Constraint Satisfaction Problems and Finite Algebras

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Abstract. Many natural combinatorial problems can be expressed as constraint satisfaction problems. This class of problems is known to be **NP**-complete in general, but certain restrictions on the form of the constraints can ensure tractability. In this paper we show that any restricted set of constraint types can be associated with a finite universal algebra. We explore how the computational complexity of a restricted constraint satisfaction problem is connected to properties of the corresponding algebra. Using these results we exhibit a common structural property of all known intractable constraint satisfaction problems. Finally, we classify all finite strictly simple surjective algebras with respect to tractability. The result is a dichotomy theorem which significantly generalises Schaefer's dichotomy for the Generalised Satisfiability problem.

1 Introduction

The constraint satisfaction problem provides a framework in which it is possible to express, in a natural way, a wide variety of combinatorial problems [1, 12, 14]. The aim in a constraint satisfaction problem is to find an assignment of values to a given set of variables, subject to constraints on the values which can be assigned simultaneously to certain specified subsets of the variables.

The mathematical framework used to describe constraint satisfaction problems has strong links with several other areas of computer science and mathematics. For example, links with relational database theory [1, 5] and with some notions of group theory [3] and universal algebra [8] have been investigated. There is a survey of these results in [15].

The constraint satisfaction problem is known to be **NP**-complete in general [11]. However, certain restrictions on the form of the constraints can ensure tractability [3, 6, 7, 9, 10]. In [9], it is proved that the time complexity of a problem involving a restricted set of constraint relations is determined by certain algebraic invariance properties of the constraint relations: namely, by the *clone of polymorphisms* of these relations. This result transforms the search for new tractable constraint types to the search for clones leading to tractability.

In this paper we extend this result by exploring further the fundamental connection between constraint relations, clones of operations, and finite algebras. The motivation for this research is that by characterising sets of relations in terms of their algebraic properties we can use powerful algebraic tools to analyse the complexity of the corresponding problems. Furthermore, in many cases the algebraic characterisation is simpler than an explicit description. As an example of the power of the algebraic approach we show that Schaefer's theorem [17] concerning the complexity of the GENERALISED SATISFIABILITY problem can be expressed very simply as a classification of two-element algebras with respect to tractability.

The paper is organised as follows. In Section 2 we give basic definitions relating to the constraint satissfaction problem, clones of polymorphisms, and finite algebras, and we define the notion of a tractable algebra. In Section 3 we prove that, for our purposes, it suffices to consider idempotent algebras, that is, algebras whose unary term operations are trivial. We also study how the tractability of a finite algebra relates to the tractability of its smaller derived algebras. In Section 4 we use the results obtained to classify all strictly simple surjective algebras. We show that such algebras give rise to constraint satisfaction problems which are either tractable or **NP**-complete. Thus we extend Schaefer's dichotomy theorem to a much larger class of problems.

2 Definitions

2.1 Constraint Satisfaction Problems

The central notion in the study of constraints and constraint satisfaction problems is the notion of a *relation*. An n-ary relation on A is a subset of the set A^n, where A^n denotes the set of all n-tuples of elements of A. The set of all finitary relations on A is denoted by R_A.

We now define the standard constraint satisfaction problem which has been widely studied [1, 10–12, 14].

Definition 1. *The* constraint satisfaction problem (**CSP**) *is the combinatorial decision problem with*

Instance: *a triple* $(V, A, \mathcal{C})$ *where* V *is a set of* variables*;* A *is a domain of* values*; and* $\mathcal{C}$ *is a set of* constraints*,* $\{C_1, \ldots, C_q\}$*.*
Each constraint $C_i \in \mathcal{C}$ *is a pair* $\langle s_i, \rho_i \rangle$*, where* s_i *is a tuple of variables of length* m_i*, called the* constraint scope*, and* ρ_i *is an* m_i*-ary relation on* A*, called the* constraint relation*.*

Question: *does there exist a* solution*, i.e. a function* f*, from* V *to* A*, such that for each constraint* $\langle s_i, \rho_i \rangle \in \mathcal{C}$*, with* $s_i = (x_{i_1}, \ldots, x_{i_m})$*, the tuple* $(f(x_{i_1}), \ldots, f(x_{i_m}))$ *belongs to* ρ_i*?*

Example 1. An instance of the standard propositional SATISFIABILITY problem [4] is specified by giving a formula in propositional logic, (that is, a conjunction of clauses), and asking whether there are values for the variables which make the formula true.

Suppose that $\phi = X_1 \wedge \cdots \wedge X_n$ is such a formula, where the X_i are clauses. The satisfiability question for ϕ can be expressed as the instance $(V, \{0,1\}, \mathcal{C})$ of **CSP**, where V is the set of all variables appearing in the clauses X_i, and $\mathcal{C} = \{\langle s_1, \rho_1 \rangle, \ldots, \langle s_n, \rho_n \rangle\}$. The constraints $\langle s_k, \rho_k \rangle$ are constructed as follows.

For every clause X_k, where $x_1^k, \ldots, x_{j_k}^k$ are the variables appearing in X_k, let $s_k = (x_1^k, \ldots, x_{j_k}^k)$ and $\rho_k = \{0,1\}^{j_k} \setminus \{(a_1, \ldots, a_{j_k})\}$ where $a_i = 0$ if x_i^k is negated and $a_i = 1$ otherwise (i.e., ρ_k consists of all j_k-tuples that make X_k true).

It is easy to show that solutions of this **CSP** instance are exactly assignments which make the formula ϕ true. Hence, any instance of SATISFIABILITY can be expressed as an instance of **CSP**.

Example 2. An instance of GRAPH UNREACHABILITY consists of a graph $G = (V, E)$ and a pair of vertices, $v, w \in V$. The question is whether there is no path in G from v to w. This can expressed as the **CSP** instance $(V, \{0,1\}, \mathcal{C})$ where

$$\mathcal{C} = \{\langle e, \{(0,0),(1,1)\}\rangle | e \in E\} \cup \{\langle (v), \{(0)\}\rangle, \langle (w), \{(1)\}\rangle\}.$$

A number of other examples of well-known problems expressed as **CSP**s can be found in [8].

The general constraint satisfaction problem is known to be **NP**-complete (as seen from Example 1). However, certain restrictions may affect the complexity of **CSP**. One of the possible natural ways to restrict **CSP** is to limit the relations which can appear in constraints.

Definition 2. *Let* Γ *be a subset of* R_A*. Denote by* $\mathbf{CSP}(\Gamma)$ *the subclass of* **CSP** *defined by the following property: any constraint relation in any instance must belong to* Γ*.*

Example 3. An instance of NOT-ALL-EQUAL SATISFIABILITY [4] consists of a collection of ternary clauses. The question is whether there is an assignment of values to the variables such that the variables in each clause do not all receive the same value.

This problem can be expressed as the problem $\mathbf{CSP}(\{N\})$ where N is the following ternary relation on $\{0,1\}$:

$$N = \{(a,b,c) \mid \{a,b,c\} = \{0,1\}\}.$$

Example 4. An instance of GRAPH q-COLOURABILITY consists of a graph G. The question is whether the vertices of G can be labelled with q colours so that adjacent vertices are assigned different colours.

This problem can be expressed as the problem $\mathbf{CSP}(\{\neq_B\})$ where B is a q-element set (of colours) and $\neq_B$ is the disequality relation on B, defined by

$$\neq_B \;= \{(a,b) \in B^2 \mid a \neq b\}.$$

It is well known (see [4]) that this problem belongs to $\mathbf{P}$ if $q \leq 2$, and is $\mathbf{NP}$-complete otherwise.

For $A = \{0,1\}$, the time complexity of $\mathbf{CSP}(\Gamma)$ was completely classified by Schaefer [17]. In particular, he proved that such a problem is either tractable (i.e., solvable in polynomial time) or $\mathbf{NP}$-complete. The classification problem for larger domains is still open and seems to be very interesting and hard [3].

Problem 1. Characterise all sets of relations Γ such that $\mathbf{CSP}(\Gamma)$ is tractable.

2.2 Operations and Clones

By an n-ary operation on a set A we mean any mapping $f : A^n \to A$. The set of all finitary operations on A is denoted by O_A. An operation f which satisfies an identity of the form $f(x_1, \ldots, x_n) = x_i$ is called a *projection*.

Definition 3 ([18]). *A set $C \subseteq O_A$ is called a* clone *on A if*

- *C contains all projections; and*
- *C is closed under composition.*

Suppose $f(x_1,\dots,x_n) \in O_A$ and let M be an $m \times n$ matrix whose entries a_{ij} are elements from A. Define

$$f(M) = \begin{pmatrix} f(a_{11},\dots,a_{1n}) \\ \vdots \\ f(a_{m1},\dots,a_{mn}) \end{pmatrix}.$$

Definition 4 ([13, 18]). *An n-ary operation $f \in O_A$* preserves *an m-ary relation $\rho \in R_A$ (or f is a* polymorphism *of ρ, or ρ is* invariant *under f) if, for any $m \times n$ matrix M whose columns belong to ρ, the column $f(M)$ belongs to ρ as well.*

For given $F \subseteq O_A$ and $\Gamma \subseteq R_A$, let

$$Inv(F) = \{\rho \in R_A \mid \rho \text{ is invariant under each operation from } F\};$$
$$Pol(\Gamma) = \{f \in O_A \mid f \text{ is a polymorphism of each relation from } \Gamma\}.$$

Example 5. Let $A = \{0,1,\dots,p-1\}$, for some prime p, and let f be the ternary operation on A given by $f(x,y,z) = x - y + z \bmod p$. The set $Inv(\{f\})$ consists of all relations which are solutions to some set of linear equations modulo p. Hence, $\mathbf{CSP}(Inv(\{f\}))$ is tractable.

Jeavons has proved a strong theorem connecting the complexity of $\mathbf{CSP}(\Gamma)$ to the clone of operations $Pol(\Gamma)$.

Theorem 1 ([9]). *Let A be a finite set, and $\Gamma, \Gamma_0 \subseteq R_A$. If Γ_0 is finite and $Pol(\Gamma) \subseteq Pol(\Gamma_0)$ then $\mathbf{CSP}(\Gamma_0)$ is reducible in polynomial time to $\mathbf{CSP}(\Gamma)$.*

Informally speaking, Theorem 1 says that the complexity of $\mathbf{CSP}(\Gamma)$ is determined by the clone $Pol(\Gamma)$. Hence it allows us to tackle problems concerning the complexity of constraint satisfaction problems using algebraic techniques.

Example 6. When $A = \{0,1\}$ all possible clones were fully described by Post [16]. Using Theorem 1, together with this description of the possible clones, it is possible to classify the complexity of any set of Boolean relations [9]. It turns out that in this case there are just 6 maximal tractable sets of relations corresponding to 6 distinct clones. For any set of relations Γ which is not contained in one of these 6 maximal sets, the corresponding problem $\mathbf{CSP}(\Gamma)$ is $\mathbf{NP}$-complete.

This algebraic approach therefore provides an alternative proof of Schaefer's well-known dichotomy result [17] concerning the complexity of the Generalised Satisfiability problem (for details see [9]).

2.3 Algebras

In this subsection we first give the basic definitions of an algebra and certain standard algebraic constructions (following [13] and [18]). We then introduce the notion of a "tractable algebra", and show how this relates to the tractability of restricted constraint satisfaction problems.

Definition 5. *An* algebra *is an ordered pair $\mathcal{A} = (A, F)$ such that A is a nonempty set and F is a family of finitary operations on A. The set A is called the* universe *of $\mathcal{A}$; the operations from F are called* basic. *An algebra with a finite universe is referred to as a* finite algebra.

Definition 6. *Let $\mathcal{A} = (A, F)$ be an algebra. The m-th* direct power *of $\mathcal{A}$ is the algebra $\mathcal{A}^m = (A^m, F)$ where we treat each operation $f_i \in F$ as acting on A^m component-wise, i.e.*

$$f_i((a_{11}, \ldots, a_{1m}), \ldots, (a_{n_i1}, \ldots, a_{n_im})) = \\ (f_i(a_{11}, \ldots, a_{n_i1}), \ldots, f_i(a_{1m}, \ldots, a_{n_im})) \ .$$

Definition 7. *Let $\mathcal{A} = (A, F)$ be an algebra and B a subset of A such that, for any $f_i \in F$, and for any $b_1, \ldots, b_{n_i} \in B$, we have $f_i(b_1, \ldots, b_{n_i}) \in B$. Then the algebra $\mathcal{B} = (B, F|_B)$, where $F|_B$ consists of the restrictions of the operations f_i to B, is called a* subalgebra *of $\mathcal{A}$.*

Definition 8. *The* finite power subuniverse system *of an algebra $\mathcal{A}$ is the set $SP_{\text{fin}}(\mathcal{A})$, where*

$$SP_{\text{fin}}(\mathcal{A}) = \{\rho \mid \rho \text{ is a universe of a subalgebra of } \mathcal{A}^m \text{ for some } m \geq 1\}.$$

Remark 1. $SP_{\text{fin}}(\mathcal{A}) = Inv(F) \subseteq R_A$.

Definition 9. *Let $\mathcal{A} = (A, F)$ be an algebra. The least clone on A containing F is called the* clone of term operations *of $\mathcal{A}$ and is denoted by $Clo(\mathcal{A})$. Two algebras with the same universe are called* term equivalent *if they have the same clone of term operations.*

In other words, the clone $Clo(\mathcal{A})$ consists of all operations that can be obtained from F and the projections by means of composition.

Theorem 2 ([18]). *For a finite algebra $\mathcal{A}$, $SP_{\text{fin}}(\mathcal{A})$ is the greatest set Γ with $Pol(\Gamma) = Clo(\mathcal{A})$.*

Corollary 1. *If finite algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are term equivalent then the equality $SP_{\text{fin}}(\mathcal{A}_1) = SP_{\text{fin}}(\mathcal{A}_2)$ holds.*

We are now in a position to link the complexity of a restricted constraint satisfaction problem $\mathbf{CSP}(\Gamma)$ to properties of a corresponding algebra. Given an arbitrary set of relations $\Gamma \subseteq R_A$, consider the algebra $\mathcal{A}_\Gamma = (A, Pol(\Gamma))$. By Theorem 2, Γ is contained in $SP_{\text{fin}}(\mathcal{A}_\Gamma)$. By Theorem 1, if $\mathbf{CSP}(\Gamma)$ is tractable, then $\mathbf{CSP}(\Gamma_0)$ is tractable for any finite subset Γ_0 of $SP_{\text{fin}}(\mathcal{A}_\Gamma)$. Hence to identify sets of relations for which $\mathbf{CSP}(\Gamma)$ is tractable, it suffices to identify when the corresponding algebra, $\mathcal{A}_\Gamma$ is tractable, in the following sense.

Definition 10. *We call an algebra $\mathcal{A}$* (globally) tractable *if $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$ is tractable, and* locally tractable[1] *if, for every finite $\Gamma_0 \subseteq SP_{\text{fin}}(\mathcal{A})$, $\mathbf{CSP}(\Gamma_0)$ is tractable.*

We call an algebra $\mathcal{A}$ **NP**-complete *if $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$ is* **NP**-*complete.*

With this definition, our original problem of identifying all sets of relations for which $\mathbf{CSP}(\Gamma)$ is tractable has been reduced to the following:

Problem 2. Characterise all tractable finite algebras.

Schaefer's result [17] (see Example 6) provides a first step towards answering this problem, because it yields a complete classification of two-element algebras with respect to tractability.

Theorem 3. *A two-element algebra $\mathcal{A} = (\{0,1\}, F)$ is* **NP**-*complete if it is term equivalent to an algebra $(\{0,1\}, G)$ where G is a permutation group on $\{0,1\}$. Otherwise $\mathcal{A}$ is tractable.*

The central idea in the remainder of this paper is to generalise this result by using the algebraic properties of an arbitrary algebra, $\mathcal{A}$, to determine the complexity of the associated constraint satisfaction problem, $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$.

3 Tractability and Algebraic Constructions

In this section we first show that when studying the tractability of finite algebras we can restrict our attention to certain special classes of algebras.

Definition 11. *We call an algebra* surjective[2] *if all of its term operations are surjective.*

[1] We must distinguish the notions of global tractability and local tractability because there is no guarantee that the following situation cannot happen: for any finite subset Γ_0 of some infinite set, Γ, of relations, there exists a particular polynomial algorithm solving $\mathbf{CSP}(\Gamma_0)$, but there is no uniform polynomial algorithm for solving all instances of $\mathbf{CSP}(\Gamma)$.

[2] Note that in [19] an algebra is said to be surjective if all of its basic operations are surjective. Such algebras can have non-surjective term operations.

It is easy to verify that a finite algebra is surjective if and only if its unary term operations form a permutation group.

Definition 12 ([20]). *Let $\mathcal{A} = (A, F)$ be an algebra, and U a non-empty subset of A. The* induced term algebra *$\mathcal{A}|_U$ is defined as $(U, (Clo\mathcal{A})|_U)$ where $(Clo\mathcal{A})|_U = \{g|_U : g \in Clo\mathcal{A}$ and g preserves $U\}$.*

It is easy to check that $Clo(\mathcal{A}|_U) = (Clo\mathcal{A})|_U$.

Theorem 4. *For any finite algebra $\mathcal{A}$ there exists a set U such that*
i) $\mathcal{A}|_U$ is surjective;
ii) If $\mathcal{A}$ is locally tractable then so is $\mathcal{A}|_U$;
iii) If $\mathcal{A}|_U$ is (locally) tractable then so is $\mathcal{A}$.

Proof. Let $\mathcal{A} = (A, F)$ be a finite algebra, and let U be a minimal subset of A which is the range of some unary operation $f \in Clo\mathcal{A}$.

Clearly, $\mathcal{A}|_U$ is surjective by the choice of U. We now prove that it satisfies properties (ii) and (iii).

(ii) Let Γ be a finite subset of $SP_{\text{fin}}(\mathcal{A}|_U)$. We will show that $\mathbf{CSP}(\Gamma)$ is linear-time reducible to $\mathbf{CSP}(\Gamma')$ where Γ' is a suitable finite subset of $SP_{\text{fin}}(\mathcal{A})$.

Let $\mathcal{P} = (V, U, \mathcal{C})$ be an instance of $\mathbf{CSP}(\Gamma)$. For any relation $\rho \in \Gamma$, with arity m, we denote by ρ' the universe of the subalgebra of $\mathcal{A}^m$ generated by ρ, i.e.,

$$\rho' = \{g(\overline{a_1}, \dots, \overline{a_k}) \mid k > 0,\ g \in Clo(\mathcal{A}) \text{ and } \overline{a_1}, \dots, \overline{a_k} \in \rho\}.$$

Let $\Gamma' = \{\rho' \mid \rho \in \Gamma\}$ and set $\mathcal{P}' = (V, A, \mathcal{C}')$ where $\mathcal{C}' = \{\langle s, \rho'\rangle \mid \langle s, \rho\rangle \in \mathcal{C}\}$. Now $\mathcal{P}'$ is a problem instance of $\mathbf{CSP}(\Gamma')$ which can be obtained from $\mathcal{P}$ in linear time. Since all the projections belong to $Clo(\mathcal{A})$, we have $\rho \subseteq \rho'$, so any solution of $\mathcal{P}$ is also a solution of $\mathcal{P}'$. Conversely, suppose ψ is a solution of $\mathcal{P}'$. We shall prove that $f\psi$ is a solution of $\mathcal{P}$. For this, we show that $f(\rho') \subseteq \rho$. Let $\overline{a} \in \rho'$. Then $f(\overline{a}) = fg(\overline{a_1}, \dots, \overline{a_k})$ for some $g \in Clo(\mathcal{A})$ and some $\overline{a_1}, \dots, \overline{a_k} \in \rho$. The operation $fg(x_1, \dots, x_k)$ preserves U and belongs to $Clo(\mathcal{A})$; therefore its restriction to U belongs to $Clo(\mathcal{A})|_U$. Since $\rho \in SP_{\text{fin}}(\mathcal{A}|_U)$, we get $f(\overline{a}) \in \rho$.

(iii) We shall show that $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$ can be reduced in linear time to $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}|_U))$. First note that, by the choice of U, $f|_U$ is a permutation, so by iterating as necessary we obtain a unary operation $f' \in Clo\mathcal{A}$ such that $f'(a) = a$ for all a in U.

Let $\mathcal{P} = (V, A, \mathcal{C})$ be an instance of $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$, where $\mathcal{C} = \{\langle s_1, \rho_1\rangle, \dots, \langle s_q, \rho_q\rangle\}$, and consider the triple $\mathcal{P}' = \langle V, U, \mathcal{C}'\rangle$, where $\mathcal{C}' = \{\langle s_1, f'(\rho_1)\rangle, \dots, \langle s_q, f'(\rho_q)\rangle\}$ and $f'(\rho_i) = \{f'(\overline{a}) \mid \overline{a} \in \rho_i\}$.

We first prove that $\mathcal{P}'$ is an instance of $\mathbf{CSP}(\mathit{SP}_{\text{fin}}(\mathcal{A}|_U))$. Since the range of f' is U, any relation of the form $f'(\rho_i)$ can be viewed as a relation on U. We need to show that, for $1 \leq i \leq q$, $f'(\rho_i)$ belongs to $\mathit{SP}_{\text{fin}}(\mathcal{A}|_U)$, i.e. for any $g(x_1, \ldots, x_m) \in Clo(\mathcal{A}|_U)$, and for any $\overline{a_1}, \ldots, \overline{a_m} \in f'(\rho_i)$, we have $\overline{a} = g(\overline{a_1}, \ldots, \overline{a_m}) \in f'(\rho_i)$. The operation g is the restriction of some $g' \in Clo\mathcal{A}$ to U such that g' preserves U. Suppose that $\overline{a_j} = f'(\overline{b_j})$ for some $\overline{b_j} \in \rho_i$, $1 \leq j \leq m$. Then

$$\overline{a} = g(\overline{a_1}, \ldots, \overline{a_m}) = g'(f'(\overline{b_1}), \ldots, f'(\overline{b_m})).$$

We have $\overline{a} \in \rho_i$, because all elements $\overline{b_j}$ belong to ρ_i, and the function $g'(f'(x_1), \ldots, f'(x_m)) \in Clo(\mathcal{A})$. All components of $\overline{a}$ belong to U, since g' preserves U. Finally, we have $\overline{a} = f'(\overline{a}) \in f'(\rho_i)$, since $f'(a) = a$ for any $a \in U$.

Since each $\rho_i \in \mathbf{CSP}(\mathit{SP}_{\text{fin}}(\mathcal{A}))$, we have $f'(\rho_i) \subseteq \rho_i$; therefore any solution of $\mathcal{P}'$ is also a solution of $\mathcal{P}$. Furthermore, if ψ is a solution of $\mathcal{P}$ then $f'\psi$ is a solution of $\mathcal{P}'$. Clearly, $\mathcal{P}'$ can be obtained from $\mathcal{P}$ in linear time. Therefore tractability of $\mathcal{A}|_U$ implies that $\mathcal{A}$ is tractable. Finally, local tractability of $\mathcal{A}|_U$ implies local tractability of $\mathcal{A}$, since if all the constraint relations in $\mathcal{C}$ are taken from some finite set $\Gamma \subseteq \mathit{SP}_{\text{fin}}(\mathcal{A})$ then every constraint relation in $\mathcal{C}'$ belongs to a finite subset $\{f'(\rho) \mid \rho \in \Gamma\}$ of $\mathit{SP}_{\text{fin}}(\mathcal{A}|_U)$. □

The next theorem shows that we need only consider surjective algebras which are *idempotent*.

Definition 13. *An operation f on A is called idempotent if $f(x, \ldots, x) = x$ for any $x \in A$. The* full idempotent reduct *of an algebra $\mathcal{A} = (A, F)$ is the algebra $(A, Clo_{id}(\mathcal{A}))$ where $Clo_{id}(\mathcal{A})$ consists of all idempotent operations from $Clo(\mathcal{A})$.*

Theorem 5. *A finite surjective algebra $\mathcal{A}$ is locally tractable if and only if its full idempotent reduct is locally tractable.*

Proof. Omitted. See the full version of this paper [2].

The next two theorems connect the tractability of a finite algebra with the tractability of its subalgebras and *homomorphic images*.

Definition 14. *Let $\mathcal{A}_1 = (A_1, F_1)$ and $\mathcal{A}_2 = (A_2, F_2)$, where $F_1 = (f_i^1 \mid i \in I)$ and $F_2 = (f_i^2 \mid i \in I)$. A map $\varphi : A_1 \to A_2$ is called a* homomorphism *from $\mathcal{A}_1$ to $\mathcal{A}_2$ if $\varphi f_i^1(a_1, \ldots, a_{n_i}) = f_i^2(\varphi(a_1), \ldots, \varphi(a_{n_i}))$ holds for all $i \in I$ and all $a_1, \ldots, a_{n_i} \in A_1$. If the map φ is onto then $\mathcal{A}_2$ is said to be a* homomorphic image *of $\mathcal{A}_1$. A homomorphic image of a subalgebra of an algebra $\mathcal{A}$ is called a* factor *of $\mathcal{A}$.*

Theorem 6. *Let $\mathcal{A}$ be a finite algebra. Then*

(i) if $\mathcal{A}$ is (locally) tractable then so is every subalgebra of $\mathcal{A}$;

(ii) if $\mathcal{A}$ has an **NP***-complete subalgebra then $\mathcal{A}$ is* **NP***-complete itself.*

Proof. Let $\mathcal{A}'$ be a subalgebra of $\mathcal{A}$. It is easy to check that $\mathit{SP}_{\mathrm{fin}}(\mathcal{A}') \subseteq \mathit{SP}_{\mathrm{fin}}(\mathcal{A})$. Hence, $\mathbf{CSP}(\mathit{SP}_{\mathrm{fin}}(\mathcal{A}'))$ can be reduced to $\mathbf{CSP}(\mathit{SP}_{\mathrm{fin}}(\mathcal{A}))$ in constant time.

Now (i) and (ii) follow immediately from this reduction. □

Theorem 7. *If a finite algebra $\mathcal{A}$ is locally tractable then so is every homomorphic image of $\mathcal{A}$.*

Proof. Let $\mathcal{B}$ be a homomorphic image of $\mathcal{A}$ and let φ be the corresponding homomorphism. We will show that for any finite $\Gamma \subseteq \mathit{SP}_{\mathrm{fin}}(\mathcal{B})$, $\mathbf{CSP}(\Gamma)$ is linear-time reducible to $\mathbf{CSP}(\Gamma')$ for some finite $\Gamma' \subseteq \mathit{SP}_{\mathrm{fin}}(\mathcal{A})$.

For $\rho \in \mathit{SP}_{\mathrm{fin}}(\mathcal{B})$, set $\varphi^{-1}(\rho) = \{\overline{a} \mid \varphi(\overline{a}) \in \rho\}$ where φ acts componentwise. It is clear that $\varphi^{-1}(\rho)$ is a relation of the same arity as ρ. It is straightforward to check that $\varphi^{-1}(\rho) \in \mathit{SP}_{\mathrm{fin}}(\mathcal{A})$. Let $\Gamma' = \{\varphi^{-1}(\rho) \mid \rho \in \Gamma\}$. Then Γ' is a finite subset of $\mathit{SP}_{\mathrm{fin}}(\mathcal{A})$.

Take an instance $\mathcal{P} = (V, B, \mathcal{C})$ of $\mathbf{CSP}(\Gamma)$ and construct the instance $\mathcal{P}' = (V, A, \mathcal{C}')$ of $\mathbf{CSP}(\Gamma')$ where $\mathcal{C}' = \{\langle s, \varphi^{-1}(\rho)\rangle \mid \langle s, \rho\rangle \in \mathcal{C}\}$.

If ψ is a solution of $\mathcal{P}'$ then $\varphi\psi$ is a solution of $\mathcal{P}$. Conversely, if ξ is a solution of $\mathcal{P}$, then any function $\psi : V \to A$ such that $\varphi\psi(v) = \xi(v)$ for any $v \in V$ is a solution of $\mathcal{P}'$. □

Corollary 2. *If $\mathcal{A}$ is a locally tractable finite algebra then so is every factor of $\mathcal{A}$.*

The following corollary from Theorems 6 and 7 seems to be quite significant, since all known forms of **NP**-complete **CSP**s (see [15, 17]) can be obtained using it. An algebra (A, G) where G is a permutation group on A is said to be a *G-set*. A G-set is said to be *non-trivial* if its universe is not a singleton.

Corollary 3. *A finite algebra $\mathcal{A}$ is* **NP***-complete if it has a factor which is term equivalent to a non-trivial G-set.*

Proof. Denote by $\mathcal{A}'$ a subalgebra of $\mathcal{A}$ such that $\mathcal{B}$ is a homomorphic image of $\mathcal{A}'$, and let B be the universe of $\mathcal{B}$.

First, consider the case when B contains only two elements, say 0 and 1. Then the relation N of Example 3 belongs to $\mathit{SP}_{\mathrm{fin}}(\mathcal{B}) = \mathit{Inv}(G)$. It follows from the proof of Theorem 7 that $\mathbf{CSP}(\{N\})$ is reducible in polynomial time to $\mathbf{CSP}(\Gamma')$ for some finite $\Gamma' \subseteq SP_{\text{fin}}(\mathcal{A}')$. Since Not-All-Equal Satisfiability which corresponds to $\mathbf{CSP}(\{N\})$ is **NP**-complete [17], we see that $\mathcal{A}'$ is also **NP**-complete. Applying Theorem 6, we conclude that $\mathcal{A}$ is **NP**-complete.

Now consider the case when $|B| \geq 3$. Then the disequality relation $\neq_B$ of Example 4 belongs to $SP_{\text{fin}}(\mathcal{B})$. Proceeding as in the previous case one can show that Graph $|B|$-Colourability (see Example 4), which is **NP**-complete [4], is reducible in polynomial time to $\mathbf{CSP}(SP_{\text{fin}}(\mathcal{A}))$. Hence, by Theorem 6, $\mathcal{A}$ is **NP**-complete in this case also. □

4 Strictly Simple Surjective Algebras

Using the results of the previous section, we now consider certain special algebras all of whose smaller factors are tractable.

Definition 15. *A finite algebra is called* strictly simple[3] *if all of its smaller factors contain one element.*

All finite strictly simple surjective algebras have been described by Szendrei [19]. Using this powerful algebraic result we are able to classify the complexity of all such algebras.

Theorem 8. *A finite strictly simple surjective algebra $\mathcal{A}$ is* **NP***-complete if it is term equivalent to a non-trivial G-set. Otherwise $\mathcal{A}$ is tractable.*

Proof. Omitted. See the full version of the paper [2].

Example 7. Let $A = \{0, 1, \ldots, m\}$, choose $k > 2$, and let $\Gamma_k = \{R_0, R_k, \{(0)\}, \{(1)\}, \ldots, \{(m)\}\}$, where

$$R_0 = \{(0,0), (1,2), (2,3), \ldots, (m-1, m), (m, 1)\};$$
$$R_k = \{(a_1, \ldots, a_k) \in A^k \mid a_i = 0 \text{ for at least one } i,\ 1 \leq i \leq k\}.$$

Using Szendrei's classification [19], it can be shown that $\mathcal{A}_{\Gamma_k}$ is a strictly simple surjective algebra, for every k. Furthermore, $\mathcal{A}_{\Gamma_k}$ contains essentially non-unary operations, and so is not term-equivalent to a G-set. Hence, by Theorem 8, $\mathbf{CSP}(\Gamma_k)$ is tractable.

Theorem 8 is a dichotomy result: it shows that any finite strictly simple surjective algebra is either tractable or **NP**-complete. Note that Schaefer's result for the Generalised Satisfiability problem (Theorem 3) is a special case of this result. Whether there is such a dichotomy in general is possibly the most intriguing open question in this area.

[3] In some papers appearing before 1990 such algebras are called *plain*.

Problem 3. Is every finite algebra either tractable or **NP**-complete?

The results obtained via the algebraic approach and, in particular, the results obtained in this paper, prompt us to make the following conjecture.

Conjecture 1. A finite algebra is **NP**-complete if it has a factor which is term equivalent to a non-trivial G-set. Otherwise it is tractable.

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