Some infinite intervals in clone lattices

Bulatov A.A.

Abstract. We apply computation with pp-formulas in relational clones to find uncountable families of clones. It is proved that the submaximal clone Pol $\varrho := \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$ on a three-element set has continuum many subclones, and that there exist continuum many finite polynomially inequivalent simple algebras with some additional properties.

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A number of results on clones and clone lattices establish the existence of large families of clones possessing a certain property. Such a family is usually produced by a clone with an infinite basis. However, applying the Galois connection between clones and sets of relations, one may obtain families of this form starting with a relational clone with an infinite basis. This paper gives two examples of such constructions.

§1. The Galois connection

Let O_A denote the set of all finitary functions on a finite set A, and let $O_A^{(n)}$ denote the set of all n-ary functions on A. A subset C of O_A is called *clone* if it is closed under superposition and contains all projections (i. e. operations of the form $e_i^n(x_1, \ldots, x_n) = x_i$). The clones on the set A form a complete algebraic lattice which is denoted by $\mathcal{L}_A$. Since $\mathcal{L}_A$ and $\mathcal{L}_B$ are isomorphic for any A, B of the same cardinality, we will write $\mathcal{L}_k$ instead of $\mathcal{L}_A$, where $k = |A|$.

The main tool for processing clones in this paper is the Galois connection between sets of functions and sets of relations. Recall some basic notions.

A function $f(x_1, \dots, x_n)$ is said to *preserve* an m-ary relation ϱ if ϱ is a subalgebra of $(A; f)^m$. The set of all m-ary relations on the set A is denoted by $R_A^{(m)}$. Also, we put $R_A = \bigcup_{m\in\mathbb{N}} R_A^{(m)}$. We will write $\mathrm{ar}\varrho = m$ to say that ϱ is m-ary. For a set $C \subseteq O_A$, let $\mathrm{Inv}\ C$ denote the set of all relations which are preserved by every function from C. Similarly, for $T \subseteq R_A$ let $\mathrm{Pol}\ T$ denote the set of all functions preserving all relations from D. These mappings generate a Galois connection between the power sets of O_A and R_A. The corresponding closed sets are those of the forms $\mathrm{Inv}\ C$ and $\mathrm{Pol}\ T$. It is well known that the sets of the form $\mathrm{Pol}\ T$ are precisely the clones. Closed sets of relations can be pepresented as sets closed with respect to some operations as well.

Before introducing these operations, recall that any closed set of relations contains so-called *trivial relations*. An n-ary relation of the form υ^σ, where σ is an equivalence relation on $\underline{n} = \{1, \dots, n\}$, and the n-tuple $(a_1, \dots, a_n)$ belongs to υ^σ if and only if $a_i = a_j$ for all $(i, j) \in \sigma$, is called *trivial*. If σ is the identity relation, then υ^σ is called the n-ary *universal* relation. Analogously, if σ is the universal relation, then υ^σ is said to be the *generalized identity* relation and denoted by ω_n.

Below some operations on relations are listed.

1) *Projection*. Let $I \subseteq \underline{n}$. For an n-tuple $\mathbf{a} = (a_1, \dots, a_n)$, the tuple $(a_i)_{i\in I}$ is denoted by $\mathbf{a}|_I$. By the *projection* of an n-ary relation ϱ onto the set I we mean the relation

$$\varrho|_I = \{\mathbf{a}|_I \mid \mathbf{a} \in \varrho\}.$$

2) *Direct product*. The direct product of ϱ and τ, where $\mathrm{ar}\varrho = n$ and $\mathrm{ar}\tau = l$, is the relation

$$\varrho \times \sigma = \{(a_1, \dots, a_n, b_1, \dots, b_l) \mid (a_1, \dots, a_n) \in \varrho, (b_1, \dots, b_l) \in \sigma\}.$$

3) *Permutation of indices*. Let ϱ be an n-ary relation and let π be a permutation of $\underline{n}$. Put

$$\pi(\varrho) = \{(a_{\pi(1)}, \dots, a_{\pi(n)}) \mid (a_1, \dots, a_n) \in \varrho\}.$$

4) *Identification of indices*. For $n \in \mathbb{N}$, $i, j \in \underline{n}$, $i < j$, by $\nabla_{i,j}$ we denote the identification of indices i and j which is defined as follows:

$$\nabla_{i,j}(\varrho) = \{(a_1, \dots, a_{n-1}) \mid (a_1, \dots, a_i, \dots, a_{j-1}, a_i, a_j, \dots, a_{n-1}) \in \varrho\},$$

where ϱ is an n-ary relation.

5) *Duplication of an index.* Let ϱ be an n-ary relation and $i \in \underline{n}$. The operation D_i, where

$$D_i(\varrho) = \{(a_1, \ldots, a_{i-1}, a_i, a_i, a_{i+1}, \ldots, a_n) \mid (a_1, \ldots, a_n) \in \varrho\},$$

is called the *duplication* of an index i.

One can easily check that every set of the form Inv C is closed under operations 1–5. It was proved in [3] that any set $T \subseteq R_A$ which is closed under projections, direct product, permutations of indices, identification of indices and contains the identity relation is of the form Inv C for a certain $C \subseteq O_A$. A closed set of relations with respect to these operations is called *a relational clone.* The relational clone generated by a set $T \subseteq R_A$ will be denoted by $\langle T \rangle$.

The clones of relations on A form a complete algebraic lattice $\mathcal{L}_A^\partial$. It is dually isomorphic to the clone lattice $\mathcal{L}_A$. We often use this fact in proofs.

Another approach to closed sets of relations uses first-order formulas. Let P_ϱ be the n-ary predicate corresponding to an n-ary relation ϱ in the natural way. Then the operations 1–5 can be written through so called $\exists\wedge$-formulas. By an $\exists\wedge$-formula[1] we mean a first-order formula built from predicate symbols, variables, conjunction symbols, and existencial quantifiers. For example, if ϱ is n-ary and $\varrho' = \varrho|_{\{1, \ldots, n-1\}}$, then

$$P_{\varrho'}(x_1, \ldots, x_{n-1}) = \exists x_n P_\varrho(x_1, \ldots, x_n).$$

Moreover, for any ϱ_α, $\alpha \in \Lambda$ for a finite set Λ, $\mathrm{ar}\varrho_\alpha = n_\alpha$, for any $n, m \in \mathbb{N}$ and arbitrary mappings $\pi\colon \underline{n} \to \underline{m}$ and $\pi_\alpha\colon \underline{n_\alpha} \to \underline{m}$, $\alpha \in \Lambda$ the relation ϱ defined through the formula

$$P_\varrho(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \exists (x_i)_{i \in \underline{m} \setminus \pi(\underline{n})} \bigwedge_{\alpha \in \Lambda} P_{\varrho_\alpha}(x_{\pi_\alpha(1)}, \ldots, x_{\pi_\alpha(n_\alpha)}) \tag{1}$$

belongs to $\langle \{\varrho_\alpha \mid \alpha \in \Lambda\} \rangle$. The main advantage of the logical approach to relational clones is the converse assertion ([12]).

Proposition 1 *Let $T \subseteq R_A$. For any $\varrho \in \langle T \rangle$ there exists a formula of the form (1) representing ϱ such that $\varrho_\alpha \in T$ for all $\alpha \in \Lambda$.*

[1] A first order formula of this form is also called a *primitive-positive formula*, or briefly a *pp-formula.*

We say that an m-tuple $(a_1, \ldots, a_m)$ is *permissible* if the statement

$$\bigwedge_{\alpha \in \Lambda} P_{\varrho_\alpha}(a_{\pi_\alpha(1)}, \ldots, a_{\pi_\alpha(n_\alpha)})$$

is true.

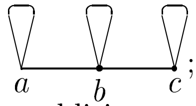
Actually we may restrict to $\exists\wedge$-formulas of a special kind. If π is not injective, say $\pi(i) = \pi(j)$, then $a_i = a_j$ is valid for any $\mathbf{a} \in \varrho$. Hence ϱ can be derived from $\varrho|_{\underline{n}\setminus\{j\}}$ by duplication of an index. So, we may assume that all the relations we deal with have no pairs of indices satisfying this condition, and the mapping π in formulas of type (1) will always be injective. Clearly, in this case it is possible to redefine mappings π, π_α in a suitable way such that $\pi(i) = i$ for all $i \in \underline{n}$. Below we assume π to be of this form.

§2. Cardinality of the subclone lattice of submaximal clones in $\mathcal{L}_3$

For $k \geq 3$ the lattice $\mathcal{L}_k$ has the cardinality of the continuum ($= \mathfrak{c}$). This fact was proved in the fifties by finding subalgebras with an infinite basis (see for instance [8]). To obtain an overview of the structure of the lattice $\mathcal{L}_k$, J. Demetrovics, L. Hannak and S.S. Marcenkov proved that all coatoms except the clone of all quasi-linear operations have continuum many subclones ([5, 6], [7], [11]). They used the knowledge of the coatoms (i.e. the maximal clones of O_A) of the lattice described by I.G. Rosenberg in [13]. The lattice of clones of quasi-linear operations is finite for k prime ([4]) and countably infinite otherwise (see for instance [10] or [14]).

Studying subclone lattices of submaximal clones (that is clones which are covered by maximal ones) is much harder. For instance, the maximal clone Pol $\leq$, where $\leq$ is a special order on an 8-element set, has no finite basis ([15]). This implies that either some clone is contained in no submaximal one, or that there exist infinitely many submaximal clones. However, for $\mathcal{L}_3$ there exist only 158 submaximal clones, and the complete list of them was given in [9]. We have no room to give the full list, but some information is gathered in the following table. Types (i)–(vii) of maximal clones in $\mathcal{L}_3$ are numbered accordingly to the theorem of [16]. The third and fourth columns state the number of maximal clones of the type and the number of submaximal ones covered by each maximal clone of this type. The last column gives the cardinality of the subclone lattice of the submaximal clone. We assume the

3-element base set to be $E_3 = \{0, 1, 2\}$. The following notation is used in the table:
σ is an nontrivial equivalence relation;

τ is a binary relation determined by the graph 
;
$\lambda = \{(a, b, c, d) \mid a + b = c + d\}$, where "+" denotes addition modulo 3 (so, the clone Pol λ is the clone of all linear operations);
π is the cyclic permutation $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$;
$\mu = \{(a, b, c) \mid |\{a, b, c\}| \leq 2\}$;
$\varrho = \begin{pmatrix} a & b & a & c \\ b & a & c & a \end{pmatrix}$, where a, b, c are distinct elements of E_3;
$C_{\max}$ and $C_{\min}$ are clones generated by all unary operations from Pol $\leq$ together with the operation $\max(x, y)$ (respectively $\min(x, y)$), where $\leq$ is a linear order;
$C_q = \{f \in O_{E_3}^{(n)} \mid \exists f_i \in O_{E_3}^{(1)} \colon f(x_1, \ldots, x_n) = f_0(f_1(x_1) + \ldots + f_n(x_n)) \pmod 2\} \cup O_{E_3}^{(1)}\}$.

no.	type of maximal clones	number of maximal clones	number of submaximal clones	cardinality
(i)	Pol $\{a\}$	3	12	finite for Pol $\{a\} \cap$ Pol λ $\mathfrak{c}$ otherwise
	Pol $\{a, b\}$	3	15	$\mathfrak{c}$
(ii)	Pol λ	1	5	finite
(iii)	Pol $\leq$	3	13	$\aleph_0$ for $C_{\max}$ and $C_{\min}$ $\mathfrak{c}$ otherwise
(iv)	Pol π	1	2	finite for Pol $\pi \cap$ Pol λ $\mathfrak{c}$ otherwise
(v)	Pol τ	3	7	$\mathfrak{c}$
(vi)	Pol σ	3	13	unknown for Pol ϱ $\mathfrak{c}$ otherwise
(vii)	Pol μ	1	5	$\aleph_0$ for C_q $\mathfrak{c}$ otherwise

Thus, the table has one blank space to fill in. In this section we prove that the clone Pol ϱ has uncountably many subclones.

A good survey and a complete proof of the results concerning the number of subclones of submaximal clones the reader can find in [2]. Here we give the part of the proof which uses the Galois connection.

Put $a = 0$, $b = 1$, $c = 2$, that is

$$\varrho := \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}.$$

Let

$$\varrho_n := \Big\{(a_1, a_2, ..., a_n) \mid \exists i :\ (a_i \in \{1,2\} \wedge (\forall j \neq i :\ a_j = 0)\Big\} \cup \Big(\{1,2\}^n \backslash \{2\}^n\Big) \tag{2}$$

for $n \geq 3$

Proposition 2 *The relational clone* $\langle \{\varrho\} \cup \{\varrho_n \mid n \geq 3\}\rangle$ *has the infinite independent basis* $\{\varrho, \varrho_3, \varrho_4, \ldots\}$.

Proof.

Since the constant 1 preserves any relation ϱ_n for $n \geq 3$, but $\varrho \notin \mathrm{Inv}\ \{1\}$, we obtain $\varrho \notin \langle \varrho_3, \varrho_4, \ldots \rangle$.

We prove that $\varrho_l \notin \langle \{\varrho\} \cup \{\varrho_n \mid n \geq 3, n \neq l\}\rangle$ for any l. For a certain index-set Λ and each $\alpha \in \Lambda$, let π_α be m_α-ary relation of $\{\varrho\} \cup \{\varrho_m \mid m \geq 3\}$. Furthermore, let φ_α be a mapping from $\underline{m_\alpha}$ into $\underline{n}$.

Suppose the relation ϱ_l, $l \geq 3$, has the representation

$$P_{\varrho_l}(x_1, \ldots, x_l) = \exists x_{l+1}, \ldots, x_n \bigwedge_{\alpha \in \Lambda} P_{\pi_\alpha}(x_{\varphi_\alpha(1)}, x_{\varphi_\alpha(2)}, ..., x_{\varphi_\alpha(m_\alpha)}) \tag{3}$$

for a certain n.

Claim 1. If a tuple $(a_1, ..., a_n) \in E_3^n$ is permissible and $a_j = 2$ for a certain $j \in \underline{n}$, then the tuple $(a_1, ..., a_{j-1}, 1, a_{j+1}, .., a_n)$ is permissible as well.

If we replace 2 with 1 in tuple $\mathbf{a}$ of ϱ_l or ϱ, we again obtain a tuple of ϱ_l or ϱ, respectively. So, Claim 1 holds.

Claim 2. There exists $\alpha \in \Lambda$ such that $\varphi_\alpha(\underline{m_\alpha}) \subseteq \underline{m}$, and $\pi_\alpha \neq \varrho$.

Suppose that

$$\forall \alpha \in \Lambda :\ (\pi_\alpha \neq \varrho \implies \varphi_\alpha(\underline{m_\alpha}) \not\subseteq \underline{l}). \tag{4}$$

By (2) and (3) there exists a permissible n-tuple of the form $\mathbf{a} = (a_1, ..., a_l, a_{l+1}, ..., a_n)$ with $a_1 = 1$ and $a_2 = ... = a_l = 2$, i.e.

$$\forall \alpha \in \Lambda : \ (a_{\varphi_\alpha(1)}, ..., a_{\varphi_\alpha(m_\alpha)}) \in \pi_\alpha. \tag{5}$$

By Claim 1, we may suppose without lost of generality that

$$\forall i \in \{l+1, l+2, ..., n\} : \ a_i \in \{0, 1\}.$$

Next, we want to prove that (4) and (5) imply the permissibility of the tuple $\mathbf{b} = (b_1, ..., b_n)$ with $b_1 = ... = b_l = 2$ and $b_i = a_i$ for all $i \in \{l+1, l+2, ..., n\}$, which leads to the contradiction $(2, 2, ..., 2) \in \varrho_l$.

The following cases are possible for $\alpha \in \Lambda$:

Case 1: $1 \notin \varphi_\alpha(m_\alpha)$.
The permissibility of $\mathbf{b}$ follows obviously from (5).

Case 2: $1 \in \varphi_\alpha(m_\alpha)$, i.e. there exists $i \leq m_\alpha$ with $\varphi_\alpha(i) = 1$.

Subcase 2.1: $\pi_\alpha = \varrho$.
W.l.o.g. let $\varphi_\alpha(1) = 1$. By (5) we have $(a_{\varphi_\alpha(1)}, a_{\varphi_\alpha(2)}) \notin \{(1,1), (1,2)\}$ hence $b_{\varphi_\alpha(2)} = a_{\varphi_\alpha(2)} = 0$ and $(b_{\varphi_\alpha(1)}, b_{\varphi_\alpha(2)}) = (2, 0) \in \varrho$.

Subcase 2.2: $\pi_\alpha = \varrho_s$.

Subcase 2.2.1: $0 \in \{a_{\varphi_\alpha(1)}, ..., a_{\varphi_\alpha(s)}\}$.
By (5) and the definition of ϱ_s we obtain

$$(a_{\varphi_\alpha(1)}, ..., a_{\varphi_\alpha(i-1)}, a_1, a_{\varphi_\alpha(i+1)}, ..., a_{\varphi_\alpha(s)}) = (0, ..., 0, 1, 0, ..., 0). \tag{6}$$

Since $a_k = b_k$ for all $k \geq 2$, and $b_1 = 2$, it follows from (6) that $(b_{\varphi_\alpha(1)}, ..., b_{\varphi_\alpha(s)}) \in \varrho_s$.

Subcase 2.2.2: $\{a_{\varphi_\alpha(1)}, ..., a_{\varphi_\alpha(s)}\} \subseteq \{1, 2\}$.
Since $\varphi_\alpha(\underline{m_\alpha}) \not\subseteq \underline{m}$, there exists an index j such that $\varphi_\alpha(j) \notin \underline{m}$, and $b_{\varphi_\alpha(j)} = a_{\varphi_\alpha(j)} = 1$, so the inclusion $\{b_{\varphi_\alpha(1)}, ..., b_{\varphi_\alpha(s)}\} \subseteq \{1, 2\}$ holds. Hence $(b_{\varphi_\alpha(1)}, ..., b_{\varphi_\alpha(s)})$ belongs to π_α, and therefore $\mathbf{b}$ is permissible.

Claim 3. Let $\varphi_\alpha(\underline{m_\alpha}) \subseteq \underline{m}$ and $\pi_\alpha \neq \varrho$. Then φ_α is bijective and $\pi_\alpha = \varrho_l$.

By assumption there exists a number $s \geq 3$ such that $\pi_\alpha = \varrho_s$, hence $m_\alpha = s$. First, we prove that φ_α is injective. Suppose φ_α were not injective. Then we consider both possible cases.

Case 1: $|\varphi_\alpha(\underline{m_\alpha})| = 1$, i.e. $\varphi_\alpha(\underline{s}) = \{j\}$ for a certain $j \leq m$.
Then for every permissible tuple $\mathbf{a} = (a_1, ..., a_s)$ we have $(a_{\varphi_\alpha(1)}, a_{\varphi_\alpha(2)}, ..., a_{\varphi_\alpha(s)}) = (a_j, a_j, ..., a_j) = (1, 1, ..., 1)$. This contradicts our assumption (5)

because each element of $\{0,1,2\}$ occurs in each row of the relation ϱ_l.
Case 2: $\exists i_1, i_2, i_3 \in \underline{s} :\ (i_1 \neq i_2)\ \wedge\ (\varphi_\alpha(i_1) = \varphi_\alpha(i_2))\ \wedge\ (\varphi_\alpha(i_3) \neq \varphi_\alpha(i_1))$. W.l.o.g. let $\varphi(i_1) = 1$. Then for the tuple $(a_1, ..., a_l) = (1, 0, 0, ..., 0) \in \varrho_l$ we obtain $(a_{\varphi_\alpha(1)}, a_{\varphi_\alpha(2)}, ..., a_{\varphi_\alpha(s)}) \in \varrho_s$, which is impossible because $a_{\varphi(i_1)} = a_{\varphi_\alpha(i_2)} = 1$ and $a_{\varphi_\alpha(i_3)} = 0$. Consequently, φ_α is injective.

The surjectivity of φ_α we can prove as follows. Suppose $\varphi_\alpha(\underline{m_\alpha}) \subset \underline{m}$ (i.e. $\varphi_\alpha(\underline{m_\alpha}) \neq \underline{m}$). Furthermore, w.l.o.g. we may suppose $1 \notin \varphi_\alpha(\underline{m_\alpha})$. Then for the tuple $\mathbf{a} = (a_1, a_2, ..., a_l) = (1, 2, 2, ..., 2) \in \varrho_l$ we have $(a_{\varphi_\alpha(1)}, ..., a_{\varphi_\alpha(s)}) = (2, 2, 2, ..., 2) \notin \varrho_s$ in contradiction to (5)

By the bijectivity of the mapping φ_α and the definition of ϱ_l, π_α must coincide with the relation ϱ_l.

Finally, Claim 3 and Proposition 1 imply $\varrho_l \notin \langle\{\varrho\} \cup \{\varrho_n \mid n \geq 3, n \neq l\}\rangle$ for any l, as desired. □

§3. The number of simple polynomially inequivalent finite algebras.

Another point of application of the computational technique with $\exists\wedge$-formulas is the problem of enumeration of termally or polynomially inequivalent finite algebras with certain properties. On the one hand it is well known, and not very hard to verify, that up to termal equivalence there exist uncountably many finite simple algebras. On the other hand an uncountable family of finite polynomially inequivalent algebras was constructed in [1]. Here we prove that there exist uncountably many polynomially unequivalent simple finite algebras.

Fix a finite set A, $|A| \geq 3$, and suppose $0, 1, 2 \in A$. If a clone C contains all constant functions, then $\omega_n \subseteq \varrho$ for every n-ary $\varrho \in \mathrm{Inv}\ C$. In other words, the n-tuple $(a, \ldots, a)$ belongs to ϱ for every $a \in A$. So we have to exhibit an uncountable family of relational clones such that for each of its clones C: 1) C contains no equivalence relations; 2) for every $\varrho \in C$ we have $\omega_{\mathrm{ar}\varrho} \subseteq \varrho$.

Take $n \geq 3$. Let θ_n be an n-ary relation such that the n-tuple $\mathbf{a}$ belongs to θ_n whenever 0 occurs among the components of $\mathbf{a}$. Further, put $\nu_n = \{(2, \ldots, 2, \underset{j}{1}, 2, \ldots, 2) \mid j \in \underline{n}\}$ and $\tau_n = \nu_n \cup \theta_n \cup \omega_n$. The next statement is straightforward.

Lemma 1 *(a) For any $I \subseteq \underline{n}$ the projection $\tau_n|_I$ is a trivial relation.*

(b) If the unary relation ϱ is derived from τ_n by identification of all indices, then $\varrho = A$.

(c) For any permutation π of $\underline{n}$ the equation $\pi(\tau_n) = \tau_n$ is valid.

The following assertion plays a key role in this section. Let $\mathbb{N}_3$ denote the set $\mathbb{N}\backslash\{1, 2, 3\}$.

Proposition 3 *For any $J \subseteq \mathbb{N}_3$, if $\tau_n \in \langle\{\tau_j \mid j \in J\}\rangle$, then $n \in J$.*

Proof. Suppose $\tau_n \in \langle\{\tau_l \mid j \in J\}\rangle$. Then by Proposition 1 there exists a finite set Λ, a set $\{n_\alpha \mid \alpha \in \Lambda\} \subseteq J$, $m \in \mathbb{N}$ and $\pi_\alpha\colon \underline{n_\alpha} \to \underline{m}$ $(\alpha \in \Lambda)$ such that

$$P_{\tau_n}(x_1, \ldots, x_n) = \exists (x_i)_{i\in\underline{m}\backslash\underline{n}} \bigwedge_{\alpha\in\Lambda} P_{\tau_{n_\alpha}}(x_{\pi_\alpha(1)}, \ldots, x_{\pi_\alpha(n_\alpha)}). \tag{7}$$

By Lemma 1(b) we may assume that all π_α are nonconstant. The rest of the proof is divided into several claims.

Claim 1. Let ϱ be defined through the formula

$$P_\varrho(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \bigwedge_{\alpha\in\Lambda'} P_{\tau_{n_\alpha}}(x_{\pi_\alpha(1)}, \ldots, x_{\pi_\alpha(n_\alpha)}), \tag{8}$$

where $\Lambda' = \{\alpha \mid \pi_\alpha(\underline{n_\alpha}) \subseteq \underline{n}\}$ and the π_α are considered as mappings into $\underline{n}$. Then $\varrho = \tau_n$.

If $\mathbf{a} \in \tau_n$, then for all $\alpha \in \Lambda'$ the predicate $P_{\tau_{n_\alpha}}(a_{\pi_\alpha(1)}, \ldots, a_{\pi_\alpha(n_\alpha)})$ is true. So $\tau_n \subseteq \varrho$. Conversely, suppose $\mathbf{a} \in \varrho$. Consider the m-tuple $\mathbf{b}$, where $\mathbf{b}|_{\underline{n}} = \mathbf{a}$ and $b_i = 0$ if $i \in \underline{m}\backslash\underline{n}$. If $\alpha \in \Lambda'$, then $P_{\tau_{n_\alpha}}(b_{\pi_\alpha(1)}, \ldots, b_{\pi_\alpha(n_\alpha)})$ is true, since $\mathbf{a}$ is permissible by (8). Otherwise, 0 occurs among $b_{\pi_\alpha(1)}, \ldots, b_{\pi_\alpha(n_\alpha)}$, hence $P_{\tau_{n_\alpha}}(b_{\pi_\alpha(1)}, \ldots, b_{\pi_\alpha(n_\alpha)})$ is true again. Thus $\mathbf{b}$ is permissible by (7) and $\mathbf{a} = \mathbf{b}|_{\underline{n}} \in \tau_n$.

Claim 2. π_α is injective for any $\alpha \in \Lambda'$. Suppose $\pi_{n_\alpha}(i_1) = \pi_{n_\alpha}(i_2) \neq \pi_{n_\alpha}(i_3)$ for some $i_1, i_2, i_3 \in \underline{n_\alpha}$. Since $\mathbf{a} = (2, \ldots, 2, \underset{\pi_{n_\alpha}(i_1)}{1}, 2, \ldots, 2) \in \tau_n$, the tuple $\mathbf{a}' = (a_{\pi_{n_\alpha}(1)}, \ldots, a_{\pi_{n_\alpha}(n_\alpha)})$ belongs to τ_{n_α}. However, $1 = a_{\pi_{n_\alpha}(i_1)} \neq a_{\pi_{n_\alpha}(i_3)} = 2$ implies $\mathbf{a}' \notin \omega_{n_\alpha}$, furthermore $a_{\pi_{n_\alpha}(i_1)} = a_{\pi_{n_\alpha}(i_2)} = 1$ implies $\mathbf{a}' \notin \nu_{n_\alpha}$, and finally, as there is no 0 among $a_{\pi_{n_\alpha}(1)}, \ldots, a_{\pi_{n_\alpha}(n_\alpha)}$ we have $\mathbf{a}' \notin \theta_{n_\alpha}$, a contradiction.

Claim 3. π_α is surjective for any $\alpha \in \Lambda'$. Since π_α is injective by Lemma 1(c) we have $\tau_{n_\alpha} = \tau_n|_{\pi_\alpha(\underline{n_\alpha})}$. If $\pi_\alpha(\underline{n_\alpha}) \neq \underline{n}$, then by Lemma 1 τ_{n_α} is trivial.

Finally, since τ_n is nontrivial, $\Lambda' \neq \varnothing$. So $n_\alpha = n$ for any $\alpha \in \Lambda'$ and $n \in J$. $\square$

Thus clones of the form Pol $\{\tau_j \mid j \in J\}$, where $J \subseteq \mathbb{N}_3$, are distinct. Suppose ϱ is a binary relation and $\varrho \in \langle \{\tau_n \mid n \in \mathbb{N}_3\} \rangle$. Then for a certain finite set Λ, a set $\{n_\alpha \mid \alpha \in \Lambda\} \subseteq \mathbb{N}_3$, $m \in \mathbb{N}$, and mappings $\pi_\alpha : \underline{n_\alpha} \to \underline{m}$ $(\alpha \in \Lambda)$ we have

$$P_{\varrho_l}(x_1, x_2) = \exists (x_i)_{i \in \underline{m} \setminus \pi(\underline{2})} \bigwedge_{\alpha \in \Lambda} P_{\pi_\alpha}(x_{\varphi_\alpha(1)}, x_{\varphi_\alpha(2)}, \ldots, x_{\varphi_\alpha(m_i)}).$$

If $\pi(1) = \pi(2)$, then ϱ is the identity relation. So assume $\pi(1) = 1$, $\pi(2) = 2$. As in Claim 1 in the proof of Proposition 3, we have

$$P_{\varrho_l}(x_1, x_2) = \bigwedge_{\alpha \in \Lambda'} P_{\pi_\alpha}(x_{\varphi_\alpha(1)}, x_{\varphi_\alpha(2)}, \ldots, x_{\varphi_\alpha(m_i)}), \tag{9}$$

where $\Lambda' = \{\alpha \mid \pi_\alpha(\underline{n_\alpha}) \subseteq \underline{2}\}$. We prove that any tuple (a_1, a_2) such that $0 \in \{a_1, a_2\}$ is permissible by (9). Indeed, if π_α is a constant mapping, then $(a_{\pi_\alpha(1)}, \ldots, a_{\pi_\alpha(n_\alpha)}) \in \tau_{n_\alpha}$ by definition of τ_{n_α}. Otherwise, 0 occurs among $a_{\pi_\alpha(1)}, \ldots, a_{\pi_\alpha(n_\alpha)}) \in \theta_{n_\alpha} \subseteq \tau_{n_\alpha}$.

Thus, any binary relation from $\langle \{\tau_n \mid n \in \mathbb{N}_3\} \rangle$ contains the relation θ_2. This means that the algebra $(A; \text{Pol } \{\tau_j \mid j \in J\})$ is simple, having no nontrivial automorphism, ordering or tolerance.

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Andrei A. Bulatov
Dept. of Algebra and Discrete Math.
Ural State University
Lenin ave. 51
620083 Ekaterinburg
Russia
e-mail: Andrei.Bulatov@usu.ru