{a} \in R$, $\mathbf{b} \in \mathrm{pr}_I R$ with $\mathrm{pr}_I\mathbf{a} \le \mathbf{b}$ there is $\mathbf{b}' \in R$ such that $\mathbf{a} \le \mathbf{b}'$ and $\mathrm{pr}_I\mathbf{b}' = \mathbf{b}$.

(2) For any $\mathbf{a} \in R$, $\mathbf{b}_1,\dots,\mathbf{b}_k \in \mathrm{pr}_I R$ with $\mathrm{pr}_I\mathbf{a} = \mathbf{b}_1 \le \mathbf{b}_2 \le \dots \le \mathbf{b}_k$, there are $\mathbf{b}'_1,\dots,\mathbf{b}'_k \in R$ such that $\mathbf{a} = \mathbf{b}'_1 \le \mathbf{b}'_2 \le \dots \le \mathbf{b}'_k$ and $\mathrm{pr}_I\mathbf{b}'_i = \mathbf{b}_i$ for $i \in [k]$.

(3) For any $\mathbf{b} \in \mathsf{max}(\mathrm{pr}_I R)$ there is $\mathbf{b}' \in \mathsf{max}(R)$ such that $\mathrm{pr}_I\mathbf{b}' = \mathbf{b}$. In particular, $\mathrm{pr}_{[n]-I}\mathbf{b}' \in \mathsf{max}(\mathrm{pr}_{[n]-I}R)$.

For a subalgebra R of $\mathbb{A}_1 \times \mathbb{A}_2$ and $a \in \mathbb{A}_1, b \in \mathbb{A}_2$ we write $R[a] = \{c \in \mathbb{A}_2 \mid (a,c) \in R\}$ and $R^{-1}[b] = \{d \in \mathbb{A}_1 \mid (d,b) \in R\}$.

LEMMA 15. *Let R be a subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2$.*

(1) If $a,b \in \mathbb{A}_1$ and $c,d \in \mathbb{A}_2$, are such that ab is thin affine or $a \le b$ and cd is a thin edge, such that $(a,c),(a,d),(b,c) \in R$, then $(b,d) \in R$.

(2) Let $a \in \mathbb{A}_1$ and $B = R[a]$. For any $b \in \mathbb{A}_1$ such that ab is thin affine or $a \le b$, and any $c \in R[b] \cap B$, $\mathrm{Ft}_B(c) \subseteq R[b]$.

LEMMA 16. *Let R be a linked subdirect product of $\mathbb{A}_1 \times \mathbb{A}_2$, and $\mathbb{A}_1$ is a simple module. Then $\mathbb{A}_1 \times \mathsf{amax}(\mathbb{A}_2) \subseteq R$.*

LEMMA 17. *Let R be a subdirect product of $\mathbb{A}_1,\dots,\mathbb{A}_n$, $I \subseteq [n]$.*

(1) For any $\mathbf{a} \in R$, $\mathbf{b} \in \mathrm{pr}_I R$ such that $(\mathrm{pr}_I\mathbf{a})\mathbf{b}$ is a thin affine edge there are $\mathbf{a}',\mathbf{b}' \in R$ such that $\mathbf{a} \sqsubseteq \mathbf{a}'$, $\mathbf{a}'\mathbf{b}'$ is a thin affine edge, and $\mathrm{pr}_I\mathbf{a}' = \mathrm{pr}_I\mathbf{a}$, $\mathrm{pr}_I\mathbf{b}' = \mathbf{b}$.

(2) For any $\mathbf{a} \in R$, and an as-path $\mathbf{b}_1,\dots,\mathbf{b}_k \in \mathrm{pr}_I R$ with $\mathrm{pr}_I\mathbf{a} = \mathbf{b}_1$ there is an as-path $\mathbf{b}'_1,\dots,\mathbf{b}'_\ell \in R$ such that $\mathbf{a} = \mathbf{b}'_1$ and $\mathrm{pr}_I\mathbf{b}'_1,\dots,\mathrm{pr}_I\mathbf{b}'_\ell$ is the path $\mathbf{b}_1,\dots,\mathbf{b}_k$ with possible repetitions.

(3) For any $\mathbf{b} \in \mathsf{amax}(\mathrm{pr}_I R)$ there is $\mathbf{b}' \in \mathsf{amax}(R)$ such that $\mathrm{pr}_I\mathbf{b}' = \mathbf{b}$. In particular, $\mathrm{pr}_{[n]-I}\mathbf{b}' \in \mathsf{amax}(\mathrm{pr}_{[n]-I}R)$.

5. Rectangularity

In this section we prove the rectangularity of relations with respect to as-components.

LEMMA 18. *Let R be a subdirect product of algebras $\mathbb{A}_1,\mathbb{A}_2$, B_1,B_2 as-components of $\mathbb{A}_1,\mathbb{A}_2$, respectively, and $a \in \mathbb{A}_1$ such that $R \cap (B_1 \times B_2) \ne \varnothing$ and $\{a\} \times B_2 \subseteq R$. Then $B_1 \times B_2 \subseteq R$.*

PROPOSITION 19. *Let R be a linked subdirect product of $\mathbb{A}_1 \times \mathbb{A}_2$, and let D_1,D_2 be as-components of $\mathbb{A}_1,\mathbb{A}_2$, respectively, such that $R \cap (D_1 \times D_2) \ne \varnothing$. Then $D_1 \times D_2 \subseteq R$.*

Proof: We prove by induction on the size of $\mathbb{A}_1,\mathbb{A}_2$ that for any as-components E_1,E_2 of $\mathbb{A}_1,\mathbb{A}_2$, respectively, such that $R \cap (E_1 \times E_2) \ne \varnothing$, there are $a_1 \in \mathbb{A}_1$, $a_2 \in \mathbb{A}_2$ such that $\{a_1\} \times E_2 \subseteq R$ and $E_1 \times \{a_2\} \subseteq R$. The result then follows from by Lemma 18. The base case of induction when $|\mathbb{A}_1| = 1$ or $|\mathbb{A}_2| = 1$ is obvious.

Take $b \in \mathbb{A}_1$ and construct two sequences of subalgebras $\mathbb{B}_1,\dots,\mathbb{B}_k$ of $\mathbb{A}_1$ and $\mathbb{C}_1,\dots,\mathbb{C}_k$ of $\mathbb{A}_2$, where $\mathbb{B}_1 = \{b\}$, $\mathbb{C}_i = R[\mathbb{B}_i]$, and $\mathbb{B}_i = R^{-1}[\mathbb{C}_{i-1}]$, such that k is the minimal number with $\mathbb{B}_k = \mathbb{A}_1$ or $\mathbb{C}_k = \mathbb{A}_2$. Such a number exists, because R is linked. Observe that for each $i \le k$ the relation $R_i = R \cap (\mathbb{B}_i \times \mathbb{C}_i)$ is linked. Therefore, there is a proper subalgebra $\mathbb{A}'_1$ of $\mathbb{A}_1$ or $\mathbb{A}'_2$ of $\mathbb{A}_2$ such that $R' = R \cap (\mathbb{A}'_1 \times \mathbb{A}_2)$ or $R' = R \cap (\mathbb{A}_1 \times \mathbb{A}'_2)$, respectively, is linked and subdirect. Suppose there is $\mathbb{A}'_1$ with the required properties. By the induction hypothesis for any as-component C_2 of $\mathbb{A}_2$ there is $a_1 \in \mathbb{A}'_1 \subseteq \mathbb{A}_1$ with $\{a_1\} \times C_2 \subseteq R' \subseteq R$. Take an as-component C_1 of $\mathbb{A}_1$. By Lemma 17(3) there is an as-component C_2 of $\mathbb{A}_2$ such that $(C_1 \times C_2) \cap R \ne \varnothing$. Let $a_1 \in \mathbb{A}'_1$ be the element satisfying $\{a_1\} \times C_2 \subseteq R' \subseteq R$. Then by Lemma 18 $C_1 \times C_2 \subseteq R$, and therefore any $a_2 \in C_2$ satisfies the condition $C_1 \times \{a_2\} \subseteq R$. □

COROLLARY 20. *Let R be a subdirect product of $\mathbb{A}_1$ and $\mathbb{A}_2$, θ_1,θ_2 the link congruences, and let B_1,B_2 be as-components of a θ_1-block and a θ_2-block, respectively, such that $R \cap (B_1 \times B_2) \ne \varnothing$. Then $B_1 \times B_2 \subseteq R$.*

6. Affine and majority: few subpowers

6.1 Few subpowers

We call algebras without semilattice edges *semilattice free*. In this section we prove two results that relate semilattice free algebras to algebras with the property to have few subpowers. The few subpowers property has been introduced in (Berman et al. 2010). Let $\mathbb{A}$ be a finite algebra. Then $s_{\mathbb{A}}(n)$ denotes the logarithm (base 2) of the number of subalgebras of $\mathbb{A}^n$; and $g_{\mathbb{A}}(n)$ is the least number

k such that for every subalgebra $\mathbb{B}$ of $\mathbb{A}^n$, $\mathbb{B}$ has a generating set containing at most k elements. Algebra $\mathbb{A}$ is said to have *few subpowers* if $s_{\mathbb{A}}(n)$ is bounded by a polynomial in n.

Having few subpowers can be characterized by the presence of an *edge* term. A term operation f in $k+1$ variables is called an edge term if the following k identities are satisfied:

$$\begin{array}{rcl} f(y,y,x,x,x,\dots,x) &=& x \\ f(y,x,y,x,x,\dots,x) &=& x \\ f(x,x,x,y,x,\dots,x) &=& x \\ f(x,x,x,x,y,\dots,x) &=& x \\ &\vdots& \\ f(x,x,x,x,x,\dots,y) &=& x. \end{array}$$

THEOREM 21 ((Berman et al. 2010)). *For a finite algebra $\mathbb{A}$ the following conditions are equivalent:*

(a) $\mathbb{A}$ has few subpowers,

(b) the variety generated by $\mathbb{A}$ has an edge term,

(c) $g_{\mathbb{A}}$ is bounded by a polynomial.

In this section we will need the property of a finite collection of algebras to have *few subproducts*. More precisely, let $\mathcal{A}$ be a finite set of similar algebras. Let $s_{\mathcal{A}}(n)$ be the maximal number of subalgebras of a direct product $\mathbb{A}_1\times\dots\times\mathbb{A}_n$, where $\mathbb{A}_1,\dots,\mathbb{A}_n\in\mathcal{A}$ are not necessarily different. Also, let $g_{\mathcal{A}}(n)$ be the least number k such that for every subalgebra $\mathbb{B}$ of $\mathbb{A}_1\times\dots\times\mathbb{A}_n$, $\mathbb{B}$ has a generating set containing at most k elements. Set $\mathcal{A}$ is said to have *few subpowers* if $s_{\mathcal{A}}(n)$ is bounded by a polynomial in n. The next statement easily follows from Theorem 21.

COROLLARY 22. *For a finite set of finite idempotent algebras $\mathcal{A}$ the following conditions are equivalent:*

(a) $\mathcal{A}$ has few subpowers,
(b) the variety generated by $\mathcal{A}$ has an edge term,
(c) $g_{\mathcal{A}}$ is bounded by a polynomial.

6.2 Semilattice free algebras have few subpowers

We now show that every finite collection of semilattice free algebras has an edge term. We use the definition of signature and representation quite similar to (Berman et al. 2010), except instead of *minority index* we use thin affine edges. Let R be a subdirect product of $\mathbb{A}_1,\dots,\mathbb{A}_n$, let every $\mathbb{A}_i$ be semilattice free, and let $\mathsf{Af}(\mathbb{A}_i)$ denote the set of thin affine edges of $\mathbb{A}_i$ (it also contains all pairs of the form (a,a)). The *signature* is the set

$$\begin{array}{rcl}\mathrm{Sig}(R) &=& \{(i,a,b)\mid i\in[n],(a,b)\in\mathsf{Af}(\mathbb{A}_i),\exists\,\mathbf{a},\mathbf{b}\in R \\ && \text{with } \mathbf{a}[i]=a,\ \mathbf{b}[i]=b, \text{ and } \mathrm{pr}_{[i-1]}\mathbf{a}=\mathrm{pr}_{[i-1]}\mathbf{b}\}.\end{array}$$

A set of tuples $R'\subseteq R$ is a *representation* of R if

(1) for each $(i,a,b)\in\mathrm{Sig}(R)$ there are $\mathbf{a},\mathbf{b}\in R'$ such that $\mathbf{a}[i]=a$, $\mathbf{b}[i]=b$, and $\mathrm{pr}_{[i-1]}\mathbf{a}=\mathrm{pr}_{[i-1]}\mathbf{b}$;
(2) for each $I\subseteq[n]$, $|I|\le 3$, and every $\mathbf{a}\in\mathrm{pr}_I R$ there is $\mathbf{b}\in R'$ such that $\mathrm{pr}_I\mathbf{b}=\mathbf{a}$.

As is easily seen, every representation R' of R contains a subset $R''\subseteq R'$ which is also a representation and has size at most

$$2|\mathrm{Sig}(R)|+\binom{n}{3}\cdot\max\{|\mathbb{A}_i|\cdot|\mathbb{A}_j|\cdot|\mathbb{A}_k|\mid i,j,k\in[n]\}.$$

THEOREM 23. *Let $\mathcal{A}$ be a finite set of finite semilattice free algebras closed under subalgebras. Then $\mathcal{A}$ has few subpowers.*

Proof: Let R be a subdirect product of $\mathbb{A}_1\times\dots\times\mathbb{A}_n$, $\mathbb{A}_1,\dots,\mathbb{A}_n\in\mathcal{A}$. We show that any representation of R generates it, which will prove that any such R has a generating set of size $O(n^3)$. Let $R'\subseteq R$ be a representation of R, and $Q=\mathsf{Sg}(R')$.

Take $\mathbf{a}\in R$; we prove by induction on $k\in[n]$ that $\mathrm{pr}_{[k]}\mathbf{a}\in\mathrm{pr}_{[k]}Q$. For $k\le 3$ it follows from property (2) of representations, so assume that $k\ge 4$. Suppose that there is $\mathbf{b}\in Q$ with $\mathrm{pr}_{[k]}\mathbf{b}=\mathrm{pr}_{[k]}\mathbf{a}$. Let $\mathbf{a}[k+1]=a$, $\mathbf{b}[k+1]=b$, and $\mathbb{B}$ the subalgebra of $\mathbb{A}_{k+1}$ generated by $\{a,b\}$. We will show that $\mathrm{pr}_{[k]}\mathbf{a}\times\mathbb{B}\subseteq\mathrm{pr}_{[k+1]}Q$, which implies the result. Note that $\mathrm{pr}_{[k]}\mathbf{a}\times\mathbb{B}\subseteq\mathrm{pr}_{[k+1]}R$. Let $C=\{c\in\mathbb{B}\mid(\mathrm{pr}_{[k]}\mathbf{a},c)\in\mathrm{pr}_{[k+1]}Q\}$. If $C\ne\mathbb{B}$ then by connectivity there are $c\in C$ and $d\in\mathbb{B}-C$ such that cd is a thin majority or affine edge. For the sake of obtaining a contradiction, replace a and b with d and c, respectively. If ba is an affine edge, then as $(\mathrm{pr}_{[k]}\mathbf{a},a)\in\mathrm{pr}_{[k+1]}R$, the triple $(k+1,b,a)\in\mathrm{Sig}(R)$. Therefore there are $\mathbf{c},\mathbf{d}\in R'$ witnessing it, and so (b,a) is in the link congruence of $\mathrm{pr}_{[k+1]}Q$. By Corollary 20 $(\mathrm{pr}_{[k]}\mathbf{a},a)\in\mathrm{pr}_{[k+1]}Q$, a contradiction.

Consider now the case when ba is a majority edge. We show that for any $J\subseteq[k]$ there is $\mathbf{c}\in Q$ such that $\mathrm{pr}_J\mathbf{c}=\mathrm{pr}_J\mathbf{a}$ and $\mathbf{c}[k+1]=a$. For subsets $|J|\le 2$ the statement follows from property (2) of representations. Take $J\subseteq[k]$, without loss of generality, $J=[\ell]$, and suppose that there are $\mathbf{a}_1,\mathbf{a}_2\in Q$ such that $\mathbf{a}_1[k+1]=\mathbf{a}_2[k+1]=a$, and $\mathrm{pr}_{J-\{\ell-1\}}\mathbf{a}_1=\mathrm{pr}_{J-\{\ell-1\}}\mathbf{a}$, $\mathrm{pr}_{J-\{\ell\}}\mathbf{a}_2=\mathrm{pr}_{J-\{\ell\}}\mathbf{a}$. Let $\mathbb{B}_1$ be the subalgebra of $\mathbb{A}_{\ell-1}$ generated by $a_1=\mathbf{a}[\ell-1]$ and $b_1=\mathbf{a}_1[\ell-1]$, and let $C_1=\{e\in\mathbb{B}_1\mid(\mathrm{pr}_{[\ell-2]}\mathbf{a},e,\mathbf{a}[\ell],a)\in\mathrm{pr}_{[\ell]\cup\{k+1\}}Q\}$. As $a_1\notin C_1$, $C_1\ne\mathbb{B}_1$, and therefore there are $c_1\in C_1$ and $d_1\in\mathbb{B}_1-C_1$ such that c_1d_1 is a thin affine or majority edge. Again, replace a_1 with d_1 and b_1 with c_1. If b_1a_1 is an affine edge, by Lemma 13(1) there is a term operation $t(x,y)$ such that $t(a,b)=a$ and $t(b_1,a_1)=a_1$. Applying $\mathbf{c}=t(\mathbf{a},\mathbf{a}_1)$ we obtain a tuple $\mathbf{c}$ such that $\mathbf{c}[i]=\mathbf{a}[i]$ for $i\in[\ell]-\{\ell-1\}$, because t is idempotent, $\mathbf{c}[\ell-1]=a_1$, and $\mathbf{c}[k+1]=a$, a contradiction.

Consider the case when b_1a_1 is a majority edge. Let $\mathbb{B}_2$ be the subalgebra of $\mathbb{A}_\ell$ generated by $a_2=\mathbf{a}[\ell]$ and $b_2=\mathbf{a}_1[\ell]$, and let $C_2=\{e\in\mathbb{B}_2\mid(\mathrm{pr}_{[\ell-1]}\mathbf{a},e,a)\in\mathrm{pr}_{[\ell]\cup\{k+1\}}Q\}$. As before, we may assume that b_2a_2 is a thin majority edge. Then by Lemma 13(3) there is a term operation g such that $g(a_1,a_1,b_1)=a_1$, $g(a_2,b_2,a_2)=a_2$, and $g(b,a,a)=a$. Therefore for $\mathbf{c}=g(\mathbf{b},\mathbf{a}_2,\mathbf{a}_1)$ we have $\mathrm{pr}_{[\ell-2]}\mathbf{c}=\mathrm{pr}_{[\ell-2]}\mathbf{a}$, $\mathbf{c}[\ell-1]=a_1$, $\mathbf{c}[\ell]=a_2$, and $\mathbf{c}[k+1]=a$. The result follows. □

COROLLARY 24. *Let $\mathcal{A}$ be a finite set of similar semilattice free algebras. Then the variety generated by $\mathcal{A}$ has an edge term.*

Proof: Let $\mathfrak{V}$ be the variety generated by $\mathcal{A}$. By Theorem 12 every finite algebra from $\mathfrak{V}$ is semilattice free. By Theorem 23 it also has few subpowers, and by (Berman et al. 2010) $\mathfrak{V}$ has an edge term. □

7. Quasi-2-decomposability

Recall that an (n-ary) relation over a set A is called *2-decomposable* if, for any tuple $\mathbf{a}\in A^n$, $\mathbf{a}\in R$ if and only if, for any $i,j\in[n]$, $\mathrm{pr}_{ij}\mathbf{a}\in\mathrm{pr}_{ij}R$. 2-decomposability is closely related to the existence of majority polymorphisms of the relation. In our case relations in general do not have a majority polymorphism, but they still have a property close to 2-decomposability. We say that a relation R, a subdirect product of $\mathbb{A}_1,\dots,\mathbb{A}_n$, is *quasi-2-decomposable*, if for any elements $a_1,\dots,a_n$, such that $(a_i,a_j)\in\mathsf{amax}(\mathrm{pr}_{ij}R)$ for any i,j, there is a tuple $\mathbf{b}\in R$ with $(\mathbf{b}[i],\mathbf{b}[j])\in\mathsf{as}(a_i,a_j)$ for all $i,j\in[n]$.

THEOREM 25. *Any relation invariant under an sm-smooth algebra $\mathbb{A}$ is quasi-2-decomposable.*

Moreover, if R is an n-ary relation, $X\subseteq[n]$, tuple $\mathbf{a}$ is such that $(\mathbf{a}[i],\mathbf{a}[j])\in\mathsf{amax}(\mathrm{pr}_{i,j}R)$ for any i,j, and $\mathrm{pr}_X\mathbf{a}\in$

$\mathsf{amax}(\mathrm{pr}_X R)$*, there is a tuple* $\mathbf{b} \in R$ *with* $(\mathbf{b}[i], \mathbf{b}[j]) \in \mathsf{as}(\mathbf{a}[i], \mathbf{a}[j])$ *for any* $i, j \in [n]$*, and* $\mathrm{pr}_X \mathbf{b} = \mathrm{pr}_X \mathbf{a}$*.*

REMARK 26. *Theorem 25, as well as Proposition 31, can be applied to general algebras, not only sm-smooth, as follows. Let* $\mathbb{A}'$ *denote the sm-smooth reduct of algebra* $\mathbb{A}$ *that exists by Theorem 6. We then can apply Theorem 25 and Proposition 31 to* $\mathbb{A}$ *replacing the graph* $\mathcal{G}'(\mathbb{A})$ *along with all paths and as-components with those of* $\mathcal{G}(\mathbb{A}')$*.*

7.1 Auxiliary lemmas

We start with an auxiliary lemma and a special case of Theorem 25.

LEMMA 27. *Let* $\mathbb{A}_1, \dots, \mathbb{A}_n$ *be similar algebras,* R *a subdirect product of these algebras,* $I \subseteq [n]$*,* $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathrm{pr}_I R$ *an as-path in* R*, and* $\mathbf{a} \in \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ *such that* $\mathrm{pr}_I \mathbf{a} = \mathbf{b}_1$*. Then there are* $\mathbf{b}'_1, \dots, \mathbf{b}'_k \in \mathsf{Sg}(R \cup \{\mathbf{a}\})$ *such that* $\mathbf{b}'_1 = \mathbf{a}$*,* $\mathrm{pr}_I \mathbf{b}'_i = \mathbf{b}_i$ *for* $i \in [k]$*, and* $\mathbf{b}'_i \mathbf{b}'_{i+1}$ *is a thin edge in* $\mathsf{Sg}(\mathbf{b}'_i, \mathbf{b}'_{i+1})$ *of the same type as* $\mathbf{b}_i \mathbf{b}_{i+1}$ *for each* $i \in [k-1]$*. In particular, if for some* $J \subseteq [n]$*,* $\mathrm{pr}_J \mathbf{a} \in \mathrm{pr}_J R$*, then* $\mathrm{pr}_J \mathbf{b}'_i \in \mathrm{Ft}^{as}_{\mathrm{pr}_J R}(\mathrm{pr}_J \mathbf{a})$*.*

LEMMA 28. *Let* R *be a subdirect product of sm-smooth* $\mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$*, and let* (a_1, a_2, a_3) *be such that* $(a_i, a_j) \in \mathsf{amax}(\mathrm{pr}_{ij} R)$ *for* $i, j \in \{1, 2, 3\}$*,* $i \neq j$*. Then there is* $(a'_1, a'_2, a'_3) \in R$ *such that* (a'_i, a'_j) *is in the as-component of* $\mathrm{pr}_{ij} R$ *containing* (a_i, a_j) *for* $i, j \in \{1, 2, 3\}$*,* $i \neq j$*.*

Proof: We proceed by induction on the size of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$. The base case of induction is when for each $i \in [3]$ either $|\mathbb{A}_i| = 2$ and $\mathbb{A}_i$ is a semilattice or a majority edge, or $\mathbb{A}_i$ is a module. By the assumption some tuples $\mathbf{a}_1 = (b_1, a_2, a_3)$, $\mathbf{a}_2 = (a_1, b_2, a_3)$, $\mathbf{a}_3 = (a_1, a_2, b_3)$ belong to R. If one of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ is a semilattice edge, say, $b_1 \leq a_1$, then again from the as-maximality of a_1, a_2, $(a_1, a_2, a_3) = \mathbf{a}_1 \cdot \mathbf{a}_2 \in R$. If one of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ is a module, say, $\mathbb{A}_1$ is, then $\mathbf{a}_1$ satisfies the requirements of the lemma. Finally, if all $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ are majority edges, then $(a_1, a_2, a_3) = g(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$. Note that if one of the $\mathbb{A}_i$ has a unique maximal element (an absorbing element, for example), the statement also holds.

Suppose that the lemma is proved for any subdirect product of $\mathbb{A}'_1 \times \mathbb{A}'_2 \times \mathbb{A}'_3$, where $\mathbb{A}'_i$ is a subalgebra or a factor of $\mathbb{A}_i$, $i \in [3]$, and at least one of them is a proper subalgebra or a factor. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in R$ be as before. Also let $\mathcal{D}$ denote the set of $(c_1, c_2, c_3) \in \mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$ such that (c_i, c_j), $i, j \in [3]$, $i \neq j$, belongs to the as-component of $\mathrm{pr}_{ij} R$ containing (a_i, a_j).

CLAIM 1. Every $\mathbb{A}_i$ can be assumed to be $\mathsf{Sg}(a_i, b_i)$ and b_i can be chosen to be an as-maximal element.

Suppose $\mathbb{A}_1 \neq \mathbb{B} = \mathsf{Sg}(a_1, b_1)$. Let (a'_1, a'_2) be an as-maximal element in $(\mathbb{B} \times \mathbb{A}_2) \cap \mathrm{pr}_{12} R$ such that $(a_1, a_2) \sqsubseteq_{as} (a'_1, a'_2)$. By Lemma 27 (a_1, a_2, a_3) can be replaced with $(a'_1, a'_2, a'_3) \in \mathcal{D}$ for an appropriate a'_3. Repeating the process for the other binary projections if necessary we obtain $(a''_1, a''_2, a''_3) \in \mathcal{D}$ such that (a''_i, a''_j) is as-maximal in $\mathrm{pr}_{ij}(R \cap (\mathbb{B} \times \mathbb{A}_2 \times \mathbb{A}_3))$. By the induction hypothesis there is $(a'''_1, a'''_2, a'''_3) \in R \cap (\mathbb{B} \times \mathbb{A}_2 \times \mathbb{A}_3)$ such that (a'''_i, a'''_j) is in the as-maximal component containing (a''_i, a''_j). Clearly, (a'''_1, a'''_2, a'''_3) is as required.

If, say, b_1 is not an as-maximal element, and $b_1 \sqsubseteq_{as} b'_1$ and b'_1 is as-maximal, then again using Lemma 27 we can obtain $(a'_1, a'_2, a'_3) \in \mathcal{D}$ such that $(b'_1, a'_2, a'_3) \in R$. Then we choose an as-path in $\mathrm{pr}_{23} R$ from (a'_2, a'_3) to (a_2, a_3). By Lemma 27 we get $(d, a_2, a_3) \in R$ such that $d \in \mathsf{amax}(\mathbb{A}_1)$.

CLAIM 2. For every $i, j \in [3]$, $\mathsf{as}(a_i) \times \mathbb{A}_j \subseteq \mathrm{pr}_{ij} R$. Therefore, $\mathsf{as}(a_i) \times \mathsf{as}(a_j) \subseteq \mathrm{pr}_{ij} R$, and $\mathsf{as}(a_i) \times \mathsf{as}(a_j)$ is an as-component of $\mathrm{pr}_{ij} R$.

Since $(a_i, a_j), (a_i, b_j) \in \mathrm{pr}_{ij} R$ and $\mathbb{A}_j = \mathsf{Sg}(a_j, b_j)$, we have $\{a_i\} \times \mathbb{A}_j \subseteq \mathrm{pr}_{ij} R$. By Lemma 18 the first part of the claim follows. The second part is obvious.

CLAIM 3. Every $\mathbb{A}_i$ can be assumed simple.

Suppose θ is a nontrivial congruence of $\mathbb{A}_1$ and $R/_\theta = \{(c_1^\theta, c_2, c_3) \mid (c_1, c_2, c_3) \in R\}$. By the induction hypothesis there is $(a''_1, a'_2, a'_3) \in R/_\theta$ satisfying the conditions of the lemma, that is, there is $(b_1, a'_2, a'_3) \in R$ such that $b_1^\theta = a''_1$, and $(a_2, a_3) \sqsubseteq_{as} (a'_2, a'_3)$, $(a_1^\theta, a_i) \sqsubseteq_{as} (a''_1, a'_i)$ for $i \in \{2, 3\}$. Let $a'_1 \in b_1^\theta$ be any element such that $a_1 \sqsubseteq_{as} a'_1$ and a'_1 is maximal in b_1^θ. Then (a'_1, a'_2, a'_3) is as required. Indeed, it suffices to observe that $(a'_i, a'_j) \in \mathsf{as}(a_i) \times \mathsf{as}(a_j)$, and therefore $(a_i, a_j) \sqsubseteq_{as} (a'_i, a'_j)$, for any $i, j \in \{1, 2, 3\}$. Since $\mathsf{Sg}(a'_1, b_1) \neq \mathbb{A}_1$, the claim follows from Claim 1.

We now prove the induction step. Suppose that $|\mathbb{A}_i| > 2$ and $\mathbb{A}_i$ is not a module for some i. For an n-ary relation $Q \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$, $j \in [n]$, and $c_j \in \mathbb{A}_j$, let $Q[c_j]_j$ denote the set $\{(c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_n) \in \mathrm{pr}_{\{1,\dots,j-1,j+1,\dots,n\}} Q \mid (c_1, \dots, c_n) \in Q\}$. We consider two cases.

CASE 1. For some $i \in [3]$ the set $R[b_i]_i$ contains $\mathsf{as}(a_j) \times \mathsf{as}(a_\ell)$, where $\{j, \ell\} = [3] - \{i\}$.

Assume $i = 1$. Since a_1 is as-maximal, by Proposition 10 there is a path P from b_1 to a_1. We prove that for any element c on this path $\{c\} \times \mathsf{as}(a_2) \times \mathsf{as}(a_3) \subseteq R$. This is true for $c = b_1$ by the assumption made. Assume the contrary, and let c be the first element in P such that this property is not true. Let also d be the element preceding c in P; we may assume $d = b_1$. If $b_1 c$ is semilattice then take any $\mathbf{c} = (c, c_2, c_3) \in R$ and set $\mathbf{c}' = \mathbf{a}_1 \cdot \mathbf{c} = (c, a'_2, a'_3)$. By the properties of multiplication $\cdot$, $(a'_2, a'_3) \in \mathsf{as}(a_2, a_3)$. Therefore by Lemma 18 $\{c\} \times \mathsf{as}(a_2) \times \mathsf{as}(a_3) \subseteq R$.

Let $b_1 c$ be a thin majority edge, $\mathbb{B} = \mathsf{Sg}(b_1, c)$, and θ a congruence witnessing that $b_1 c$ a majority edge; in particular, $\mathbb{B} = b_1^\theta \cup c^\theta$ by Theorem 6. If $\mathbb{B} = \mathbb{A}_1$ then θ is the equality relation, as $\mathbb{A}_1$ is simple, and so $|\mathbb{A}_1| = 2$, a contradiction. Suppose $\mathbb{B} \neq \mathbb{A}_1$. Consider $R' = R \cap (\mathbb{B} \times \mathbb{A}_2 \times \mathbb{A}_3)$ and take any $e \in \mathsf{max}(\mathbb{B}) \cap c^\theta$. For the tuple (e, a_2, a_3) we have $(e, a_i) \in \mathrm{pr}_{1i} R'$ for $i \in \{2, 3\}$ by Claim 2 and $(a_2, a_3) \in \mathrm{pr}_{23} R'$ by the assumption made. By the induction hypothesis there is $(e', a'_2, a'_3) \in R'$ with $e' \in \mathsf{as}(e)$ (and so $e' \in c^\theta$) and $a'_i \in \mathsf{as}(a_i)$, $i \in \{2, 3\}$. Since $b_1 c$ is a thin edge, $e'' = g(b_1, e', e')$ is such that $b_1 e''$ is also a thin majority edge and $\mathsf{Sg}(b_1, e'') = \mathsf{Sg}(b_1, c)$. Moreover, $(e'', a'_2, a'_3) \in R'$. Let $Q = R'[e'']_1 = \{(c_2, c_3) \in \mathrm{pr}_{23} R \mid (e'', c_2, c_3) \in R'\}$ and $Q' = Q \cap (\mathsf{as}(a_2) \times \mathsf{as}(a_3))$. We show that $Q' = \mathsf{as}(a_2) \times \mathsf{as}(a_3)$. As (a'_2, a'_3) shows, $Q' \neq \varnothing$. Suppose there are $\mathbf{e} \in Q'$, $\mathbf{e}' \in (\mathsf{as}(a_2) \times \mathsf{as}(a_3)) - Q'$ such that $\mathbf{e}\mathbf{e}'$ is a semilattice of a thin affine edge. Then by Lemma 13(1) there is a term operation t such that $t(e'', b_1) = e''$ and $t(\mathbf{e}, \mathbf{e}') = \mathbf{e}'$. This means $\mathbf{e}' \in Q'$, a contradiction.

Since $\mathsf{Sg}(b_1, e'') = \mathsf{Sg}(b_1, c)$, $c = r(b_1, e'')$ for some term operation r. It remains to notice that

$$\begin{pmatrix} c \\ a'_2 \\ a'_3 \end{pmatrix} = r\left(\begin{pmatrix} b_1 \\ a'_2 \\ a'_3 \end{pmatrix}, \begin{pmatrix} e'' \\ a'_2 \\ a'_3 \end{pmatrix}\right) \in R'$$

for any $a'_2 \in \mathsf{as}(a_2)$, $a'_3 \in \mathsf{as}(a_3)$, a contradiction.

Finally, let $b_1 c$ be a thin affine edge, and let η be the link congruence of $\mathrm{pr}_{23} R'$ when R' is viewed as a subdirect product of $\mathbb{B} = \mathsf{Sg}(b_1, c)$ and $\mathrm{pr}_{23} R'$. By the assumption $\mathsf{as}(a_2) \times \mathsf{as}(a_3)$ belongs to a η-block. By Lemma 15 $\{c\} \times \mathsf{as}(a_2) \times \mathsf{as}(a_3) \subseteq R'$. Thus, $\{a_1\} \times \mathsf{as}(a_2) \times \mathsf{as}(a_3) \subseteq R'$, and $(a_1, a_2, a_3) \in R$.

CASE 2. For all $i \in [3]$, $\mathsf{as}(a_j) \times \mathsf{as}(a_\ell) \not\subseteq R(b_i)$, where $\{j, \ell\} = [3] - \{i\}$.

Let η_i be the link congruence of $\mathrm{pr}_{j\ell}R$ when R is viewed as a subdirect product of $\mathbb{A}_i$ and $\mathrm{pr}_{j\ell}R$; and let θ_i be the link congruence of $\mathbb{A}_i$. Since b_i is as-maximal, if θ_i is the total congruence, then by the assumption $\mathsf{as}(a_j) \times \mathsf{as}(a_\ell) \subseteq R[b_i]_i$, a contradiction with the assumption made. Therefore θ_i is the equality relation for all $i \in [3]$. Consider the η_i-block $Q = R[a_i]_i$. By Claim 2 Q is a subdirect product of $\mathbb{A}_j \times \mathbb{A}_\ell$. It is not linked, as otherwise $\mathsf{as}(a_j) \times \mathsf{as}(a_\ell) \subseteq Q$, and since $\mathbb{A}_2, \mathbb{A}_3$ are simple, Q is the graph of a bijection. Thus, $\mathbb{A}_j$ and $\mathbb{A}_\ell$ are isomorphic. In a similar way $\mathbb{A}_i$ and $\mathbb{A}_j$ are isomorphic. In particular, $|\mathbb{A}_1| = |\mathbb{A}_2| = |\mathbb{A}_3| = k$. Therefore, $\mathrm{pr}_{j\ell}R$ contains k η_i-blocks of size k each. This means $|\mathrm{pr}_{j\ell}R| = k^2$, and so $\mathrm{pr}_{j\ell}R = \mathbb{A}_j \times \mathbb{A}_\ell$, which is isomorphic to $\mathbb{A}_j^2$. By (Kearnes 1996) this implies $\mathbb{A}_j$, and therefore all of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ are modules, and the result follows by the base case of induction. □

7.2 Proof of Theorem 25

Let $\mathbf{a}$ be a tuple satisfying the conditions of quasi-2-decomposability. By induction on ideals of the power set of $[n]$ (i.e. subsets of the power set closed under taking subsets) we prove that for any ideal I there is $\mathbf{a}'$ such that for all $i, j \in [n]$, $(\mathbf{a}'[i], \mathbf{a}'[j]) \in \mathsf{as}(\mathbf{a}[i], \mathbf{a}[j])$, and for any $U \in I$ $\mathrm{pr}_U\mathbf{a}' \in \mathsf{amax}(\mathrm{pr}_U R)$. The base case, the ideal that consists of all at most 2-elements sets, set X, and its subsets, is given by the tuple $\mathbf{a}$.

Suppose that the claim is true for an ideal I, take a set W such that it does not belong to I, but all its proper subsets do. Let $\mathcal{D}$ be the set of all tuples $\mathbf{c}$ such that $\mathrm{pr}_U\mathbf{c} \in \mathsf{amax}(\mathrm{pr}_U R)$ for every $U \in I$, and $\mathrm{pr}_X\mathbf{c} \in \mathsf{as}(\mathrm{pr}_X\mathbf{a})$. If a tuple belongs to $\mathcal{D}$ it is said to *support* I. We show that $\mathcal{D}$ contains a tuple $\mathbf{b}$ with $\mathrm{pr}_W\mathbf{b} \in \mathsf{amax}(\mathrm{pr}_W R)$.

Assume that $W = [\ell]$ and fix $\mathbf{b} \in \mathcal{D}$. We prove the following statement:

Let $\mathbf{c} \in \mathcal{D}$ be such that $\mathrm{pr}_U\mathbf{c} \in \mathsf{as}(\mathrm{pr}_U\mathbf{b})$ for all $U \in I$ and Q a subalgebra of $\mathrm{pr}_W R$ such that for any $U \subset W$ there is $\mathbf{c}_U \in R$ with $\mathrm{pr}_U\mathbf{c}_U = \mathrm{pr}_U\mathbf{c}$ and $\mathrm{pr}_W\mathbf{c}_U \in Q' = Q \cap \mathsf{max}(\mathrm{pr}_W R)$. Then there is $\mathbf{d}$ supporting I and such that $\mathrm{pr}_W\mathbf{d} \in Q'$ and $\mathrm{pr}_U\mathbf{d} \in \mathsf{as}(\mathrm{pr}_U\mathbf{b})$ for $U \in I$.

Note that setting $\mathbf{c} = \mathbf{b}$, if $Q = \mathrm{pr}_W R$ then the statement implies that $\mathcal{D}$ contains a tuple $\mathbf{d}$ with $\mathrm{pr}_W\mathbf{d} \in \mathsf{max}(\mathrm{pr}_W R)$ and $\mathrm{pr}_U\mathbf{d} \in \mathsf{as}(\mathrm{pr}_U\mathbf{b})$ for $U \in I$. This would prove the induction step. We proceed by induction on the sizes of unary projections of Q. If one of them is 1-element then the statement follows from the assumption $\mathrm{pr}_U\mathbf{c} \in Q$ for U including all coordinates whose projections contain more than 1 element. Suppose that the statement is proved for all relations with unary projections smaller than Q.

By the assumption there are $\mathbf{c}_1, \ldots, \mathbf{c}_\ell \in Q$ with $\mathrm{pr}_{W-\{i\}}\mathbf{c}_i = \mathrm{pr}_{W-\{i\}}\mathbf{c}$. By Lemma 27 these tuples can be chosen such that $\mathbf{c}_i$ is as-maximal. This may require changing the tuple $\mathbf{c}$. Let $\mathbf{c}'$ be the new tuple. Note that $\mathbf{c}'$ supports I and $(\mathbf{c}'[i], \mathbf{c}'[j]) \in \mathsf{as}(\mathbf{c}[i], \mathbf{c}[j])$.

Suppose that for some i $\mathrm{pr}_i Q \neq \mathsf{Sg}(\mathbf{c}[i], \mathbf{c}_i[i])$. Let $i = 1$. Set

$$Q'' = Q \cap \left(\mathsf{Sg}(\mathbf{c}[1], \mathbf{c}_1[1]) \times \prod_{i \in W-\{1\}} \mathrm{pr}_i Q \right).$$

We show that $\mathbf{c}$ can be changed so that Q'' satisfies the conditions of the statement. If $\mathrm{pr}_{W-\{1\}}\mathbf{c}$ is not as-maximal in $\mathrm{pr}_{W-\{1\}}Q'$ then using Lemma 27 replace $\mathbf{c}$ with $\mathbf{c}'$ such that $\mathrm{pr}_{W-\{1\}}\mathbf{c}'$ is as-maximal. Repeat this process with other projections if necessary. The resulting tuple $\mathbf{d}$ supports I and $\mathrm{pr}_U\mathbf{d} \in \mathsf{as}(\mathrm{pr}_U\mathbf{c})$ for $U \in I$. Then just apply the induction hypothesis.

Let $\mathbf{c}_i$, $i \in W$, be chosen such that $\mathsf{Sg}(\mathbf{c}[i], \mathbf{c}_i[i])$ are minimal possible. Observe that it suffices to prove that there is $\mathbf{d} \in \mathsf{Sg}(\mathbf{c}_1, \ldots, \mathbf{c}_\ell)$ such that $\mathrm{pr}_U\mathbf{c} \sqsubseteq_{as} \mathrm{pr}_U\mathbf{d}$ in $\mathsf{Sg}(\mathbf{c}_1, \ldots, \mathbf{c}_\ell)$ for all $U \subset W$. Indeed, if this is the case, let $U = [\ell - 1]$ (recall that $W = [\ell]$). Then $\mathrm{pr}_U\mathbf{c} \sqsubseteq_{as} \mathrm{pr}_U\mathbf{d}$. Choose an as-path from $\mathrm{pr}_U\mathbf{c}$ to $\mathrm{pr}_U\mathbf{d}$, by Lemma 27 $\mathbf{c}$ can be replaced with some $\mathbf{c}'$ that still supports I, but $\mathrm{pr}_U\mathbf{c}' = \mathrm{pr}_U\mathbf{d}$. Note that $\mathbf{c}'[\ell]$ is in the same as-component of $\mathrm{pr}_\ell Q$ as $\mathbf{d}[\ell]$, therefore, $\mathbf{c}'[\ell] \sqsubseteq_{as} \mathbf{d}[\ell]$. If $\mathsf{Sg}_{\mathbb{A}_\ell}(\mathbf{c}'[\ell], \mathbf{d}[\ell]) = \mathrm{pr}_\ell Q$, then there is an as-path from $\mathrm{pr}_W\mathbf{c}'$ to $\mathbf{d}$ in $\mathsf{Sg}(Q \cup \mathrm{pr}_W\mathbf{c}')$ (note that $\mathrm{pr}_W\mathbf{c}''$ and $\mathbf{d}$ differ only at component ℓ). Again by Lemma 27 change $\mathbf{c}'$ such that the resulting tuple $\mathbf{c}''$ supports I, and $\mathrm{pr}_W\mathbf{c}'' = \mathrm{pr}_W\mathbf{d}$, implying $\mathbf{c}''$ supports $I \cup \{W\}$. If $\mathbb{B} = \mathsf{Sg}_{\mathbb{A}_\ell}(\mathbf{c}'[\ell], \mathbf{d}[\ell]) \neq \mathrm{pr}_\ell Q$, then set $Q' = Q \cap (\mathrm{pr}_{[\ell-1]}Q \times \mathbb{B})$. As before, we may assume that all the proper projections of $\mathbf{c}'$ are maximal in Q'. Since $\mathbf{c}'$ satisfies all the conditions for Q', we obtain the result by inductive hypothesis.

We will prove that a tuple $\mathbf{d} \in \mathsf{Sg}(\mathbf{c}_1, \ldots, \mathbf{c}_\ell)$ with the required properties exists. First, replace Q with $\mathsf{Sg}(\mathbf{c}_1, \ldots, \mathbf{c}_\ell)$. Then replacing Q with the relation

$$Q'(x, y, z) \;=\; \exists x_4, \ldots, x_n (Q(x, y, z, x_4, \ldots, x_\ell) \wedge (x_4 = \mathbf{c}[4]) \wedge \ldots \wedge (x_\ell = \mathbf{c}[\ell])$$

Q can be assumed ternary. Now Q and $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$ satisfy the conditions of Lemma 28, the result follows.

To finish the proof note that, since $X \in I$ already in the base case, the resulting tuple $\mathbf{d}'$ is such that $\mathrm{pr}_X\mathbf{a} \sqsubseteq \mathrm{pr}_X\mathbf{d}'$. By Lemma 27 $\mathbf{d}'$ can be changed to a tuple $\mathbf{d}''$ satisfying the same requirements and such that $\mathrm{pr}_X\mathbf{d}'' = \mathrm{pr}_X\mathbf{a}$.

Theorem 25 is proved.

8. Semilattice and majority: bounded width

8.1 Bounded width

Let $\mathcal{P} = (V, \delta, \mathcal{C})$ be a CSP and $W \subseteq V$. The *restriction* of $\mathcal{P}$ to W is the CSP $\mathcal{P}_W = (W, \delta|_W, \mathcal{C}_W)$, where $\delta|_W$ is the restriction of δ on W, and for every $C = \langle \mathbf{s}, R \rangle$ the set $\mathcal{C}_W$ contains the constraint $C_W = \langle \mathbf{s} \cap W, \mathrm{pr}_{\mathbf{s} \cap W} R \rangle$, where $\mathbf{s} \cap W$ is the subtuple of $\mathbf{s}$ containing all the elements from W in $\mathbf{s}$, say, $\mathbf{s} \cap W = (i_1, \ldots, i_k)$, and $\mathrm{pr}_{\mathbf{s} \cap W} R$ stands for $\mathrm{pr}_{\{i_1, \ldots, i_k\}} R$. A solution of $\mathcal{P}_W$ is called a *partial solution* of $\mathcal{P}$ on W. The set of all partial solutions on W is denoted by $\mathcal{S}_W$. A set $\mathcal{F}$ of pairs (W, φ), where $\varphi \in \mathcal{S}_W$ is said to be a (k, ℓ)-*strategy* if the following conditions hold:

- $|W| \leq \ell$ for all $(W, \varphi) \in \mathcal{F}$;
- for any $(W, \varphi) \in \mathcal{F}$ and any $U \subseteq W$, it holds $(U, \varphi|_U) \in \mathcal{F}$;
- for any $(W, \varphi) \in \mathcal{F}$ and $V \supseteq U \supseteq W$ such that $|W| \leq k$ and $|U| \leq \ell$, there is $(U, \psi) \in \mathcal{F}$ with $\psi|_W = \varphi$.

CSP $\mathcal{P}$ is said to be (k, ℓ)-*consistent* if it has a nonempty (k, ℓ)-strategy. Checking if a CSP is (k, ℓ)-consistent can be done in polynomial time (Dechter 2003). Finally, $\mathrm{CSP}(\mathbf{B})$ (or $\mathrm{CSP}(\mathcal{A})$) is said to have width (k, ℓ) if every (k, ℓ)-consistent instance of this problem has a solution. A problem is said to have *bounded width* if it has width (k, ℓ) for some k, ℓ. Every CSP of bounded width has a polynomial time solution algorithm.

Problem $\mathrm{CSP}(\mathcal{A})$ has bounded width if and only if $\mathcal{A}$ omits types **1** and **2** (Larose and Zádori 2007; Bulatov 2009; Barto and Kozik 2014; Barto 2014). Moreover, a CSP has bounded width if and only if it has width $(2, 3)$ (Bulatov 2009; Barto 2014). There is also an alternative characterization in terms of types of edges

PROPOSITION 29 ((Bulatov 2004)). *For a class $\mathcal{A}$ of similar idempotent algebras omitting type* **1** *the following two conditions are equivalent.*

(1) The variety generated by $\mathcal{A}$ omits types **1** *and* **2**.

(2) Algebras from $\mathcal{A}$ have no affine edges.

We also give a prove of the following theorem (see (Bulatov 2009; Barto and Kozik 2014; Barto 2014)).

THEOREM 30. *Let $\mathcal{A}$ be a class of similar algebras without affine edges. Then $\mathrm{CSP}(\mathcal{A})$ has bounded width. More precisely, every (2,3)-consistent instance of $\mathrm{CSP}(\mathcal{A})$ has a solution.*

8.2 Reduction to as-components

We start with a useful reduction that works for all CSPs over algebras omitting type **1**, not only those without affine edges.

Let $\mathcal{P} = (V;\delta;\mathcal{C})$ be a (2,3)-consistent problem instance. For $u, v, w \in V$ by $\mathcal{S}_u, \mathcal{S}_{uv}, \mathcal{S}_{uvw}$ we denote sets of partial solutions of $\mathcal{P}$ on $\{u\}, \{u,v\}, \{u,v,w\}$, respectively. We show that $\mathcal{P}$ can be transformed to a (2,3)-consistent problem instance whose domains are generated by a single as-component.

PROPOSITION 31. *Let $\mathcal{P} = (V;\delta;\mathcal{C})$ be a (2,3)-consistent problem instance from $\mathrm{CSP}(\mathcal{A})$, where algebras from $\mathcal{A}$ are sm-smooth. Let $v \in V$ and B be an as-component of $\mathbb{A}_{\delta(v)}$. Then the problem instance $\mathcal{P}_{v,B} = (V;\delta;\mathcal{C}')$, where*

- *for each $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ there is $C' = \langle \mathbf{s}, R'\rangle \in \mathcal{C}'$ where R' contains all tuples $\mathbf{a}$ from R such that for any $u \in \mathbf{s}$ there is $c \in \mathbb{B} = \mathsf{Sg}(B)$ with $(c, \mathbf{a}[u]) \in \mathcal{S}_{vu}$,*

is (2,3)-consistent and if $\mathcal{P}_{v,B}$ has a solution, then $\mathcal{P}$ does.

Proof: The second claim of the proposition is straightforward from the construction.

For variables $x, y, z \in V$, the set of partial solutions of $\mathcal{P}_{v,B}$ on $\{x\}$, $\{x, y\}$, and $\{x, y, z\}$ will be denoted by $\mathcal{S}'_x, \mathcal{S}'_{x,y}$, and $\mathcal{S}'_{x,y,z}$, respectively. We present a $(2,3)$-strategy for $\mathcal{P}_{v,B}$, that is, a collection of subsets of $\mathcal{S}'_{xy}$ and $\mathcal{S}'_{xyz}$ for $x, y, z \in V$ satisfying the conditions of a strategy. For $x, y \in V$, let $\mathcal{S}''_x = \mathsf{amax}(\mathrm{pr}_x(\mathcal{S}_{vx} \cap (\mathbb{B} \times \mathbb{A}_x))$ and $\mathcal{S}''_{xy} = \mathsf{amax}(\mathrm{pr}_{xy}(\mathcal{S}_{vxy} \cap (\mathbb{B} \times \mathbb{A}_x \times \mathbb{A}_y))$.

CLAIM 1. If an as-maximal element $a \in \mathbb{A}_x$ belongs to $\mathcal{S}''_x$, then $\mathsf{as}(a) \subseteq \mathcal{S}''_x$. If an as-maximal pair $(a, b) \in \mathcal{S}_{xy}$ belongs to $\mathcal{S}''_{xy}$, then $\mathsf{as}(a, b) \subseteq \mathcal{S}''_{xy}$.

By Lemma 17(2) either $\mathcal{S}_{vxy}$ contains a subdirect product of $\mathbb{B}$ and $\mathsf{as}(a, b)$, or $(B \times \mathsf{as}(a, b)) \cap \mathcal{S}_{vxy} = \varnothing$. Since $(a, b) \in \mathcal{S}''_{xy}$ the former option holds. For the first part of the claim observe that $\mathrm{pr}_x \mathsf{as}(a, b) = \mathsf{as}(a)$.

CLAIM 2. For any $x, y \in V$, $\mathcal{S}''_{xy}$ is a subdirect product of $\mathcal{S}''_x \times \mathcal{S}''_y$.

Let $a \in \mathcal{S}''_x$, then there is $d \in B$ with $(d, a) \in \mathcal{S}_{vx}$. Since $\mathcal{P}$ is $(2,3)$-consistent, there is $b \in \mathbb{A}_y$ with $(d, a, b) \in \mathcal{S}_{vxy}$. By Lemma 17(3) b can be chosen as-maximal; that is $(a, b) \in \mathcal{S}''_{xy}$.

CLAIM 3. For any $x, y, z \in V - \{v\}$ and any $(a, b) \in \mathcal{S}''_{xy}$ there is c such that $(a, c) \in \mathcal{S}''_{xz}$ and $(b, c) \in \mathcal{S}''_{yz}$.

Consider the following relation

$$R(x_1, x_2, x_3, x_4) = \exists u(\mathcal{S}_{vxz}(x_1, x_2, u) \wedge \mathcal{S}_{vyz}(x_3, x_4, u)).$$

Let d be the element of B such that $(d, a, b) \in \mathcal{S}_{vxy}$, and let $\mathbf{a} = (d, a, d, b)$. We show that $\mathrm{pr}_{i,j}\mathbf{a} \in \mathrm{pr}_{i,j}R$ for any $i, j \in [4]$. If $i = 2$, $j = 4$ or the other way round then we set u to be an extension e of (a, b) in $\mathcal{S}_{xyz}$, and x_1, x_3 to extensions of (a, e) and (b, e) in $\mathcal{S}_{vxz}$ and $\mathcal{S}_{vyz}$, respectively. If $i = 1$, $j = 2$ or $i = 3$, $j = 4$ then set u to be an extension e of (d, a) or (d, b) in $\mathcal{S}_{vxz}$ and $\mathcal{S}_{vyz}$, respectively. Then extend e to a tuple from $\mathcal{S}_{vyz}$ or $\mathcal{S}_{vxz}$, respectively. If $i = 1$, $j = 4$ or $i = 3$, $j = 2$ then extend (d, b) or (d, a) by an element e to a tuple in $\mathcal{S}_{vyz}$ or $\mathcal{S}_{vxz}$, respectively. Then set x_2 (resp., x_4) to be a value extending (d, e) in $\mathcal{S}_{vzx}$ (resp., $\mathcal{S}_{vzy}$), and x_3 (resp., x_1) to be a value extending (d, b) (resp., (d, a)) to a tuple in $\mathcal{S}_{vzy}$ (resp., $\mathcal{S}_{vzx}$). Finally, if $i = 1$, $j = 3$ then choose e so that $(d, e) \in \mathcal{S}_{vz}$ and extend this pair to tuples from $\mathcal{S}_{vzx}$ and $\mathcal{S}_{vyz}$.

By Theorem 25 there is $\mathbf{b} \in R$ such that $\mathbf{b}[2] = \mathbf{a}[2] = a$, $\mathbf{b}[4] = \mathbf{a}[4] = b$, $\mathbf{b}[1], \mathbf{b}[3] \in \mathsf{as}(d) = B$, and $(\mathbf{b}[i], \mathbf{b}[j]) \in \mathsf{as}((\mathbf{a}[i], \mathbf{a}[j]))$. Therefore there is c such that $(\mathbf{b}[1], a, c) \in \mathcal{S}_{vxz}$ and $(\mathbf{b}[3], b, c) \in \mathcal{S}_{vyz}$, which implies $(a, c) \in \mathcal{S}''_{xz}$ and $(b, c) \in \mathcal{S}''_{yz}$. The claim is proved.

CLAIM 4. (1) For any $x, y \in V - \{v\}$ and any $(a, b) \in \mathcal{S}''_{xy}$, there is mapping $\varphi : V \to \bigcup_{w \in V} \mathbb{A}_{\delta(w)}$ with $\varphi(w) \in \mathbb{A}_{\delta(w)}$ such that $\varphi(x) = a, \varphi(y) = b, \varphi(v) \in B$, and $(\varphi(u), \varphi(w)) \in \mathcal{S}''_{uw}$ for any $u, w \in V$.

(2) For any $x, y \in V$, $\mathcal{S}''_{xy} \subseteq \mathcal{S}'_{xy}$.

(1) Let $V = \{v_1, \dots, v_n\}$ and $v = v_1$, $x = v_2$, $y = v_3$. By induction on i we prove that a mapping φ_i can be found on $I = \{v_1, \dots, v_i\}$ that satisfies the conditions of the claims for all $u, w \in I$. For $i = 3$ the existence of φ_3 follows from the assumptions. So, suppose φ_i exists. Let $\mathcal{S}^+_{xy} = \mathsf{Sg}(\mathcal{S}''_{xy})$. Take φ_i satisfying the conditions on I and consider the relation given by

$$R(x_1, \dots, x_i) = \exists y \bigwedge_{j=1}^{i} \mathcal{S}^+_{v_j v_{i+1}}(x_j, y).$$

By the inductive hypothesis and Claim 2, for any $j, k \in [i]$ we have $(\varphi_i(v_j), \varphi_i(v_k)) \in \mathrm{pr}_{j,k}R$. By Theorem 25 there is $\mathbf{a} \in R$ such that $\mathbf{a}[2] = a$, $\mathbf{a}[3] = b$, and $(\mathbf{a}[j], \mathbf{a}[k]) \in \mathsf{as}((\varphi_i(v_j), \varphi_i(v_k)))$ for any $j, k \in [i]$. This means that there is c such that $(\mathbf{a}[j], c) \in \mathcal{S}^+_{v_j, v_{i+1}}$ for all $j \in [i]$. Observe that c can be chosen to be an as-maximal element of $\mathcal{S}_{v_{i+1}}$. Since $\mathbf{a}[1] \in B$, $c \in \mathcal{S}''_{v_{i+1}}$. The mapping φ_{i+1} on $I' = I \cup \{i+1\}$ given by $\varphi_{i+1}(v_j) = \mathbf{a}[j]$ for $j \in [i]$ and $\varphi_{i+1}(v_{i+1}) = c$ satisfies the required conditions.

(2) We need to show that for any $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, any $x, y \in \mathbf{s}$, and any $(a, b) \in \mathcal{S}''_{x,y}$ there is a tuple $\mathbf{a} \in R$ such that $\mathbf{a}[z] \in \mathcal{S}''_z$ for $z \in \mathbf{s}$, and $\mathbf{a}[x] = a, \mathbf{a}[y] = b$. By part (1) of Claim 4, there is $\mathbf{b} = (\mathbf{b}[z])_{z \in \mathbf{s}}$ such that $(\mathbf{b}[z], \mathbf{b}[t]) \in \mathcal{S}''_{zt}$ and $\mathbf{b}[x] = a, \mathbf{b}[y] = b$. Since $\mathcal{S}''_{zt} \subseteq \mathrm{pr}_{zt}R$ for any $z, t \in \mathbf{s}$, Theorem 25 implies that there is $\mathbf{a} \in R$ with the required properties.

We now show that every $(a, b) \in \mathcal{S}''_{xy}$, $x, y \in V$, can be extended to $(a, b, c) \in \mathcal{S}'_{xyz}$ such that $(a, c) \in \mathcal{S}''_{xz}, (b, c) \in \mathcal{S}''_{yz}$ for any $z \in V$. Let $W = \{x, y, z\}$. By Claim 3, and Theorem 25 there is $c \in \mathcal{S}''_z$ such that $(a, b, c) \in \mathcal{S}_{xyz}$. Let $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ be such that $\mathbf{s} \cap W \neq \varnothing$. Let $\mathbf{s} = (v_1, \dots, v_\ell)$. If $|\mathbf{s} \cap W| < 3$ then by Claim 4(2) there is $\mathbf{a} \in R$ such that $\mathbf{a}[i] \in \mathcal{S}''_{v_i}$ for $i \in [\ell]$ and such that $\mathbf{a}[i] = a$ if $v_i = x$, and $\mathbf{a}[i] = b$ if $v_i = y$, $\mathbf{a}[i] = c$ if $v_i = z$. Suppose $W \subseteq \mathbf{s}$ and $v_1 = x$, $v_2 = y$, $v_3 = z$. Again, by Claim 4(2) there is $\mathbf{a} \in R$ with $\mathbf{a}[i] \in \mathcal{S}''_{v_i}$ for $i \in [\ell]$ and such that $\mathbf{a}[1] = a, \mathbf{a}[2] = b$. By Theorem 25, setting $X = [3]$, there is also $\mathbf{b} \in R$ with the same property and such that $\mathrm{pr}_{[3]}\mathbf{b} = (a, b, c)$. Therefore $(a, b, c) \in \mathcal{S}'_{xyz}$. □

8.3 Omitting affine edges and bounded width

We need one more auxiliary statement.

LEMMA 32. *Let $R \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ be such that $\mathbb{A}_i$ has no affine edges and $\mathbb{A}_1 \times \mathsf{max}(\mathbb{A}_i) \subseteq \mathrm{pr}_{1i}R$ for $i \in [n]$, and $\mathbb{A}_1$ is simple and generated by any of its maximal components. Then $\mathbb{A}_1 \times \mathsf{max}(\mathrm{pr}_{\{2,\dots,n\}}R) \subseteq R$.*

Now we are in a position to prove Theorem 30.

Proof:[of Theorem 30] By Theorem 6 we may assume all algebras from $\mathcal{A}$ are sm-smooth. Let $\mathcal{P} = (V;\delta;\mathcal{C})$ be a $(2,3)$-consistent problem instance. We prove by induction on the number of elements in $\mathbb{A}_{\delta(v)}$, $v \in V$, that $\mathcal{P}$ has a solution. If all $\mathbb{A}_{\delta(v)}$, $v \in V$, are 1-element, the result holds trivially.

Suppose that the theorem holds for problem instances $\mathcal{P}' = (V;\delta';\mathcal{C}')$ where $|\mathbb{A}_{\delta'(v)}| \le |\mathbb{A}_{\delta(v)}|$ for $v \in V$ (here $\mathbb{A}_{\delta'(v)}$ denotes the set of partial solutions to $\mathcal{P}'$ on $\{v\}$) and at least one inequality is strict. By Proposition 31 we may assume that for all $v \in V$ algebra $\mathbb{A}_{\delta(v)}$ is generated by any of its maximal components.

For some $u \in V$, take a maximal congruence θ of $\mathbb{A}_{\delta(u)}$. Note that for any $v \in V - \{u\}$, $\mathcal{S}^{\theta}_{uv} = \{(a^{\theta}, b) \mid (a,b) \in \mathcal{S}_{uv}\}$ is either the direct product $\mathbb{A}_{\delta(u)}/_{\theta} \times \mathbb{A}_{\delta(v)}$, or the graph of a surjective mapping $\pi_v\colon \mathbb{A}_{\delta(v)} \to \mathbb{A}_{\delta(u)}/_{\theta}$. Indeed, if the link congruence of $\mathbb{A}' = \mathbb{A}_{\delta(u)}/_{\theta}$ is the equality relation, $\mathcal{S}^{\theta}_{uv}$ is the graph of a mapping. Otherwise, since $\mathbb{A}'$ is simple, $\mathcal{S}^{\theta}_{uv}$ is linked. By Proposition 19, for any maximal components B_1, B_2 of $\mathbb{A}'$ and $\mathbb{A}_{\delta(v)}$ such that $(B_1 \times B_2) \cap \mathcal{S}^{\theta}_{uv} \neq \varnothing$, we have $B_1 \times B_2 \subseteq \mathcal{S}^{\theta}_{uv}$ Since B_1 generates $\mathbb{A}'$ and B_2 generates $\mathbb{A}_{\delta(v)}$, it holds $\mathbb{A}' \times \mathbb{A}_{\delta(v)} \subseteq \mathcal{S}^{\theta}_{uv}$.

Let W denote the set consisting of u and all $v \in V$ such that $\mathcal{S}^{\theta}_{u,v}$ is the graph of π_v, and let θ_v denote $\ker \pi_v$, the congruence of $\mathbb{A}_{\delta_v}$ which is the kernel of π_v, for $v \in W$ and let θ_v denote the equality relation for $v \in V - W$; also let $\theta_u = \theta$. Since $\mathcal{P}$ is $(2,3)$-consistent, for any $v, w \in W$ there is a bijective mapping $\pi_{vw} : \mathbb{A}_{\delta(v)}/_{\theta_v} \to \mathbb{A}_{\delta(w)}/_{\theta_w}$ such that whenever $(a,b) \in \mathcal{S}_{vw}$, $\pi_{vw}(a^{\theta_v}) = b^{\theta_w}$. Take a maximal (as an element of $\mathbb{A}'$) θ-block $B \subseteq \mathbb{A}_{\delta(u)}$ and let $\mathcal{P}'$ to be the problem $(V; \delta'; \mathcal{C}')$ given by

$$\mathbb{A}_{\delta'(v)} = \begin{cases} \pi_{uv}(B) & \text{if } v \in W, \\ \mathbb{A}_{\delta(v)} & \text{otherwise,} \end{cases}$$

and for each $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ there is $C' = \langle \mathbf{s}, R'\rangle \in \mathcal{C}'$ such that $\mathbf{a} \in R'$ if and only if $\mathbf{a} \in R$ and $a_v \in \pi_{uv}(B)$ for all $v \in W \cap \mathbf{s}$.

Since $|\mathbb{A}_{\delta'(u)}| < |\mathbb{A}_{\delta(u)}|$, we just have to show that $\mathcal{P}'$ is $(2,3)$-consistent. Let $u_1, u_2, u_3 \in V$ and $S_i = \mathbb{A}_{\delta'(u_i)}$. We show that for any $(a,b) \in \mathsf{max}(\mathcal{S}_{u_1u_2}) \cap (S_1 \times S_2)$ there is $c \in S_3$ such that $(a,b,c) \in \mathcal{S}'_{u_1u_2u_3}$, where $\mathcal{S}'_{u_1u_2u_3}$ is the set of solutions of $\mathcal{P}'$. Note also that $\mathsf{max}(\mathcal{S}_{u_1u_2}) \cap (S_1 \times S_2) \neq \varnothing$ by construction.

Let $U = \{u_1, u_2\}$ and take $\langle \mathbf{s}, R\rangle \in \mathcal{C}$. Clearly, $\mathrm{pr}_{U\cap\mathbf{s}}(a,b) \in \mathrm{pr}_{U\cap\mathbf{s}} R$, let $\mathbf{a} \in \mathsf{max}(R)$ be such that $\mathrm{pr}_{U\cap\mathbf{s}}\mathbf{a} = \mathrm{pr}_{U\cap\mathbf{s}}(a,b)$. If $U \cap W \cap \mathbf{s} \neq \varnothing$ then, for any $v \in s \cap W$, $\mathbf{a}[v] \in \pi_{uv}(B)$, and therefore $\mathbf{a} \in R'$. If $\mathbf{s} \cap W = \varnothing$ then $R' = R$, and again $\mathbf{a} \in R'$. Otherwise choose $v \in \mathbf{s} \cap W$ and set $X = (\mathbf{s} - W) \cup \{v\}$, $Q = \mathrm{pr}_X R$ and $Q' = \{(a^{\theta_w}_w)_{w\in X} \mid (a_w)_{w\in X} \in Q\}$. Since $\mathbb{A}_{\delta(v)}$ is generated by any of its maximal components, by Lemma 32, $\mathbb{A}_{\delta(v)}/_{\theta_v} \times \mathsf{max}(\mathrm{pr}_{\mathbf{s}-W} Q') \subseteq Q'$. This means that there is $\mathbf{c} \in R$ such that $\mathrm{pr}_{\mathbf{s}-W}\mathbf{c} = \mathrm{pr}_{\mathbf{s}-W}\mathbf{a}$ and $\mathbf{c}[v] \in \pi_{uv}(B)$. Therefore, $\mathbf{c}[w] \in \pi_{uw}(B)$ for any $w \in \mathbf{s} \cap W$, and hence $\mathbf{c} \in R'$. □

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