Some complex sublattices of clone lattices

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As it is seen from the title the main object of our consideration will be lattices of clones of finite-valued functions. All functions occured in this paper are defined on the set $E_k = \{0, \ldots, k-1\}$. Let us remind that a clone is a set of k-valued functions containing all projections (i.e. functions of the form $e_i^n(x_1, \ldots, x_n) = x_i$) and closed under all superpositions. All clones of k-valued functions form a complete lattice which usually denoted by $\mathcal{L}_k$. Along with the term "clone" similar concept of "iterative algebra" ([1]), i.e. a set of k-valued functions closed under superposition, permutations, and identifications of variables as well as adding fictive arguments is often used. Iterative algebras which contain all projections are precisely clones. Although all results in this paper are formulated for clone lattices it will be convinient to prove them in terms of iterative algebras. The lattice of all iterative algebras of k-valued functions is denoted by $\mathcal{L}'_k$. The lattice $\mathcal{L}_k$ is isomorfic to the principal filter of $\mathcal{L}'_k$ generated by the algebra of all projections. The iterative algebra of all k-valued functions is denoted by P_k.

When we investigate such a complex lattice as $\mathcal{L}_k$ we can pose many various natural problems. One of them is the problem of finding families of its sublattices of some special kinds. We can regard this problem solved for the lattice $\mathcal{L}_2$ because there exists full description of $\mathcal{L}_2$ ([2]). There are only few advancements in decision of this problem for $\mathcal{L}_k$, if $k \geq 3$. In particular, in [3] it was proved that a power set of a countable set can be embedded as an interval into $\mathcal{L}_3$ (and hence into $\mathcal{L}_k$ for $k \geq 3$ because $\mathcal{L}_3$ is isomorfic to some interval in $\mathcal{L}_k$). This fact implies embeddability of any finite or countable distributive lattice into $\mathcal{L}_3$. In [4] it was noted that for any finite lattice L there exist k (which may be rather large) such that L is a sublattice of $\mathcal{L}_k$. Further, as it was proved in [5] the subsemigroup lattice of free semigroup of countable rank is isomorfically embeddable into $\mathcal{L}_4$. Using this fact and some results of the papers [6], [7] we can conclude that a wide class of finite lattices is embeddable into $\mathcal{L}_k$ for $k \geq 4$.

The starting-point of this paper is the question: is an arbitrary finite lattice embeddable into $\mathcal{L}_k$ for some fixed k? A stronger assertion may be proved. It implies the solution for the problem of finding of all finite sublattices in $\mathcal{L}_k$. The main resalt is the next statement.

Theorem 1 *Any lattice being at most countable direct product of finite lattices is isomorfically embeddable into $\mathcal{L}_4$.*

The theorem gives us examples of continual lattices embeddable into $\mathcal{L}_4$. If we consider only countable lattices of pointed out kind only then we get the well known class of countable residually finite lattices.

Corollary 1 *Any countable residually finite lattice is isomorfically embeddable into $\mathcal{L}_4$.*

Let $\mathcal{X}$ be a lattice variety generated by its finite members. It is well known that the $\mathcal{X}$-free lattice of countable rank is residually finite. This fact gives the followig example of sublattice of $\mathcal{L}_4$ for every variety $\mathcal{X}$ of mentioned kind.

Corollary 2 *The $\mathcal{X}$-free lattice of countable rank is isomorfically embeddable into $\mathcal{L}_4$.*

An embeddability of the absolutely free lattice of countable rank follows from results of [5], [6], [7].

Corollary 3 *Any finite lattice is isomorfically embeddable into $\mathcal{L}_4$.*

Corollary 3 also gives information on some important property of clone lattices.

Corollary 4 *The lattice $\mathcal{L}_4$ does not satisfy any nontrivial lattice quasy-identity.*

Similar assertion for nontrivial identities in $\mathcal{L}_3$ was proved in [5].

We note one corollary from the proof of the theorem. Thi assertion is connected with so-called monoidal intervals. Let us remaind that a monoidal interval ($IntM$) for a transformation monoid M on the set E_k is the family of all clones such that every its members has M as the set of its unary operations (this family is an interval in $\mathcal{L}_k$). In literature (see for example [8]) questions connected with structure (in particular, with cardinality) of monoidal intervlas were considered. All clones occured in embedding constructed in the proof of the theorem have the same set M of unary operations on E_4, where $M = \{e_1^1, f\}$, $f(2) = 2$, $f(x) = 0$, if $x \neq 2$. Therefore the following statemetnt holds.

Corollary 5 *Any lattice being an most countable direct product of finite lattices can be isomorfically embedded into* $IntM$.

The author suggests that the embedding in the proof of the theorem can be modified so that the property from corollary will be true not only for the mentioned monoid M but for some other monoids.

The question on truth of the theorem for the lattice $\mathcal{L}_3$ is of very interest. In other words, is the restriction $k \geq 4$ in the theorem of principle or it produced only by defects of authors technique? It is not known any essential distinctions between lattice properties of lattices $\mathcal{L}_3$ and $\mathcal{L}_k$ for $k \geq 4$. The author has no hypothesis on this question, however.

Problem 1 *Can an arbitrary finite lattice be embedded into* $\mathcal{L}_3$*?*

The following problem is closely connected with previous one and is of large interest.

Problem 2 *Does the lattice* $\mathcal{L}_3$ *satisfy any nontrivial quasy-identity?*

It would be very interesting to get some infomation about an embeddability of lattices $\mathcal{L}_k$ one into another. Now, we know that $\mathcal{L}_k$ can be embedded (more then, it can be embedded as an interval) into $\mathcal{L}_m$, if $m \geq k$ ([10]). But, is the inverse situation possible? More precisely, are there two integers k and m, $k < m$ such that $\mathcal{L}_m$ is embeddable into $\mathcal{L}_k$? We formulate a more concrete question.

Problem 3 *Does the integer* k *with the property:* $\mathcal{L}_m$ *can be embedded into* $\mathcal{L}_k$ *for any* m*, exist? (it would be very gladly, if* k *equals to 3 or 4)*

By the corollary 3 any finite lattice is isomorfic to some sublattice of $\mathcal{L}_4$. But it is doubtful whether an analogous fact is true for countable lattices. All lattices embedded into $\mathcal{L}_4$ by the theorem have many congruences. So, we ought to find an unembeddable lattice among infinite lattices with few congruences, for example, among prime lattices. The most known lattices of that kind are the lattice of equivalence relations on a coutable set and the lattice M_ω i.e. the prime lattice of height 2 with a countable set of incomparable elements.

Problem 4 *Is the lattice M_ω or the lattice of equivalences on a countable set embeddable into $\mathcal{L}_k$ for some k (in particular, for $k = 3, 4$)?*

The mentioned results of [3] give us the example of an embedding as an interval or a presentation by an interval in clone lattices. So, we can reformulate the problems marked above for embedding of such restricted kind. Only in the problem 1 we have to ask about all lattices $\mathcal{L}_k$ but not only about $\mathcal{L}_3$.

At the end of the paper we are saying a few words about the proof of the theorem. The proof is rather long and sophisticated so we are doing to describe the concrete lattice embedded into $\mathcal{L}_4$ and to give rough sketch of the proof. As usual, the lattice of all sublattices (lattice of all $0,1$-sublattices) of the lattice L is denoted by $\mathrm{Sub}\, L$ (respectively, $\mathrm{Sub}_{0,1}\, L$). It was showed in [9] that any finite lattice can be isomofically embedded into $\mathrm{Sub}\, P(A)$, where $P(A)$ is the power set of a suitable finite set A. It is easy to see that $\mathrm{Sub}\, P(A)$ is embeddable into $\mathrm{Sub}_{0,1}\, P(B)$, where $B = A \cup \{b, c\}, b, c \notin A$. Therefore any finite lattice can be isomorfically embedded into the lattice of all $0,1$-sublattices of a proper finite boolean algebra. Thus it is sufficient to embed a direct products of lattices of the form $\mathrm{Sub}_{0,1}\, P(A)$, where A is a finite set. Since $A \subseteq B$ implies that $\mathrm{Sub}_{0,1}\, P(A)$ is a sublattice of $\mathrm{Sub}_{0,1}\, P(B)$ we see that it is sufficient to construct embedding into $\mathcal{L}_4$ of the lattice $\prod_{l \in \mathbf{N}} \mathrm{Sub}_{0,1}\, P(A_l)$, where $A_l = \{0,1\}^l$.

Now, we must construct an embedding of the pointed out lattice. For it we define the special function $f_B(x, y, z_1, \ldots, z_l)$ for each $B \subseteq \{0,1\}^l, l \in \mathbf{N}$. These functions have many useful properties. For example, for any two sets $B_1, B_2 \subseteq \{0,1\}^l, l \in \mathbf{N}$ the following identities are true:

$$f_{B_1}(f_{B_2}(x, y, z_1, \ldots, z_l), y, z_1, \ldots z_l) =$$

$$
\begin{array}{lrl}
(1\cap) & = & f_{B_2}(f_{B_1}(x,y,z_1,\dots,z_l),y,z_1,\dots,z_l) = \\
 & = & f_{B_1\cap B_2}(x,y,z_1,\dots,z_l) \\
 & & f_{B_1}(x,f_{B_2}(x,y,z_1,\dots,z_l),z_1,\dots,z_l) = \\
(1\cup) & = & f_{B_2}(x,f_{B_1}(x,y,z_1,\dots,z_l),z_1,\dots,z_l) = \\
 & = & f_{B_1\cup B_2}(x,y,z_1,\dots,z_l).
\end{array}
$$

So, superpositions of the such form correspond to the operations of the lattice $P(\{0,1\}^l$. On the other hand if we apply some other "bad" superpositions or identifications of variables then we get a "bad" function. Therefore, any $0,1$-sublattice Q of $P(\{0,1\}^l)$ is put into correspondence with an iterative algebra generated by functions which closely connected with functions of the set $\{f_B \mid B \in Q\}$. Finally, every element $(Q_1, Q_2, \dots)$ of $\prod_{l\in\mathbf{N}} \mathrm{Sub}_{0,1} P(\{0,1\}^l)$ corresponds to the lattice union of all images of $0,1$-sublattices $Q_1, Q_2, \dots$. The main part of the proof is devoted to checking stability and bijectivity of the constructed mapping.

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