Boolean Max-Co-Clones

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Abstract. In our ISMVL 2012 paper we introduced the notion of max-co-clone as a set of relations closed under a new type of quantification, max-quantification. This new concept was motivated by its connections to approximation complexity of counting constraint satisfaction problems. In this paper we go beyond scattered examples of max-co-clones and describe all max-co-clones on a 2-elements set (Boolean max-co-clones). It turns out that there are infinitely many Boolean max-co-clones and that all of them are regular co-clones, although it is not true for larger sets. Also there are many usual co-clones that are not closed under max-quantification, and therefore are not max-co-clones.

1. Introduction

The study of various closure operators on the set of relations can be traced back to the seminal work by Post [22]. A number of closure operators have been investigated since then, including intersection (conjunction, if we treat relations as predicates), projections (existential quantification), union (disjunction), universal quantification, etc., defined on various types of relations (see [21] or [3, 4] for a survey).

Most of these types of relations and closure operators have been motivated by certain Galois correspondences that allowed for better understanding the structure of closed sets of functions of various types. Recently, the study of co-clones has received a strong additional motivation from computer science. More precisely it was shown that the usual closure operators of co-clones — intersection and projection — preserve the complexity of constraint satisfaction problems (CSPs) [18], and therefore the complexity of this problem is a property of a certain co-clone. Later a similar connection was discovered for quantified CSPs (QCSPs) and co-clones additionally closed under universal quantification [5]. Another connection exists for partial co-clones (closed only under intersections) and the approximation complexity of counting CSPs [7, 10]. Certain analogs of co-clones for weighted CSPs were introduced in [8, 12] Finally the connection from [7, 10] was strengthened by introducing the max-quantification construction that also preserves the approximation complexity of counting CSPs [9]. In that paper we only established that max-quantification preserves the approximation complexity of counting CSPs and

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gave several examples. For completeness we should also mention the recent work [19] that introduces CSPs with counting quantifiers and the corresponding type of co-clones.

In this paper we embark on a systematic study of max-co-clones. Intuitively, applying max-quantification to a relation $R(x_1, \dots, x_n, y_1, \dots, y_k)$ results in the relation $\exists_{\max}(y_1, \dots, y_k) R(x_1, \dots, x_n, y_1, \dots, y_k)$ that contains those tuples $(a_1, \dots, a_n)$ that have a maximal number of extensions $(a_1, \dots, a_n, b_1, \dots, b_k)$ such that $R(a_1, \dots, a_n, b_1, \dots, b_k)$ is satisfied. As a first step we give a complete description of Boolean max-co-clones, that is, ones on a 2-element set. We show that there are infinitely many of them (although countably many). Each of the Boolean max-co-clones is also a regular co-clone, although this is a coincidence, since there are max-co-clones on larger sets that are not co-clones. Not every co-clone is a max-co-clone; in some cases whole infinite hierarchies of co-clones collapse into a single max-co-clone, see Fig. 2. We also give some generating sets (in terms of max-quantification) of all Boolean max-co-clones.

2. Preliminaries

By $[n]$ we denote the set $\{1, \dots, n\}$. For a set D, by D^n we denote the set of all *n-tuples* of elements of D. An n-ary relation is any set $R \subseteq D^n$. The number n is called the *arity* of R and denoted $\mathsf{ar}(R)$. Tuples will be denoted in boldface, say, $\mathbf{a}$, and their entries will be denoted by $\mathbf{a}[1], \dots, \mathbf{a}[n]$. For $I = (i_1, \dots, i_k) \subseteq [n]$ by $\mathrm{pr}_I \mathbf{a}$ we denote the tuple $(\mathbf{a}[i_1], \dots, \mathbf{a}[i_k])$, and we use $\mathrm{pr}_I R$ to denote $\{\mathrm{pr}_I \mathbf{a} \mid \mathbf{a} \in R\}$. We will also need predicates corresponding to relations. To simplify the notation we use the same symbol for a relation and the corresponding predicate, for instance, for an n-ary relation R the corresponding predicate $R(x_1, \dots, x_n)$ is given by $R(\mathbf{a}[1], \dots, \mathbf{a}[n]) = 1$ if and only if $\mathbf{a} \in R$. Relations and predicates are used interchangeably.

For a set of relations Γ over a set D, the set $\langle\langle \Gamma \rangle\rangle$ includes all relations that can be expressed (as a predicate) using (a) relations from Γ, together with the binary equality relation $=_D$ on D, (b) conjunctions, and (c) existential quantification. This set is called the *co-clone generated by* Γ.

Partial co-clone generated by Γ is obtained in a similar way by disallowing existential quantification. $\langle \Gamma \rangle$ includes all relations that can be expressed using (a) relations from Γ, together with $=_D$, and (b) conjunctions.

If $\Gamma = \langle \Gamma \rangle$ or $\Gamma = \langle\langle \Gamma \rangle\rangle$, the set Γ is said to be a *partial co-clone*, and a *co-clone*, respectively.

Sometimes there is no need to apply even conjunction to produce a new relation. For instance, $Q(x, y) = R(x, y, y)$ defines a binary relation from a ternary one. Therefore it is often convenient, especially for technical purposes, to group manipulations with variables of a relation into a separate category. More formally, for a relation $R(x_1, \dots, x_n)$ and a mapping $\pi \colon \{x_1, \dots, x_n\} \to V$, where V is some set of variables, πR denotes the relation $R(\pi(x_1), \dots, \pi(x_n))$. We will understand by (partial) co-clones sets of relations closed under manipulation with variables, conjunction, and existential quantification (respectively, closed under manipulation with variables and conjunction).

Let R be a (k-ary) relation on a set D, and $f\colon D^n \to D$ an n-ary function on the same set. Function f *preserves* R, or is a *polymorphism* of R, if for any n tuples $\mathbf{a}_1, \ldots, \mathbf{a}_n \in R$ the tuple $f(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ obtained by component-wise application of f also belongs to R. Relation R in this case is said to be *invariant* with respect to f. The set of all functions that preserve every relation from a set of relations Γ is denoted by $\mathsf{Pol}(\Gamma)$, the set of all relations invariant with respect to a set of functions C is denoted by $\mathsf{Inv}(C)$.

Operators Inv and Pol form a Galois connection between sets of functions and sets of relations. Sets of the form $\mathsf{Inv}(C)$ are precisely co-clones [17, 2]; on the functional side there is another type of closed sets.

A set of functions is said to be a *clone* of functions if it is closed under superpositions and contain all the *projection* functions, that is functions of the form $f(x_1, \ldots, x_n) = x_i$. Sets of functions of the form $\mathsf{Pol}(\Gamma)$ are exactly clones of functions [20] .

3. Approximate counting and max-quantification

Let D be a set, and let Γ be a finite set of relations over D. An instance of the counting Constraint Satisfaction Problem, $\#\mathrm{CSP}(\Gamma)$, is a pair $\mathcal{P} = (V, \mathcal{C})$ where V is a set of *variables*, and $\mathcal{C}$ is a set of *constraints*. Every constraint is a pair $\langle \mathbf{s}, R\rangle$, in which R is a member of Γ, and $\mathbf{s}$ is a tuple of variables from V of length $\mathsf{ar}(R)$ (possibly with repetitions). A *solution* to $\mathcal{P}$ is a mapping $\varphi : V \to D$ such that $\varphi(\mathbf{s}) \in R$ for every constraint $\langle \mathbf{s}, R\rangle \in \mathcal{C}$. The objective in $\#\mathrm{CSP}(\Gamma)$ is to find the number $\#\mathcal{P}$ of solutions to a given instance $\mathcal{P}$.

We are interested in the complexity of this problem depending on the set Γ. The complexity of the exact counting problem (when we are required to find the exact number of solutions) is settled in [6, 11] by showing that for any finite D and any set Γ of relations over D the problem is polynomial time solvable or is complete in a natural complexity class $\#P$. One of the key steps in that line of research is the following result: For a relation R and a set of relations Γ over D, if R belongs to the co-clone generated by Γ, then $\#\mathrm{CSP}(\Gamma \cup \{R\})$ is polynomial time reducible to $\#\mathrm{CSP}(\Gamma)$. This results emphasizes the importance of co-clones in the study of constraint problems.

A situation is different when we are concerned about approximating the number of solutions. We will need some notation and terminology. Let A be a counting problem. A randomized algorithm Alg is said to be an *approximation algorithm* for A with relative error ε (which may depend on the size of the input) if it is polynomial time and for any instance $\mathcal{P}$ of A it outputs a certain number $\mathsf{Alg}(\mathcal{P})$ such that $\mathsf{Alg}(\mathcal{P}) = 0$ if $\mathcal{P}$ has no solution and

$$\mathsf{Prob}\left[\frac{|\#\mathcal{P} - \mathsf{Alg}(\mathcal{P})|}{\#\mathcal{P}} < \varepsilon\right] > \frac{2}{3}$$

otherwise, where $\#\mathcal{P}$ denotes the exact number of solutions to $\mathcal{P}$.

The following framework is viewed as one of the most realistic models of efficient computations. A *fully polynomial approximation scheme* (FPRAS, for short) for a problem A is an algorithm Alg such that: It takes as input an instance $\mathcal{P}$ of A and a real number $\varepsilon > 0$, the relative error of Alg on the input $(\mathcal{P}, \varepsilon)$ is less than ε, and Alg is polynomial time in the size of $\mathcal{P}$ and $\log(\frac{1}{\varepsilon})$.

To determine the approximation complexity of problems approximation preserving reductions are used. Suppose A and B are two counting problems whose complexity (of approximation) we want to compare. An *approximation preserving reduction* or *AP-reduction* [15] from A to B is an algorithm Alg, using B as an oracle, that takes as input a pair $(\mathcal{P}, \varepsilon)$ where $\mathcal{P}$ is an instance of A and $0 < \varepsilon < 1$, and satisfies the following three conditions: (i) every oracle call made by Alg is of the form $(\mathcal{P}', \delta)$, where $\mathcal{P}'$ is an instance of B, and $0 < \delta < 1$ is an error bound such that $\log\left(\frac{1}{\delta}\right)$ is bounded by a polynomial in the size of $\mathcal{P}$ and $\log\left(\frac{1}{\varepsilon}\right)$; (ii) the algorithm Alg meets the specifications for being an FPRAS for A whenever the oracle meets the specification for being an FPRAS for B; and (iii) the running time of Alg is polynomial in the size of $\mathcal{P}$ and $\log(\frac{1}{\varepsilon})$. If an approximation preserving reduction from A to B exists we denote it by $A \le_{\mathrm{AP}} B$, and say that A is *AP-reducible to* B.

In [9] we introduced the following closure operator of max-quantification. Let $R(x_1, \dots, x_n, y_1, \dots, y_m)$ be a relation on a set D. By $\exists_{\max}(y_1, \dots, y_m) R(x_1, \dots, x_n, y_1, \dots, y_m)$ we denote the relation $Q(x_1, \dots, x_n)$ on the same set given by the rule: $\mathbf{a} \in Q$ if and only if there are M tuples $\mathbf{b} \in D^m$ such that $(\mathbf{a}, \mathbf{b}) \in R$, where M is the maximal number of elements in the set $E(\mathbf{a}) = \{\mathbf{b} \mid (\mathbf{a}, \mathbf{b}) \in Q\}$ over all $\mathbf{a} \in D^n$. A set of relations Γ over D is said to be a *max-co-clone* if it contains the equality relations, and closed under conjunctions and max-quantification. The smallest max-co-clone containing a set of relations Γ is called the *max-co-clone generated by* Γ and denoted $\langle\Gamma\rangle_{\max}$.

Theorem 1 ([9]). *If $R \in \langle\Gamma\rangle_{\max}$, then there is an approximation preserving reduction from $\#\mathrm{CSP}(\Gamma \cup \{R\})$ to $\#\mathrm{CSP}(\Gamma)$.*

Note that unlike existential and universal quantification max-quantification over a collection of variables cannot be substituted with a sequence of single-variable quantifications, see [9].

4. The lattice of Boolean max-co-clones

In this section we give a description of all max-co-clones on $\{0, 1\}$. We will use the description of usual Boolean co-clones from [22] and *plain bases* of

Boolean co-clones found in [14]. Recall that plain basis of a co-clone C is a set Γ of relations such that the closure of Γ with respect to manipulation of variables and conjunction is C.

To state the results of [14] and then to proceed with the proof, we need some definitions and notation. A relation $R(x_1,\ldots,x_n)$ is said to be *trivial* if it can be specified by giving a set of variables that are equal to 0 (to 1) in every tuple from R, and a collection of conditions of the form $x_i = x_j$. More formally, there are sets $Z, W \subseteq [n]$ and an equivalence relation $\sim$ on $[n] - (Z \cup W)$ such that $\mathbf{a} \in R$ if and only if $\mathbf{a}[i] = 0$ whenever $i \in Z$, $\mathbf{a}[i] = 1$ whenever $i \in W$, and $\mathbf{a}[i] = \mathbf{a}[j]$ whenever $i \sim j$. A relation is called *monotone* if it is invariant with respect to $\vee$, the Boolean disjunction operation, or $\wedge$, the Boolean conjunction operation. Relation R is called *self-complement* if along with any tuple $\mathbf{a} \in R$ it also contains its *complement*, the tuple $\neg\mathbf{a}$ such that $\neg\mathbf{a}[i] = 1$ if and only if $\mathbf{a}[i] = 0$. Finally, relation R is called *affine* if it is the set of solutions to a system of linear equations over $GF(2)$. Addition in $GF(2)$ we will denote by $\oplus$.

For $I \subseteq [n]$ we denote by $\mathbf{a}_I$ the assignment to $x_1,\ldots,x_n$ in which $\mathbf{a}[i] = 1$ if $i \in I$ and $\mathbf{a}[i] = 0$ otherwise. We will use the following notation: δ_0, δ_1 denote the unary *constant* relations $\{(0)\}, \{(1)\}$, respectively. EQ is the binary *equality* relation $\{(0,0),(1,1)\}$; while NEQ is the binary *disequality* relation $\{(0,1),(1,0)\}$. $\mathsf{IMP}^k(x_1,\ldots,x_k,y)$ is the Horn $(k+1)$-ary relation given by the formula $\neg x_1 \vee \ldots \vee \neg x_k \vee y$, that is, $\mathbf{a} \in R$ if and only if $(\mathbf{a}[1],\ldots,\mathbf{a}[k],\mathbf{a}[k+1])$ satisfies the formula. By NIMP^k we denote the anti-Horn relation given by the formula $x_1 \vee \ldots \vee x_k \vee \neg y$. OR^k denotes the relation $\{0,1\}^k - \{(0,\ldots,0)\}$, and NAND^k denotes the relation $\{0,1\}^k - \{(1,\ldots,1)\}$. Finally, $\mathsf{Compl}_{k,\ell}$ is the $(k+\ell)$-ary relation $\{0,1\}^{k+\ell} - \{(0,\ldots,0,1,\ldots,1),(1,\ldots,1,0,\ldots,0)\}$, where the first of the two excluded tuples contains k zeros and ℓ ones, while the second contains k ones and ℓ zeros.

Fig. 1 shows the lattice of Boolean co-clones (borrowed from [14]), and Table 1 lists plain bases of Boolean co-clones. Table 1 is also taken from [14] only with notation changed to match the one used here.

The next theorem states the main result of this section.

Theorem 2. *The lattice of Boolean max-co-clones is shown in Fig 2. Some generating sets of these max-co-clones are given in Table 2.*

The theorem will follow from a sequence of auxiliary statements. In Section 4.1 we show that using the $\exists_{\max}$ quantifier we can define various relations, and that any relation can be defined by any two nontrivial binary relations. Then we show, Lemma 6, that any proper max-co-clone must contain only monotone, or only self-complement, or only affine relations. We consider these three cases. In the case of affine relations we show that the max-co-clones of such relations are exactly regular co-clones, Lemma 8. Then we show, Proposition 17, that there is only one max-co-clone of self-complement relations which contains a non-affine relation, IN_2. Then we show, Lemmas 10,11, that

Co-clone	Plain basis
IBF	$\{\mathsf{EQ}\}$
IR_0	$\{\mathsf{EQ}, \delta_0\}$
IR_1	$\{\mathsf{EQ}, \delta_1\}$
IR_2	$\{\mathsf{EQ}, \delta_0, \delta_1\}$
IM	$\{\mathsf{IMP}\}$
IM_0	$\{\mathsf{IMP}, \delta_0\}$
IM_1	$\{\mathsf{IMP}, \delta_1\}$
IM_2	$\{\mathsf{IMP}, \delta_0, \delta_1\}$
IS_0^k	$\{\mathsf{EQ}\} \cup \{\mathsf{OR}^\ell \mid \ell \leq k\}$
IS_0	$\{\mathsf{EQ}\} \cup \{\mathsf{OR}^\ell \mid \ell \in \mathbb{N}\}$
IS_1^k	$\{\mathsf{EQ}\} \cup \{\mathsf{NAND}^\ell \mid \ell \leq k\}$
IS_1	$\{\mathsf{EQ}\} \cup \{\mathsf{NAND}^\ell \mid \ell \in \mathbb{N}\}$
IS_{02}^k	$\{\mathsf{EQ}, \delta_0\} \cup \{\mathsf{OR}^\ell \mid \ell \leq k\}$
IS_{02}	$\{\mathsf{EQ}, \delta_0\} \cup \{\mathsf{OR}^\ell \mid \ell \in \mathbb{N}\}$
IS_{12}^k	$\{\mathsf{EQ}, \delta_1\} \cup \{\mathsf{NAND}^\ell \mid \ell \leq k\}$
IS_{12}	$\{\mathsf{EQ}, \delta_1\} \cup \{\mathsf{NAND}^\ell \mid \ell \in \mathbb{N}\}$
IS_{01}^k	$\{\mathsf{IMP}\} \cup \{\mathsf{OR}^\ell \mid \ell \leq k\}$
IS_{01}	$\{\mathsf{IMP}\} \cup \{\mathsf{OR}^\ell \mid \ell \in \mathbb{N}\}$
IS_{11}^k	$\{\mathsf{IMP}\} \cup \{\mathsf{NAND}^\ell \mid \ell \leq k\}$
IS_{11}	$\{\mathsf{IMP}\} \cup \{\mathsf{NAND}^\ell \mid \ell \in \mathbb{N}\}$
IS_{00}^k	$\{\mathsf{IMP}, \delta_0\} \cup \{\mathsf{OR}^\ell \mid \ell \leq k\}$
IS_{00}	$\{\mathsf{IMP}, \delta_0\} \cup \{\mathsf{OR}^\ell \mid \ell \in \mathbb{N}\}$
IS_{10}^k	$\{\mathsf{IMP}, \delta_1\} \cup \{\mathsf{NAND}^\ell \mid \ell \leq k\}$
IS_{10}	$\{\mathsf{IMP}, \delta_1\} \cup \{\mathsf{NAND}^\ell \mid \ell \in \mathbb{N}\}$
ID	$\{\mathsf{EQ}, \mathsf{NEQ}\}$
ID_1	$\{\mathsf{EQ}, \mathsf{NEQ}, \delta_0, \delta_1\}$
ID_2	$\{\delta_0, \delta_1, \mathsf{OR}, \mathsf{IMP}, \mathsf{NAND}\}$
IL	$\{x_1 \oplus \ldots \oplus x_k = 0 \mid k \text{ even}\}$
IL_0	$\{x_1 \oplus \ldots \oplus x_k = 0 \mid k \in \mathbb{N}\}$
IL_1	$\{x_1 \oplus \ldots \oplus x_k = c \mid k \in \mathbb{N}, k \equiv c \pmod 2, c \in \{0,1\}\}$
IL_2	$\{x_1 \oplus \ldots \oplus x_k = c \mid k \in \mathbb{N}, c \in \{0,1\}\}$
IL_3	$\{x_1 \oplus \ldots \oplus x_k = c \mid k \text{ even}, c \in \{0,1\}\}$
IV	$\{\mathsf{IMP}^k \mid k \geq 1\}$
IV_0	$\{\mathsf{IMP}^k \mid k \geq 1\} \cup \{\delta_0\}$
IV_1	$\{\mathsf{OR}^k \mid k \in \mathbb{N}\} \cup \{\mathsf{IMP}^k \mid k \geq 1\}$
IV_2	$\{\mathsf{OR}^k \mid k \in \mathbb{N}\} \cup \{\mathsf{IMP}^k \mid k \geq 1\} \cup \{\delta_0\}$
IE	$\{\mathsf{NIMP}^k \mid k \geq 1\}$
IE_0	$\{\mathsf{NAND}^k \mid k \in \mathbb{N}\} \cup \{\mathsf{NIMP}^k \mid k \geq 1\}$
IE_1	$\{\mathsf{NIMP}^k \mid k \geq 1\} \cup \{\delta_1\}$
IE_2	$\{\mathsf{NAND}^k \mid k \in \mathbb{N}\} \cup \{\mathsf{NIMP}^k \mid k \geq 1\} \cup \{\delta_1\}$
IN	$\{\mathsf{Compl}_{k,\ell} \mid k, \ell \geq 1\}$
IN_2	$\{\mathsf{Compl}_{k,\ell} \mid k, \ell \in \mathbb{N}\}$
II	$\{x_1 \vee \ldots \vee x_k \vee \neg y_1 \vee \ldots \vee \neg x_\ell \mid k, \ell \geq 1\}$
II_0	$\{x_1 \vee \ldots \vee x_k \vee \neg y_1 \vee \ldots \vee \neg x_\ell \mid k, \ell \geq 1\} \cup \{\delta_0\}$
II_1	$\{x_1 \vee \ldots \vee x_k \vee \neg y_1 \vee \ldots \vee \neg x_\ell \mid k, \ell \geq 1\} \cup \{\delta_1\}$
II_2	$\{x_1 \vee \ldots \vee x_k \vee \neg y_1 \vee \ldots \vee \neg x_\ell \mid k, \ell \geq 1\} \cup \{\delta_0, \delta_1\}$

TABLE 1. Plain bases of Boolean co-clones

Max-co-clone	Max-basis
IBF	$\{\mathsf{EQ}\}$
IR_0	$\{\mathsf{EQ}, \delta_0\}$
IR_1	$\{\mathsf{EQ}, \delta_1\}$
IR_2	$\{\mathsf{EQ}, \delta_0, \delta_1\}$
IM_2	$\{\mathsf{IMP}\}$
IS_0^k	$\{\mathsf{EQ}\} \cup \{\mathsf{OR}^k\}$
IS_0	$\{\mathsf{EQ}\} \cup \{\mathsf{OR}^\ell \mid \ell \in \mathbb{N}\}$
IS_1^k	$\{\mathsf{EQ}\} \cup \{\mathsf{NAND}^k\}$
IS_1	$\{\mathsf{EQ}\} \cup \{\mathsf{NAND}^\ell \mid \ell \in \mathbb{N}\}$
IS_{02}^k	$\{\mathsf{EQ}, \delta_0, \mathsf{OR}^k\}$
IS_{02}	$\{\mathsf{EQ}, \delta_0\} \cup \{\mathsf{OR}^\ell \mid \ell \in \mathbb{N}\}$
IS_{12}^k	$\{\mathsf{EQ}, \delta_1\} \cup \{\mathsf{NAND}^\ell \mid \ell \leq k\}$
IS_{12}	$\{\mathsf{EQ}, \delta_1\} \cup \{\mathsf{NAND}^\ell \mid \ell \in \mathbb{N}\}$
ID	$\{\mathsf{EQ}, \mathsf{NEQ}\}$
ID_1	$\{\mathsf{EQ}, \mathsf{NEQ}, \delta_0, \delta_1\}$
IL	$\{x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 0\}$
IL_0	$\{x_1 \oplus x_3 \oplus x_3 = 0\}$
IL_1	$\{x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 0, x_1 \oplus x_3 \oplus x_3 = 1\}$
IL_2	$\{x_1 \oplus x_3 \oplus x_3 = c \mid c \in \{0,1\}\}$
IL_3	$\{x_1 \oplus x_2 \oplus x_3 \oplus x_4 = c \mid c \in \{0,1\}\}$
IN_2	$\{\mathsf{Compl}_{3,0}\}$
II_2	$\{\mathsf{IMP}, \mathsf{OR}\}$

TABLE 2. Max-bases of Boolean max-co-clones

there is only one proper, that is, not II_2, the set of all relations, max-co-clone containing IMP, and this max-co-clone is IM_2. Finally, we consider the four remaining infinite chains of co-clones. In Lemma 12 we introduce a property that defines them. Then we show, Lemma 13, and 15, that there are no other max-co-clones containing OR (for NAND a dual result holds). Finally, we show that each of these co-clones is a max-co-clone.

4.1. Some implementations. We start with several useful observations.

Lemma 3. *(1)* $\delta_0, \delta_1 \in \langle \mathsf{IMP} \rangle_{\max}$*;*
(2) $\delta_0 \in \langle \mathsf{NEQ}, \delta_1 \rangle_{\max}$*,* $\delta_1 \in \langle \mathsf{NEQ}, \delta_0 \rangle_{\max}$*;*
(3) $\mathsf{NAND}^k \in \langle \mathsf{NAND}^m \rangle_{\max}$ *for any* $k \leq m$*;*
(4) $\mathsf{OR}^k \in \langle \mathsf{OR}^m \rangle_{\max}$ *for any* $k \leq m$*.*

Proof. (1) As is easily seen, $\delta_0(x) = \exists_{\max} y\, \mathsf{IMP}(x, y)$, and $\delta_1(x) = \exists_{\max} y\, \mathsf{IMP}(y, x)$.

(2) The first inclusion follows from $\delta_0(x) = \exists_{\max} y(\mathsf{NEQ}(x, y) \wedge \delta_1(y))$; the second one is similar.

(3) This claim follows from $\mathsf{NAND}^{m-1}(x_1, \ldots, x_{m-1}) = \exists_{\max} x_m \mathsf{NAND}^m(x_1, \ldots, x_m)$.

(4) is similar to (3). ☐

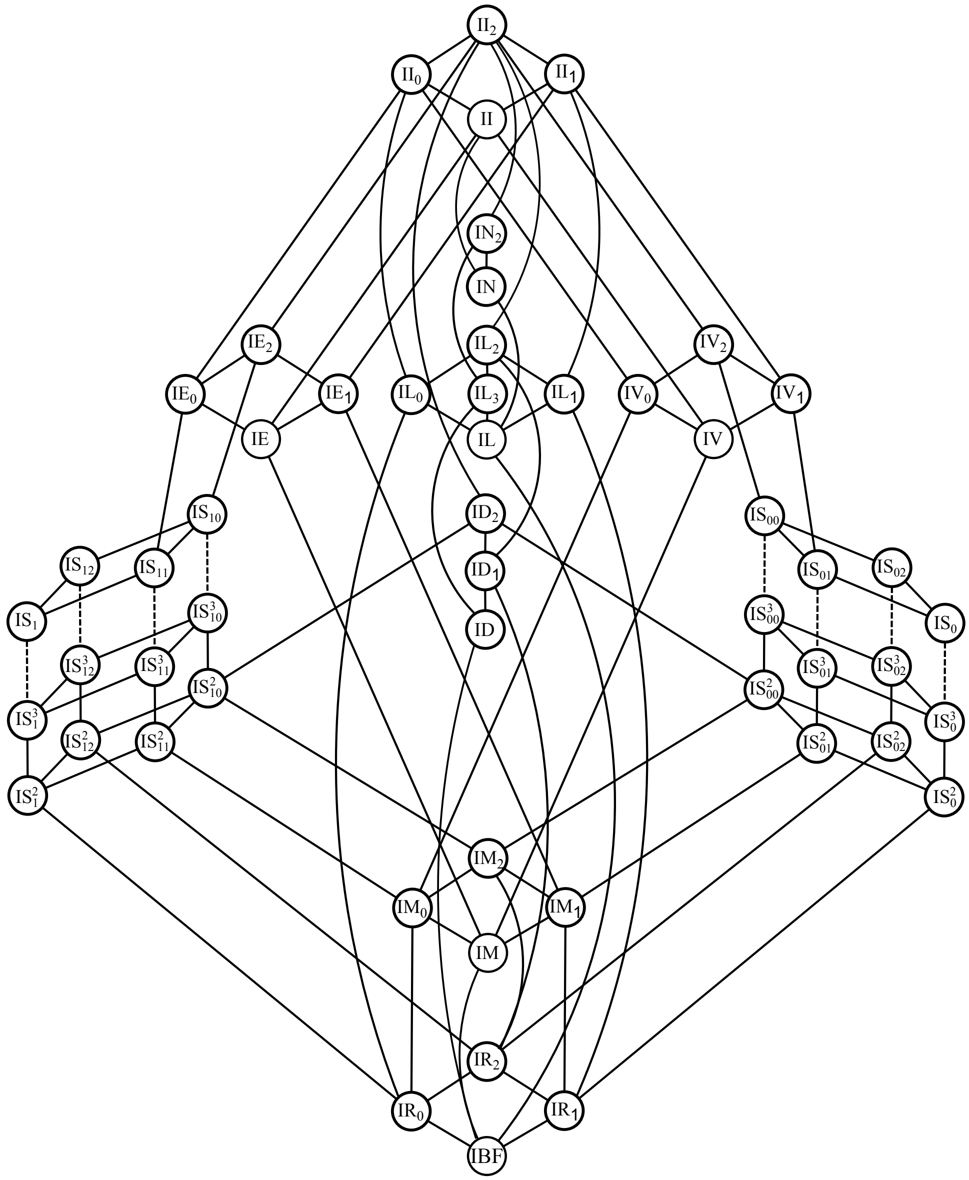

FIGURE 1. The lattice of Boolean co-clones

Lemma 4. *For any two different relations* $R, R' \in \{\mathsf{NEQ}, \mathsf{IMP}, \mathsf{OR}, \mathsf{NAND}\}$, $\langle R, R' \rangle_{\max} = II_2$, *the set of all relations on* $\{0, 1\}$.

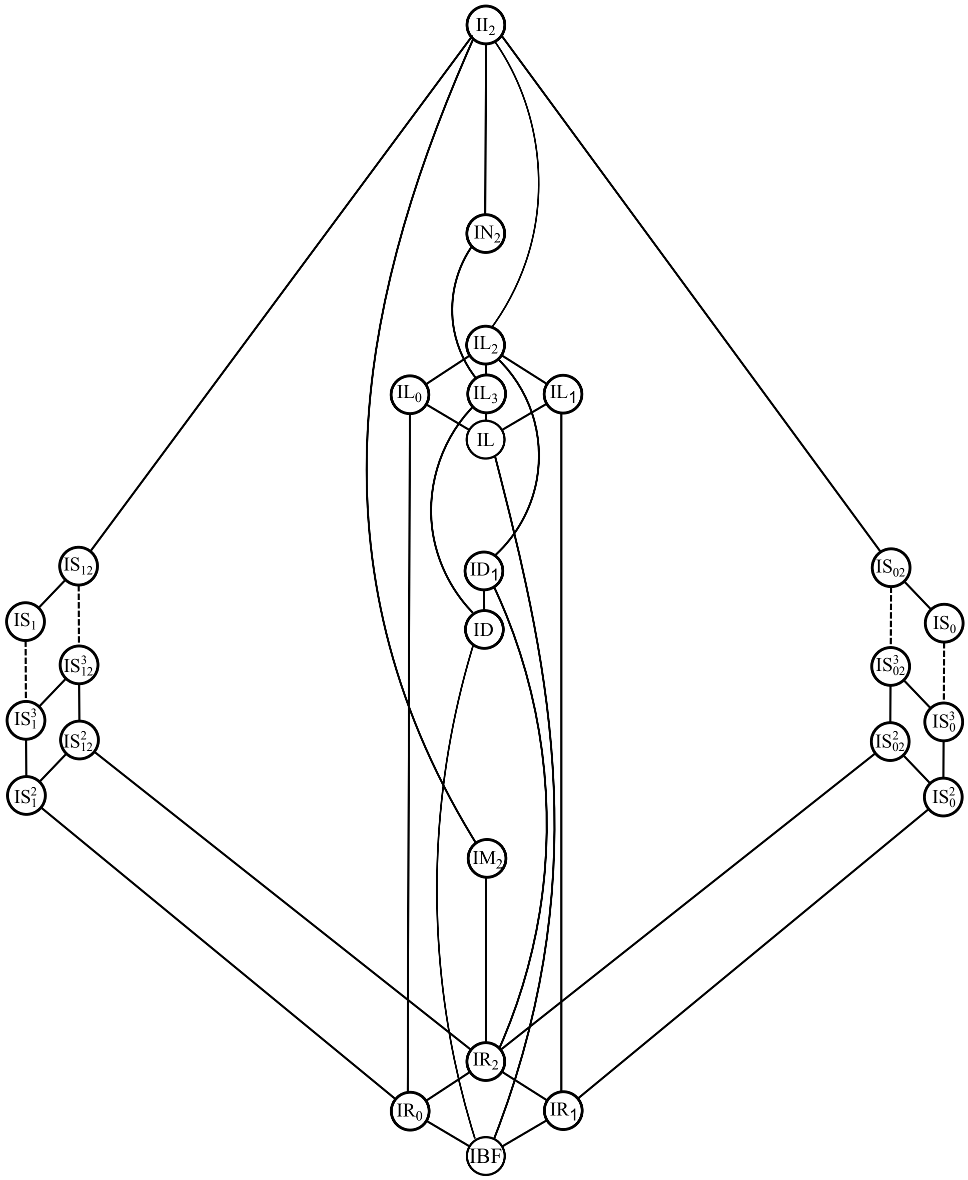

FIGURE 2. The lattice of Boolean max-co-clones

Proof. Observe first that

$$\begin{aligned}
\mathsf{OR} \cap \mathsf{NAND} &= \mathsf{NEQ}, \\
\mathsf{IMP}(x,y) &= \exists_{\max} z(\mathsf{OR}(z,y) \wedge \mathsf{NEQ}(z,x)) \\
&= \exists_{\max} z(\mathsf{NAND}(x,z) \wedge \mathsf{NEQ}(z,y)) \\
\mathsf{OR}(x,y) &= \exists_{\max} z(\mathsf{IMP}(z,y) \wedge \mathsf{NEQ}(z,x)) \\
&= \exists_{\max} z,t(\mathsf{NAND}(z,t) \wedge \mathsf{NEQ}(z,x) \wedge \mathsf{NEQ}(t,y)) \\
\mathsf{NAND}(x,y) &= \exists_{\max} z(\mathsf{IMP}(x,z) \wedge \mathsf{NEQ}(z,x)) \\
&= \exists_{\max} z,t(\mathsf{OR}(z,t) \wedge \mathsf{NEQ}(z,x) \wedge \mathsf{NEQ}(t,y)).
\end{aligned}$$

Also in the relation $Q(x,y,z,t) = \mathsf{OR}(x,y) \wedge \mathsf{IMP}(x,z) \wedge \mathsf{IMP}(y,t)$ assignments $(0,1)$ and $(1,0)$ to x,y are extendible in two ways, while $(1,1)$ is extendible in only one way. Therefore

$$\begin{aligned} \mathsf{NEQ}(x,y) &= \exists_{\max}(z,t)(\mathsf{OR}(x,y) \wedge \mathsf{IMP}(x,z) \wedge \mathsf{IMP}(y,t)), \quad \text{and, similarly,} \\ \mathsf{NEQ}(x,y) &= \exists_{\max}(z,t)(\mathsf{NAND}(x,y) \wedge \mathsf{IMP}(z,x) \wedge \mathsf{IMP}(t,y)). \end{aligned}$$

Thus $\{\mathsf{NEQ}, \mathsf{IMP}, \mathsf{OR}, \mathsf{NAND}\} \subseteq \langle R, R'\rangle_{\max}$, and it suffices to show that $\langle \mathsf{NEQ}, \mathsf{IMP}, \mathsf{OR}, \mathsf{NAND}\rangle_{\max} = II_2$.

The rest of the proof is derived from that of Lemma 8.1 [12], only it does not have to deal with weights.

Let $R(x_1, \ldots, x_n)$ be any relation. For each $I \subseteq [n]$ with $\mathbf{a}_I \in R$ introduce a new variable z_I. Consider the relation given by

$$Q = \bigwedge_{I \subseteq [n], \mathbf{a}_I \in R} \left(\bigwedge_{i \in I} \mathsf{IMP}(z_I, x_i) \wedge \bigwedge_{i \notin I} \mathsf{NAND}(z_I, x_i) \right).$$

Every assignment $\mathbf{a}_I \in R$ can be extended to the variables z_J in two ways: with $z_I = 0$ and $z_I = 1$. Any other assignment can be extended in only one way. Therefore

$$R(x_1, \ldots, x_n) = \exists_{\max}(z_I)_{I \subseteq [n], \mathbf{a}_I \in R} Q,$$

which completes the proof. □

Lemma 5. *Let R be a non-affine relation and $a \in \{0,1\}$. Then $\langle R, \mathsf{NEQ}, \delta_a\rangle_{\max} = II_2$.*

Proof. By Lemma 4 it suffices to prove that one of $\mathsf{IMP}, \mathsf{OR}$, or NAND belongs to $\langle R, \mathsf{NEQ}, \delta_a\rangle_{\max}$. Observe first that we can always assume that the all-zero tuple $\mathbf{a}_\varnothing \in R$. Indeed, if for some $I \subseteq [n]$ we have $\mathbf{a}_I \in R$ then the relation

$$R'(x_1, \ldots, x_n) = \exists_{\max}(z_i)_{i \in I} \left(R(x_1, \ldots, x_n) \wedge \bigwedge_{i \in I} \mathsf{NEQ}(z_i, x_i) \right)$$

contains $\mathbf{a}_\varnothing$. As $R \notin IL_2$, by Lemma 4.10 of [13], there are tuples $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{d} = \mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \notin R$. (Equivalently, R is not invariant under $x \oplus y \oplus z$.) Observing that $\mathbf{e} \in R$ if and only if $\mathbf{e} \oplus \mathbf{a}_I \in R'$, we have that $\mathbf{a} \oplus \mathbf{a}_I, \mathbf{b} \oplus \mathbf{a}_I, \mathbf{c} \oplus \mathbf{a}_I \in R'$, but $\mathbf{d} \oplus \mathbf{a}_I = (\mathbf{a} \oplus \mathbf{a}_I) \oplus (\mathbf{b} \oplus \mathbf{a}_I) \oplus (\mathbf{c} \oplus \mathbf{a}_I) \notin R$. Hence R' is not affine as well. Also, if $b \in \{0,1\}$ is such that $\{0,1\} = \{a,b\}$ then by Lemma 3(2) $\delta_0, \delta_1 \in \langle R, \mathsf{NEQ}, \delta_a\rangle_{\max}$.

Again we use Lemma 4.10 of [13] to find to find tuples $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{d} = \mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \notin R$. Note that $\mathbf{a}$ can be chosen to be the all-zero tuple $\mathbf{a}_\varnothing$. After rearranging variables these tuples can be represented as follows

$\mathbf{a}$	$0\ldots0$	$0\ldots0$	$0\ldots0$	$0\ldots0$	$\in R$
$\mathbf{b}$	$0\ldots0$	$0\ldots0$	$1\ldots1$	$1\ldots1$	$\in R$
$\mathbf{c}$	$0\ldots0$	$1\ldots1$	$0\ldots0$	$1\ldots1$	$\in R$
$\mathbf{d}$	$0\ldots0$	$1\ldots1$	$1\ldots1$	$0\ldots0$	$\notin R$
	$x\ldots x$	$y\ldots y$	$z\ldots z$	$t\ldots t$	

Denote by R' the relation obtained from R by identifying variables as shown in the last row of the table. Relation R' contains tuples $(0,0,0,0),(0,0,1,1)$, $(0,1,0,1)$ but does not contain $(0,1,1,0)$, and so does not belong to IL_2. Replacing R' with

$$R''(x,y,z) = \exists_{\max} t(R(t,x,y,z) \wedge \delta_0(t)),$$

we obtain a relation R'' such that $(0,0,0),(0,1,1),(1,0,1) \in R''$ but $(1,1,0) \notin R''$.

We now proceed depending on which of the 4 remaining tuples (a) $(1,0,0)$, (b) $(0,1,0)$, (c) $(0,0,1)$, and (d) $(1,1,1)$ relation R'' contains. If it contains none of (a)–(d) then $\mathsf{NAND}(x,y) = \exists_{\max} z R''(x,y,z)$. If it contains (a) or (b) but not (d) then NAND is obtained by identifying y and z, or x and z, respectively. If R'' contains (c) but not (d) then $\mathsf{NAND}(x,y) = \exists_{\max} z(R''(x,y,z) \wedge \delta_1(z))$. If it contains (d) but not (a) then $\mathsf{IMP}(x,y) = R''(x,y,y)$. In the case R'' contains (a), (d), but does not contain (b) $\mathsf{IMP} = R''(y,x,y)$. If R'' contains (a), (d), and (b) $\mathsf{OR}(x,y) = \exists_{\max} z(R''(x,y,z) \wedge \delta_1(z))$. Finally, if the relation contains all of (a)–(d) $\mathsf{IMP}(x,y) = R''(x,x,y)$. □

Next we show that every max-co-clone is a subset of IL_2, IN_2, IV_2, or IE_2.

Lemma 6. *Let Γ be a set of relations, which is not affine, monotone, or self-complement. Then $\langle\Gamma\rangle_{\max} = II_2$.*

Proof. Let $R(x_1,\dots,x_n) \in \Gamma$ be a non-self-complement relation. Then after suitable rearrangement of variables there is $i \in \{0,\dots,n\}$ such that $\mathbf{a}_{[i]} \in R$, while $\mathbf{a}_{[n]-[i]} \notin R$. If $0 < i < n$ then identifying variables $x_1,\dots,x_i$ and $x_{i+1},\dots,x_n$ we obtain a binary relation R' that contains $(1,0)$ but does not contain $(0,1)$. As is easily seen either $\exists_{\max} x R'$ or $\exists_{\max} y R'$ is a constant relation. In the case $i=0$ or $i=n$, identifying all variables of R we obtain a constant relation. Thus either $\delta_0 \in \langle\Gamma\rangle_{\max}$ or $\delta_1 \in \langle\Gamma\rangle_{\max}$.

Suppose $\delta_1 \in \langle\Gamma\rangle_{\max}$. The case $\delta_0 \in \langle\Gamma\rangle_{\max}$ is similar. By Lemma 5.30 of [13] for any non-affine relation $R \in \Gamma$, the set $\langle R,\delta_1\rangle \subseteq \langle R,\delta_1\rangle_{\max}$ contains one of the following relations: $\mathsf{OR}, \mathsf{IMP}, \mathsf{NAND}$. If $\mathsf{NAND} \in \langle R,\delta_1\rangle_{\max}$ then $\delta_0(x) = \mathsf{NAND}(x,x)$, and we can make all the arguments below for δ_0 and NAND. Therefore we have two cases to consider. Suppose first that $\mathsf{OR} \in \langle R,\delta_1\rangle_{\max}$. There is a relation $Q \in \Gamma$ that is not invariant under the $\vee$ operation. Therefore for some tuples $\mathbf{a},\mathbf{b} \in Q$ the tuple $\mathbf{a} \vee \mathbf{b}$ does not belong to Q. After an appropriate rearrangement of variables these tuples can be represented as follows

$\mathbf{a}$	$0\dots0$	$0\dots0$	$1\dots1$	$1\dots1$	$\in Q$
$\mathbf{b}$	$0\dots0$	$1\dots1$	$0\dots0$	$1\dots1$	$\in Q$
$\mathbf{d}$	$0\dots0$	$1\dots1$	$1\dots1$	$1\dots1$	$\notin Q$
	$x\dots x$	$y\dots y$	$z\dots z$	$t\dots t$	

Denote by Q' the relation obtained from Q by identifying variables as shown in the last row of the table. Relation Q' contains tuples $(0,0,1,1),(0,1,0,1)$ but does not contain $(0,1,1,1)$. Then, relation $Q''(x,y,z) = \exists_{\max} t(Q'(x,y,z,t) \wedge \delta_1(t) \wedge \mathsf{OR}(y,z))$ contains tuples $(0,0,1),(0,1,0)$ but does not contain $(0,1,1)$, $(0,0,0),(1,0,0)$. We have several cases depending on the 3 remaining tuples (a) $(1,1,0)$, (b) $(1,0,1)$, (c) $(1,1,1)$. If none of (a)–(c) is in Q'' then $\mathsf{NEQ}(x,y) = \exists_{\max} z Q''(z,x,y)$. If Q'' contains (a) but not (c) (or (b) but not (c)), then $\mathsf{NEQ}(x,y) = Q''(x,x,y)$ (respectively, $\mathsf{NEQ}(x,y) = Q''(x,y,x)$). If it contains (c) but does not contain (a) and (b) then $\mathsf{IMP}(x,y) = \exists_{\max} z\, Q''(x,y,z)$. If Q'' contains both (b) and (c) then $\mathsf{IMP}(x,y) = \exists_{\max} z(Q''(y,x,z) \wedge \delta_1(z))$. Finally if Q'' contains (a),(c), but not (b), then $\mathsf{IMP}(x,y) = \exists_{\max} z(Q''(y,z,x) \wedge \delta_1(z))$.

In either case $\langle\Gamma\rangle_{\max}$ contains a constant relation, either NEQ or IMP, and contains one of $\mathsf{OR}, \mathsf{IMP}, \mathsf{NAND}$. If it contains NEQ, we are done by Lemma 4. So suppose $\mathsf{IMP} \in \langle\Gamma\rangle_{\max}$. Then we also have $\delta_0, \delta_1 \in \langle\Gamma\rangle_{\max}$. Since Γ is not monotone, as before we can derive relations $S_1, S_2 \in \langle\Gamma\rangle_{\max}$ such that $(0,0,1,1),(0,1,0,1) \in S_1, S_2$, but $(0,1,1,1) \notin S_1$, $(0,0,0,1) \notin S_2$. Now it is easy to see that $\mathsf{NEQ} = S'_1 \wedge S'_2$, where $S'_i(x,y) = \exists_{\max} z \exists_{\max} t(S_i(z,x,y,t) \wedge \delta_0(z) \wedge \delta_1(t)$. □

4.2. Affine relations. Recall that the set of affine relations, that is, (n-ary) relations that can be represented as the set of solutions to a system of linear equations over GF(2), is denoted by IL_2. The next lemma follows from basic linear algebra, as sets of extensions of tuples are cosets of the same vector subspace. For the sake of completeness we give a proof of this lemma.

Lemma 7. *Let R be an (n-ary) affine relation. Then for any $I \subseteq [n]$ any two tuples $\mathbf{a}, \mathbf{b} \in \mathrm{pr}_I R$ have the same number of extensions to tuples from R.*

Proof. Let R be the set of solutions of a system of linear equations $A \cdot \mathbf{x} = \mathbf{c}$, where A is a $\ell \times n$-matrix over $GF(2)$, $\mathbf{x} = (x_1, \ldots, x_n)^\top$, and $\mathbf{c} \in \{0,1\}^\ell$. Without loss of generality $I = [k]$. Then A can be represented as $A = [A_1 \mid A_2]$, where A_1 is a $\ell \times k$-matrix and A_2 is a $\ell \times (n-k)$-matrix; $\mathbf{x}$ can be represented as $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)^\top$, where $\mathbf{x}^1 = (x_1, \ldots, x_k)$, $\mathbf{x}^2 = (x_{k+1}, \ldots, x_n)$. Fix $\mathbf{a} \in \mathrm{pr}_{[k]} R$ and set $\mathbf{c_a} = \mathbf{c} \oplus (A_1 \cdot \mathbf{a})$. The set of extensions of $\mathbf{a}$ is the set of solutions of the system $A_2 \cdot \mathbf{x}^2 = \mathbf{c_a}$. Clearly, the number of solutions this system does not depend on $\mathbf{a}$, provided the system is consistent. □

Lemma 8. *Let $\Gamma \subseteq IL_2$. Then Γ is a max-co-clone if and only if it is a co-clone.*

Proof. Lemma 7 implies that for any (n-ary) relation $R \in IL$ and any set $J = \{i_1, \ldots, i_k\} \subseteq [n]$ the max-quantification $\exists_{\max}(x_{i_1}, \ldots, x_{i_k})$ is equivalent to a sequence of ordinary existential quantifiers $\exists x_{i_1} \ldots \exists x_{i_k}$. □

4.3. Monotone relations. Recall that a relation is said to be monotone if it is invariant with respect to $\wedge$ or $\vee$. In this section we consider relations invariant under $\vee$. A proof in the case of relations invariant under $\wedge$ is similar. A monotone relation is called *nontrivial* if it does not belong to IR_2.

Lemma 9. *Let R be a nontrivial relation invariant under $\vee$. Then either* $\mathsf{IMP} \in \langle R\rangle_{\max}$*, or* $\mathsf{OR} \in \langle R\rangle_{\max}$*. In particular, if the all-zero tuple belongs to* R *then* $\mathsf{IMP} \in \langle R\rangle_{\max}$.

Proof. Observe that R is not self-complement, because as it follows from [22] (see also Fig. 1) all self complement monotone relations are trivial. Also if the all-one tuple does not belong to R, since R is invariant under $\vee$, some variables of R equal 0 in all tuples from R. Such variables can be quantified away, and the resulting relation is nontrivial as R is nontrivial. We may assume the all-one tuple is in R.

Suppose first that the all-zero tuple belongs to R. Therefore there is a tuple $\mathbf{a}_I \in R$, $I \neq [\mathsf{ar}(R)]$, such that its complement does not belong to R. After a suitable rearrangement of variables $\mathbf{a} = (0,\dots,0,1,\dots,1)$. Identify variables that take 1 in $\mathbf{a}$ and also variables that take 0 in $\mathbf{a}$. The resulting relation is IMP.

Suppose now that the all-zero tuple does not belong to R. Then $\delta_1(x) = R(x,\dots,x)$. We also assume that R is a nontrivial relation of the minimal arity from $\langle R\rangle_{\max}$. Let $x_1,\dots,x_n$ be the variables R depends on. We introduce a partial order on $[n]$ as follows: $i \le_R j$ iff for any $\mathbf{a} \in R$ $\mathbf{a}[i] = 1$ implies $\mathbf{a}[j] = 1$. If $i \le_R j$ for no $i,j \in [n]$, then for any $i \in [n]$ $R' = \exists_{\max} x_i (R(x_1,\dots,x_n) \wedge \delta_1(x_i))$ is a trivial relation, none of its projections equal $\{1\}$, and therefore the all-zero tuple belongs to R'. Hence $\mathbf{a}_{\{i\}} \in R$ where $\mathbf{a}_{\{i\}}[i] = 1$ and $\mathbf{a}_{\{i\}}[j] = 0$ for $j \neq i$. Since R is invariant under $\vee$, this implies that $R = \mathsf{OR}^n$, and $\mathsf{OR} \in \langle R\rangle_{\max}$ by Lemma 3(4).

Next, consider the case when $i \le_R j$ for some $i,j \in [n]$. This means there are tuples $\mathbf{a},\mathbf{b},\mathbf{c} \in R$ such that $\mathbf{a}[i] = \mathbf{a}[j] = 0$ (since the projection of R on each variable is $\{0,1\}$), $\mathbf{b}[i] = 0$, $\mathbf{b}[j] = 1$ (due to the minimality of R, there must be a tuple $\mathbf{b}$ with $\mathbf{b}[i] \neq \mathbf{b}[j]$), and $\mathbf{c}$ is the all-one tuple, in particular $\mathbf{c}[i] = \mathbf{c}[j] = 1$. Moreover, as R is invariant under $\vee$, we may assume that $\mathbf{b}[\ell] = 1$ whenever $\mathbf{a}[\ell] = 1$. After rearranging variables these tuples can be represented as follows

$$\begin{array}{c|ccc|c}
\mathbf{a} & 0\dots0 & 0\dots0 & 1\dots1 & \in R\\
\mathbf{b} & 0\dots0 & 1\dots1 & 1\dots1 & \in R\\
\mathbf{c} & 1\dots1 & 1\dots1 & 1\dots1 & \in R\\
\hline
 & x\dots x & y\dots y & z\dots z &
\end{array}$$

Denote by R' the relation obtained from R by identifying variables as shown in the last row of the table. Relation R' contains tuples $\mathbf{a}' = (0,0,1), \mathbf{b}' = (0,1,1), \mathbf{c}' = (1,1,1)$. Observe that for no $\mathbf{d} \in R'$ we have $\mathbf{d}[1] = 1$ and $\mathbf{d}[2] = 0$. Therefore $\mathsf{IMP}(x,y) = \exists_{\max} u (R'(x,y,u) \wedge \delta_1(u))$. □

We first study max-co-clones not containing OR. By Lemma 3(1) and [14] (see also Table 1) $\langle \mathsf{IMP}\rangle_{\max} = IM_2$.

Lemma 10. *IM_2, IR_2, IR_0, IR_1 are max-co-clones.*

Proof. Since IR_2, IR_0, IR_1 essentially contain only unary relations, the lemma for these co-clones is straightforward.

For IM_2 the result actually follows from Lemma 4.1 of [12]. However, as [12] uses a different framework, we give a short proof of this result here. Our proof can be derived from the one from [12]. Observe first that IMP satisfies the property of log-supermodularity. A function $f\colon \{0,1\}^n \to \mathbb{R}$ is said to be *log-supermodular* if for any $\mathbf{a}, \mathbf{b}$

$$f(\mathbf{a}) \cdot f(\mathbf{b}) \le f(\mathbf{a} \vee \mathbf{b}) \cdot f(\mathbf{a} \wedge \mathbf{b}).$$

Here $\wedge$ and $\vee$ denote componentwise conjunction and disjunction. This definition can be extended to relations if they are treated as predicates, that is, functions with values $0, 1$. As is easily seen, a relation is log-supermodular if and only if it is invariant under $\wedge$ and $\vee$. First we show that if Γ is a set of log-supermodular relations then every relation from $\langle\Gamma\rangle_{\max}$ is log-supermodular. The property of log-supermodularity is obviously preserved by manipulations with variables and conjunction, because it is equivalent to the existence of certain polymorphisms. Suppose $R(x_1, \dots, x_n, y_1, \dots, y_m)$ is log-supermodular and $Q(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m)R(x_1, \dots, x_n, y_1, \dots, y_m)$. We associate every tuple $(\mathbf{a}, \mathbf{b}) \in \{0,1\}^{n+m}$ with the set of ones in this tuple, and therefore can view R as a function on the power set of $[n+m]$. Take $\mathbf{a}, \mathbf{a}' \in \{0,1\}^n$ and prove that $Q(\mathbf{a}) \cdot Q(\mathbf{a}') \le Q(\mathbf{a} \vee \mathbf{a}') \cdot Q(\mathbf{a} \wedge \mathbf{a}')$. Let A be the set of tuples of the form $(\mathbf{a}, \mathbf{b}) \in \{0,1\}^{n+m}$ and A' the set of tuples of the form $(\mathbf{a}', \mathbf{b}) \in \{0,1\}^{n+m}$ viewed as subsets of $[n+m]$. Also, let $R(C) = \sum_{(\mathbf{c},\mathbf{d})\in C} R(\mathbf{c}, \mathbf{d})$ for a subset C of the power set of$[n+m]$ and $f(x_1, \dots, x_n) = \sum_{y_1,\dots,y_m} R(x_1, \dots, x_n, y_1, \dots, y_n)$. Denote by $A \vee A'$ and $A \wedge A'$ the sets $A \vee A' = \{\mathbf{c} \vee \mathbf{c}' \mid \mathbf{c} \in A \text{ and } \mathbf{c}' \in A'\}$ and $A \wedge A' = \{\mathbf{c} \wedge \mathbf{c}' \mid \mathbf{c} \in A \text{ and } \mathbf{c}' \in A'\}$. Note that $f(\mathbf{a} \vee \mathbf{a}') = R(A \vee A')$ and $f(\mathbf{a} \wedge \mathbf{a}) = R(A \wedge A')$. Since R is log-supermodular, we know that $R(\mathbf{c}, \mathbf{d}) \cdot R(\mathbf{c}', \mathbf{d}') \le R(\mathbf{c} \vee \mathbf{c}', \mathbf{d} \vee \mathbf{d}') \cdot R(\mathbf{c} \wedge \mathbf{c}', \mathbf{d} \wedge \mathbf{d}')$ for all $(\mathbf{c}, \mathbf{d}), (\mathbf{c}', \mathbf{d}') \in \{0,1\}^{n+m}$. Thus, applying the Ahlswede-Daykin Four-Functions Theorem [1] with $\alpha = \beta = \gamma = \delta = R$,

$$f(\mathbf{a}) \cdot f(\mathbf{a}') = R(A) \cdot R(A') \le R(A \vee A') \cdot R(A \wedge A') = f(\mathbf{a} \vee \mathbf{a}') \cdot f(\mathbf{a} \wedge \mathbf{a}'). \quad (4.1)$$

Now suppose $\mathbf{a}, \mathbf{a}' \in Q$. This means that $f(\mathbf{a}) = f(\mathbf{a}')$ and this number is the maximal number of extensions of a tuple from $\{0,1\}^n$ to tuples from R. By (4.1) $f(\mathbf{a} \vee \mathbf{a}'), f(\mathbf{a} \wedge \mathbf{a}') \ne 0$ and either $f(\mathbf{a} \vee \mathbf{a}') \ge f(\mathbf{a})$ or $f(\mathbf{a} \wedge \mathbf{a}') \ge f(\mathbf{a}')$. However, as $f(\mathbf{a})$ is the maximal number of extensions, strict inequality is impossible, and we get $f(\mathbf{a} \vee \mathbf{a}') = f(\mathbf{a} \wedge \mathbf{a}') = f(\mathbf{a})$. Therefore $(\mathbf{a} \vee \mathbf{a}'), (\mathbf{a} \wedge \mathbf{a}') \in Q$, and so $Q(\mathbf{a}) \cdot Q(\mathbf{a}') \le Q(\mathbf{a} \vee \mathbf{a}') \cdot Q(\mathbf{a} \wedge \mathbf{a}')$.

Thus $\langle IM_2\rangle_{\max}$ contains only log-supermodular relations. However, as it was observed above, log-supermodularity of relations is equivalent to invariance under $\wedge$ and $\vee$. Since, IM_2 is the class of all relations invariant under this two operations, we have $\langle IM_2\rangle_{\max} = IM_2$. □

Lemma 11. *Let $R \notin IM_2$. Then $\langle R, \mathsf{IMP}\rangle_{\max} = II_2$.*

Proof. If R is not invariant under $\vee$ and $\wedge$ then the result follows by Lemma 6, since IMP is not affine or self-complement. Suppose R is invariant with respect $\vee$.

Recall that a relation $Q(x_1, \dots, x_n)$ is called *2-decomposable* if any tuple $\mathbf{a}$ such that $(\mathbf{a}[i], \mathbf{a}[j]) \in \mathrm{pr}_{\{i,j\}}Q$ for all $i, j \in [n]$ belongs to Q.

Case 1. R is not 2-decomposable.

Let $I \subseteq [n]$ be a minimal set such that $\mathrm{pr}_I R$ is not 2-decomposable, clearly, $|I| \geq 3$. Let $R' = \mathrm{pr}_I R$. There is $\mathbf{a} \in \{0,1\}^{|I|}$ such that for any $i \in I$ $\mathbf{a}_i \in R'$, where $\mathbf{a}_i$ denotes the tuple such that $\mathbf{a}_i[i] \neq \mathbf{a}[i]$ and $\mathbf{a}_i[j] = \mathbf{a}[j]$ for $i \neq j$. Choose $i_1, i_2, i_3 \in I$, and set $I - \{i_1, i_2, i_3\} = \{i_4, \dots, i_k\}$ and

$$Q = \exists_{\max} x_{i_4} \dots \exists_{\max} x_{i_k} (R(x_1, \dots, x_n) \wedge \delta_{\mathbf{a}[i_4]}(x_{i_4}) \wedge \dots \wedge \delta_{\mathbf{a}[i_k]}(x_{i_k})).$$

As is easily seen, Q is not 2-decomposable, and moreover, $\mathrm{pr}_{\{i_1,i_2,i_3\}}Q$ is not 2-decomposable. Let $Q' = \mathrm{pr}_{\{i_1,i_2,i_3\}}Q$. There is $\mathbf{a} \in \{0,1\}^3$ such that for any $i \in I$ $\mathbf{a}_i \in Q'$, where $\mathbf{a}_i$ denotes the tuple such that $\mathbf{a}_i[i] \neq \mathbf{a}[i]$ and $\mathbf{a}_i[j] = \mathbf{a}[j]$ for $i \neq j$. Observe that there are at most one 1 among components of $\mathbf{a}$. Indeed, if, say, $\mathbf{a} = (1,1,0)$ then $\mathbf{a} = \mathbf{a}_1 \vee \mathbf{a}_2 \in Q'$. Suppose first that $\mathbf{a}$ is the all-zero tuple. Then after rearranging variables these tuples can be represented as follows

$\mathbf{a}_1$	1	0	0	$0\dots0$	$0\dots0$	$0\dots0$	$1\dots1$	$0\dots0$	$1\dots1$	$1\dots1$	$1\dots1$	$\in R$
$\mathbf{a}_2$	0	1	0	$0\dots0$	$0\dots0$	$1\dots1$	$0\dots0$	$1\dots1$	$0\dots0$	$1\dots1$	$1\dots1$	$\in R$
$\mathbf{a}_3$	0	0	1	$0\dots0$	$1\dots1$	$0\dots0$	$0\dots0$	$1\dots1$	$1\dots1$	$0\dots0$	$1\dots1$	$\in R$
$\mathbf{a}$	0	0	0	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$\notin R$
	x	y	z	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	

Denote by Q'' the relation obtained from Q by identifying variables as shown in the last row of the table. Then set

$$S(x,y,z,t,u,v) = \exists_{\max} t_1 \exists_{\max} t_8 (Q''(x,y,z,t_1,z,y,x,t,u,v,t_8) \wedge \delta_0(t_1) \wedge \delta_1(t_8)).$$

Relation S contains tuples $\mathbf{b}_1 = (1,0,0,1,1,0), \mathbf{b}_2 = (0,1,0,1,0,1), \mathbf{b}_3 = (0,0,1,0,1,1)$ but does not contain $(0,0,0,a,b,c)$ for any $a,b,c \in \{0,1\}$. Next we set $S'(x,y,z) = \exists_{\max} t,u,v(S(x,y,z,t,u,v) \wedge \delta_1(t) \wedge \delta_1(u) \wedge \delta_1(v))$. Since S is invariant under $\vee$, it contains $\mathbf{b}_1 \vee \mathbf{b}_2, \mathbf{b}_2 \vee \mathbf{b}_3, \mathbf{b}_3 \vee \mathbf{b}_1$, and therefore S' contains tuples $(1,1,0),(1,0,1),(0,1,1),(1,1,1)$, but does not contain $(0,0,0)$. Let also $S''(x,y,z) = S'(x,y,z) \wedge S'(z,x,y) \wedge S'(y,z,x)$. As is easily seen S'' is either OR^3 or $\{(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$. In the former case we are done, while in the latter case we just observe that $\mathsf{OR}(x,y) = \exists_{\max} z(S''(x,y,z) \wedge \delta_1(z))$.

Now suppose $\mathbf{a} = (0,0,1)$. As before we can construct a relation S such that $\mathbf{b}_1 = (0,0,0,1,1,1), \mathbf{b}_2 = (0,1,1,0,0,1), \mathbf{b}_3 = (1,0,1,0,1,0)$ belong to S, but $(0,0,1,a,b,c)$ does not belong to S for any $a,b,c \in \{0,1\}$. Since R is invariant under $\vee$ tuples $\mathbf{b}_2 \vee \mathbf{b}_1, \mathbf{b}_3 \vee \mathbf{b}_1, \mathbf{b}_2 \vee \mathbf{b}_3 \vee \mathbf{b}_1$ also belong to S. Hence $(0,0,0,1),(0,1,1,1),$
$(1,0,1,1),(1,1,1,1) \in S'(x,y,z,t) = S(x,y,z,t,t,t)$, and $(0,0,1,1) \notin S'$.

Therefore
$\mathsf{OR}(x,y) = \exists_{\max} z \exists_{\max} t (S'(x,y,z,t) \wedge \delta_1(z) \wedge \delta_1(t))$.

CASE 2. R is 2-decomposable.

Since $\langle \mathsf{IMP} \rangle_{\max}$ contains IM_2 and therefore all 2-decomposable relations whose binary projections are either trivial relations or IMP, relation R has to have a binary projection which is not one of them. As it and all its projections are invariant under $\vee$, the only nontrivial binary projections it may have are IMP and OR. Therefore for some $i,j \in [n]$ $\mathrm{pr}_{\{i,j\}} R = \mathsf{OR}$. There are $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{a}[i] = \mathbf{b}[j] = 0$ and $\mathbf{a}[j] = \mathbf{b}[i] = \mathbf{c}[i] = \mathbf{c}[j] = 1$, but for no $\mathbf{d} \in R$ $\mathbf{d}[i] = \mathbf{d}[j] = 0$. Note also that $\mathbf{c}$ can be replaced with $\mathbf{c} \vee \mathbf{a} \vee \mathbf{b}$. After rearranging variables these tuples can be represented as follows

$\mathbf{a}$	0	1	$0\dots0$	$0\dots0$	$0\dots0$	$1\dots1$	$1\dots1$	$\in R$
$\mathbf{b}$	1	0	$0\dots0$	$0\dots0$	$1\dots1$	$0\dots0$	$1\dots1$	$\in R$
$\mathbf{c}$	1	1	$0\dots0$	$1\dots1$	$1\dots1$	$1\dots1$	$1\dots1$	$\in R$
$\mathbf{d}$	0	0	$*$	$*$	$*$	$*$	$*$	$\notin R$
	x	y	$z_1\dots z_1$	$z_2\dots z_2$	$z_3\dots z_3$	$z_4\dots z_4$	$z_5\dots z_5$	

Denote by R' the relation obtained from R by identifying variables as shown in the last row of the table. Then set

$$Q(x,y,z) = \exists_{\max} z_1 \exists_{\max} z_5 (Q(x,y,z_1,z,x,y,z_5) \wedge \delta_0(z_1) \wedge \delta_1(z_5)).$$

Relation Q contains tuples $(0,1,0),(1,0,0),(1,1,1)$, and $(1,1,0)$, as it is invariant under $\vee$, but does not contain $(0,0,a)$ for any $a \in \{0,1\}$. Then $\mathsf{OR}(x,y) = \exists_{\max} z (Q(x,y,z) \wedge \delta_0(z))$. $\square$

Next we consider max-co-clones containing OR, but not IMP.

Let $R(x_1,\dots,x_n)$ be a relation. If $i,j \in [n]$ are such that $\mathbf{a}[i] = \mathbf{a}[j]$ for any $\mathbf{a} \in R$, we write $i \sim_R j$. Clearly, $\sim_R$ is an equivalence relation on $[n]$; its class containing i will be denoted by $S_R(i)$ or $S_R(x_i)$. Let also O_R denote the set of variables x_j such that there is $\mathbf{b} \in R$ with $\mathbf{b}[j] = 1$. An n-tuple $\mathbf{a}$ is said to be $\sim_R$*-conforming* if (a) $\mathbf{a}[i] = \mathbf{a}[j]$ whenever $i \sim_R j$, and (b) $\mathbf{a}[i] = 0$ whenever $i \notin O_R$. When considered ordered with respect to the natural componentwise order $(0 \le 1)$, $\sim_R$-conforming tuples form a poset isomorphic to $\{0,1\}^{k_R}$, where k_R is the number of $\sim_R$-classes except for the class $[n] - O_R$. In what follows $\le$ and $<$ will denote relations on the set of $\sim_R$-conforming tuples for appropriate R. We say that a relation $R(x_1,\dots,x_n)$ satisfies the *filter property* if for any $\mathbf{a} \in R$ any $\sim_R$-conforming tuple $\mathbf{a}'$ with $\mathbf{a} \le \mathbf{a}'$ belongs to R. The filter property implies that if R is considered as a subset of the ordered set $\{0,1\}^{k_R}$, then it is an order filter in this set. In particular, it is completely determined by its minimal (with respect to $\le$) elements, or equivalently by the maximal elements not belonging to R. We say that R satisfies the r-*filter property*, if it satisfies the filter property, and every maximal tuple not belonging to R contains zeros in at most r classes of $\sim_R$ from O_R.

Lemma 12. *(1) A relation R belongs to IS_{12} if and only if it satisfies the filter property.*
(2) A relation R belongs to IS_{12}^r if and only if it satisfies the r-filter property.

Proof. (1) Suppose $R(x_1, \ldots, x_n) \in IS_{12}$. Then by Proposition 3 of [14] relations $\mathsf{EQ}, \delta_0, \delta_1$ and OR^m, $m \geq 2$ form a plain basis of IS_{12}, and therefore R can be represented by a conjunctive formula Φ containing variables $x_1, \ldots, x_n$, relations $\mathsf{EQ}, \delta_0, \delta_1$, and OR^m. Let $\mathbf{a} \in R$, and let $\mathbf{b}$ be a $\sim_R$-conforming tuple such that $\mathbf{a} \leq \mathbf{b}$. We show that it belongs to R. Clearly, $\mathbf{b}$ satisfies all the δ_1 relations. Also, it satisfies all the δ_0 relations, if $\delta_0(x_j)$ belongs to Φ then $j \notin O_R$ and $\mathbf{b}[j] = 0$. Since $\mathbf{b}$ contains 0 only in the positions $\mathbf{a}$ does, every relation OR^m is satisfied by $\mathbf{b}$. Finally, if $\mathsf{EQ}(x_{j_1}, x_{j_2})$ belongs to Φ, then $j_1 \sim_R j_2$, therefore all the EQ relations remain satisfied by $\mathbf{b}$.

Suppose now that $R(x_1, \ldots, x_n)$ satisfies the filter property. Let $W, Z \subseteq [n]$ be the sets of variables such that for all $\mathbf{a} \in R$ $\mathbf{a}[i] = 1$ (respectively, $\mathbf{a}[i] = 0$) for $i \in W$ ($i \in Z$). Let also $\mathbf{a}_1, \ldots, \mathbf{a}_\ell$ be the maximal tuples not from R. By Z_j we denote the set of $i \in O_R$ such that $\mathbf{a}_j[i] = 0$. Suppose Z_j contains elements from m_j classes of $\sim_R$. We construct a formula Φ using variables $x_1, \ldots, x_n$ and relations $\mathsf{EQ}, \delta_0, \delta_1, \mathsf{OR}^m$, and prove that it represents R. Formula Φ includes
(1) $\delta_0(x_i)$ for each $i \in Z$ and $\delta_1(x_i)$ for each $i \in W$;
(2) $\mathsf{EQ}(x_i, x_j)$ for any pair x_i, x_j, $i \sim_R j$;
(3) $\mathsf{OR}^{m_j}(x_{i_1}, \ldots, x_{i_{m_j}})$ for any $\mathbf{a}_j$, $j \in [\ell]$, and any $i_1, \ldots, i_{m_j}$ such that $i_1, \ldots, i_{m_j}$ belong to different $\sim_R$-classes from Z_j.
Let the resulting relation be denoted by Q. By what is proved above Q satisfies the filter property. It is straightforward that $O_Q = O_R$ and the maximal tuples not in Q are the same as those of R. Therefore $Q = R$.

(2) Suppose first that R satisfies the r-filter property. Then it can be represented by a formula Φ as in part (1) and for every relation OR^m used $m \leq r$. Therefore $R \in IS_{12}^r$.

Let now $R(x_1, \ldots, x_n) \in IS_{12}^r$, and therefore can be represented by a formula Φ in $x_1, \ldots, x_n$, and relations $\mathsf{EQ}, \delta_0, \delta_1$, and OR^m for $m \leq r$. We need to study the structure of maximal tuples from the complement of R. We use the notation from part (1). Let $\mathbf{a}$ be such a tuple. It is $\sim_R$-conforming, so, $\mathbf{a}[i] = 0$ for all $i \in Z$, and $\mathbf{a}[i] = \mathbf{a}[j]$ for any $i \sim_R j$. This means that $\mathbf{a}$ satisfies all the δ_0 and EQ relations in Φ. If $\mathbf{a}$ violates a relation δ_1 and there is $i \notin W$ such that $\mathbf{a}[i] = 0$ then $\mathbf{a}$ is not maximal in the complement of R. Therefore $\mathbf{a}[i] = 0$ if and only if $i \in W$, and W is a single $\sim_R$-class. Suppose $\mathbf{a}$ violates a relation $\mathsf{OR}^m(x_{i_1}, \ldots, x_{i_m})$, and let $D = S(i_1) \cup \ldots \cup S(i_m)$. If there is $i \in O_R - D$ such that $\mathbf{a}[i] = 0$ then the tuple $\mathbf{b}$ given by $\mathbf{b}[j] = 1$ if $j \in S(i)$ and $\mathbf{b}[j] = \mathbf{a}[j]$ otherwise does not belong to R and $\mathbf{a} < \mathbf{b}$, a contradiction. Therefore the set of zeros of any maximal tuple from the complement of R spans at most r classes of $\sim_R$, as required. □

Let Γ be a max-co-clone of monotone relations. By $\mathsf{or}(\Gamma)$ we denote the maximal m such that $\mathsf{OR}^m \in \langle\Gamma\rangle_{\max}$. If a maximal number m does not exist we set $\mathsf{or}(\Gamma) = \infty$.

Lemma 13. *For any set $\Gamma \subseteq IS_{12}$ of monotone relations*
$\langle\Gamma\rangle_{\max} = \langle\{\mathsf{OR}^m \mid m \le \mathsf{or}(\Gamma)\}\rangle_{\max}$ *or*
$\langle\Gamma\rangle_{\max} = \langle\{\mathsf{OR}^m \mid m \le \mathsf{or}(\Gamma)\}\rangle_{\max} \cup \{\delta_0\}$.

Proof. It suffices to show that if Γ contains a relation R with a maximal tuple that spans k classes of $\sim_R$, then $\mathsf{OR}^k \in \langle\Gamma\rangle_{\max}$. Let R be such a relation. Applying $\exists_{\max}$ we may assume that the sets W and Z for R are empty; applying identification of variables we may assume that every set $S(i)$ is a singleton. Now let $\mathbf{a}$ be a maximal tuple that spans k classes of $\sim_R$, and I the set of positions such that $\mathbf{a}[i] = 0$ if and only if $i \in I$; without loss of generality assume $I = [k]$. Since R satisfies the filter property, for any $(b_1, \dots, b_k) \in \mathrm{pr}_{[k]} R$ the tuple $(b_1, \dots, b_k, 1, \dots, 1)$ belongs to R. Observe that identifying all the variables of R we make sure that $\delta_1 \in \langle\Gamma\rangle_{\max}$. Therefore the relation given by

$$Q(x_1, \dots, x_k) = \exists_{\max}(x_{k+1}, \dots, x_n)(R(x_1, \dots, x_n) \wedge \delta_1(x_{k+1}) \wedge \dots \wedge \delta_1(x_n))$$

belongs to $\langle\Gamma\rangle_{\max}$. It remains to show that $Q = \mathsf{OR}^k$. By the filter property of R for any $b_1, \dots, b_k$ that are not all zeros $(b_1, \dots, b_k, 1, \dots, 1) \in R$. Therefore $(b_1, \dots, b_k) \in Q$. On the other hand, $(0, \dots, 0, 1, \dots, 1) \notin R$.

It remains to show that for any $R(x_1, \dots, x_n) \in IS_{12}$ such that $\mathbf{a}_{[n]} \notin R$ (the all-ones tuple), $\delta_0 \in \langle R\rangle_{\max}$. By the filter property of R if $\mathbf{a}_{[n]} \notin R$ there is $i \in [n]$ such that $\mathbf{a}[i] = 0$ for all $\mathbf{a} \in R$. Let $I \subseteq [n]$ be the set of all such coordinate positions; without loss of generality we may assume that $I = [m]$. Since $\delta_1 \in \langle R\rangle_{\max}$, we have

$$\delta_0(x) = \exists_{\max} y(R(x, \dots, x, y, \dots, y) \wedge \delta_1(y)),$$

where x is in the first m positions. □

Lemma 14. *Every co-clone $IS_1, IS_{12}, IS_1^r, IS_{12}^r$ for $r \in \{2, 3, \dots\}$ is a max-co-clone.*

Proof. First we show that IS_{12}, IS_{12}^r are max-co-clones. By Lemma 12 it suffices to prove that if every relation from Γ satisfies the filter or r-filter property, then so does every relation from $\langle\Gamma\rangle_{\max}$. These properties are preserved by manipulations with variables and conjunction, because IS_{12}, IS_{12}^r are co-clones. It remains to show that they are also preserved by max-quantification.

Suppose $R(x_1, \dots, x_n, y_1, \dots, y_m)$ satisfies the filter property and $Q(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m) R(x_1, \dots, x_n, y_1, \dots, y_m)$. Observe that we may assume that for any x_i the set $S(x_i)$ does not contain any variable y_j. Indeed, if $\mathbf{a}[i] = \mathbf{b}[j]$ for any assignment $(\mathbf{a}, \mathbf{b})$ that satisfies R, then we can identify these two variables, and denote the new variable by x_i. The number of extensions of any assignment to $x_1, \dots, x_n$ does not change, therefore the relation Q defined in the same way from the new relation does not change.

Choose a representation Φ of Q that uses OR^r, EQ, δ_0, δ_1. Such a representation exists as the listed relations constitute a plain basis for IS_{12} by [14] (see Table 1). Take $\mathbf{a} \in Q$ and $x_i \in O_Q$; let $\mathbf{a}'$ be a tuple such that $\mathbf{a} \leq \mathbf{a}'$. It suffices to verify that every extension $\mathbf{b}$ of $\mathbf{a}$ is also an extension of $\mathbf{a}'$. Indeed, if this is the case, since $\mathbf{a}$ has the maximum number of extensions, so does $\mathbf{a}'$, and thus $\mathbf{a}' \in Q$. Suppose $(\mathbf{a}, \mathbf{b}) \in R$. Then $(\mathbf{a}', \mathbf{b})$ satisfies every relation OR^r from Φ, as this tuple contains 1 in every position $(\mathbf{a}, \mathbf{b})$ does. It also satisfies every relation EQ, because there is no relation of the form $\mathsf{EQ}(x_\ell, y_j)$, and $\mathbf{a}'[i] = \mathbf{a}'[j]$ whenever $i \sim_R j$. Finally, δ_0 and δ_1 are also satisfied, because no value is changed in the scopes of the former, and no value is changed to 0 in the scope of the latter.

Next we prove that the number of $\sim_R$-classes spanned by zeros of maximal tuples from the complement of Q does not exceed that of R. More precisely we show that (1) $S_R(x_i) \cap \{x_1, \dots, x_n\} \subseteq S_Q(x_i)$ for any $i \in [n]$, and (2) for every maximal tuple $\mathbf{a} \notin Q$ there is $\mathbf{b} \in \{0,1\}^m$ such that $(\mathbf{a}, \mathbf{b})$ is a maximal tuple not belonging to R.

The first claim is obvious, as $Q \subseteq \mathrm{pr}_{[n]} R$ and therefore if $\mathbf{a}[i] = \mathbf{a}[j]$ for any $(\mathbf{a}, \mathbf{b}) \in R$ then $\mathbf{c}[i] = \mathbf{c}[j]$ for any $\mathbf{c} \in Q$. Observe that we may assume that $\mathrm{pr}_j R = \{0,1\}$ for any $j \in \{n+1, \dots, n+m\}$, since otherwise such a variable does not affect the number of extensions of tuples from $\mathrm{pr}_{[n]} R$. For the second claim let $\mathbf{a}$ be a maximal tuple not belonging to Q. Suppose first that $\mathbf{a} \notin \mathrm{pr}_{[n]} R$. Since for any $\mathbf{a}' \in \mathrm{pr}_{[n]} R$ the tuple $(\mathbf{a}', 1, \dots, 1)$ belongs to R, the tuple $(\mathbf{a}, 1, \dots, 1)$ is a maximal tuple not belonging to R. Next assume $\mathbf{a} \in \mathrm{pr}_{[n]} R$. Let $E(\mathbf{c})$ denote the set of extensions of a tuple $\mathbf{c} \in \mathrm{pr}_{[n]} R$ to a tuple from R. Due to the filter property of R and the assumption that no set $S(x_i)$ contains any y_j, if $\mathbf{c} \leq \mathbf{c}'$ then $E(\mathbf{c}) \subseteq E(\mathbf{c}')$. As $\mathbf{a}$ is a maximal tuple not belonging to Q, the number of extensions of any tuple $\mathbf{a}'$, $\mathbf{a} < \mathbf{a}'$, is the same, including the all-one tuple $\mathbf{a}_{[n]}$. However, for any such tuple $\mathbf{a}'$, $E(\mathbf{a}') \subseteq E(\mathbf{a}_{[n]})$ and yet $|E(\mathbf{a}')| = |E(\mathbf{a}_{[n]})|$ implying $E(\mathbf{a}') = E(\mathbf{a}_{[n]})$. Since $|E(\mathbf{a})| < |E(\mathbf{a}')|$ for any tuple $\mathbf{a}'$, $\mathbf{a} < \mathbf{a}'$, there is $\mathbf{b}$ such that $(\mathbf{a}, \mathbf{b}) \notin R$ and $(\mathbf{a}', \mathbf{b}) \in R$ for any tuple $\mathbf{a}'$, $\mathbf{a} < \mathbf{a}'$. Choose a maximal $\mathbf{b}'$, $\mathbf{b} \leq \mathbf{b}'$, with this property. We need to show that $(\mathbf{a}, \mathbf{b}')$ is a maximal tuple not belonging to R. For any $\mathbf{b}'' > \mathbf{b}'$ the tuple $(\mathbf{a}, \mathbf{b}'') \in R$, because, by the choice of $\mathbf{b}'$, it is a maximal tuple such that $(\mathbf{a}, \mathbf{b}') \notin R$. For any $\mathbf{a}'$, $\mathbf{a} < \mathbf{a}'$, the tuple $(\mathbf{a}', \mathbf{b})$ belongs to R, and therefore $(\mathbf{a}', \mathbf{b}') \in R$.

Next we show that $\langle IS_1^r \rangle_{\max} = IS_1^r$. Co-clone IS_1^r contains all relations from IS_{12}^r invariant under the constant function 1. So, we prove that any relation $R \in \langle IS_1 \rangle_{\max}$ contains the all-one tuple. Relations EQ, δ_1, and OR^r satisfy this condition. Manipulations with variables and conjunction preserves this property. It remains to verify that $\exists_{\max}$ also preserves this property in IS_{12}. Let $R(x_1, \dots, x_n, y_1, \dots, y_m) \in IS_{12}$ and $(1, \dots, 1, 1, \dots, 1) \in R$. Let also $Q(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m) R(x_1, \dots, x_n, y_1, \dots, y_m)$. As before we may assume that for any x_i the set $S(x_i)$ does not contain any variable y_j.

Then since $E(\mathbf{a}) \subseteq E(\mathbf{a}_{[n]})$, where $\mathbf{a}_{[n]}$ is the all-one tuple, for any $\mathbf{a} \in \mathrm{pr}_{[n]} R$, $\mathbf{a}_{[n]} \in Q$. □

Lemma 15. *Let $R \notin IS_{12}$, then $\langle R, \mathsf{OR}\rangle_{\max} = II_2$.*

Proof. First of all R can be assumed to be closed under $\vee$. Indeed, OR is not self-complement, affine, or closed under $\wedge$; so if R is not closed under $\vee$ the result follows from Lemma 6. We as well may assume that every unary projection of R contains two elements. Next, observe that we can also assume that for each variable x of R the set $S(x)$ contains only one element. Indeed, R can be replaced with a relation R' constructed by identifying all variables in every set of the form $S(x)$. It now suffices to verify that $R' \notin IS_{12}$ whenever $R \notin IS_{12}$. To see this note that R can be obtained from R' through adding new variables and imposing equality relations.

If R contains the all-zero tuple then by Lemma 9 $\mathsf{IMP} \in \langle R\rangle_{\max}$ and the result follows from Lemma 4.

Suppose that the all-zero tuple does not belong to R. We show that either R satisfies the filter property, and therefore belongs to IS_{12}, or there is a nontrivial relation $Q \in \langle R\rangle_{\max}$ containing the all-zero tuple. By what is proved above it implies the result.

For $\mathbf{a} \in R$ we denote by $R_{\mathbf{a}}$ the relation obtained as follows. Let $O(\mathbf{a})$ denote the set of coordinate positions in which $\mathbf{a}$ equals 1. Then

$$R_{\mathbf{a}} = \exists_{\max}(x_i)_{i\in O(\mathbf{a})}(R(x_1,\ldots,x_n \wedge \bigwedge_{i\in O(\mathbf{a})} \delta_1(x_i)).$$

If $R_{\mathbf{a}}$ is a nontrivial relation then we are done, since it does not satisfy the filter property and the all-zero tuple belongs to $R_{\mathbf{a}}$. Therefore assume that every relation $R_{\mathbf{a}}$ is trivial. Observe that since $\mathbf{a} \vee \mathbf{b} \in R$ for any $\mathbf{b} \in R$ and $\mathrm{pr}_{[n]-O(\mathbf{a})}(\mathbf{a} \vee \mathbf{b}) = \mathrm{pr}_{[n]-O(\mathbf{a})}\mathbf{b}$, we have $R_{\mathbf{a}} = \mathrm{pr}_{[n]-O(\mathbf{a})} R$. Therefore every set of the form $S(x)$ for $R_{\mathbf{a}}$ is 1-element. Hence $R_{\mathbf{a}} = \{0,1\}^{n-|O(\mathbf{a})|}$. In particular, for any $\mathbf{a} \in R$ and any $i \notin O(\mathbf{a})$ the tuple $\mathbf{b}$ obtained from $\mathbf{a}$ by changing $\mathbf{a}[i]$ to 1 belongs to R. Thus R satisfies the filter property, a contradiction. □

Proposition 16. *Every max-co-clone of monotone relations containing a nontrivial relation equals one of IS_1, IS_{12}, IS_1^i, IS_{12}^i for $i \in \{2,3,\ldots\}$, IM_2.*

Proof. By Lemmas 10 and 14 all these sets are max-co-clones. By Lemma 11 and the observation that $\langle\mathsf{IMP}\rangle_{\max} = IM_2$, max-co-clone IM_2 is the only max-co-clone containing IMP. By Lemma 15 IS_{12} is the greatest max-co-clone containing OR. Thus it remains to prove that there are no max-co-clones containing OR and different from $IS_1, IS_{12}, IS_1^i, IS_{12}^i$ for $i \in \{2,3,\ldots\}$. It follows from Lemma 13. □

4.4. Self-complement max-co-clones. In this section we consider the remaining case of self-complement max-co-clones.

Proposition 17. *There is only one max-co-clone of self-complement relations that is not a subclone of IL_2. It is IN_2, the clone of all self-complement relations.*

The proposition follows from the following four lemmas.

Lemma 18. *IN_2 is a max-co-clone.*

Proof. We need to prove that IN_2 is closed under manipulations with variables, conjunction, and max-implementation. Since IN_2 is a co-clone, it is closed under the first two operations. Let $R(x_1, \dots, x_n, y_1, \dots, y_m) \in IN_2$ and $Q(x_1, \dots, x_n) = \exists_{\max}(y_1, \dots, y_m) R(x_1, \dots, x_n, y_1, \dots, y_m)$. Let $\mathbf{a} \in Q$ and let $\neg\mathbf{a}$ denote its complement. Then for each extension $(\mathbf{a}, \mathbf{c}) \in R$ of $\mathbf{a}$ the tuple $(\neg\mathbf{a}, \neg\mathbf{c})$ belongs to R, as R is self-complement, and $(\neg\mathbf{a}, \neg\mathbf{c})$ is an extension of $\neg\mathbf{a}$. Therefore $\neg\mathbf{a}$ has the same number of extensions as $\mathbf{a}$, and so $\neg\mathbf{a} \in Q$. Thus, Q is self-complement. □

Lemma 19. *Let R be a self-complement relation that does not belong to IL_2 (that is, non-affine), then $\mathsf{Compl}_{3,0} \in \langle R \rangle_{\max}$ or $\mathsf{Compl}_{1,2} \in \langle R \rangle_{\max}$.*

Proof. Let $R(x_1, \dots, x_n)$ satisfy the conditions of the lemma. There are two cases.

Case 1. R does not contain the all-zero tuple.

Observe first that in this case $\langle R \rangle_{\max}$ contains the disequality relation. Indeed, let $\mathbf{a} \in R$ and let $I \subseteq [n]$ be the set of indices such that $\mathbf{a}[i] = 0$ if and only if $i \in I$. Since the all-zero tuple does not belong to R, $I \neq [n]$. Without loss of generality let $I = [m]$. Then it is easy to see that

$$R(\underbrace{x, \dots, x}_{m \text{ times}}, y, \dots, y)$$

is the disequality relation.

As $R \notin IL_2$, by Lemma 4.10 of [13] there are tuples $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R$ such that $\mathbf{d} = \mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \notin R$. Rearranging the variables these tuples can be represented as shown in the table below.

$\mathbf{a}$	$0\dots0$	$0\dots0$	$0\dots0$	$0\dots0$	$1\dots1$	$1\dots1$	$1\dots1$	$1\dots1$	$\in R$
$\mathbf{b}$	$0\dots0$	$0\dots0$	$1\dots1$	$1\dots1$	$0\dots0$	$0\dots0$	$1\dots1$	$1\dots1$	$\in R$
$\mathbf{c}$	$0\dots0$	$1\dots1$	$0\dots0$	$1\dots1$	$0\dots0$	$1\dots1$	$0\dots0$	$1\dots1$	$\in R$
$\mathbf{d}$	$0\dots0$	$1\dots1$	$1\dots1$	$0\dots0$	$1\dots1$	$0\dots0$	$0\dots0$	$1\dots1$	$\notin R$
	$x\dots x$	$y\dots y$	$z\dots z$	$s\dots s$	$t\dots t$	$u\dots u$	$v\dots v$	$w\dots w$	

Denote by R' the relation obtained from R by identifying variables as shown in the last row of the table, and then set

$$\begin{aligned} R''(x,y,z,t) &= \exists_{\max} s \exists_{\max} u \exists_{\max} v \exists_{\max} w (R'(x,y,z,s,t,u,v,w) \\ &\quad \wedge \mathsf{NEQ}(x,w) \wedge \mathsf{NEQ}(y,v) \wedge \mathsf{NEQ}(z,u) \wedge \mathsf{NEQ}(t,s)). \end{aligned}$$

Relation R'' contains tuples $(0,0,0,1), (0,0,1,0), (0,1,0,0)$ but does not contain $(0,1,1,1)$, and so does not belong to IL_2.

There are 16 cases depending on whether or not tuples (a) $(0,0,1,1)$, (b) $(0,1,0,1)$, (c) $(0,1,1,0)$, and (d) $(0,0,0,0)$ belong to R'' (remember, this relation is self complement). If none of them belongs to R'' then $\mathsf{Compl}_{3,0}(x,y,z) = \exists_{\max} t R''(t,x,y,z)$. Suppose first $(0,0,0,0) \notin R''$. If (a) belongs to R'' then $\mathsf{Compl}_{3,0}(x,y,z) = R''(x,x,y,z)$; if (b) is in R'' then $\mathsf{Compl}_{3,0}(x,y,z) = R''(x,y,x,z)$; finally, if (c) is in R'' then $\mathsf{Compl}_{3,0}(x,y,z) = R''(x,y,z,x)$. Suppose now (d) belongs to R. If (a) is not there then $\mathsf{Compl}_{1,2}(x,y,z) = R''(x,x,y,z)$. If (a) is also in R, then $\mathsf{Compl}_{1,2}(x,y,z) = R''(x,y,z,z)$.

Case 2. The all-zero tuple belongs to R.

Again by Lemma 4.10 of [13] there are tuples $\mathbf{a},\mathbf{b},\mathbf{c} \in R$ such that $\mathbf{d} = \mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \notin R$, but $\mathbf{a}$ can be chosen to be the all-zero tuple. Then after rearranging variables these tuples can be represented as follows

$$\begin{array}{c|cccc|c}
\mathbf{a} & 0\ldots0 & 0\ldots0 & 0\ldots0 & 0\ldots0 & \in R\\
\mathbf{b} & 0\ldots0 & 0\ldots0 & 1\ldots1 & 1\ldots1 & \in R\\
\mathbf{c} & 0\ldots0 & 1\ldots1 & 0\ldots0 & 1\ldots1 & \in R\\
\hline
\mathbf{d} & 0\ldots0 & 1\ldots1 & 1\ldots1 & 0\ldots0 & \notin R\\
\hline
 & x\ldots x & y\ldots y & z\ldots z & t\ldots t &
\end{array}$$

Denote by R' the relation obtained from R by identifying variables as shown in the last row of the table. Relation R' contains tuples $(0,0,0,0),(0,0,1,1)$, $(0,1,0,1)$ but does not contain $(0,1,1,0)$, and so does not belong to IL_2.

There are 16 cases depending on whether or not tuples (a) $(0,0,0,1)$, (b) $(0,0,1,0)$, (c) $(0,1,0,0)$, and (d) $(1,0,0,0)$ belong to R'. If none of the tuples belong to R' or all of them belong to R', then $\mathsf{Compl}_{2,1}(x,y,z) = \exists_{\max} t R'(t,x,y,z)$. In the first case every tuple has exactly one extension, and in the second case every tuple has exactly 2 extensions. If exactly one of (a) and (b) belongs to R' then up to permutation of variables $\mathsf{Compl}_{1,2}(x,y,z) = R'(x,x,y,z)$. If exactly one of (a) and (d) belongs to R' then up to permutation of variables $\mathsf{Compl}_{1,2}(x,y,z) = R'(x,y,y,z)$. Finally, if exactly one of (c) and (d) belongs to R' then up to permutation of variables $\mathsf{Compl}_{1,2}(x,y,z) = R'(x,y,z,z)$. $\square$

Lemma 20. *If $k+\ell \geq 3$ then $\langle \mathsf{Compl}_{k,\ell}\rangle_{\max} = IN_2$.*

Proof. Observe first that

$$\begin{aligned}
\mathsf{Compl}_{k,\ell}(x_1,\ldots,x_{k+\ell}) &= \exists_{\max} y \mathsf{Compl}_{k,\ell+1}(x_1,\ldots,x_{k+\ell},y),\\
\mathsf{Compl}_{k,\ell}(x_1,\ldots,x_{k+\ell}) &= \exists_{\max} y (\mathsf{Compl}_{k+1,\ell-1}(x_1,\ldots,x_k,y,x_{k+2},x_{k+\ell}) \qquad (4.2)\\
&\quad \wedge \mathsf{NEQ}(y,x_{k+1})), \quad \text{and}\\
\mathsf{Compl}_{k,0}(x_1,\ldots,x_k) &= \exists_{\max} y \mathsf{Compl}_{k+1,0}(x_1,\ldots,x_k,y).
\end{aligned}$$

Also,

$$\begin{aligned}
&\mathsf{Compl}_{k,\ell}(x_1,\ldots,x_{k+\ell})\\
&\quad = \exists_{\max} y_1,\ldots,y_k \mathsf{Compl}_{k+\ell,0}(y_1,\ldots,y_k,x_{k+1},\ldots,x_{k+\ell+1})\\
&\qquad \wedge \mathsf{NEQ}(y_1,x_1) \wedge \ldots \wedge \mathsf{NEQ}(y_k,x_k)).
\end{aligned}$$

Since $\mathsf{NEQ} = \mathsf{Compl}_{2,0}$, the equalities above imply that if $k' + \ell' \le k + \ell$ then $\mathsf{Compl}_{k',\ell'} \in \langle \mathsf{Compl}_{k,\ell}\rangle_{\max}$.

Now it suffices to show that $\mathsf{Compl}_{2k,0} \in \langle \mathsf{Compl}_{k+1,0}\rangle_{\max}$. We start with the relation given by the following formula

$$\begin{aligned}\Phi(x_1,\dots,x_{2k},y_1,\dots,y_{\binom{k}{2k}}) \quad = & \bigwedge_{I=\{i_1,\dots,i_k\}\subseteq[2k]} \mathsf{Compl}_{k+1,0}(x_{i_1},\dots,x_{i_k},y_{j_I}) \\ & \wedge \bigwedge_{I\subseteq[2k],|I|=k} \mathsf{NEQ}(y_{j_I},y_{j_{\overline{I}}}).\end{aligned}$$

Here j_I is some enumeration of the k-element subsets of $[2k]$. We are interested in assignments of $x_1,\dots,x_{2k}$ and the number of ways such an assignment can be extended to a satisfying assignment of Φ. First, observe that the only assignments of $x_1,\dots,x_{2k}$ that can not be extended are the all-zero and all-one assignment. Second, since Φ is symmetric with respect of permutations of $\{x_1,\dots,x_{2k}\}$ in the sense that for any permutation of this set there is a permutation of the y_i's that keeps the formula unchanged, the number of extensions of an assignment of $x_1,\dots,x_{2k}$ depends only on the number of 0's in the assignment. We will denote this number by $N_\Phi(m)$, where m is the number of zeros. Notice that Φ defines a self-complement relation, therefore, we always assume that the number of zeros is at least k. As is easily seen, if a tuple $\mathbf{a}$ has $m \ge k$ zeros, it can be extended in $N_\Phi(m) = 2^{\frac{1}{2}\binom{2k}{k}-\binom{m}{k}}$ ways. Indeed, y_I is uniquely defined by $\mathbf{a}$ if I or $\overline{I}$ is a subset of the set of zeros of $\mathbf{a}$. Otherwise it can take any value independently of the values of other variables, except that $y_{j_I} \ne y_{j_{\overline{I}}}$.

Let $Q(x_1,\dots,x_k,y)$ be the relation given by: if $x_1 = \ldots = x_k$ then y can be any, otherwise $y = x_1$. Relation Q is an intersection of some relations $\mathsf{Compl}_{k',\ell'}$ with $k'+\ell' = k+1$. Therefore by (4.2) it belongs to $\langle \mathsf{Compl}_{k+1,0}\rangle_{\max}$. Set

$$\Phi'(x_1,\dots,x_{2k},z_1,\dots,z_{\binom{2k}{k}}) = \bigwedge_{I=\{i_1,\dots,i_k\}\subseteq[2k]} Q(x_{i_1},\dots,x_{i_k},z_{j_I}),$$

and consider $\Psi = \Phi \wedge \Phi'$, where Φ, Φ' have the same variables x_i, but the sets of the auxiliary variables y_i, z_i are disjoint. Observe that $N_\Psi(m) = N_\Phi(m) \cdot N_{\Phi'}(m)$. Similarly to Φ, $N_{\Phi'}(m) = 2^{\binom{m}{k}}$, provided $m \ge k$. Indeed, variable z_{j_I} can be assigned any value if $x_i = 0$ for all $i \in I$; otherwise z_{j_I} can take only one value. Therefore for any $m \ne 0$

$$N_\Psi(m) = 2^{\frac{1}{2}\binom{k}{2k}-\binom{k}{m}} \cdot 2^{\binom{k}{m}} = 2^{\frac{1}{2}\binom{k}{2k}}$$

and $N_\Psi(0) = 0$. Thus $\mathsf{Compl}_{2k,0} = \exists_{\max}(y_1,\dots,y_{\binom{k}{2k}})\Psi$.

It now remains to apply Proposition 3 of [14] that claims, in particular, that the relation $\mathsf{Compl}_{k,\ell}$ constitute a plain basis of IN_2. □

5. Conclusion

The results of the previous section can be used to reprove some complexity results, namely, that of [16]. If for counting problems A and B there are approximation preserving reductions from A to B, and from B to A, we denote it by $A =_{AP} B$. The problem #CSP(IMP) plays a special role in this result. This problem can also be interpreted as the problem of counting the number of independent sets in a bipartite graph, $\#BIS$, or as the problem of counting antichains in a partially ordered set [15]. The problem of counting the number of satisfying assignments to a CNF, $\#SAT$, is predictably the most difficult problem among counting CSPs.

Theorem 21 ([16])**.** *Let Γ be a set of relations over $\{0,1\}$. If every relation in Γ is affine then $\#CSP(\Gamma)$ is solvable exactly in polynomial time. Otherwise if every relation in Γ is in IM_2 then $\#CSP(\Gamma) =_{AP} \#BIS$. Otherwise $\#CSP(\Gamma) =_{AP} \#SAT$.*

Proof. The #CSP over affine relations can be solved exactly in polynomial time, as it is proved in [13]. If Γ contains OR or NAND, the problem #CSP(Γ) is interreducible with $\#SAT$ by Theorem 3 of [15] (observe that the problem #IS of counting the number of independent sets in a graph can be represented as #CSP(NAND)). By Theorems 1 and 2 this leaves only two max-co-clones to consider, IM_2 and IN_2. Since IM_2 is generated by IMP and by Lemma 9, for any $\Gamma \subseteq IM_2$ the problem #CSP(Γ) is either polynomial time solvable, or is interreducible with $\#BIS$. The remaining max-co-clone, IN_2 is generated by $\mathsf{Compl}_{3,0}$ that contains all tuples such that not all their entries are equal; this is why it is sometimes called the Not-All-Equal relation, or NAE. Therefore for any $\Gamma \subseteq IN_2$ such that $\Gamma \not\subseteq IL_3$ the problem #CSP(Γ) is interreducible with #CSP(NAE). By [23] the decision problem CSP(NAE) is NP-complete. Therefore by Theorem 1 of [15] #CSP(NAE) is AP-interreducible with $\#SAT$. $\square$

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