Graphs of finite algebras: maximality, rectangularity, and decomposition

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Abstract. In this paper we continue the study of edge-colored graphs associated with finite idempotent algebras initiated in [A.Bulatov, "Local structure of idempotent algebras I", CoRR, abs/2006.09599, 2020.]. We prove stronger connectivity properties of such graphs that will allows us to demonstrate several useful structural features of subdirect products of idempotent algebras such as rectangularity and 2-decomposition.

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1. Introduction and Preliminaries

We continue the study of graphs associated with algebras that was initiated in the first part of this paper [6] (see also [5]). The vertices of the graph $\mathcal{G}(\mathbb{A})$ associated with an idempotent algebra $\mathbb{A}$ are the elements of $\mathbb{A}$, and the edges are pairs of vertices that have certain properties with respect to term operations of $\mathbb{A}$, see the definitions below. Two kinds of edges were introduced, 'thick' and 'thin', where the thin version is a more technical kind, perhaps less intuitive, but also more suitable as a tool for the results of this part. Thin edges are also directed converting $\mathcal{G}(\mathbb{A})$ into a digraph. In the second part we first focus on the connectivity properties of this digraph, in particular, we show that more vertices are connected by directed paths of thin edges than one might expect (Theorem 21). Then we study the structure of subdirect products of algebras. An important role here is played by so-called as-components of algebras, which are subsets of algebras defined through certain connectivity properties. We prove that subdirect products when restricted to as-components have the property of rectangularity similar to Mal'tsev algebras, and also similar to the Absorption Theorem (see,

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e.g. [2]). Finally, we show that, again, modulo as-components any subdirect product is 2-decomposable, similar to Baker-Pixley Theorem [1].

1.1. Thick edges

We start with recalling the main definitions and results from [6]. Operation $x-y+z$ of a module $\mathbb{M}$ is said to be *affine*. Let $\mathbb{A}$ be a finite algebra with universe A. Recall that for $B\subseteq A$ the subalgebra of $\mathbb{A}$ generated by B is denoted $\mathsf{Sg}_{\mathbb{A}}(B)$, or just $\mathsf{Sg}(B)$ if $\mathbb{A}$ is clear from the context. Edges of $\mathbb{A}$ are defined as follows. A pair ab of vertices is an *edge* if and only if there exists a congruence θ of $\mathsf{Sg}(a,b)$ such that either $\mathsf{Sg}(a,b)/_\theta$ is a set, or it is term equivalent to the full idempotent reduct of a module (we will simply say that $\mathsf{Sg}(a,b)/_\theta$ is a module) and there is a term operation of $\mathbb{A}$ such that $f/_\theta$ is an affine operation of $\mathsf{Sg}(a,b)/_\theta$, or there exists a term operation of $\mathbb{A}$ such that $f/_\theta$ is a semilattice operation on $\{a/_\theta,b/_\theta\}$, or $f/_\theta$ is a majority operation on $\{a/_\theta,b/_\theta\}$.

If there exists a congruence and a term operation of $\mathbb{A}$ such that $f/_\theta$ is a semilattice operation on $\{a/_\theta,b/_\theta\}$ then ab is said to have the *semilattice* type. Edge ab is of the *majority* type if there are a congruence θ and $f\in\mathsf{Term}(\mathbb{A})$ such that $f/_\theta$ is a majority operation on $\{a/_\theta,b/_\theta\}$, but there is no term operation of $\mathbb{A}$ which is semilattice on this set. Pair ab has the *affine type* if there are a congruence θ and $f\in\mathsf{Term}(\mathbb{A})$ such that $\mathsf{Sg}(a,b)/_\theta$ is a module and $f/_\theta$ is its affine operation. Finally, ab is of the *unary type* if $\mathsf{Sg}(a,b)/_\theta$ is a set. In all cases we say that congruence θ *witnesses* the type of edge ab. The set $\{a/_\theta,b/_\theta\}$ will often be referred to as a *thick* edge.

In this paper we assume that $\mathbb{A}$ does not have edges of the unary type, which is equivalent to the statement that $\mathsf{var}(\mathbb{A})$, the variety generated by $\mathbb{A}$, omits type **1** (Theorem 5(2) of [6]). We restate the relevant results from [6].

Theorem 1 (Theorem 5(2,3) of [6]). *Let $\mathbb{A}$ be an idempotent algebra $\mathbb{A}$ such that $\mathsf{var}(\mathbb{A})$ omits type **1**. Then*

(1) *$\mathbb{A}$ contains no edges of the unary type, and any two elements of $\mathbb{A}$ are connected by a sequence of edges of the semilattice, majority, and affine types;*
(2) *$\mathsf{var}(\mathbb{A})$ omits type **2** if and only if $\mathbb{A}$ contains no edges of the unary and affine types.*

Algebra $\mathbb{A}$ is said to be *smooth* if for every edge ab of the semilattice or majority type, $a/_\theta\cup b/_\theta$, where θ is a congruence witnessing that ab is an edge, is a subalgebra of $\mathbb{A}$.

Theorem 2 (Theorem 12 of [6]). *For any idempotent algebra $\mathbb{A}$ such that $\mathbb{A}$ does not contain edges of the unary type there is a reduct $\mathbb{A}'$ of $\mathbb{A}$ that is smooth and does not contain edges of the unary type.*

Moreover, if $\mathbb{A}$ does not contain edges of the affine types, $\mathbb{A}'$ can be chosen such that it does not contain edges of the affine type.

In the case of smooth algebras the operations involved in the definition of edges can be significantly unified.

Theorem 3 (Theorem 21, Corollary 22 of [6]). *Let $\mathcal{K}$ be a finite set of similar smooth idempotent algebras. There are term operations f, g, h of $\mathcal{K}$ such that for every edge ab of $\mathbb{A} \in \mathcal{K}$, where θ is a congruence of $\mathsf{Sg}(a,b)$ witnessing that ab is an edge and $B = \{a/_\theta, b/_\theta\}$*

- **(i).** *$f|_B$ is a semilattice operation if ab is a semilattice edge, and it is the first projection if ab is a majority or affine edge;*
- **(ii).** *$g|_B$ is a majority operation if ab is a majority edge, it is the first projection if ab is an affine edge, and $g|_B(x,y,z) = f|_B(x, f|_B(y,z))$ if ab is semilattice;*
- **(iii).** *$h|_{\mathsf{Sg}(a,b)/_\theta}$ is an affine operation operation if ab is an affine edge, it is the first projection if ab is a majority edge, and $h|_B(x,y,z) = f|_B(x, f|_B(y,z))$ if ab is semilattice.*

Operations f, g, h from Theorem 3 above can be chosen to satisfy certain identities.

Lemma 4 (Lemma 23 of [6]). *Operations f, g, h identified in Theorem 3 can be chosen such that*

(1) *$f(x, f(x,y)) = f(x,y)$ for all $x, y \in \mathbb{A} \in \mathcal{K}$;*
(2) *$g(x, g(x,y,y), g(x,y,y)) = g(x,y,y)$ for all $x, y \in \mathbb{A} \in \mathcal{K}$;*
(3) *$h(h(x,y,y), y, y) = h(x,y,y)$ for all $x, y \in \mathbb{A} \in \mathcal{K}$.*

Proposition 5 (Lemmas 8,10 of [6]). *Let $\mathbb{A}$ be an idempotent algebra. Then*

(1) *if ab is an edge in $\mathbb{A}$, it is an edge of the same type in any subalgebra $\mathbb{B}$ of $\mathbb{A}$ containing a, b;*
(2) *if α is a congruence of $\mathbb{A}$ and $a/_\alpha b/_\alpha$, $a, b \in \mathbb{A}$, is an edge in $\mathcal{G}(\mathbb{A}/_\alpha)$ then ab is also an edge in $\mathcal{G}(\mathbb{A})$ of the same type.*

Note that, as Example 9 from [6] shows that if ab is an edge in $\mathbb{A}$ and $\alpha \in \mathsf{Con}(\mathbb{A})$, then $a/_\alpha b/_\alpha$ does not have to be an edge even if $a/_\alpha \neq b/_\alpha$.

1.2. Thin edges

Thin edges, also introduced in [6], offer a better technical tool. Here we generalize this concept to algebras that are not necessarily smooth.

Let $\mathcal{K}$ be a finite class of smooth idempotent algebras closed under taking subalgebras and homomorphic images and $\mathcal{V}$ the class of finite algebras from the variety it generates, that is, the pseudovariety generated by $\mathcal{K}$. We will slightly abuse the terminology and call $\mathcal{V}$ the variety generated by $\mathcal{K}$. If we are interested in a particular algebra $\mathbb{A}$, set $\mathcal{K} = \mathsf{HS}(\mathbb{A})$. Fix operations f, g, h satisfying the conditions of Theorem 3 and Lemma 4. For $\mathbb{A} \in \mathcal{K}$ and $a, b \in \mathbb{A}$, the pair ab is called a *thin semilattice edge* if the equality relation witnesses that it is a semilattice edge; or in other words if $f(a,b) = f(b,a) = b$. The binary operation f from Theorem 3 can be chosen to satisfy a special property.

Proposition 6 (Proposition 24, [6]). *Let $\mathcal{K}$ be a finite class of similar smooth idempotent algebras. There is a binary term operation f of $\mathcal{K}$ such that f is a semilattice operation on every thick semilattice edge of every $\mathbb{A} \in \mathcal{K}$ and* *for any $a, b \in \mathbb{A}$, $\mathbb{A} \in \mathcal{K}$, either $a = f(a,b)$ or the pair $(a, f(a,b))$ is a thin semilattice edge.*

We assume that operation f satisfying the conditions of Proposition 6 is fixed, and use $\cdot$ to denote it (think multiplication). If ab is a thin semilattice edge, that is, $a \cdot b = b \cdot a = b$, we write $a \le b$.

Let $\mathbb{A} \in \mathcal{V}$, $a, b \in \mathbb{A}$, $\mathbb{B} = \mathsf{Sg}(a,b)$, and θ a congruence of $\mathbb{B}$. Pair ab is said to be *minimal* with respect to θ if for any $b' \in b/_\theta$, $b \in \mathsf{Sg}(a,b')$. A ternary term g' is said to satisfy the *majority condition* with respect to $\mathcal{K}$ if it satisfies the conditions of Lemma 4(2), and g' is a majority operation on every thick majority edge of every algebra from $\mathcal{K}$. A ternary term operation h' is said to satisfy the *minority condition* if it satisfies the conditions of Lemma 4(3), and h' is a Mal'tsev operation on every thick affine edge of every algebra from $\mathcal{K}$. By Theorem 3 and Lemma 4 operations satisfying the majority and minority conditions exist.

Let $\mathbb{A} \in \mathcal{V}$ and $a, b \in \mathbb{A}$. The pair ab is a *thin semilattice edge* if the term $\cdot$ of $\mathcal{V}$ is a semilattice operation on $\{a,b\}$ and $ab = b$. It is said to be a *thin majority edge* if

(*) for any term operation g' satisfying the majority condition with respect to $\mathcal{K}$, the subalgebras $\mathsf{Sg}(a, g'(a,b,b)), \mathsf{Sg}(a, g'(b,a,b)), \mathsf{Sg}(a, g'(b,b,a))$ contain b.

If in addition to the condition above ab is also a majority edge, a congruence θ witnesses that, and ab is a minimal pair with respect to θ, we say that ab is a *special majority edge*. The pair ab is called a *thin affine edge* if for any term operation h' satisfying the minority condition *with respect to $\mathcal{K}$*

(**) $h(b,a,a) = b$ and $b \in \mathsf{Sg}(a, h'(a,a,b))$.

The operations g, h from Theorem 3 do not have to satisfy any specific conditions on the set $\{a,b\}$, when ab is a thin majority or affine edge, except what follows from their definition. Also, both thin majority and thin affine edges are directed, since a, b in the definition occur asymmetrically. Note also, that which pairs of an algebra $\mathbb{A}$ are thin majority and affine edges depends not only on the algebra itself, but also on the underlying class $\mathcal{K}$ and the choice of operations f, g, h.

It was shown in [6] that in smooth algebras every edge has a thin edge associated with it.

Lemma 7 (Corollaries 25,29,33 Lemmas 28,32, [6]). *Let $\mathbb{A} \in \mathcal{K}$ and let ab be a semilattice (majority, affine) edge, θ a congruence of $\mathsf{Sg}(a,b)$ that witnesses this, and $c \in a/_\theta$. If ab is a semilattice or majority edge, then for any $d \in b/_\theta$ such that cd is a minimal pair with respect to θ the pair cd is a thin semilattice or special majority edge. If ab is affine then for any $d \in b/_\theta$ such that ad is a minimal pair with respect to θ and $h(d,a,a) = d$ the pair ad is a thin affine edge. Moreover, $d \in b/_\theta$ satisfying these conditions exists.*

First of all we observe that Proposition 6 generalizes to algebras from $\mathcal{V}$. To prove Proposition 8 below it suffices to observe that the property of the operation $\cdot$ stated in Proposition 6 can be expressed as identities

$$x \cdot (x \cdot y) = (x \cdot y) \cdot x = x \cdot y.$$

Proposition 8. *For every $\mathbb{A} \in \mathcal{V}$ and for any $a, b \in \mathbb{A}$ either $a = a \cdot b$ or the pair $(a, a \cdot b)$ is a thin semilattice edge.*

We will need statements similar to Lemmas 31, 35 from [6] for algebras in $\mathcal{V}$.

Lemma 9. (1) *Let ab and cd be thin affine edges in $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{V}$. Then there is an operation h' such that $h'(b, a, a) = b$ and $h'(c, c, d) = d$. In particular, for any thin affine edge ab there is an operation h' such that $h'(b, a, a) = h'(a, a, b) = b$.*

(2) *Let ab and cd be thin edges in $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{V}$. If they have different types there is a binary term operation p such that $p(b, a) = b$, $p(c, d) = d$.*

Proof. (1) Let R be the subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2$ generated by $(b, c), (a, c), (a, d)$. By the definition of thin affine edges,

$$\begin{pmatrix} b \\ d' \end{pmatrix} = h\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}\right) \in R,$$

where h is the operation fixed in the beginning of the section. Then as $d \in \mathsf{Sg}_{\mathbb{A}_2}(c, h(c, c, d))$, there is a term operation $r(x, y)$ such that $d = r(c, d')$. Therefore

$$\begin{pmatrix} b \\ d \end{pmatrix} = r\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} b \\ d' \end{pmatrix}\right) \in R.$$

The result follows.

(2) Let R be the subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2$ generated by $(b, c), (a, d)$.

If ab is majority and $c \le d$, let $g'(x, y, z) = g(x, y \cdot x, z \cdot x)$. Since $x \cdot y = x$ on every (thick) majority edge of every algebra from $\mathcal{K}$, the operation g' is a majority operation on every thick majority edge of any algebra from $\mathcal{K}$. We use the construction from the proof of Lemma 4(2) (Lemma 16(2) of [6]). Consider the unary operation $g_x(y) = g'(x, y, y)$. Clearly, for some n, the operation g_x^n is idempotent for every $\mathbb{B} \in \mathcal{K}$ and $x \in \mathbb{B}$. Set $g''(x, y, z) = g_x^{n-1}(g'(x, y, z))$. Then it is not hard to see that g'' is a majority operation on every thick majority edge and that $g''(x, g''(x, y, y), g''(x, y, y)) = g''(x, y, y)$ is an identity in $\mathcal{V}$ (see the proof of Lemma 16(2) of [6] for details). Also, as $g'(d, c, c) = g(d, cd, cd) = d$, we have $g''(d, c, c) = d$.

Therefore the operation g'' satisfies the majority condition. By the definition of thin majority edges there is a binary term operation r such that $b = r(a, g''(a, b, b))$. Then

$$\begin{pmatrix} b \\ d \end{pmatrix} = r\left(\begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} g''(a, b, b) \\ d \end{pmatrix}\right) = r\left(\begin{pmatrix} a \\ d \end{pmatrix}, g''\left(\begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}\right)\right) \in R.$$

Therefore $p(x, y) = r(y, g''(y, x, x))$ satisfies the conditions required.

If ab is affine and $c \le d$, let $h'(x, y, z) = h(x, y \cdot x, z \cdot x)$. Since $x \cdot y = x$ on every (thick) affine edge of every algebra from $\mathcal{K}$, the operation h' is a Mal'tsev operation on every thick affine edge of every $\mathbb{B} \in \mathcal{K}$. We use the construction from the proof of Lemma 4(3) (Lemma 23(3) of [6]). Let $h_y(x) = h'(x, y, y)$. Clearly, for some n, the operation h_y^n is idempotent for every $\mathbb{B} \in \mathcal{K}$ and $y \in \mathbb{B}$. Let $h'_0(x, y, z) = h'(x, y, z)$ and $h'_{i+1}(x, y, z) = h'_i(h(x, y, y), y, z)$ for $i \geq 0$. Then $h'_i(x, y, y) = h_y^i(x)$. Hence $h'_n(h'_n(x, y, y), y, y) = h'_n(x, y, y)$ is an identity in $\mathcal{V}$. It is not hard to see that $h''(x, y, z) = h'_n(x, y, z)$ is a Mal'tsev operation on every thick affine edge of every $\mathbb{B} \in \mathcal{K}$. Also, as $h'(d, d, c) = h(d, dd, cd) = d$, we have $h''(d, d, c) = d$.

Thus, h'' satisfies the minority condition. Hence by the definition of thin affine edges there is a binary term operation r such that $b = r(a, h''(a, a, b))$. Then

$$\begin{pmatrix} b \\ d \end{pmatrix} = r\left(\begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} h''(a,a,b) \\ d \end{pmatrix}\right) = r\left(\begin{pmatrix} a \\ d \end{pmatrix}, h''\left(\begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}\right)\right) \in R.$$

Therefore $p(x, y) = r(y, h''(y, y, x))$ satisfies the conditions required.

If ab is affine and cd is majority, then set

$$\begin{pmatrix} b' \\ d' \end{pmatrix} = h\left(\begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}\right) \in R.$$

By the definition of thin affine edges there is a binary term operation r such that $b = r(a, h(a, a, b))$. Consider the operation

$$g'(z, y, x) = g(r(z, h(z, y, x)), r(y, h(y, z, x)), x).$$

It satisfies the following condition.

$$\begin{aligned} g'(b, a, a) &= g(r(a, h(a, a, b)), r(a, h(a, a, b)), b) \\ &= g(r(a, b'), r(a, b'), b) = g(b, b, b) = b. \end{aligned}$$

Also, on any thick majority edge $\{c'/_\theta, d'/_\theta\}$ of an algebra $\mathbb{B} \in \mathcal{K}$, where θ witnesses that $c'd'$ is a majority edge, $h(x, y, z) = x$, therefore

$$\begin{aligned} g'(x, y, z) &= g(r(z, h(z, y, x)), r(y, h(y, z, x)), x) \\ &= g(r(z, z), r(y, y), x) = g(z, y, x) \end{aligned}$$

on $\mathsf{Sg}_{\mathbb{B}}(c', d')/_\theta$.

Next, we use the argument from the beginning of item (2). Consider the unary operation $g_x(y) = g'(x, y, y)$. Let n be such that the operation g_y^n is idempotent for every $\mathbb{B} \in \mathcal{K}$ and $y \in \mathbb{B}$, and set $g''(x, y, z) = g_x^{n-1}(g'(x, y, z))$. Then g'' is a majority operation on every thick majority edge of any $\mathbb{B} \in \mathcal{K}$, and $g''(x, g''(x, y, y), g''(x, y, y)) = g''(x, y, y)$ is an identity in $\mathcal{V}$. Also, as $g'(b, a, a) = b = g_b(a) = g_b(b)$, we have $g_b^n(a) = b$, and therefore $g''(b, a, a) = b$.

Therefore the operation g'' satisfies the majority condition, and since cd is a thin majority edge, there exists a binary term operation s such that $s(c, g''(c, d, d)) = d$. Thus,

$$\begin{pmatrix} b \\ d \end{pmatrix} = s\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} b \\ g''(c,d,d) \end{pmatrix}\right) = s\left(\begin{pmatrix} b \\ c \end{pmatrix}, g''\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}\right)\right) \in R.$$

The result follows. □

1.3. Paths and filters in smooth algebras

Let $\mathbb{A} \in \mathcal{K}$ be a smooth algebra. A *path* in $\mathbb{A}$ is a sequence $a_0, a_1, \dots, a_k$ such that $a_{i-1}a_i$ is a thin edge for all $i \in [k]$ (note that thin edges are always assumed to be directed). We will distinguish paths of several types depending on what types of edges are allowed. If $a_{i-1} \leq a_i$ for all $i \in [k]$ then the path is called a *semilattice* or *s-path*. If for every $i \in [k]$ either $a_{i-1} \leq a_i$ or $a_{i-1}a_i$ is a thin affine edge then the path is called *affine-semilattice* or *as-path*. The path is called *asm-path* when all types of edges are allowed. If there is a path $a = a_0, a_1, \dots, a_k = b$ which is arbitrary (semilattice, affine-semilattice) then a is said to be *asm-connected* (or *s-connected*, or *as-connected*) to b. We will also say that a is *connected* to b if it is asm-connected. We denote this by $a \sqsubseteq^{asm} b$ (for asm-connectivity), $a \sqsubseteq b$, and $a \sqsubseteq^{as} b$ for s-, and as-connectivity, respectively. If all the thin majority edges in an asm-path are special, we call such path special. The following is a direct implication of Theorem 1 and Lemma 7.

Corollary 10. *Any two elements of a smooth algebra $\mathbb{A} \in \mathcal{K}$ are connected by an oriented path consisting of thin edges (i.e. a path in which some edges can be traversed backwards).*

Let $\mathcal{G}_s(\mathbb{A})$ $(\mathcal{G}_{as}(\mathbb{A}), \mathcal{G}_{asm}(\mathbb{A}))$ denote the digraph whose nodes are the elements of $\mathbb{A}$, and the edges are the thin semilattice edges (thin semilattice and affine edges, arbitrary thin edges, respectively). The strongly connected component of $\mathcal{G}_s(\mathbb{A})$ containing $a \in \mathbb{A}$ will be denoted by $\mathsf{s}(a)$. The set of strongly connected components of $\mathcal{G}_s(\mathbb{A})$ are ordered in the natural way (if $a \leq b$ then $\mathsf{s}(a) \leq \mathsf{s}(b)$), the elements belonging to maximal ones will be called *maximal*, and the set of all maximal elements from $\mathbb{A}$ will be denoted by $\max(\mathbb{A})$.

The strongly connected component of $\mathcal{G}_{as}(\mathbb{A})$ containing $a \in \mathbb{A}$ will be denoted by $\mathsf{as}(a)$. A maximal strongly connected component of this graph is called an *as-component*, an element from an as-component is called *as-maximal*, and the set of all as-maximal elements is denoted by $\mathsf{amax}(\mathbb{A})$.

Finally, the strongly connected component of $\mathcal{G}_{asm}(\mathbb{A})$ containing $a \in \mathbb{A}$ will be denoted by $\mathsf{asm}(a)$. A maximal strongly connected component of this graph is called an *universally maximal component* (or *u-maximal component* for short), an element from a u-component is called *u-maximal*, and the set of all u-maximal elements is denoted by $\mathsf{umax}(\mathbb{A})$.

Alternatively, maximal, as-maximal, and u-maximal elements can be characterized as follows: an element $a \in \mathbb{A}$ is maximal (as-maximal, u-maximal) if for every $b \in \mathbb{A}$ such that $a \sqsubseteq b$ ($a \sqsubseteq^{as} b$, $a \sqsubseteq^{asm} b$) it also holds that $b \sqsubseteq a$ ($b \sqsubseteq^{as} a$, $b \sqsubseteq^{asm} a$). Sometimes it will be convenient to specify what the algebra is, in which we consider maximal components, as-components, or u-maximal components, and the corresponding connectivity. In such cases we will specify it by writing $\mathsf{s}_{\mathbb{A}}(a)$, $\mathsf{as}_{\mathbb{A}}(a)$, or $\mathsf{asm}_{\mathbb{A}}(a)$. For connectivity we will use $a \sqsubseteq_{\mathbb{A}} b$, $a \sqsubseteq^{as}_{\mathbb{A}} b$, and $a \sqsubseteq^{asm}_{\mathbb{A}} b$.

By $\mathrm{Ft}_{\mathbb{A}}(a) = \{b \in \mathbb{A} \mid a \sqsubseteq_{\mathbb{A}} b\}$ we denote the set of elements a is connected to (in terms of semilattice paths); similarly, by $\mathrm{Ft}^{as}_{\mathbb{A}}(a) = \{b \in \mathbb{A} \mid$

$a \sqsubseteq^{as}_{\mathbb{A}} b\}$ and $\mathrm{Ft}^{asm}_{\mathbb{A}}(a) = \{b \in \mathbb{A} \mid a \sqsubseteq^{asm}_{\mathbb{A}} b\}$ we denote the set of elements a is as-connected and asm-connected to. Also, $\mathrm{Ft}_{\mathbb{A}}(C) = \bigcup_{a\in C} \mathrm{Ft}_{\mathbb{A}}(a)$ ($\mathrm{Ft}^{as}_{\mathbb{A}}(C) = \bigcup_{a\in C} \mathrm{Ft}^{as}_{\mathbb{A}}(a)$, $\mathrm{Ft}^{asm}_{\mathbb{A}}(C) = \bigcup_{a\in C} \mathrm{Ft}^{asm}_{\mathbb{A}}(a)$, respectively) for $C \subseteq \mathbb{A}$. Note that if a is a maximal (as-maximal or u-maximal) element then $\mathsf{s}(a) = \mathrm{Ft}_{\mathbb{A}}(a)$ ($\mathsf{as}(a) = \mathrm{Ft}^{as}_{\mathbb{A}}(a)$, and $\mathsf{umax}(a) \subseteq \mathrm{Ft}^{asm}_{\mathbb{A}}(b)$).

1.4. Paths in non-smooth algebras

Given the notion of thin edges, (s-, as-, asm-) paths, (as-, asm-) maximal elements in algebras from $\mathcal{V}$ can be defined in the same way as for algebras in $\mathcal{K}$, even if those algebras are not smooth. We use the same notation $\leq$, $\sqsubseteq$, $\sqsubseteq^{as}$, $\sqsubseteq^{asm}$, $\mathsf{s}(a)$, $\mathsf{as}(a)$, $\mathsf{asm}(a)$, $\max(\mathbb{A})$, $\mathsf{amax}(\mathbb{A})$, $\mathsf{umax}(\mathbb{A})$ as before. In this section we study properties of such paths and maximal elements and the connections between (as-, asm-) maximal elements of an algebra with those in a quotient algebra or subdirect product.

Lemma 11. *Let $\mathbb{A} \in \mathcal{V}$ and $\theta \in \mathsf{Con}(\mathbb{A})$.*

(1) *If $\overline{a}\overline{b}$ is a thin edge in $\mathbb{A}/_\theta$ and $a \in \overline{a}$, then there is $b \in \overline{b}$ such that ab is a thin edge in $\mathbb{A}$ of the same type. Morever, if $\mathbb{A} \in \mathcal{K}$ and $\overline{a}\overline{b}$ is a special thin majority edge, then so is ab.*
(2) *If ab is a thin edge in $\mathbb{A}$, then $a/_\theta b/_\theta$ is a thin edge in $\mathbb{A}/_\theta$.*

Remark 12. *Note that in Lemma* 11*(2) if ab is a special thin majority edge, there is no guarantee that $a/_\theta b/_\theta$ is also a special edge.*

Proof. Pick an arbitrary $b \in \overline{b}$ such that the pair ab is minimal with respect to θ. Let $\mathbb{B} = \mathsf{Sg}_{\mathbb{A}}(a, b)$.

First, if $\overline{a}\overline{b}$ is a thin semilattice edge, then $a \cdot b \in \overline{a} \cdot \overline{b} = \overline{b}$. As $a \neq b$, by Proposition 8 (a, ab) is a semilattice edge.

Now suppose that $\overline{a}\overline{b}$ is a thin majority edge. Let g' be an operation satisfying the majority property with respect to $\mathcal{K}$. Let $c = g'(a, b, b)$. Since $\overline{a}\overline{b}$ is a thin majority edge, there is a binary term operation t such that $t(\overline{a}, c/_\theta) = \overline{b}$. Note that $t(a, c) \in \mathbb{B}$, as well. By the choice of b, it holds that $b \in \mathsf{Sg}(a, t(a, c))$. Therefore, $b \in \mathsf{Sg}(a, g'(a, b, b))$. That $b \in \mathsf{Sg}(a, g'(b, a, b))$ and $b \in \mathsf{Sg}(a, g'(b, b, a))$ can be proved in the same way. Suppose in addition that $\mathbb{A} \in \mathcal{K}$ and $\overline{a}\overline{b}$ is a special thin majority edge, that is, it is a majority edge in $\mathbb{A}/_\theta$ and it is witnessed by a congruence η of $\mathsf{Sg}_{\mathbb{A}/_\theta}(\overline{a}, \overline{b})$. It is then straightforward that ab is a majority edge as witnessed by the congruence $\eta/_\theta = \{(d, e) \in \mathbb{B}^2 \mid d/_\theta \stackrel{\eta}{\equiv} e/_\theta\}$.

Finally, suppose that $\overline{a}\overline{b}$ is a thin affine edge, and set $b' = h(b, a, a)$, where h is the operation identified in Theorem 3(iii). By Lemma 4(3) it holds that $h(b', a, a) = b'$. Note that ab' is a minimal pair as well. Consider $c = h'(a, a, b')$, for any h' satisfying the minority condition with respect to K. Since $\overline{a}\overline{b}$ is a thin affine edge, $\overline{b} \in \mathsf{Sg}_{\mathbb{A}/_\theta}(\overline{a}, c/_\theta)$. This means that there is $d \in \mathsf{Sg}_{\mathbb{A}}(a, c)$ such that $d \in \overline{b}$. Since $d \in \mathsf{Sg}_{\mathbb{A}}(a, b')$ and ab' is a minimal pair with respect to θ, we obtain $b' \in \mathsf{Sg}_{\mathbb{A}}(a, d) \subseteq \mathsf{Sg}_{\mathbb{A}}(a, c)$. Thus ab' is a thin affine edge.

(2) If $a \stackrel{\theta}{\equiv} b$, the statement of the lemma is trivial. Otherwise if $a \le b$, then $a/_\theta \cdot b/_\theta = b/_\theta \cdot a/_\theta = b/_\theta$ showing that $a/_\theta \le b/_\theta$. If ab is a thin majority edge, then consider an abitrary g' satisfying the majority condition. Since $b \in \mathsf{Sg}_{\mathbb{A}}(a, g'(a,b,b))$, we also have $b/_\theta \in \mathsf{Sg}_{\mathbb{A}/_\theta}(a/_\theta, g'(a/_\theta, b/_\theta, b/_\theta))$. The argument for the rest of condition (*) is similar. If ab is a thin affine edge, then $h(b,a,a) = b$, implying $h(b/_\theta, a/_\theta, a/_\theta) = b/_\theta$. Also, for any h' satisfying the minority condition, as $b \in \mathsf{Sg}_{\mathbb{A}}(a, h'(a,a,b))$, we have $b/_\theta \in \mathsf{Sg}_{\mathbb{A}}(a/_\theta, h'(a/_\theta, a/_\theta, b/_\theta))$. □

The next statement straightforwardly follows from Lemma 11.

Corollary 13. *Let $\mathbb{A} \in \mathcal{V}$ and $\theta \in \mathsf{Con}\mathbb{A}$.*

(1) *If $\overline{a}_1, \dots, \overline{a}_k$ is an s- (as-,asm-) path in $\mathbb{A}/_\theta$ and $a \in \overline{a}_1$, then there are $a_i \in \overline{a}_i$ such that $a_1 = a$ and $a_1, \dots, a_k$ is an s- (as-, asm-) path in $\mathbb{A}$. Morever, if $\mathbb{A} \in \mathcal{K}$ and $\overline{a}_1, \dots, \overline{a}_k$ is a special asm-path, then $a_1, \dots, a_k$ is also a special asm-path.*

(2) *If $a_1, \dots, a_k$ is an s- (as-, asm-) path in $\mathbb{A}$, then $a_1/_\theta, \dots, a_\ell/_\theta$ is an s- (as-, asm-) path in $\mathbb{A}/_\theta$.*

Corollary 14. *Let $\mathbb{A} \in \mathcal{V}$ and $\theta \in \mathsf{Con}\mathbb{A}$.*

(1) *If $\overline{a} \in \mathbb{A}/_\theta$ is maximal (as-maximal, u-maximal) in $\mathbb{A}/_\theta$, then there is $a \in \overline{a}$ that is maximal (as-maximal, u-maximal) in $\mathbb{A}$.*

(2) *If a is maximal (as-maximal, u-maximal) in $\mathbb{A}$, then $a/_\theta$ is maximal (as-maximal, u-maximal) in $\mathbb{A}/_\theta$.*

Proof. (1) We will use notation $c \sqsubseteq^x d$, $x \in \{s, as, asm\}$, to denote that there is an x-path from c to d. Pick an arbitrary $a' \in \overline{a}$ and let b be any x-maximal element of $\mathbb{A}$ such that $a' \sqsubseteq^x b$. This means that there is an x-path from a' to b, and by Corollary 13(2) there is also an x-path from $a'/_\theta$ to $b/_\theta$ in $\mathbb{A}/_\theta$. Since by the assumption $\overline{a}$ is x-maximal, there is also an x-path in $\mathbb{A}/_\theta$ from $b/_\theta$ to $\overline{a}$. By Corollary 13(1) there is an x-path in $\mathbb{A}$ from b to some element $a'' \in \overline{a}$. Since b is an x-maximal element, so is a''.

(2) Suppose that $a/_\theta \sqsubseteq^x b/_\theta$ for some $b \in \mathbb{A}$. It suffices to show that in this case $b/_\theta \sqsubseteq^x a/_\theta$. By Corollary 13(1) there is an x-path in $\mathbb{A}$ from a to some $b' \in b/_\theta$. Since a is x-maximal, there also exists an x-path $b' = b_1, \dots, b_k = a$ from b' to a. By Corollary 13(2) $b/_\theta = b_1/_\theta, \dots, b_k/_\theta = a/_\theta$ is an x-path in $\mathbb{A}/_\theta$ from $b/_\theta$ to $a/_\theta$. □

Next we study connectivity in subalgebras of direct products.

Lemma 15. *Let R be a subalgebra of $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$, $\mathbb{A}_1, \dots, \mathbb{A}_n \in \mathcal{V}$, $I \subseteq [n]$.*

(1) *For any $\mathbf{a} \in R$, $\mathbf{a}^* = \mathrm{pr}_I \mathbf{a}$, $\mathbf{b} \in \mathrm{pr}_I R$ such that $\mathbf{a}^*\mathbf{b}$ is a thin edge, there is $\mathbf{b}' \in R$, $\mathrm{pr}_I \mathbf{b}' = \mathbf{b}$, such that $\mathbf{a}\mathbf{b}'$ is a thin edge of the same type.*

(2) *If $\mathbf{ab}$ is a thin edge in R then $\mathrm{pr}_I \mathbf{a}\, \mathrm{pr}_I \mathbf{b}$ is a thin edge in $\mathrm{pr}_I R$ of the same type (including the possibility that $\mathrm{pr}_I \mathbf{a} = \mathrm{pr}_I \mathbf{b}$).*

Proof. Observe that we can consider $\mathrm{pr}_I R$ as the quotient algebra $R/_{\eta_I}$, where η_I is the projection congruence of R, that is, $(\mathbf{c}, \mathbf{d}) \in \eta_I$ if and only if $\mathrm{pr}_I \mathbf{c} =$

$\mathrm{pr}_I\mathbf{d}$. Then item (1) can be rephrased as follows: For any $\mathbf{a} \in R$ and $\bar{b} \in R/_{\eta_I}$ such that $\mathbf{a}/_{\eta_I}\bar{b}$ is a thin edge in $R/_{\eta_I}$, there is $\mathbf{b}' \in \bar{b}$ such that $\mathbf{ab}'$ is a thin edge of the same type. It clearly follows from Lemma 11(1). Similarly, item (2) can be rephrased as: If $\mathbf{ab}$ is a thin edge in R, then $\mathbf{a}/_{\eta_I}\mathbf{b}/_{\eta_I}$ is a thin edge in $R/_{\eta_I}$ of the same type. It follows straightforwardly from Lemma 11(2). $\square$

The next statement follows from Lemma 15.

Corollary 16. *Let R be a subalgebra of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $\mathbb{A}_1, \dots, \mathbb{A}_n \in \mathcal{V}$, and $I \subseteq [n]$.*

(1) *For any $\mathbf{a} \in R$, and an s- (as-, asm-) path $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathrm{pr}_I R$ with $\mathrm{pr}_I\mathbf{a} = \mathbf{b}_1$, there is an s- (as-, asm-) path $\mathbf{b}'_1, \dots, \mathbf{b}'_k \in R$ such that $\mathbf{b}'_1 = \mathbf{a}$ and $\mathrm{pr}_I\mathbf{b}'_i = \mathbf{b}_i$, $i \in [k]$. Moreover, if $\mathbf{b}_1, \dots, \mathbf{b}_k$ is a special asm-path, so is $\mathbf{b}'_1, \dots, \mathbf{b}'_k$.*

(2) *If $\mathbf{a}_1, \dots, \mathbf{a}_k$ is an s- (as-, asm-) path in R, then $\mathrm{pr}_I\mathbf{a}_1, \dots, \mathrm{pr}_I\mathbf{a}_k$ is an s- (as-, asm-) path in $\mathrm{pr}_I R$.*

There is a connection between maximal (as-maximal, u-maximal) elements of a subdirect product and its projections.

Corollary 17. *Let R be a subalgebra of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $\mathbb{A}_1, \dots, \mathbb{A}_n \in \mathcal{V}$, and $I \subseteq [n]$.*

(1) *For any maximal (as-maximal, u-maximal) (in $\mathrm{pr}_I R$) element $\mathbf{b} \in \mathrm{pr}_I R$, there is $\mathbf{b}' \in R$ which is maximal (as-maximal, u-maximal) in R and such that $\mathrm{pr}_I\mathbf{b}' = \mathbf{b}$. In particular, $\mathrm{pr}_{[n]-I}\mathbf{b}'$ is a maximal (as-maximal, u-maximal) in $\mathrm{pr}_{[n]-I}R$.*

(2) *If $\mathbf{a}$ is maximal (as-maximal, u-maximal) in R, then $\mathrm{pr}_I\mathbf{a}$ is maximal (as-maximal, u-maximal) in $\mathrm{pr}_I R$.*

Proof. (1) We again can use the isomorphism between $\mathrm{pr}_I R$ and the quotient algebra $R/_{\eta_I}$. Then the statement of the lemma translates into: If $\bar{a} \in R/_{\eta_I}$ is maximal (as-maximal, u-maximal) in $R/_{\eta_I}$, then there is $a \in \bar{a}$ that is maximal (as-maximal, u-maximal) in R. The latter statement is true by the second part of this corollary.

(2) Using the isomorphism between $\mathrm{pr}_I R$ and the quotient algebra $R/_{\eta_I}$, the statement follows from Corollary 14(2). Indeed, as $\mathbf{a}$ is maximal (as-maximal, u-maximal) in R, the element $\mathbf{a}/_{\eta_I}$ that corresponds to $\mathrm{pr}_I\mathbf{a}$ is maximal (as-maximal, u-maximal) in $R/_{\eta_I}$. $\square$

The following lemma considers a special case of maximal components of various types in subdirect products.

Lemma 18. *Let R be a subdirect product of $\mathbb{A}_1 \times \mathbb{A}_2$, B, C maximal components (as-components, u-components) of $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{V}$, respectively, and $B \times C \subseteq R$. Then $B \times C$ is a maximal component (as-component, u-component) of R.*

Proof. We need to show that for any $\mathbf{a}, \mathbf{b} \in B \times C$ there is an s-path (as-path, asm-path) from $\mathbf{a}$ to $\mathbf{b}$. By the assumption there is a (s-,as-,asm-) path $\mathbf{a}[1] = a_1, a_2, \dots, a_k = \mathbf{b}[1]$ from $\mathbf{a}[1]$ to $\mathbf{b}[1]$ and a (s-,as-,asm-) path $\mathbf{a}[2] = b_1, b_2, \dots, b_\ell = \mathbf{b}[2]$ from $\mathbf{a}[2]$ to $\mathbf{b}[2]$. Then the sequence

$\mathbf{a} = (a_1, b_1), (a_2, b_1), \ldots, (a_k, b_1), (a_k, b_2), \ldots, (a_k, b_\ell) = \mathbf{b}$ is a (s-,as,asm-) path from $\mathbf{a}$ to $\mathbf{b}$. Thus, $B \times C$ is a connected component. It remains to observe that by Corollary 17 it contains a maximal (as-maximal, u-maximal) element, and therefore is a maximal component (as-component, u-component) of R. □

2. Connectivity

Recall that $\mathcal{K}$ is a finite class of finite smooth idempotent algebras omitting type $\mathbf{1}$, and $\mathcal{V}$ is the variety it generates.

2.1. General connectivity

The main result of this section is that all maximal elements are connected to each other. The undirected connectivity easily follows from the definitions, and Theorem 1, so the challenge is to prove directed connectivity, as defined above. We start with an auxiliary lemma.

Let $R \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_k$ be a relation. Also, let $\mathsf{tol}_i(R)$ (or simply tol_i if R is clear from the context), $i \in [k]$, denote the *link* tolerance

$$\{(a_i, a_i') \in \mathbb{A}_i^2 \mid (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_k),\\ (a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_k) \in R, \text{ for some } (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k)\}.$$

Recall that a tolerance is said to be *connected* if its transitive closure is the full relation. The transitive closure $\mathsf{lk}_i(R)$ of $\mathsf{tol}_i(R)$, $i \in [k]$, is called the *link congruence*, and it is, indeed, a congruence.

If R is binary, that is, a subdirect product of $\mathbb{A}_1, \mathbb{A}_2$, then by $R[c], R^{-1}[c']$ for $c \in \mathbb{A}_1, c' \in \mathbb{A}_2$ we denote the sets $\{b \mid (c, b) \in R\}, \{a \mid (a, c') \in R\}$, respectively, and for $C \subseteq \mathbb{A}_1, C' \subseteq \mathbb{A}_2$ we use $R[C] = \bigcup_{c \in C} R[c]$, $R^{-1}[C'] = \bigcup_{c' \in C'} R^{-1}[c']$, respectively. Relation R is said to be *linked* if the link congruences $\mathsf{lk}_1(R), \mathsf{lk}_2(R)$ are full congruences.

Lemma 19. *Let $\mathbb{A} \in \mathcal{V}$ and let S be a tolerance of $\mathbb{A}$. Suppose that (a, b), $a, b \in \mathbb{A}$, belongs to the transitive closure of S, that is, there are $d_1, \ldots, d_{k-1}$ such that $(d_i, d_{i+1}) \in S$ for $i \in \{0, 1, \ldots, k-1\}$, where $d_0 = a, d_k = b$. If for some $i \in \{0, 1, \ldots, k\}$ there is $d_i' \in \mathbb{A}$ such that $d_i \sqsubseteq d_i'$, then there are $d_j' \in \mathbb{A}$ for $j \in \{0, 1, , \ldots, k\} - \{i\}$ such that $d_j \sqsubseteq d_j'$ for $j \in \{0, 1, \ldots, k\}$, and $(d_j', d_{j+1}') \in S$ for $j \in \{0, 1, \ldots, k-1\}$.*

Moreover, if $d_0, \ldots, d_{i-1}$ are maximal, there are $d_0'', \ldots, d_k''$ such that $d_j'' = d_j$ for $j \in \{0, \ldots, i-1\}$, $d_i \sqsubseteq d_i' \sqsubseteq d_i''$, $d_j \sqsubseteq d_j''$ for $j \in \{i+1, \ldots, k\}$, and $(d_j'', d_{j+1}'') \in S$ for $j \in \{0, \ldots, k-1\}$.

Proof. Let $d_i = d_i^1 \le \ldots \le d_i^s = d_i'$ be an s-path from d_i to d_i'. For each $j \in \{0, 1, \ldots, k\} - \{i\}$, we construct a sequence $d_j = d_j^1 \le \ldots \le d_j^s$ by setting

$$d_j^q = d_j^{q-1} \cdot d_i^q.$$

Now, we prove by induction that $(d_j^q, d_{j+1}^q) \in S$ for all $q \in [s]$ and $j \in \{0, 1, \ldots, k-1\}$. For $q = 1$ it follows from the assumptions of the lemma. If

$(d_j^q, d_{j+1}^q) \in S$, then $(d_j^{q+1}, d_{j+1}^{q+1}) = (d_j^q \cdot d_i^{q+1}, d_{j+1}^q \cdot d_i^{q+1}) \in S$, since S is a tolerance. The first part of the lemma is proved.

For the second statement we apply the same construction, but only to elements $d_{i-1}, \dots, d_k$, and do not change $d_0, \dots, d_{i-2}$, that is, we set $d'_j = d_j$ for $j \in \{0, \dots, i-2\}$. Then, as before, we have $(d'_j, d'_{j+1}) \in S$ for $j \in \{i-1, \dots, k\}$ and for $j \in \{0, \dots, i-3\}$. However, there is no guarantee that $(d'_{i-2}, d'_{i-1}) \in S$. To remedy this we again apply the same construction as follows. Since d_{i-1} is a maximal element and $d_{i-1} \sqsubseteq d'_{i-1}$, it also holds $d'_{i-1} \sqsubseteq d_{i-1}$. Let $d'_{i-1} = e_{i-1}^1 \le \dots \le e_{i-1}^\ell = d_{i-1}$ be an s-path connecting d'_{i-1} to d_{i-1}. We set $d''_j = d_j$ for $j \in \{0, \dots, i-2\}$, and $e_j^1 = d'_j$, $e_j^{q+1} = e_j^q \cdot e_{i-1}^{q+1}$ and $d''_j = e_j^\ell$ for $j \in \{i-1, \dots, k\}$. Then, as before, $(d''_j, d''_{j+1}) \in S$ for $j \in \{i-1, \dots, k-1\}$ and for $j \in \{0, \dots, i-3\}$. However, since $d''_{i-2} = d_{i-2}$ and $d''_{i-1} = d_{i-1}$, we also have $(d''_{i-2}, d''_{i-1}) \in S$. The result follows. $\square$

Corollary 20. *Let $\mathbb{A} \in \mathcal{V}$ and S be a tolerance of $\mathbb{A}$. Suppose that (a, b), $a, b \in \mathbb{A}$, belongs to the transitive closure of S, that is, there are $d_1, \dots, d_{k-1}$ such that $(d_i, d_{i+1}) \in S$ for $i \in \{0, 1, \dots, k-1\}$, where $d_0 = a, d_k = b$. If a, b are maximal in $\mathbb{A}$, then there are $d'_0, \dots, d'_k$ such that $d'_0 = a$, each d'_j is maximal in $\mathbb{A}$, $(d'_j, d'_{j+1}) \in S$ for $j \in \{0, \dots, k-1\}$, and $d'_k \in \mathsf{s}_{\mathbb{A}}(b)$.*

Proof. We show by induction on i that there are $d'_0, \dots, d'_k$ such that $d'_0 = a$, $(d'_j, d'_{j+1}) \in S$ for $j \in \{0, \dots, k-1\}$, $d'_k \in \mathsf{s}_{\mathbb{A}}(b)$ and $d'_0, \dots, d'_i$ are maximal. The base case of induction, $i = 0$, follows from the conditions of the lemma. Suppose some d'_js with the required properties exist for $i \in \{0, 1, \dots, k-1\}$. Let d^*_{i+1} be a maximal element with $d'_{i+1} \sqsubseteq d^*_{i+1}$. By Lemma 19 there are $d''_{i+1}, \dots, d''_k$ such that $d^*_{i+1} \sqsubseteq d''_{i+1}$, $d'_j \sqsubseteq d''_j$ for $j \in \{i+2, \dots, k\}$, $(d'_i, d''_{i+1}) \in S$, and $(d''_j, d''_{j+1}) \in S$ for $j \in \{i+1, \dots, k-1\}$. Elements $d'_0, \dots, d'_i, d''_{i+1}, \dots, d''_k$ satisfy the required conditions. $\square$

The next theorem is the main result of this section. Note that we consider smooth algebras first.

Theorem 21. *Let $\mathbb{A} \in \mathcal{K}$. Any $a, b \in \max(\mathbb{A})$ (or $a, b \in \mathsf{amax}(\mathbb{A})$, or $a, b \in \mathsf{umax}(\mathbb{A})$) are asm-connected. Moreover if $a, b \in \max(\mathbb{A})$ or $a, b \in \mathsf{amax}(\mathbb{A})$, they are connected by a special path.*

Proof. We start by showing asm-connectivity by a special path (most of the time we will not mention that we are looking for a special path, except when it is essential) for maximal elements, so let $a, b \in \max(\mathbb{A})$. We proceed by induction on the size of $\mathbb{A}$ through a sequence of claims. In the base case of induction, when $\mathbb{A} = \{a, b\}$, elements a, b are connected by a thin majority or affine edge, as they are both maximal, and the claim is straightforward.

Claim 1. $\mathbb{A}$ can be assumed to be $\mathsf{Sg}(a, b)$.

If $a, b \in \max(\mathbb{B})$, $\mathbb{B} = \mathsf{Sg}(a, b)$, and $\mathbb{B} \neq \mathbb{A}$, then we are done by the induction hypothesis. Suppose one of them is not maximal in $\mathbb{B}$, and let $c, d \in \max(\mathbb{B})$ be such that $a \sqsubseteq c$ and $b \sqsubseteq d$. By the induction hypothesis c is asm-connected to d. As $a \sqsubseteq c$, a is asm-connected to c. It remains to show that d is asm-connected to b. This, however, follows straightforwardly from the assumption that b is maximal. Indeed, it implies $d \in \mathsf{s}_{\mathbb{A}}(b)$, and hence $d \sqsubseteq b$ in $\mathbb{A}$.

Claim 2. $\mathbb{A}$ can be assumed to be simple.

Suppose $\mathbb{A}$ is not simple and α is its maximal congruence. Let $\mathbb{B} = \mathbb{A}/_{\alpha}$. By the induction hypothesis $a/_{\alpha}$ is asm-connected to $b/_{\alpha}$ with a special path, that is, there is a sequence $a/_{\alpha} = \overline{a}_0, \overline{a}_1, \dots, \overline{a}_k = b/_{\alpha}$ such that $\overline{a}_i \leq \overline{a}_{i+1}$ or $\overline{a}_i\overline{a}_{i+1}$ is a thin affine or special majority edge in $\mathbb{B}$. By Corollary 13(1) there is a special path $a_1, \dots, a_k$ such that $a_1 = a$ and $a_i \in \overline{a}_i$ in $\mathbb{A}$.

It remains to show that a_k is asm-connected to b. Since b is maximal, it suffices to take elements a', b' maximal in $b/_{\alpha}$ and such that $a_k \sqsubseteq_{\overline{a}_k} a'$ and $b \sqsubseteq_{\overline{a}_k} b'$. Then $a_k \sqsubseteq_{\overline{a}_k} a'$, element a' is asm-connected to b' in $\overline{a}_k$ by the induction hypothesis, and b' is asm-connected to b in $\mathbb{A}$, as $b' \in \mathsf{s}_{\mathbb{A}}(b)$.

Claim 3. $\mathsf{Sg}(a, b)$ can be assumed to be equal to $\mathsf{Sg}(a', b')$ for any $a' \in \mathsf{s}_{\mathbb{A}}(a)$, $b' \in \mathsf{s}_{\mathbb{A}}(b)$.

If $\mathsf{Sg}(a', b') \subset \mathsf{Sg}(a, b)$ for some $a' \in \mathsf{s}_{\mathbb{A}}(a)$, $b' \in \mathsf{s}_{\mathbb{A}}(b)$, then by the induction hypothesis a'' is asm-connected to b'' for some $a'' \in \mathsf{s}_{\mathbb{A}}(a)$, $b'' \in \mathsf{s}_{\mathbb{A}}(b)$. Therefore a is also asm-connected to b.

From now on we assume $\mathbb{A}$ and a, b to satisfy all the conditions of Claims 1–3.

We say that elements $c, d \in \mathbb{A}$ with $c \in \max(\mathbb{A})$ are connected by proper subalgebras of $\mathbb{A}$ if there are subalgebras $\mathbb{B}_1, \dots, \mathbb{B}_\ell$ such that $\mathbb{B}_i \neq \mathbb{A}$ for $i \in [\ell]$, $c \in \mathbb{B}_1$, $d \in \mathbb{B}_\ell$, and $\mathbb{B}_i \cap \mathbb{B}_{i+1} \cap \max(\mathbb{A}) \neq \emptyset$ for $i \in [\ell - 1]$.

Claim 4. Let R_{ab} be the subalgebra of $\mathbb{A}^2$ generated by $(a, b), (b, a)$. Then either R_{ab} is the graph of an automorphism φ of $\mathbb{A}$ such that $\varphi(a) = b$, $\varphi(b) = a$, or a, b are connected by proper subalgebras of $\mathbb{A}$.

Suppose that R_{ab} is not the graph of a mapping, or, in other words, there is no automorphism of $\mathbb{A}$ that maps a to b and b to a. We consider the link tolerance $S = \mathsf{tol}_1(R_{ab})$. Since $\mathbb{A}$ is simple and R is not the graph of a mapping, the transitive closure of S is the full relation. Suppose first that for every $e \in \mathbb{A}$ the set $R_{ab}^{-1}[e] = \{d' \mid (d', e) \in R\}$ (which is a subalgebra of $\mathbb{A}$) does not equal $\mathbb{A}$. There are $e_1, \dots, e_k \in \mathbb{A}$ such that the subalgebras $\mathbb{B}_1, \dots, \mathbb{B}_k$, $\mathbb{B}_i = R_{ab}^{-1}[e_i]$ are such that $a \in \mathbb{B}_1$, $b \in \mathbb{B}_k$, and $\mathbb{B}_i \cap \mathbb{B}_{i+1} \neq \emptyset$ for every $i \in [k-1]$. Choose $d_i \in \mathbb{B}_i \cap \mathbb{B}_{i+1}$ for $i \in [k-1]$ and note that $(d_i, d_{i+1}) \in S$ and also $(a, d_1), (d_{k-1}, b) \in S$. By Corollary 20 it is possible to choose $d'_1, \dots, d'_k \in \mathbb{A}$ such that $(a, d'_1) \in S$, $(d'_i, d'_{i+1}) \in S$ for $i \in [k-1]$, all the d'_i are maximal, and $b \sqsubseteq b' = d'_k$. The conditions $(a, d'_1) \in S$, $(d'_i, d'_{i+1}) \in S$ mean that there are $e'_0, e'_1, \dots, e'_k \in \mathbb{A}$ such that $d'_i, d'_{i+1} \in R^{-1}[e'_i]$ for $i \in \{2, \dots, k\}$ and $c, d'_1 \in R_{ab}^{-1}[e'_1]$. Therefore elements a, b' are connected by proper subalgebras of $\mathbb{A}$. Since b is maximal, $b' \sqsubseteq b$, and so b' is connected to b with proper subalgebras as well, which are the thin semilattice edges in the path from b' to b.

Suppose that there is $e \in \mathbb{A}$ such that $\mathbb{A} \times \{e\} \subseteq R$. If $e \notin \max(\mathbb{A})$, choose $e' \in \max(\mathbb{A})$ with $e \sqsubseteq e'$. By Corollary 16 the path from e to e' can be extended to paths from $(a,e),(b,e)$ to some $(a',e'),(b',e')$, respectively. Then, as $a \sqsubseteq a'$ and $a \in \max(\mathbb{A})$, we also have $a' \sqsubseteq a$. The path from a' to a can again be extended to a path from (a',e') to (a,e'') for some e''. Also, the path from e' to e'' can be extended to a path from (b',e') to (b'',e''). As is easily seen, $b'' \in \mathsf{s}(b)$ and e'' is maximal. Since $\mathbb{A} = \mathsf{Sg}(a,b'')$ by Claim 3, we have $\mathbb{A} \times \{e''\} \subseteq R$. Thus, e can be assumed to be maximal. We have therefore $(a,b),(a,e),(b,e),(b,a) \in R$. If both $\mathsf{Sg}(b,e)$ and $\mathsf{Sg}(e,a)$ are proper subalgebras of $\mathbb{A}$, then a,b are connected by proper subalgebras of $\mathbb{A}$. Otherwise suppose $\mathsf{Sg}(b,e) = \mathbb{A}$. This means $\{a\} \times \mathbb{A} \subseteq R_{ab}$, and, in particular, $(a,a) \in R_{ab}$. Therefore there is a binary term operation f such that $f(a,b) = f(b,a) = a$. This means $b \le a$ or $a \le b$, where the latter case is possible if there is also another semilattice operation f' on $\{a,b\}$ with $f'(a,b) = f'(b,a) = b$, and this operation is picked for the relation $\le$. If $\mathsf{Sg}(a,e) = \mathbb{A}$ then by a similar argument we get $a \le b$ or $b \le a$. However, $a \le b$ or $b \le a$ is an impossibility, because in this case $\mathbb{A} = \{a,b\}$, and only one of a,b is maximal.

Claim 5. Assuming the induction hypothesis, if $c \in \max(\mathbb{A})$ and $d \in \mathbb{A}$ are connected by proper subalgebras, then c and d' for some d', $d \sqsubseteq d'$, are asm-connected.

There are proper subalgebras $\mathbb{B}_1, \dots, \mathbb{B}_k$ of $\mathbb{A}$ such that $c \in \mathbb{B}_1$, $d \in \mathbb{B}_k$, and $\mathbb{B}_i \cap \mathbb{B}_{i+1} \cap \max(\mathbb{A}) \neq \emptyset$ for every $i \in [k-1]$. Let $e_i \in B_i \cap B_{i+1}$ be an element maximal in $\mathbb{A}$. For each $i \in [k-1]$ choose $c_i, c_i' \in \max(\mathbb{B}_i)$ with $e_{i-1} \sqsubseteq_{\mathbb{B}_i} c_i$ and $e_i \sqsubseteq_{\mathbb{B}_i} c_i'$. By the induction hypothesis c_i is asm-connected to c_i' (in $\mathbb{B}_i$). Then clearly e_{i-1} is asm-connected to c_i, and, as e_i is maximal in $\mathbb{A}$ and $c_i' \in \mathsf{s}_{\mathbb{A}}(e_i)$, c_i' is asm-connected to e_i, as well. The element d' is then any with $d' \in \max(\mathbb{B}_k)$ and $d \sqsubseteq d'$.

Let $R = R_{ab}$. We consider two cases.

Case 1. R is not the graph of a mapping.

By Claim 4 a,b are connected by proper subalgebras. By Claim 5 a is asm-connected to some $b' \in \mathsf{s}_{\mathbb{A}}(b)$, which is s-connected to b.

Case 2. R is the graph of a mapping, or, in other words, there is an automorphism of $\mathbb{A}$ that maps a to b and b to a.

We again consider several cases.

Subcase 2a. There is no nonmaximal element $c \le a'$ or $c \le b'$ for any $a' \in \mathsf{s}_{\mathbb{A}}(a)$, $b' \in \mathsf{s}_{\mathbb{A}}(b)$. In other words, whenever $c \sqsubseteq a$ or $c \sqsubseteq b$, we have $c \in \mathsf{s}_{\mathbb{A}}(a)$ or $c \in \mathsf{s}_{\mathbb{A}}(b)$.

Suppose first that there is a maximal d such that $d' \le d$ for some non-maximal $d' \in \mathbb{A}$ (that is, $\mathbb{A} \neq \max(\mathbb{A})$). In this case there is no automorphism of $\mathbb{A}$ that swaps a and d or b and d, because there is no nonmaximal a' [and b'] with $a' \le a$ [respectively, $b' \le b$], while $d' \le d$. Therefore we are in the conditions of Case 1, and a is asm-connected to d and d is asm-connected to b.

Suppose all elements in $\mathbb{A}$ are maximal. By Theorem 1 there are $a = a_1, a_2, \dots, a_k = b$ such that for any $i \in [k-1]$ the pair $a_i a_{i+1}$ or $a_{i+1} a_i$ is a semilattice, affine or majority edge (not a thin edge). We need to show that a_i is asm-connected to a_{i+1}. Let θ be a congruence of $\mathbb{B} = \mathsf{Sg}(a_i, a_{i+1})$ witnessing that $a_i a_{i+1}$ or $a_{i+1} a_i$ is a semilattice, affine, or majority edge. Except for the case when $a_{i+1} a_i$ is a semilattice edge, by Lemma 7 there is $b \in \mathbb{C} = a_{i+1}/_\theta$ such that $a_i b$ is a thin edge. Then take $c, d \in \max(\mathbb{C})$ such that $b \sqsubseteq_\mathbb{C} c$ and $a_{i+1} \sqsubseteq_\mathbb{C} d$. By the induction hypothesis c is asm-connected to d in $\mathbb{C}$. Finally, as all elements in $\mathbb{A}$ are maximal, d is asm-connected with a_{i+1} in $\mathbb{A}$ by a semilattice path. If $a_{i+1} a_i$ is a semilattice edge, by Lemma 7 there is $b \in \mathbb{D} = a_i/_\theta$ such that $a_{i+1} \le b$. Since $b \in \mathsf{s}_\mathbb{A}(a_{i+1})$, there is a semilattice path from b to a_{i+1}. Then again we choose $c, d \in \max(\mathbb{D})$ with $b \sqsubseteq_\mathbb{D} c$ and $a_i \sqsubseteq_\mathbb{D} d$. By the induction hypothesis d is asm-connected to c. Since $a_i \sqsubseteq_\mathbb{D} d$ and $c \in \mathsf{s}_\mathbb{A}(b)$, element a_i is asm-connected to a_{i+1}. Subcase 2a is thus completed.

Claim 6. Let $c \le b$, $c \notin \max(\mathbb{A})$. Then a is asm-connected to an element d such that $c \sqsubseteq d$.

If $\mathsf{Sg}(a, c) = \mathbb{B} \ne \mathbb{A}$, then there are $a', d \in \mathbb{B}$ such that $a \sqsubseteq a'$ and $c \sqsubseteq d$ and by the induction hypothesis a' is asm-connected to d. So, assume that $\mathsf{Sg}(a, c) = \mathbb{A}$.

Consider the relation $R = R_{ac}$, that is, the binary relation generated by (a, c) and (c, a). Since a is maximal and c is not, there is no automorphism of $\mathbb{A}$ that swaps a and c. Therefore $\mathsf{tol}_1(R)$ is a nontrivial tolerance, in particular, (a, b) is in its transitive closure. If $R^{-1}[e] = \mathbb{A}$ for no $e \in \mathbb{A}$, by Corollary 20 a and b' for some $b' \in \mathsf{s}_\mathbb{A}(b)$ are connected by proper subalgebras. By the induction hypothesis and Claim 5 a and b' are asm-connected, thus taking d to be b' we obtain the claim. So, suppose there is $e \in \mathbb{A}$ such that $\mathbb{A} \times \{e\} \subseteq R$.

By Corollary 16 there are $a' \in \mathsf{s}_\mathbb{A}(a)$ and $e'' \in \max(\mathbb{A})$ such that $(a', e'') \in R$ and $(a, e) \sqsubseteq (a', e'')$ in R. Again by Corollary 16 there is $e' \in \max(\mathbb{A})$ such that $(a, e') \in R$ and $(a, e) \sqsubseteq (a', e'') \sqsubseteq (a, e')$ in R. Let $(a, e) = (a_1, e_1) \le \dots \le (a_k, e_k) = (a, e')$ in R. Consider the sequence $c = c_1, \dots, c_k$ given by $c_{i+1} = c_i \cdot a_{i+1}$ for $i \in [k-1]$. Then, since $(c, e) \in R$, we have $(c_1, e_1) \in R$, and

$$\begin{pmatrix} c_{i+1} \\ e_{i+1} \end{pmatrix} = \begin{pmatrix} c_i \\ e_i \end{pmatrix} \cdot \begin{pmatrix} a_{i+1} \\ e_{i+1} \end{pmatrix} \in R, \qquad \text{for } i \in [k-1]$$

implies that $(a, e'), (c', e') \in R$, where $c' = c_k$ and $e' = e_k$.

We consider several cases. First, suppose that $\mathbb{B} = \mathsf{Sg}(a, c') \ne \mathbb{A}$. Then let $a'', c'' \in \max(\mathbb{B})$ be such that $a \sqsubseteq_\mathbb{B} a''$, $c' \sqsubseteq_\mathbb{B} c''$. By the induction hypothesis a'' is asm-connected to c'', and, hence, a is asm-connected to c''. Since $c \sqsubseteq c''$, take c'' for d and the result follows.

Suppose $\mathsf{Sg}(a, c') = \mathbb{A}$. Then $(c, e') \in R$, as $(a, e'), (c', e') \in R$. Therefore $(a, c), (a, e'), (c, e'), (c, a) \in R$. If both $\mathbb{B}_1 = \mathsf{Sg}(a, e')$ and $\mathbb{B}_2 = \mathsf{Sg}(e', c)$ are not equal to $\mathbb{A}$, then a and c are connected with proper subalgebras. By the induction hypothesis and Claim 5 a and d are asm-connected some d with $c \sqsubseteq d$. The result holds in this case as well.

Suppose $\mathsf{Sg}(a, e') = \mathbb{A}$ or $\mathsf{Sg}(e', c) = \mathbb{A}$. Then $(c, c) \in R$ or $(a, a) \in R$, which means there is a binary operation f such that $f(a, c) = f(c, a) = c$ or $f(a, c) = f(c, a) = a$, implying ac is a thin semilattice edge. Since c is not maximal, we have $c \le a$, and the result follows.

Elements a, b are said to be *v-connected* if there is $c \in \mathsf{Sg}(a, b)$ such that $c \sqsubseteq a$ and $c \sqsubseteq b$.

Subcase 2b. Elements a, b are v-connected.

Recall that there is an automorphism of $\mathbb{A}$ that swaps a and b. The depth of element c, denoted $\mathsf{dep}(c)$, is defined to be the maximal number of s-components on an s-path $c = c_1 \le \cdots \le c_k$ such that c_k is maximal. We proceed by induction on the size of $\mathsf{Sg}(a, b)$ and minimal $\mathsf{dep}(c)$, such that $c \sqsubseteq a$, $c \sqsubseteq b$. Let $c = a_1 \le a_2 \le \cdots \le a_k = a$ and $c = b_1 \le b_2 \le \ldots \le b_m = b$.

Suppose first that $k = m = 2$, that is, $c \le a$, $c \le b$. As $c \in \mathsf{Sg}(a, b)$, there is a binary term operation f such that $f(a, b) = c$. Let $d = f(b, a)$. Since there is an automorphism swapping a and b, $d \le a$ and $d \le b$. Set $r(x, y, z) = (f(y, x) \cdot f(y, z)) \cdot f(x, z)$. We have

$$\begin{aligned} r(a, a, b) &= (ac)c = a, \\ r(a, b, a) &= (dd)a = a, \\ r(b, a, a) &= (ca)d = a. \end{aligned}$$

Since a and b are swapped by an automorphism, r is a majority operation on $\{a, b\}$. Therefore, ab is not only a thin majority edge, but also a majority edge, as is witnessed by the equality relation. Since $\mathbb{A}$ is smooth, $\mathsf{Sg}(a, b) = \{a, b\}$, and this case is in fact impossible.

Next we consider two cases that cover both the base and inductive steps. Let $k > 2$ or $m > 2$. We may assume $m > 2$ and $d = b_{m-1}$ to be a nonmaximal element. Indeed, if $b_{m-1} \in \mathsf{s}_{\mathbb{A}}(b)$ then we can replace b with b_{m-1}.

Subsubcase I. $c \notin \mathsf{s}_{\mathbb{A}}(d)$.
Suppose the result is proved for all algebras and pairs of elements v-connected through an element of depth less than $\mathsf{dep}(c)$. Note that $\mathsf{dep}(d) < \mathsf{dep}(c)$, because d is on an s-path from c to a maximal element. First we observe that there is $e \in \max(\mathbb{A})$ such that a is asm-connected to e and $d \sqsubseteq_{\mathbb{A}} e$. By Claim 6 a is asm-connected to an element d' such that $d \sqsubseteq_{\mathbb{A}} d'$. Then choose $e \in \max(\mathbb{A})$ such that $d' \sqsubseteq_{\mathbb{A}} e$; clearly a is asm-connected to e. Elements e and b are v-connected through d. Let $\mathbb{C} = \mathsf{Sg}(e, b)$. If $\mathbb{C} = \mathbb{A}$, the result follows by the induction hypothesis, since $\mathsf{dep}(d) < \mathsf{dep}(c)$. Otherwise the induction hypothesis does not apply directly, as e, b may not be maximal in $\mathbb{C}$ and $\mathsf{dep}(d)$ may change. We choose $e', b' \in \max(\mathbb{C})$ and such that $e \sqsubseteq_{\mathbb{C}} e'$, $b \sqsubseteq_{\mathbb{C}} b'$. By the induction hypothesis e', b' are asm-connected in $\mathbb{C}$, which implies e, b are asm-connected in $\mathbb{A}$, since e', b' belong to $\mathsf{s}_{\mathbb{A}}(e)$ and $\mathsf{s}_{\mathbb{A}}(b)$, respectively.

Subsubcase II. $c \in \mathsf{s}_{\mathbb{A}}(d)$. This covers in particular the base case.
Let $\mathbb{B} = \mathsf{Sg}(a, d)$. We show that there is $e \in \mathbb{B}$ such that a is asm-connected to e and either $e \in \mathsf{s}_{\mathbb{A}}(d)$ (this includes $e = d$) or $d \le e$. Observe that this implies

that a is asm-connected to b. Indeed, if $e \in \mathsf{s}_{\mathbb{A}}(d)$ then a is asm-connected to e, which is asm-connected to b. Note that this includes the case $e = a$. Otherwise if $e \in \max(\mathbb{A})$, as $d \le e$ and $d \le b$, we argue as in the beginning of Subcase 2b. Suppose $e \notin \max(\mathbb{A})$. Take any $e' \in \max(\mathbb{A})$ with $e \sqsubseteq_{\mathbb{A}} e'$, a is asm-connected to e'. For b and e' we consider connections $d \sqsubseteq b$ and $d \le e \le \dots \le e'$. Since $e \notin \mathsf{s}_{\mathbb{A}}(d)$, we have $\mathsf{dep}(e) < \mathsf{dep}(d)$, and we are in the conditions of Subsubcase I (with d playing the role of c, e' and b playing the role of b and a, respectively and some element on the s-path from e to e' playing the role of d).

Suppose first that for some $a' \in \mathsf{s}_{\mathbb{A}}(a)$, $\mathsf{Sg}(a', d) \ne \mathbb{A}$. We may assume that $a' = a$. Then by the induction hypothesis, a is asm-connected to all maximal elements of $\mathbb{B}$. Let $\mathbb{C}$ be a minimal subalgebra of $\mathbb{B}$ of the form $\mathsf{Sg}(d, e'')$, where a is asm-connected to e'' and $d \sqsubseteq_{\mathbb{B}} e''$. Take $e \in \max(\mathbb{C})$ such that $d \sqsubseteq_{\mathbb{C}} e$. Then by the inductive hypothesis a is asm-connected to e. If $e \in \mathsf{s}_{\mathbb{A}}(d)$, we are done. Otherwise $\mathsf{Sg}(d, e) = \mathbb{C}$ by the choice of e''. We show that $d \le e$. Consider the relation Q generated by $(e, d), (d, e)$. It suffices to show that $(e, e) \in Q$. Let $d = d_1 \le d_2 \le \dots \le d_x = e$ be an s-path in $\mathbb{C}$, and $(d, e) = (d_1, e_1) \le (d_2, e_2) \le \dots \le (d_x, e_x)$ its extension in Q. Then $d_x = e$, $e_x \in \mathsf{s}_{\mathbb{C}}(e)$. By the choice of $\mathbb{C}$ and e, $e \in \mathsf{Sg}(d, e_x) = \mathbb{C}$. Therefore the pairs $(e, d), (e, e_x) \in Q$ generate (e, e).

Suppose now that $\mathsf{Sg}(a', d) = \mathbb{A}$ for all $a' \in \mathsf{s}_{\mathbb{A}}(a)$. Then $d \sqsubseteq_{\mathbb{B}} a$, because $d \sqsubseteq_{\mathbb{A}} c$, and by the above argument $d \le a$.

Subcase 2c. Elements a, b are not v-connected

Note first that we may assume that, for any $b' \in \mathsf{s}_{\mathbb{A}}(b)$, there is an automorphism that sends b' to a and a to b', as otherwise we are in the conditions of Case 1. Recall that we also assume $\mathsf{Sg}(a, b') = \mathbb{A}$. Because of this and the automorphism swapping a and b, without loss of generality we may assume that there is nonmaximal $c \le b$. By Claim 6 a is asm-connected to some d such that d is v-connected with b. The result follows from Subcase 2b.

In the remaining statements of the theorem, when $a, b \in \mathsf{amax}(\mathbb{A})$ or $a, b \in \mathsf{umax}(\mathbb{A})$, we let $a', b' \in \mathbb{A}$ be maximal elements of $\mathbb{A}$ such that $a \sqsubseteq a'$ and $b \sqsubseteq b'$. Then by what is proved above a' is asm-connected to b', and so a is asm-connected to b'. Finally, as $b' \in \mathsf{as}(b)$ [respectively, $b' \in \mathsf{umax}(\mathbb{A})$], we have $b' \sqsubseteq_{as} b$ [respectively, $b' \sqsubseteq_{asm} b$], and b' is connected to b. Note that in the case of u-maximal elements the path showing that $b' \sqsubseteq_{asm} b$ cannot be assumed to be special. $\square$

Theorem 21 can be extended to algebras from the variety $\mathcal{V}$ generated by $\mathcal{K}$.

Corollary 22. *Let $\mathbb{A} \in \mathcal{V}$. Any $a, b \in \max(\mathbb{A})$ (or $a, b \in \mathsf{amax}(\mathbb{A})$, or $a, b \in \mathsf{umax}(\mathbb{A})$) are asm-connected.*

Proof. We need to prove that the property in the corollary is true for subalgebras of direct products of algebras from $\mathcal{K}$ and is preserved by taking quotient algebras.

Let $\mathbb{A} \in \mathcal{V}$ be such that any $a, b \in \max(\mathbb{A})$ (or $a, b \in \mathsf{amax}(\mathbb{A})$, or $a, b \in \mathsf{umax}(\mathbb{A})$) are asm-connected, and $\theta \in \mathsf{Con}(\mathbb{A})$. Take any maximal (as-maximal, u-maximal) $\overline{a}, \overline{b} \in \mathbb{A}/_\theta$. By Corollary 14(1) there are $a \in \overline{a}, b \in \overline{b}$ that are maximal (as-maximal, u-maximal) in $\mathbb{A}$. Then there is an asm-path $a = a_1, \ldots, a_k = b$ connecting a and b in $\mathbb{A}$. By Corollary 13(1) $\overline{a} = a_1/_\theta, \ldots, a_n/_\theta = \overline{b}$ is an asm-path in $\mathbb{A}/_\theta$.

Now let R be a subalgebra of the direct product of $\mathbb{A}_1, \ldots, \mathbb{A}_n \in \mathcal{K}$. Since $\mathcal{K}$ is closed under taking subalgebras, we may assume that R is subdirect. We proceed by induction on n. The base case $n = 1$ is given by Theorem 21, suppose the claim is true for $n-1$. Take maximal (as-maximal, u-maximal) $\mathbf{a}, \mathbf{b} \in R$. By Corollary 17(2) $\mathbf{a}' = \mathrm{pr}_{[n-1]}\mathbf{a}, \mathbf{b}' = \mathrm{pr}_{[n-1]}\mathbf{b}$ are maximal (as-maximal, u-maximal) in $\mathrm{pr}_{[n-1]}R$. By the induction hypothesis $\mathbf{a}', \mathbf{b}'$ are connected with an asm-path $\mathbf{a}' = \mathbf{a}'_1, \ldots, \mathbf{a}'_k = \mathbf{b}'$. By Corollary 16(1) this path can be expanded to an asm-path $\mathbf{a} = (\mathbf{a}'_1, \mathbf{a}[n]) = (\mathbf{a}'_1, a_1), \ldots, (\mathbf{a}'_k, a_k) = (\mathbf{b}', a_k)$. Consider $B = \{b \mid (\mathbf{b}', b) \in R\}$. This set contains a_k and $\mathbf{b}[n]$. Choose maximal elements $c, d \in B$ with $a_k \sqsubseteq_B c$ and $\mathbf{b}[n] \sqsubseteq_B d$. By Theorem 21 a_k is asm-connected to d implying that $\mathbf{a}$ is asm-connected to $(\mathbf{b}', d)$ in R. Then also $\mathbf{b} \sqsubseteq (\mathbf{b}', d)$, and, as $\mathbf{b}$ is maximal (as-maximal, u-maximal), $(\mathbf{b}', d) \sqsubseteq \mathbf{b}$ (respectively, $(\mathbf{b}', d) \sqsubseteq^{as} \mathbf{b}$, $(\mathbf{b}', d) \sqsubseteq^{asm} \mathbf{b}$. □

As by Theorem 21 any u-maximal elements are connected by an asm-path, we have the following

Corollary 23. *Any algebra $\mathbb{A} \in \mathcal{V}$ has a unique u-maximal component.*

3. Rectangularity

In this section we prove a result that, on one hand, is a generalization of the results by the author [3] (Lemma 3.5), [4] (Lemma 4.2, Proposition 5.4), and, on the other hand, is analogous to the Rectangularity Lemma from [2].

We will need two auxiliary lemmas.

Lemma 24. *Let R be a subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2$, $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{V}$.*
(1) Let $a, b \in \mathbb{A}_1$, $c, d \in \mathbb{A}_2$ be such that $(a, c), (a, d), (b, c) \in R$ and ab, cd are thin edges that are not both thin majority edges. Then $(b, d) \in R$.
(2) Let $B = R[a]$. For any $b \in \mathbb{A}_1$ such that ab is thin edge, and any $c \in R[b] \cap B$, $\mathrm{Ft}^{as}_B(c) \subseteq R[b]$.

Proof. (1) Without loss of generality assume that cd is not majority. Suppose that ab is semilattice or majority, or ab is affine and cd is semilattice. Then by Lemma 9(2) there is a term operation p such that $p(a, b) = b$ and $p(d, c) = d$ (if both ab and cd are of the semilattice type then p can be chosen to be $\cdot$). Then

$$\begin{pmatrix} b \\ d \end{pmatrix} = p\left(\begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}\right) \in R.$$

If both ab and cd are affine, then by Lemma 9(1) there is a term operation h' such that $h'(a,a,b)=b$ and $h'(d,c,c)=d$. Then

$$\binom{b}{d} = h'\left(\binom{a}{d}, \binom{a}{c}, \binom{b}{c}\right) \in R.$$

(2) Let $D = \mathrm{Ft}^{as}_B(c) \cap R[b]$. The set D is nonempty, as $c \in D$. If $D \neq \mathrm{Ft}^{as}_B(c)$, there are $b_1 \in D$ and $b_2 \in \mathrm{Ft}^{as}_B(c) - D$ such that b_1b_2 is a thin semilattice or affine edge. By item (1) $(b, b_2) \in R$, a contradiction with the choice of b_2. □

Lemma 25. *Let R be a subdirect product of algebras $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{V}$, let B_1, B_2 be as-components (maximal components) of $\mathbb{A}_1, \mathbb{A}_2$, respectively, and $a \in \mathbb{A}_1$ such that $R \cap (B_1 \times B_2) \neq \emptyset$ and $\{a\} \times B_2 \subseteq R$. Then $B_1 \times B_2 \subseteq R$.*

Proof. We prove the lemma for as-components; for maximal components the proof is nearly identical.

Let $(b,c) \in R \cap (B_1 \times B_2) \neq \emptyset$. For every $b' \in \mathbb{A}'_1 = \mathsf{Sg}(a,b)$ we have $(b',c) \in R$. By Lemma 24 this means that $\{b'\} \times B_2 \subseteq R$ for all $b' \in \mathrm{Ft}^{asm}_{\mathbb{A}'_1}(a)$. Indeed, let C be the set of all elements b' from $\mathrm{Ft}^{asm}_{\mathbb{A}'_1}(a)$ such that $\{b'\} \times B_2 \subseteq R$. The set C is nonempty, as $a \in C$. If $C \neq \mathrm{Ft}^{asm}_{\mathbb{A}'_1}(a)$, there are $b' \in C$ and $b'' \in \mathrm{Ft}^{asm}_{\mathbb{A}'_1}(a) - C$ such that $b'b''$ is a thin edge. Then by Lemma 24 $\mathrm{Ft}^{as}_{R[b']}(c) \subseteq R[b'']$. Since $B_2 \subseteq \mathrm{Ft}^{as}_{R[b']}(c)$, we have a contradiction.

By Theorem 21 there is an asm-path from a to every maximal element from $\mathbb{A}'_1$. Therefore, the set $\mathrm{Ft}^{asm}_{\mathbb{A}'_1}(a)$ contains all the maximal elements of $\mathbb{A}'_1$ including some b' such that $b \sqsubseteq^{as}_{\mathbb{A}'_1} b'$. Thus, $\{b'\} \times B_2 \subseteq R$ for some $b' \in B_1$. By Corollary 16(1) for every $b'' \in B_1$ we have $R[b''] \cap B_2 \neq \emptyset$. Then Lemma 24 implies that $B_2 \subseteq R[b'']$ for every $b'' \in B_1$. □

Proposition 26. *Let $R \leq \mathbb{A}_1 \times \mathbb{A}_2$, $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{V}$, be a linked subdirect product and let B_1, B_2 be as-components (maximal components) of $\mathbb{A}_1, \mathbb{A}_2$, respectively, such that $R \cap (B_1 \times B_2) \neq \emptyset$. Then $B_1 \times B_2 \subseteq R$.*

Proof. We prove by induction on the size of $\mathbb{A}_1, \mathbb{A}_2$ that for any as-components C_1, C_2 of $\mathbb{A}_1, \mathbb{A}_2$, respectively, such that $R \cap (C_1 \times C_2) \neq \emptyset$, there are $a_1 \in \mathbb{A}_1$, $a_2 \in \mathbb{A}_2$ such that $\{a_1\} \times C_2 \subseteq R$ and $C_1 \times \{a_2\} \subseteq R$. The result then follows by Lemma 25. The base case of induction when $|\mathbb{A}_1| = 1$ or $|\mathbb{A}_2| = 1$ is obvious.

Take $b \in C_1$ and construct two sequences of subalgebras $\mathbb{B}_1, \dots, \mathbb{B}_k$ of $\mathbb{A}_1$ and $\mathbb{C}_1, \dots, \mathbb{C}_k$ of $\mathbb{A}_2$, where $\mathbb{B}_1 = \{b\}$, $\mathbb{C}_i = R[\mathbb{B}_i]$, and $\mathbb{B}_i = R^{-1}[\mathbb{C}_{i-1}]$, such that k is the minimal number with $\mathbb{B}_k = \mathbb{A}_1$ or $\mathbb{C}_k = \mathbb{A}_2$. Such a number exists, because R is linked. Observe that for each $i \leq k$ the relation $R_i = R \cap (\mathbb{B}_i \times \mathbb{C}_i)$ is linked. Therefore, there is a proper subalgebra $\mathbb{A}'_1$ of $\mathbb{A}_1$ or $\mathbb{A}'_2$ of $\mathbb{A}_2$ such that $R' = R \cap (\mathbb{A}'_1 \times \mathbb{A}_2)$ or $R' = R \cap (\mathbb{A}_1 \times \mathbb{A}'_2)$, respectively, is linked and subdirect. Without loss of generality suppose there is $\mathbb{A}'_1$ with the required properties. By the induction hypothesis for any as-component C_2 of $\mathbb{A}_2$ there is $a_1 \in \mathbb{A}'_1 \subseteq \mathbb{A}_1$ with $\{a_1\} \times C_2 \subseteq R' \subseteq R$. By Lemma 25 $C_1 \times C_2 \subseteq R$, and therefore for any $a_2 \in C_2$ we have $C_1 \times \{a_2\} \subseteq R$.

The argument above also works in the case when B_1, B_2 are maximal components. $\square$

Corollary 27. *Let R be a subdirect product of $\mathbb{A}_1$ and $\mathbb{A}_2$ from $\mathcal{V}$, $\mathsf{lk}_1, \mathsf{lk}_2$ the link congruences, and let B_1, B_2 be as-components (maximal components) of an lk_1-block and an lk_2-block, respectively, such that $R \cap (B_1 \times B_2) \neq \emptyset$. Then $B_1 \times B_2 \subseteq R$.*

Proof. Let C_1, C_2 be the lk_1- and lk_2-blocks containing B_1 and B_2, respectively, and $Q = (C_1 \times C_2) \cap R$. By definition Q is a subdirect product of $C_1 \times C_2$, as R is subdirect, and Q is linked. The result follows by Proposition 26. $\square$

Proposition 28. *Let R be a subdirect product of $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{V}$, $\mathsf{lk}_1, \mathsf{lk}_2$ the link congruences, and let B_1 be an as-component of an lk_1-block and $B_2' = R[B_1]$; let $B_2 = \mathsf{umax}(B_2')$. Then $B_1 \times B_2 \subseteq R$.*

Proof. Let B_2' be a subset of a lk_2-block C. By Corollary 17(1) for any $a_0 \in B_2'$ the set B_2' contains an as-maximal element a of C such that $a_0 \sqsubseteq^{as} a$. By Corollary 27 $B_1 \times \{a\} \subseteq R$. It then suffices to show that $B_1 \times \mathrm{Ft}^{asm}_{B_2'}(a) \subseteq R$.

Suppose for $D \subseteq \mathrm{Ft}^{asm}_{B_2'}(a)$ it holds $B_1 \times D \subseteq R$. If $D \neq \mathrm{Ft}^{asm}_{B_2'}(a)$, there are $b_1 \in D$ and $b_2 \in \mathrm{Ft}^{asm}_{B_2'}(a) - D$ such that $b_1 b_2$ is a thin edge. By Lemma 24 $B_1 \times \{b_2\} \subseteq R$; the result follows. $\square$

4. Quasi-2-decomposability

An (n-ary) relation over a set A is called *2-decomposable* if, for any tuple $\mathbf{a} \in A^n$, $\mathbf{a} \in R$ if and only if, for any $i, j \in [n]$, $\mathrm{pr}_{ij}\mathbf{a} \in \mathrm{pr}_{ij} R$ [1, 9]. 2-decomposability is closely related to the existence of majority polymorphisms of the relation. Relations over general smooth algebras do not have a majority polymorphism, but they still have a property close to 2-decomposability. We say that a relation R, a subdirect product of $\mathbb{A}_1, \ldots, \mathbb{A}_n \in \mathcal{V}$, is *quasi-2-decomposable*, if for any elements $a_1, \ldots, a_n$, such that $(a_i, a_j) \in \mathrm{pr}_{ij} R$ for any i, j, there is a tuple $\mathbf{b} \in R$ with $(a_i, a_j) \sqsubseteq^{as} (\mathbf{b}[i], \mathbf{b}[j])$ for all $i, j \in [n]$. In particular, if $(a_i, a_j) \in \mathsf{amax}(\mathrm{pr}_{ij} R)$ for any i, j, then $\mathbf{b} \in R$ can be chosen such that $(\mathbf{b}[i], \mathbf{b}[j]) \in \mathsf{as}(a_i, a_j)$, $i, j \in [n]$.

Theorem 29. *Let $\mathbb{A}_1, \ldots, \mathbb{A}_n \in \mathcal{K}$. Then any subdirect product R of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is quasi-2-decomposable.*

Moreover, if J is a collection of subsets of $[n]$ containing all the 2-element subsets, $X \in J$, tuple $\mathbf{a}$ is such that $\mathrm{pr}_Y \mathbf{a} \in \mathrm{pr}_Y R$ for every $Y \in J$, and $\mathrm{pr}_X \mathbf{a} \in \mathsf{amax}(\mathrm{pr}_X R)$, there is a tuple $\mathbf{b} \in R$ with $\mathrm{pr}_Y \mathbf{a} \sqsubseteq^{as} \mathrm{pr}_Y \mathbf{b}$ for $Y \in J$, and $\mathrm{pr}_X \mathbf{b} = \mathrm{pr}_X \mathbf{a}$.

4.1. The ternary case

We start with ternary relations assuming every (a_i, a_j) is as-maximal.

Lemma 30. *Let R be a subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$, and let (a_1, a_2, a_3) be such that $(a_i, a_j) \in \mathsf{amax}(\mathrm{pr}_{ij} R)$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. Then there is $(a'_1, a'_2, a'_3) \in R$ such that (a'_i, a'_j) is in the as-component of $\mathrm{pr}_{ij} R$ containing (a_i, a_j) for $i, j \in \{1, 2, 3\}$, $i \neq j$.*

Proof. By replacing $\mathbb{A}_i$ with $\mathrm{pr}_I R$ for $i = 1, 2, 3$, the algebra R can be assumed to be a subdirect product of $\mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$. We proceed by induction on the size of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$. The base case of induction is comprised of the following situations: (1) for some $i \in [3]$, $|\mathbb{A}_i| = 2$ and $\mathbb{A}_i$ is a semilattice edge; (2) for some $i \in [3]$, $\mathbb{A}_i$ is a module (not necessarily 2-element); and (3) $|\mathbb{A}_i| = 2$ and $\mathbb{A}_i$ is a majority edge for all $i \in [3]$. By the assumption some tuples $\mathbf{a}_1 = (b_1, a_2, a_3)$, $\mathbf{a}_2 = (a_1, b_2, a_3)$, $\mathbf{a}_3 = (a_1, a_2, b_3)$ belong to R. If one of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ is a semilattice edge, say, $b_1 \leq a_1$, then from the as-maximality of a_1, a_2, a_3, we obtain $(a_1, a'_2, a_3) = \mathbf{a}_1 \cdot \mathbf{a}_2 \in R$, $a'_2 = a_2 \cdot b_2$. As is easily seen, this tuple satisfies the requirements of the lemma. If one of $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ is a module, say, $\mathbb{A}_1$ is, then $\mathbf{a}_1$ satisfies the requirements of the lemma. Finally, if all $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ are majority edges, then $(a_1, a_2, a_3) = g(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, where g is the operation from Theorem 3.

Suppose that the lemma is proved for any subdirect product of $\mathbb{A}'_1 \times \mathbb{A}'_2 \times \mathbb{A}'_3$, where $\mathbb{A}'_i$ is a subalgebra or a factor of $\mathbb{A}_i$, $i \in [3]$, and at least one of them is a proper subalgebra or a factor. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in R$ be as before. Also let $\mathcal{D}$ denote the set of $(c_1, c_2, c_3) \in \mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$ such that (c_i, c_j), $i, j \in [3]$, belongs to the as-component of $\mathrm{pr}_{ij} R$ containing (a_i, a_j), for $i \neq j$. The set $\mathcal{D}$ is nonempty, as $\mathbf{a} = (a_1, a_2, a_3) \in \mathcal{D}$.

Claim 1. Every $\mathbb{A}_i$ can be assumed to be $\mathsf{Sg}(a_i, b_i)$, and b_i can be chosen to be an as-maximal element.

Suppose $\mathbb{A}_1 \neq \mathbb{B} = \mathsf{Sg}(a_1, b_1)$. Let (a'_1, a'_2) be an as-maximal element in $Q = (\mathbb{B} \times \mathbb{A}_2) \cap \mathrm{pr}_{12} R$ such that $(a_1, a_2) \sqsubseteq^{as} (a'_1, a'_2)$ in Q. Let also $(a_1, a_2) = (c_1^1, c_2^1), \dots, (c_1^k, c_2^k) = (a'_1, a'_2)$ be an as-path from (a_1, a_2) to (a'_1, a'_2) in Q. By Corollary 16(1) it can be extended to an as-path $\mathbf{c}_1, \dots, \mathbf{c}_k \in R$ with $\mathbf{c}_1 = (a_1, a_2, b_3)$ in $R' = (\mathbb{B} \times \mathbb{A}_2 \times \mathbb{A}_3) \cap R$. Using Lemma 15(1) we define a sequence $\mathbf{c}'_1, \dots, \mathbf{c}'_k$ in $\mathcal{D}$ as follows: $\mathbf{c}'_1 = \mathbf{a}$, and $\mathbf{c}'_{i+1}$ is such that $(\mathbf{c}'_{i+1}[1], \mathbf{c}'_{i+1}[2]) = (c_1^{i+1}, c_2^{i+1})$ and $\mathbf{c}'_i \mathbf{c}'_{i+1}$ is a thin semilattice or affine edge in $\mathsf{Sg}_{\mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3}(\mathbf{c}'_i, \mathbf{c}_{i+1})$. Note that $(\mathbf{c}'_i[u], \mathbf{c}'_i[v])(\mathbf{c}'_{i+1}[u], \mathbf{c}'_{i+1}[v])$ is a thin edge of the same type as $\mathbf{c}'_i \mathbf{c}'_{i+1}$ for $u, v \in [3]$. In particular, $(a_u, a_v) \sqsubseteq^{as} (\mathbf{c}'_{i+1}[u], \mathbf{c}'_{i+1}[v])$ in $\mathrm{pr}_{uv} R'$. Replace $\mathbf{a}$ with $\mathbf{c}'_k$. Repeating the process for the other binary projections if necessary we obtain $(a'_1, a'_2, a'_3) \in \mathcal{D}$ such that (a'_i, a'_j) is as-maximal in $\mathrm{pr}_{ij} R'$. By the induction hypothesis there is $(a''_1, a''_2, a''_3) \in R \cap (\mathbb{B} \times \mathbb{A}_2 \times \mathbb{A}_3)$ such that (a''_i, a''_j) is in the as-maximal component containing (a'_i, a'_j). Clearly, (a''_1, a''_2, a''_3) is as required.

If, say, b_1 is not an as-maximal element, then choose an as-path $b_1 = c_1 \leq \dots c_k$ and its extension $\mathbf{c}_1, \dots, \mathbf{c}_k \in R$, $\mathbf{c}_1 = \mathbf{a}_1$, such that c_k is a maximal element and $\mathbf{c}_k = (c_k, a'_2, a'_3)$. Then we choose an as-path in $\mathrm{pr}_{23} R$ from (a'_2, a'_3) to (a_2, a_3). Extending this path as before we get $(d, a_2, a_3) \in R$ such that $d \in \mathsf{amax}(\mathbb{A}_1)$.

Claim 2. For every $i, j \in [3]$, it holds that $\mathsf{as}(a_i) \times \mathbb{A}_j \subseteq \mathrm{pr}_{ij} R$ and $\mathsf{as}(a_i) \times \mathsf{as}(a_j), \mathsf{as}(a_i) \times \mathsf{as}(b_j)$ are as-components of $\mathrm{pr}_{ij} R$.

Since $(a_i, a_j), (a_i, b_j) \in \mathrm{pr}_{ij} R$ and $\mathbb{A}_j = \mathsf{Sg}(a_j, b_j)$, we have $\{a_i\} \times \mathbb{A}_j \subseteq \mathrm{pr}_{ij} R$. By Lemma 25 $\mathsf{as}(a_i) \times \mathsf{as}(a_j), \mathsf{as}(a_i) \times \mathsf{as}(b_j) \subseteq \mathrm{pr}_{ij} R$. Therefore, for any $c \in \mathsf{as}(a_i)$ it holds that $(c, a_j), (c, b_j) \in \mathrm{pr}_{ij} R$, implying $\mathsf{as}(a_i) \times \mathbb{A}_j \subseteq \mathrm{pr}_{ij} R$. The second statement is obvious.

Claim 3. Every $\mathbb{A}_i$ can be assumed to be simple.

Suppose θ is a nontrivial congruence of $\mathbb{A}_1$ and $R/_\theta = \{(c_1/_\theta, c_2, c_3) \mid (c_1, c_2, c_3) \in R\}$. By the induction hypothesis there is $(a_1'', a_2', a_3') \in R/_\theta$ satisfying the conditions of the lemma, that is, there is $(b_1, a_2', a_3') \in R$ such that $b_1/_\theta = a_1''$, and $(a_2, a_3) \sqsubseteq^{as} (a_2', a_3')$, $(a_1/_\theta, a_i) \sqsubseteq^{as} (a_1'', a_i')$ for $i \in \{2, 3\}$, where the latter as-paths are in $\mathrm{pr}_{1i} R/_\theta$. Let $a_1' \in b_1/_\theta$ be any element such that $a_1 \sqsubseteq^{as} a_1'$ and a_1' is maximal in $b_1/_\theta$. Such an element exists, because $a_1/_\theta \sqsubseteq^{as} b_1/_\theta$. Then for (a_1', a_2', a_3') we have $(a_i', a_j') \in \mathsf{as}(a_i) \times \mathsf{as}(a_j) \subseteq \mathrm{pr}_{ij} R$, for any $i, j \in \{1, 2, 3\}$, where the last inclusion is by Claim 2. Therefore $(a_i, a_j) \sqsubseteq^{as} (a_i', a_j')$. Since $\mathsf{Sg}(a_1', b_1) \neq \mathbb{A}_1$, the claim follows by the induction hypothesis.

We now prove the induction step. Suppose now that $|\mathbb{A}_i| > 2$ for some i and $\mathbb{A}_i$ is not a module for any i. For an n-ary relation $Q \leq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $j \in [n]$, and $c_j \in \mathbb{A}_j$, let $Q[c_j]$ denote the set $\{(c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n) \in \mathrm{pr}_{\{1,\ldots,j-1,j+1,\ldots,n\}} Q \mid (c_1, \ldots, c_n) \in Q\}$. We still use the tuples $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in R$. There are two cases to consider.

Case 1. For some $i \in [3]$ the set $R[b_i]$ contains $\mathsf{as}(a_j) \times \mathsf{as}(a_\ell)$, where $\{j, \ell\} = [3] - \{i\}$.

Assume $i = 1$. Since a_1 is as-maximal, by Theorem 21 there is a special asm-path P from b_1 to a_1. We prove that for any element c on this path $\{c\} \times \mathsf{as}(a_2) \times \mathsf{as}(a_3) \subseteq R$. This is true for $c = b_1$ by the assumption made. Assume the contrary, and let c be the first element in P for which this property is not true. Let also d be the element preceding c in P; we may assume $d = b_1$. If $b_1 c$ is semilattice or affine, then by Lemma 15(1) applied to (b_1, a_2, a_3) and c there is $\mathbf{c} = (c, a_2', a_3') \in R$ such that $(a_2', a_3') \in \mathsf{as}(a_2, a_3)$. Therefore by Lemma 24 $\{c\} \times \mathsf{as}(a_2) \times \mathsf{as}(a_3) \subseteq R$.

Let $b_1 c$ be a special thin majority edge, $\mathbb{B} = \mathsf{Sg}(b_1, c)$, and θ a congruence witnessing that $b_1 c$ a majority edge; in particular, $\mathbb{B} = b_1/_\theta \cup c/_\theta$, as $\mathbb{A}_1$ is smooth. Suppose first that $\mathbb{B} = \mathbb{A}_1$. Then θ is the equality relation, as $\mathbb{A}_1$ is simple, and so $|\mathbb{A}_1| = 2$. In this case $c = a_1$ and $\mathsf{as}(a_1) = \{a_1\}$. We prove that $\mathsf{as}(a_1) \times \mathsf{as}(a_3) \subseteq R[b_2]$. By Claim 2 $\mathsf{as}(b_2) \times \mathsf{as}(a_3) \subseteq \mathrm{pr}_{23} R$ and $(a_1, b_2, a_3) \in R$. Extending an as-path in $\mathrm{pr}_{23} R$ from (b_2, a_3) to an arbitrary $(d_2, d_3) \in \mathsf{as}(b_2) \times \mathsf{as}(a_3)$ Corollary 16 implies that $(a_1, d_2, d_3) \in R$. A similar argument shows that $\mathsf{as}(a_1) \times \mathsf{as}(a_2) \subseteq R[b_3]$. Therefore, either $|\mathbb{A}_1| = |\mathbb{A}_2| = |\mathbb{A}_3| = 2$, which is the base case, or we may assume that $\mathbb{B} \neq \mathbb{A}_1$.

Consider $R' = R \cap (\mathbb{B} \times \mathbb{A}_2 \times \mathbb{A}_3)$. Take any $e \in \mathsf{amax}(\mathbb{B}) \cap c/_\theta$. Such an element exists, because by Corollary 13(2) any as-path that starts in $c/_\theta$ remains inside $c/_\theta$. For the tuple (e, a_2, a_3) we have the following. By Claim 2

$(e, a_i) \in \mathrm{pr}_{1i} R'$ for $i \in \{2,3\}$. Also, $(a_2, a_3) \in \mathrm{pr}_{23} R'$, as $(b_1, a_2, a_3) \in R$ by the assumption made. By the induction hypothesis there is $(e', a_2', a_3') \in R'$ with $e' \in \mathsf{as}_{\mathbb{B}}(e)$ (and so $e' \in c/_{\theta}$) and $a_i' \in \mathsf{as}_{\mathbb{A}_i}(a_i)$, $i \in \{2,3\}$. Let $e'' = g(b_1, e', e')$, where g is the operation satisfying the majority condition from Theorem 3. Then $g(b_1, e'', e'') = e''$. Since $b_1 c$ is a minimal pair with respect to θ, it holds that $c \in \mathsf{Sg}(b_1, e'')$. By Lemma 7 $b_1 e''$ is also a thin majority edge. Moreover,

$$\begin{pmatrix} e'' \\ a_2' \\ a_3' \end{pmatrix} = g\left(\begin{pmatrix} b_1 \\ a_2' \\ a_3' \end{pmatrix}, \begin{pmatrix} e' \\ a_2' \\ a_3' \end{pmatrix}, \begin{pmatrix} e' \\ a_2' \\ a_3' \end{pmatrix} \right) \in R'.$$

By Lemma 24

$$\mathsf{as}(a_2) \times \mathsf{as}(a_3) \subseteq \mathrm{Ft}^{as}_{\mathrm{pr}_{23} R'}((a_2', a_3')) \subseteq R[e''].$$

Since $\mathsf{Sg}(b_1, e'') = \mathsf{Sg}(b_1, c)$, $c = r(b_1, e'')$ for some term operation r. It remains to notice that

$$\begin{pmatrix} c \\ a_2'' \\ a_3'' \end{pmatrix} = r\left(\begin{pmatrix} b_1 \\ a_2'' \\ a_3'' \end{pmatrix}, \begin{pmatrix} e'' \\ a_2'' \\ a_3'' \end{pmatrix} \right) \in R'$$

for any $a_2'' \in \mathsf{as}(a_2)$, $a_3'' \in \mathsf{as}(a_3)$, a contradiction with the choice of c.

Case 2. For all $i \in [3]$, $\mathsf{as}(a_j) \times \mathsf{as}(a_\ell) \not\subseteq R[b_i]$, where $\{j, \ell\} = [3] - \{i\}$.

Let $\mathsf{lk}_{j\ell}$ be the link congruence of $\mathrm{pr}_{j\ell} R$ when R is viewed as a subdirect product of $\mathbb{A}_i$ and $\mathrm{pr}_{j\ell} R$; and let lk_i be the link congruence of $\mathbb{A}_i$. Since b_i is as-maximal, if lk_i is the total congruence, then by Proposition 26 and Claim 2 $\mathsf{as}(a_j) \times \mathsf{as}(a_\ell) \subseteq R[b_i]$, a contradiction with the assumption made. Therefore lk_i is the equality relation for all $i \in [3]$. Consider the $\mathsf{lk}_{j\ell}$-block $Q = R[a_i]$. By Claim 2 Q is a subdirect product of $\mathbb{A}_j \times \mathbb{A}_\ell$. If Q is linked, $\mathsf{as}(b_j) \times \mathsf{as}(a_\ell) \subseteq Q$. In this case, if $|\mathsf{as}(a_i)| = 1$ then $\mathsf{as}(a_i) \times \mathsf{as}(a_\ell) \subseteq R[b_j]$, a contradiction with the assumptions of Case 2. If $|\mathsf{as}(a_i)| > 1$, for any $c_i \in \mathbb{A}_i$ such that $a_i c_i$ is a thin semilattice or affine edge by Lemma 15(1) $(c_j, c_\ell) \in R[c_i]$ for some $(c_j, c_\ell) \in \mathsf{as}(b_j) \times \mathsf{as}(a_\ell)$, a contradiction with the assumption that lk_i is the equality relation. Therefore Q is not linked, and, since $\mathbb{A}_2, \mathbb{A}_3$ are simple, Q is the graph of a bijection. Thus, $\mathbb{A}_j$ and $\mathbb{A}_\ell$ are isomorphic and there is an isomorphism that maps a_j to b_ℓ and b_j to a_ℓ. In a similar way $\mathbb{A}_i$ and $\mathbb{A}_j$ are isomorphic.

Next, we prove that for any $i \in [3]$, $\mathsf{as}(a_i)$ and $\mathsf{as}(b_i)$ are subalgebras of $\mathbb{A}_i$. Let $\mathbb{B}_i = \mathsf{Sg}(\mathsf{as}(a_i))$ and $\mathbb{C}_i = \mathsf{Sg}(\mathsf{as}(b_i))$. Consider $R'' = R \cap (\mathbb{B}_1 \times \mathbb{B}_2 \times \mathbb{C}_3)$. Since $(a_1, a_2, b_3) \in R$, by Corollary 16 R'' is a subdirect product, and by Claim 2 $\mathbb{B}_2 \times \mathbb{C}_3 \subseteq \mathrm{pr}_{23} R''$. Let $\mathsf{lk}_1'', \mathsf{lk}_{23}''$ denote the link congruences of $\mathbb{B}_1, \mathbb{B}_2 \times \mathbb{C}_3$ when R'' is viewed as a subdirect product of $\mathbb{B}_1$ and $\mathbb{B}_2 \times \mathbb{C}_3$. Clearly, $\mathsf{lk}_1'' \subseteq \mathsf{lk}_1$, and so, lk_1'' is the equality relation, and therefore lk_{23}'' is a proper congruence of $\mathbb{B}_2 \times \mathbb{C}_3$. Moreover, $R''[a_1]$ is a block of this congruence that, as we proved above is the graph of an isomorphism φ from $\mathbb{B}_2$ to $\mathbb{C}_3$. Replacing $\mathbb{C}_3$ with a copy of $\mathbb{B}_2$ and φ with the identity mapping, we see that $\mathbb{B}_2^2$ has a congruence one of whose blocks is $\Delta = \{(c, c) \mid c \in \mathbb{B}_2\}$. Therefore

$\mathbb{B}_2$ is Abelian and, as $\mathbb{B}_2$ omits type **1**, by Theorem 9.6 of [8] and the results of [7] $\mathbb{B}_2$ is term equivalent to a module. This means that for any $c \in \mathbb{B}_2$ the pair a_2c is a thin affine edge implying $c \in \mathsf{as}(a_2)$. A similar argument shows that $\mathsf{as}(a_i)$ and $\mathsf{as}(b_j)$ are isomorphic modules for any $i, j \in [3]$.

If, say, $a_1 \in \mathsf{as}(b_1)$ then $\mathbb{A}_1$ is a module and the result follows from the base case. If $a_1 \notin \mathsf{as}(b_1)$, then, as a_1 is as-maximal, by Theorem 21 there is a special thin majority edge dc such that $d \in \mathsf{as}(b_1)$ and $c \notin \mathsf{as}(b_1)$. We show that this imples that lk_1 is the full relation contradicting the results above. Let $\mathbb{B} = \mathsf{Sg}(d, c)$, and θ a congruence witnessing that dc a majority edge; in particular, $\mathbb{B} = d/_\theta \cup c/_\theta$. The case $\mathbb{B} = \mathbb{A}_1$ is considered in Case 1, so, assume $\mathbb{B} \neq \mathbb{A}_1$. As in Case 1 consider $R' = R \cap (\mathbb{B} \times \mathbb{A}_2 \times \mathbb{A}_3)$ and take any $e \in \mathsf{amax}(\mathbb{B}) \cap c/_\theta$. It was proved in the previous paragraph that $R \cap (\mathbb{C}_1 \times \mathbb{B}_2 \times \mathbb{B}_3)$ is subdirect, therefore $(d, a_2', a_3') \in R$ for some $(a_2', a_3') \in \mathsf{as}(a_2) \times \mathsf{as}(a_3)$. Then as before by Claim 2 $(e, a_i') \in \mathrm{pr}_{1i}R'$ for $i \in \{2, 3\}$, and $(a_2', a_3') \in \mathrm{pr}_{23}R'$. By the induction hypothesis there is $(e', a_2'', a_3'') \in R'$ with $e' \in \mathsf{as}_\mathbb{B}(e)$ (and so $e' \in c/_\theta$) and $a_i'' \in \mathsf{as}_{\mathbb{A}_i}(a_i)$, $i \in \{2, 3\}$. Note that as $\mathsf{as}(b_1)$ is a module, $e' \notin \mathsf{as}(b_1)$. On the other hand, as $R \cap (\mathbb{C}_1 \times \mathbb{B}_2 \times \mathbb{B}_3)$ is subdirect, there exists $b' \in \mathsf{as}(b_1)$ with $(b', a_2'', a_3'') \in R$, implying that lk_1 is not the equality relation. □

4.2. Proof of Theorem 29

In this section we prove Theorem 29.

Proof of Theorem 29. Let $\mathbf{a}$ be a tuple satisfying the conditions of quasi-2-decomposability and such that $\mathrm{pr}_Y\mathbf{a} \in \mathrm{pr}_Y R$, for $Y \in J$, and $\mathrm{pr}_X\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_X R)$. By induction on ideals of the power set of $[n]$ (i.e. subsets of the power set closed under taking subsets) we prove that for any ideal I there is $\mathbf{a}'$ such that $\mathrm{pr}_Y\mathbf{a} \sqsubseteq^{as} \mathbf{a}'$ for any $Y \in J$, $\mathrm{pr}_X\mathbf{a}' \in \mathsf{as}(\mathrm{pr}_X\mathbf{a})$ (in particular, $\mathrm{pr}_X\mathbf{a} \sqsubseteq^{as} \mathrm{pr}_X\mathbf{a}'$), and for any $U \in I$ it holds $\mathrm{pr}_U\mathbf{a}' \in \mathrm{pr}_U R$. Note that we cannot claim here that $\mathrm{pr}_U\mathbf{a} \sqsubseteq^{as} \mathrm{pr}_U\mathbf{a}'$, because $\mathrm{pr}_U\mathbf{a} \notin \mathrm{pr}_U R$, and it is not even clear what relation such a path may belong to. Instead we introduce a relation $\mathcal{E}(I)$, specific for each I that contains such paths, but only for sets $U \in I$. Then if this statement is proved for the entire power set, $\mathbf{a}' = \mathrm{pr}_{[n]}\mathbf{a}' \in \mathrm{pr}_{[n]}R = R$ implies the result. The base case, where the ideal consists of all sets from J and their subsets (this includes all at most 2-element sets and the set X and its subsets) is given by the assumptions of the theorem and tuple $\mathbf{a}$.

Suppose that the claim is true for an ideal I, the set W does not belong to I, but all its proper subsets do. Let $\mathcal{E}(I)$ be the set of all tuples $\mathbf{c}$, not necessarily from R such that $\mathrm{pr}_U\mathbf{c} \in \mathrm{pr}_U R$ for every $U \in I$. Clearly, $R \subseteq \mathcal{E}(I)$ and $\mathcal{E}$ is a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$. By $\mathcal{D}(I)$ we denote the set of all tuples $\mathbf{c} \in \mathcal{E}(I)$ such that $\mathrm{pr}_Y\mathbf{a} \sqsubseteq^{as} \mathrm{pr}_Y\mathbf{c}$, for $Y \in J$. It does not have to be a subalgebra of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$. If a tuple belongs to $\mathcal{D}(I)$ it is said to *support* I. We show that $\mathcal{D}(I)$ contains a tuple $\mathbf{b}$ such that $\mathrm{pr}_W\mathbf{b} \in \mathrm{pr}_W R$, that is, $\mathbf{b}$ supports $I \cup \{W\}$.

Assume that $W = [\ell]$. For a subalgebra Q of $\mathrm{pr}_W R$ a tuple $\mathbf{c} \in \mathcal{D}(I)$ is said to be Q-*approximable* if $\mathrm{pr}_U \mathbf{c} \in \mathrm{pr}_U Q$. If $\mathbf{c}$ is Q-approximable, for every $U \subset W$ we choose $\mathbf{c}_U \in R$ such that $\mathrm{pr}_U \mathbf{c} = \mathrm{pr}_U \mathbf{c}_U$. We prove the following statement:

> Let Q be a subalgebra of $\mathrm{pr}_W R$. If there exists a Q-approximable tuple from $\mathcal{D}(I)$, then there is $\mathbf{d} \in \mathcal{D}(I)$ such that $\mathrm{pr}_W \mathbf{d} \in Q$.

Note that if $Q = \mathrm{pr}_W R$, then any tuple in $\mathcal{D}(I)$ is Q-approximable. Therefore the statement implies that $\mathcal{D}(I)$ contains a tuple $\mathbf{d}$ with $\mathrm{pr}_W \mathbf{d} \in \mathrm{pr}_W R$, which would prove the induction step. We prove the statement by induction on the sum of sizes of unary projections of Q. If one of these projections is 1-element then the statement trivially follows from the assumption that for some $\mathbf{d} \in \mathcal{D}(I)$ it holds that $\mathrm{pr}_U \mathbf{d} \in \mathrm{pr}_U Q$ for U including all coordinate positions whose projections contain more than 1 element, and so $\mathrm{pr}_W \mathbf{d} \in \mathrm{pr}_W Q$. So suppose that the statement is proved for all relations with unary projections smaller than Q. Also let $\mathbf{d}$ be a Q-approximable tuple from $\mathcal{D}(I)$. We need the following auxiliary statements.

Claim 1. Let $\mathbf{d} \in \mathcal{D}(I)$ and $\mathbf{e}_1, \dots, \mathbf{e}_k$ an as-path in $\mathcal{E}(I)$. Then every tuple from an as-path $\mathbf{e}'_1, \dots, \mathbf{e}'_k$ satisfying the following conditions belongs to $\mathcal{D}(I)$: $\mathbf{e}'_1 = \mathbf{d}$; if $\mathbf{e}_i\mathbf{e}_{i+1}$ is a semilattice edge then $\mathbf{e}'_{i+1} = \mathbf{e}'_i \cdot \mathbf{e}_{i+1}$ (in which case either $\mathbf{e}'_i = \mathbf{e}'_{i+1}$ or $\mathbf{e}'_i\mathbf{e}'_{i+1}$ is a semilattice edge by Proposition 6); and if $\mathbf{e}_i\mathbf{e}_{i+1}$ is a thin affine edge then $\mathbf{e}'_i\mathbf{e}'_{i+1}$ is some thin affine edge in $\mathsf{Sg}_{\mathbb{A}_1 \times \dots \times \mathbb{A}_n}(\mathbf{e}'_i, \mathbf{e}_{i+1})$.

Since $\mathbf{d}, \mathbf{e}_1, \dots, \mathbf{e}_k \in \mathcal{E}(I)$, we also have $\mathbf{e}'_1, \dots, \mathbf{e}'_k \in \mathcal{E}(I)$. Thus, we only need to check that $\mathrm{pr}_Y \mathbf{a} \sqsubseteq^{as} \mathrm{pr}_Y \mathbf{e}'_s$ for any $s \in [k]$ and $Y \in J$. However, this condition follows from the assumption $\mathbf{d} \in \mathcal{D}(I)$ — thus, $\mathrm{pr}_Y \mathbf{a} \sqsubseteq^{as} \mathrm{pr}_Y \mathbf{e}'_1$ for $Y \in J$, — and that $\mathrm{pr}_Y \mathbf{e}'_1, \dots, \mathrm{pr}_Y \mathbf{e}'_k$ is an as-path for each $Y \in J$ by Corollary 16.

Claim 2. Let $\mathbf{c} \in \mathcal{D}(I)$ be Q-approximable, $U \subset W$, and let $\mathbf{e} \in \mathrm{pr}_U Q$ be such that $\mathrm{pr}_U \mathbf{c} \sqsubseteq^{as} \mathbf{e}$ in $\mathrm{pr}_U Q$. Then there is $\mathbf{c}' \in \mathcal{D}(I)$ such that it is Q-approximable, $\mathrm{pr}_U \mathbf{c}' = \mathbf{e}$, and $\mathbf{c} \sqsubseteq^{as} \mathbf{c}'$ in $\mathcal{E}(I)$.

Let $\mathrm{pr}_U \mathbf{c} = \mathbf{b}_1, \dots, \mathbf{b}_k = \mathbf{e}$ be an as-path in $\mathrm{pr}_U Q$. Since $\mathbf{b}_i \in \mathrm{pr}_U Q$ for each $i \in [k]$, by Corollary 16(1) this path can be extended to an as-path $\mathrm{pr}_W \mathbf{c}_U = \mathbf{b}'_1, \dots, \mathbf{b}'_k$ in Q. Then, since $\mathbf{b}'_i \in \mathrm{pr}_W R$ for each $i \in [k]$, applying again Corollary 16(1) the as-path $\mathbf{b}'_1, \dots, \mathbf{b}'_k$ can be extended to an as-path $\mathbf{b}''_1, \dots, \mathbf{b}''_k$ in R such that $\mathrm{pr}_W \mathbf{b}''_i = \mathbf{b}'_i \in Q$, $\mathrm{pr}_U \mathbf{b}''_i = \mathbf{b}_i$ for each $i \in [k]$. We define a sequence $\mathbf{d}_1, \dots, \mathbf{d}_k$ as follows: $\mathbf{d}_1 = \mathbf{c}$, if $\mathbf{b}''_i\mathbf{b}''_{i+1}$ is a thin semilattice edge then set $\mathbf{d}_{i+1} = \mathbf{d}_i \cdot \mathbf{b}''_{i+1}$, and if $\mathbf{b}''_i\mathbf{b}''_{i+1}$ is a thin affine edge then choose $\mathbf{d}_{i+1}$ such that $\mathbf{d}_i\mathbf{d}_{i+1}$ is a thin affine edge in $\mathsf{Sg}_{\mathbb{A}_1 \times \dots \times \mathbb{A}_n}(\mathbf{d}_i, \mathbf{b}''_{i+1})$ and $\mathrm{pr}_U \mathbf{d}_{i+1} = \mathbf{b}_{i+1}$. Note that, since $\mathrm{pr}_U \mathbf{d}_i = \mathbf{b}_i$ and $\mathrm{pr}_U \mathbf{b}''_{i+1} = \mathbf{b}_{i+1}$ such a $\mathbf{d}_{i+1}$ exists by Lemma 15(1). Now, set $\mathbf{c}' = \mathbf{d}_k$. By Claim 1 $\mathbf{c}'$ belongs to $\mathcal{D}(I)$.

To show that $\mathbf{c}'$ is Q-approximable, for any $V \subset W$ consider the sequence $\mathrm{pr}_V \mathbf{c} = \mathrm{pr}_V \mathbf{c}_V = \mathbf{e}_1, \dots, \mathbf{e}_k = \mathrm{pr}_V \mathbf{d}_k = \mathrm{pr}_V \mathbf{e}$, where $\mathbf{e}_i = \mathrm{pr}_V \mathbf{d}_i$. By construction this is an as-path in $\mathrm{pr}_V Q$, so by Corollary 16(1) it can be extended to an as-path $\mathrm{pr}_W \mathbf{c}_V = \mathbf{e}'_1, \dots, \mathbf{e}'_k$ in Q witnessing that $\mathrm{pr}_V \mathbf{c}' \in \mathrm{pr}_V Q$.

In particular, Claim 2 implies that a Q-approximable tuple $\mathbf{c} \in \mathcal{D}(I)$ can be chosen such that for any $i \in W$ the element $\mathbf{c}[i]$ is as-maximal in $\mathrm{pr}_i Q$. We will assume it from now on. For $U = W - \{i\}$, $i \in W$, we denote $\mathbf{c}_U$ by $\mathbf{c}_i$. Suppose that for some $i \in W$ the unary projection $\mathrm{pr}_i Q \neq \mathsf{Sg}(\mathbf{c}[i], \mathbf{c}_i[i])$. Assume $i = 1$. Then set

$$Q' = Q \cap \left(\mathsf{Sg}(\mathbf{c}[1], \mathbf{c}_1[1]) \times \prod_{i \in W - \{1\}} \mathrm{pr}_i Q \right).$$

Note that in this case $\mathbf{c}$ is Q'-approximable, and the result follows by the inductive hypothesis. Let $\mathbf{c}_i$, $i \in W$, be chosen such that $\mathsf{Sg}(\mathbf{c}[i], \mathbf{c}_i[i]) = \mathrm{pr}_i Q$. It is also clear that Q can be chosen to be $\mathsf{Sg}(\mathrm{pr}_W \mathbf{c}_1, \dots, \mathrm{pr}_W \mathbf{c}_\ell)$.

Claim 3. For any $i \in W$ there is $\mathbf{d} \in Q$ such that $\mathbf{c}[i] \sqsubseteq^{as} \mathbf{d}[i]$ and $\mathrm{pr}_U \mathbf{c} \sqsubseteq^{as} \mathrm{pr}_U \mathbf{d}$ in $\mathrm{pr}_U Q$ where $U = W - \{i\}$.

Without loss of generality assume $i = 1$ and $U = W - \{i\}$. Consider the relation

$$Q'(x, y, z) = \exists x_4, \dots, x_\ell (Q(x, y, z, x_4, \dots, x_\ell) \wedge (x_4 = \mathbf{c}[4]) \wedge \dots \wedge (x_\ell = \mathbf{c}[\ell])).$$

Obviously, $\mathrm{pr}_{\{1,2,3\}} \mathbf{c}_1, \mathrm{pr}_{\{1,2,3\}} \mathbf{c}_2, \mathrm{pr}_{\{1,2,3\}} \mathbf{c}_3 \in Q'$. We show that Q' contains a tuple $\mathbf{d}$ such that $\mathrm{pr}_{12} \mathbf{c} \sqsubseteq^{as} \mathrm{pr}_{12} \mathbf{d}$, $\mathrm{pr}_{13} \mathbf{c} \sqsubseteq^{as} \mathrm{pr}_{13} \mathbf{d}$, $\mathrm{pr}_{23} \mathbf{c} \sqsubseteq^{as} \mathrm{pr}_{23} \mathbf{d}$. This would imply the claim, because, $\mathrm{pr}_{23} \mathbf{c} \sqsubseteq^{as} \mathrm{pr}_{23} \mathbf{d}$ means $\mathrm{pr}_U \mathbf{c} \sqsubseteq^{as} \mathrm{pr}_U \mathbf{d}$, and any of the first two connections means that $\mathbf{c}[i] \sqsubseteq^{as} \mathbf{d}[i]$.

If, say, the pair $(\mathbf{c}[1], \mathbf{c}[2])$ is not as-maximal in $\mathrm{pr}_{12} Q'$, choose an as-path $(\mathbf{c}[1], \mathbf{c}[2]) = \mathbf{e}_1, \dots, \mathbf{e}_s$ in $\mathrm{pr}_{12} Q'$ such that $\mathbf{e}_s$ is as-maximal in $\mathrm{pr}_{12} Q'$. By Corollary 16(1) this as-path can be extended to an as-path $\mathrm{pr}_{\{1,2,3\}} \mathbf{c}_3 = \mathbf{e}'_1, \dots, \mathbf{e}'_s$ in Q'. Now, as in the proof of Claim 2 we construct a sequence $\mathrm{pr}_{\{1,2,3\}} \mathbf{c} = \mathbf{e}''_1, \dots, \mathbf{e}''_s$, that is not necessarily from Q', as follows. If $\mathbf{e}'_i \leq \mathbf{e}'_{i+1}$, set $\mathbf{e}''_{i+1} = \mathbf{e}''_i \cdot \mathbf{e}'_{i+1}$. If $\mathbf{e}'_i \mathbf{e}'_{i+1}$ is a thin affine edge, then set $\mathbf{e}''_{i+1}$ to be any tuple in $\mathsf{Sg}(\mathbf{e}''_i, \mathbf{e}'_{i+1})$ such that $\mathbf{e}''_i \mathbf{e}''_{i+1}$ is a thin affine edge and $\mathrm{pr}_{12} \mathbf{e}''_{i+1} = \mathrm{pr}_{12} \mathbf{e}'_{i+1}$. The resulting tuple $\mathbf{e}''_s$ satisfies the following conditions: $\mathrm{pr}_{12} \mathbf{e}''_s = \mathbf{e}_s$ and

$$\mathrm{pr}_{13} \mathbf{c} \sqsubseteq^{as} \mathrm{pr}_{13} \mathbf{e}''_s \in \mathrm{pr}_{13} Q', \qquad \mathrm{pr}_{23} \mathbf{c} \sqsubseteq^{as} \mathrm{pr}_{23} \mathbf{e}''_s \in \mathrm{pr}_{23} Q'.$$

Repeating the procedure above for projections on $\{1,3\}$ and $\{2,3\}$ if necessary, we obtain a tuple $\mathbf{c}'$ such that $\mathrm{pr}_{ij} \mathbf{c}'$ is an as-maximal tuple in $\mathrm{pr}_{ij} Q'$ for $i, j \in [3]$. Relation Q' and the tuples $\mathrm{pr}_{ij} \mathbf{c}'$, $i, j \in [3]$, satisfy the conditions of Lemma 30. Therefore, Q' contains a tuple $\mathbf{d}$ such that $\mathrm{pr}_{ij} \mathbf{c} \sqsubseteq^{as} \mathrm{pr}_{ij} \mathbf{d}$ in $\mathrm{pr}_{ij} Q'$ for $i, j \in [3]$. The result follows.

To complete the proof let $U = [\ell - 1]$ (recall that $W = [\ell]$) and $\mathbf{d}$ the tuple obtained in Claim 3 for $i = \ell$. Then $\mathrm{pr}_U \mathbf{c} \sqsubseteq^{as} \mathrm{pr}_U \mathbf{d}$ in $\mathrm{pr}_U Q$. By Claim 2 $\mathbf{c}$ can be amended so that the new tuple $\mathbf{c}'$ still supports I, but $\mathrm{pr}_U \mathbf{c}' = \mathrm{pr}_U \mathbf{d}$. Note that $\mathbf{c}'[\ell]$ is in the same as-component of $\mathrm{pr}_\ell Q$ as $\mathbf{d}[\ell]$, therefore, $\mathbf{c}'[\ell] \sqsubseteq^{as} \mathbf{d}[\ell]$ in $\mathrm{pr}_\ell Q$. If $\mathsf{Sg}_{\mathbb{A}_\ell}(\mathbf{c}'[\ell], \mathbf{d}[\ell]) = \mathrm{pr}_\ell Q$, then there is an as-path from $\mathrm{pr}_W \mathbf{c}'$ to $\mathbf{d}$ in $\mathrm{pr}_W \mathcal{E}(I)$: this is because $\mathrm{pr}_U \mathbf{c}' = \mathrm{pr}_U \mathbf{d}$ and $\{\mathrm{pr}_U \mathbf{d}\} \times \mathrm{pr}_\ell Q \subseteq \mathrm{pr}_W \mathcal{E}(I)$ in this case. Extend this path to a path in $\mathcal{E}(I)$ as before, and change $\mathbf{c}'$ using this path as in Claim 1. The resulting tuple $\mathbf{c}''$ supports I, and $\mathrm{pr}_W\mathbf{c}'' = \mathbf{d} \in Q$ as required. If $\mathbb{B} = \mathsf{Sg}_{\mathbb{A}_\ell}(\mathbf{c}'[\ell], \mathbf{d}[\ell]) \neq \mathrm{pr}_\ell Q$, then set $Q'' = Q \cap (\mathrm{pr}_{[\ell-1]}Q \times \mathbb{B})$. Since $\mathbf{c}'$ is Q''-approximable, the result follows from the inductive hypothesis.

To finish the proof of Theorem 29 it suffices to take care of the requirement that the resulting tuple $\mathbf{b}$ is such that $\mathrm{pr}_X\mathbf{b} = \mathrm{pr}_X\mathbf{a}$. Since $X \in I$ already in the base case, the resulting tuple $\mathbf{b}$ is such that $\mathrm{pr}_X\mathbf{a} \sqsubseteq^{as} \mathrm{pr}_X\mathbf{b}$. By Corollary 16(1) there is also a tuple $\mathbf{b}'$ satisfying the same requirements and such that $\mathrm{pr}_X\mathbf{b}' = \mathrm{pr}_X\mathbf{a}$. $\square$

Now we generalize Theorem 29 to the variety $\mathcal{V}$ generated by $\mathcal{K}$.

Corollary 31. *Let $\mathbb{A}_1, \dots, \mathbb{A}_n \in \mathcal{V}$. Then any subdirect product R of $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$ is quasi-2-decomposable.*

Proof. We first prove that quasi-2-decomposability holds when every $\mathbb{A}_i$ is a subalgebra of a direct product of algebras from $\mathcal{K}$. Let $\mathbb{A}_i$ be a subalgebra of $\mathbb{A}_{i1} \times \dots \times \mathbb{A}_{1r_i}$, where $\mathbb{A}_{i1}, \dots, \mathbb{A}_{ir_i} \in \mathcal{K}$; we may assume $\mathbb{A}_i$ is subdirect. Let also R be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$ and $\mathbf{a} \in \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ satisfying the premise of quasi-2-decomposability. Then R can be represented as a subdirect product of

$$\prod_{i=1}^{n}\prod_{j=1}^{r_i} \mathbb{A}_{ij},$$

and $\mathbf{a}$ as a tuple from this product. Let $R', \mathbf{a}'$ denote such representations. We then apply Theorem 29 to R', $\mathbf{a}'$, and a collection J that is derived from 2-element subsets of $[n]$, that is,

$$J = \{\{(i,1), \dots, (i, r_i), (j,1), \dots, (j, r_j)\} \mid i, j \in [n]\}.$$

Next, let $\mathbb{A}_i = \mathbb{A}'_i/_{\theta_i}$, $\theta_i \in \mathsf{Con}(\mathbb{A}'_i)$, and quasi-2-decomposability hold for any subdirect product of the $\mathbb{A}'_i$. For a subdirect product $R \subseteq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ and $\mathbf{a} \in \mathbb{A}_1 \times \dots \times \mathbb{A}_n$ satisfying the premise of quasi-2-decomposability, let

$$R' = \{(b_1, \dots, b_n) \in \mathbb{A}'_1 \times \dots \times \mathbb{A}'_n \mid (b_1/_{\theta_1}, \dots, b_n/_{\theta_n}) \in R\}$$

and $\mathbf{a}' \in \mathbb{A}'_1 \times \dots \times \mathbb{A}'_n$ be such that $\mathbf{a}[i] = \mathbf{a}'[i]/_{\theta_i}$, $i \in [n]$. Then by Theorem 29 R' is quasi-2-decomposable and there is $\mathbf{b}' \in R'$ such that $(\mathbf{a}'[i], \mathbf{a}'[j]) \sqsubseteq^{as}_{\mathrm{pr}_{ij}R'} (\mathbf{b}'[i], \mathbf{b}'[j])$. Then $\mathbf{b} = (\mathbf{b}'[1]/_{\theta_1}, \dots, \mathbf{b}'[n]/_{\theta_n}) \in R$ and by Corollary 13 satisfies the requirements of quasi-2-decomposability. $\square$

Another consequence of Theorem 29 is the existence of a very useful term.

Theorem 32. *There is a term operation* maj *of $\mathcal{V}$ such that for any $\mathbb{A} \in \mathcal{V}$ and any $a, b \in \mathbb{A}$,* $\mathrm{maj}(a,a,b), \mathrm{maj}(a,b,a), \mathrm{maj}(b,a,a) \in \mathrm{Ft}^{as}_{\mathbb{A}}(a)$.

In particular, if a is as-maximal, the elements $\mathrm{maj}(a,a,b)$, $\mathrm{maj}(a,b,a)$, $\mathrm{maj}(b,a,a)$ *belong to the as-component of $\mathbb{A}$ containing a.*

Proof. Since $\mathcal{V}$ is finitely generated, it suffices to prove the result for finite subsets of $\mathcal{V}$ and then apply the compactness argument. So, let $\mathcal{K}' \subseteq \mathcal{V}$ be a finite set of finite algebras.

Let $\{a_1, b_1\}, \dots, \{a_n, b_n\}$ be a list of all pairs of elements from algebras of $\mathcal{K}'$, let $a_i, b_i \in \mathbb{A}_i$. Define a relation R to be the subalgebra of the product of $\mathbb{A}_1^3 \times \dots \times \mathbb{A}_n^3$ generated by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, where for every $i \in [n]$, $\mathrm{pr}_{3i-2,3i-1,3i}\mathbf{a}_1 = (a_i, a_i, b_i)$, $\mathrm{pr}_{3i-2,3i-1,3i}\mathbf{a}_2 = (a_i, b_i, a_i)$, $\mathrm{pr}_{3i-2,3i-1,3i}\mathbf{a}_3 = (b_i, a_i, a_i)$. In other words the triples $(\mathbf{a}_1[3i-2], \mathbf{a}_2[3i-2], \mathbf{a}_3[3i-2])$, $(\mathbf{a}_1[3i-1], \mathbf{a}_2[3i-1], \mathbf{a}_3[3i-1])$, $(\mathbf{a}_1[3i], \mathbf{a}_2[3i], \mathbf{a}_3[3i])$ have the form $(a_i, a_i, b_i), (a_i, b_i, a_i), (b_i, a_i, a_i)$, respectively. Therefore it suffices to show that R contains a tuple $\mathbf{b}$ such that $a_i \sqsubseteq^{as} \mathbf{b}[j]$, where $j \in \{3i, 3i-1, 3i-2\}$. However, since $(a_{i_1}, a_{i_2}) \in \mathrm{pr}_{j_1 j_2} R$ for any $i_1, i_2 \in [n]$ and $j_1 \in \{3i_1, 3i_1 - 1, 3i_1 - 2\}$, $j_2 \in \{3i_2, 3i_2 - 1, 3i_2 - 2\}$, this follows from Corollary 31. $\square$

A function maj satisfying the properties from Theorem 32 will be called a *quasi-majority function.*

5. Rectangularity for maximal components

In this section we show a stronger rectangularity property — involving multiary relations — than that in Proposition 26, but for maximal components, rather than as-components.

An algebra $\mathbb{A}$ is said to be *maximal generated* if it is generated by one of its maximal components.

5.1. Simple maximal generated algebras

We start with several auxiliary statements.

Lemma 33. *Let R be a subdirect product of simple maximal generated algebras $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3 \in \mathcal{V}$, generated by their maximal components C_1, C_2, C_3, respectively. If $\mathbb{A}_i \times \mathbb{A}_j \subseteq \mathrm{pr}_{ij} R$ for every $i, j \in [3]$ and $R \cap (C_1 \times C_2 \times C_3) \neq \emptyset$, then $R = \mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$.*

Proof. Note that if $|\mathbb{A}_i| = 1$ the statement is trivial, so we assume $|C_i| > 1$ for $i \in [3]$. We argue as in the proof of Lemma 30. Consider R as a subdirect product of $\mathbb{A}_1$ and $\mathbb{A}_2 \times \mathbb{A}_3$. Since $\mathbb{A}_1$ is simple, the link congruence lk_1 is either the equality relation or the full congruence. In the latter case, as $R \cap (C_1 \times C_2 \times C_3) \neq \emptyset$, by Proposition 26 $C_1 \times C_2 \times C_3 \subseteq R$ and the result follows. Suppose that lk_1 is the equality relation.

Recall that for $a \in \mathbb{A}_1$ by $R[a]$ we denote the set $R[a] = \{(b_2, b_3) \mid (a, b_2, b_3) \in R\}$. Notice that, for every $a \in \mathbb{A}_1$, $R[a]$ is a subalgebra of $\mathrm{pr}_{23} R$, and, since $\mathrm{pr}_{12} R = \mathbb{A}_1 \times \mathbb{A}_2$, $\mathrm{pr}_{13} R = \mathbb{A}_1 \times \mathbb{A}_3$, the algebra $R[a]$ is a subdirect product of $\mathbb{A}_2, \mathbb{A}_3$. Since both $\mathbb{A}_2, \mathbb{A}_3$ are simple, $R[a]$ is either linked or the graph of a bijective mapping. Let $(a_1, a_2, a_3) \in R \cap (C_1 \times C_2 \times C_3)$. If $R[a_1]$ is linked, by Proposition 26 $C_2 \times C_3 \subseteq R[a_1]$, which contradicts the assumption that lk_1 is the equality relation. Therefore $R[a_1]$ is the graph of a mapping $\varphi : \mathbb{A}_2 \to \mathbb{A}_3$. The mapping φ is an isomorphism between $\mathbb{A}_2$ and

$\mathbb{A}_3$. Replacing $\mathbb{A}_3$ with an isomorphic copy of $\mathbb{A}_2$, the link congruence lk_{23} (when considering R as a subdirect product of $\mathbb{A}_1$ and $\mathbb{A}_2\times\mathbb{A}_3$) is a nontrivial congruence of $\mathbb{A}_2^2$ one of whose blocks is $\Delta=\{(a,a)\mid a\in\mathbb{A}_2\}$. This implies that $\mathbb{A}_2$ is a module. In particular, $|C_2|=1$, a contradiction. □

Observe that maximal components in Lemma 33 (and hence in the remaining results from this section) cannot be replaced with as-components. Indeed, that would include subdirect products of modules, for which Lemma 33 is not true.

The next lemma generalizes Lemma 33 to products of multiple algebras.

Lemma 34. *Let R be a subdirect product of simple maximal generated algebras $\mathbb{A}_1,\dots,\mathbb{A}_n\in\mathcal{V}$, say, $\mathbb{A}_i$ is generated by a maximal component C_i. If $\mathbb{A}_i\times\mathbb{A}_j\subseteq \mathrm{pr}_{ij}R$ for every $i,j\in[n]$ and $R\cap(C_1\times\dots\times C_n)\neq\emptyset$, then $R=\mathbb{A}_1\times\ldots\times\mathbb{A}_n$.*

Proof. We prove the lemma by induction. The base case of induction $n=3$ has been proved in Lemma 33. Suppose that the lemma holds for each number less than n. Take $a\in C_1$ and recall that $R[a]=\{(b_2,\dots,b_n)\mid(a,b_2,\dots,b_n)\in R\}$. By Lemma 33, $\mathbb{A}_1\times\mathbb{A}_i\times\mathbb{A}_j\subseteq\mathrm{pr}_{1,i,j}R$ for any $2\le i,j\le n$. Then $\mathbb{A}_i\times\mathbb{A}_j\subseteq\mathrm{pr}_{ij}R[a]$. Therefore by induction hypothesis $R[a]=\mathbb{A}_2\times\ldots\times\mathbb{A}_n$. The lemma then follows from the assumption that C_1 generates $\mathbb{A}_1$. □

Lemma 34 allows one to describe the structure of subdirect products of simple maximal generated algebras.

Definition 1. *A relation $R\subseteq\mathbb{A}_1\times\ldots\times\mathbb{A}_n$ is said to be* almost trivial *if there exists an equivalence relation θ on the set $[n]$ with classes $I_1,\dots,I_k$, such that*

$$R=\mathrm{pr}_{I_1}R\times\ldots\times\mathrm{pr}_{I_k}R$$

where $\mathrm{pr}_{I_j}R=\{(a_{i_1},\pi_{i_2}(a_{i_1}),\dots,\pi_{i_l}(a_{i_1}))\mid a_{i_1}\in\mathbb{A}_{i_1}\}$, $I_j=\{i_1,\dots,i_l\}$, for certain bijective mappings $\pi_{i_2}\colon\mathbb{A}_{i_1}\to\mathbb{A}_{i_2},\dots,\pi_{i_l}\colon\mathbb{A}_{i_1}\to\mathbb{A}_{i_l}$.

Lemma 35. *Let R be a subdirect product of simple maximal generated algebras $\mathbb{A}_1,\dots,\mathbb{A}_n$, say, $\mathbb{A}_i$ is generated by a maximal component C_i; and let $R\cap(C_1\times\ldots\times C_n)\neq\emptyset$. Then R is an almost trivial relation.*

Proof. We prove the lemma by induction on n. When $n=1$ the result holds trivially.

We now prove the induction step. By Proposition 26, for any pair $i,j\in[n]$ the projection $\mathrm{pr}_{ij}R$ is either $\mathbb{A}_i\times\mathbb{A}_j$, or the graph of a bijective mapping. Assume that there exist i,j such that $\mathrm{pr}_{ij}R$ is the graph of a mapping $\pi\colon\mathbb{A}_i\to\mathbb{A}_j$. By the inductive hypothesis $\mathrm{pr}_{[n]-\{j\}}R$ is almost trivial, and therefore can be represented in the form

$$\mathrm{pr}_{[n]-\{j\}}R=\mathrm{pr}_{I_1}R\times\ldots\times\mathrm{pr}_{I_k}R$$

where $I_1\cup\ldots\cup I_k=[n]-\{j\}$. Suppose, for simplicity, that i is the last coordinate position in I_1, that is,

$$\begin{aligned}\mathrm{pr}_{I_1}R &= \{(a_{i_1},\dots,a_{i_k},a_i)\mid a_{i_1}\in\mathbb{A}_{i_1},\ a_{i_s}=\pi_{s1}(a_{i_1})\\ &\quad\text{for } s\in\{2,\dots,k\},\ a_i=\pi_i(a_{i_1})\}.\end{aligned}$$

Then

$$\begin{aligned}\mathrm{pr}_{I_1\cup\{j\}}R &= \{(a_{i_1},\dots,a_{i_k},a_i,a_j) \mid a_{i_1}\in\mathbb{A}_{i_1},\ a_{i_s}=\pi_{s1}(a_{i_1})\\ &\qquad \text{for } s\in\{2,\dots,k\},\ a_i=\pi_i(a_{i_1}),\ a_j=\pi\pi_i(a_{i_1})\},\end{aligned}$$

and we have $R=\mathrm{pr}_{I_1\cup\{j\}}R\times\ldots\times\mathrm{pr}_{I_k}R$, as required.

Finally, if $\mathrm{pr}_{ij}R=\mathbb{A}_i\times\mathbb{A}_j$ for all $i,j\in\underline{n}$, then the result follows by Lemma 34. □

5.2. General maximal generated algebras

Here we consider the case when factors of a subdirect product are maximal generated, but not necessarily simple.

Lemma 36. *Suppose that R is a subdirect product of maximal generated algebras $\mathbb{A}_1,\dots,\mathbb{A}_n\in\mathcal{V}$, where $\mathbb{A}_1$ is simple. Let also $\mathbb{A}_1$ be generated by a maximal component C_1, $\mathrm{pr}_{2,\dots,n}R$ is maximal generated, say, by a maximal component Q, $R\cap(C_1\times Q)\neq\emptyset$, and $\mathrm{pr}_{1i}R=\mathbb{A}_1\times\mathbb{A}_i$ for $i\in\{2,\dots,n\}$. Then $R=\mathbb{A}_1\times\mathrm{pr}_{2,\dots,n}R$.*

Proof. We prove the lemma by induction on n. The case $n=2$ is obvious. Consider the case $n=3$. We use induction on $|\mathbb{A}_1|+|\mathbb{A}_2|+|\mathbb{A}_3|$. The trivial case $|\mathbb{A}_1|+|\mathbb{A}_2|+|\mathbb{A}_3|=3$ gives the base case of induction. Let Q be a maximal component of $\mathrm{pr}_{23}R$ generating it. If both $\mathbb{A}_2,\mathbb{A}_3$ are simple, then the result follows from Lemma 35. Otherwise, suppose that $\mathbb{A}_3$ is not simple.

Note that by Corollary 16(1) R contains some subdirect product of C_1 and Q. Take a maximal congruence θ of $\mathbb{A}_3$, fix a θ-class D such that $\mathrm{pr}_3Q\cap D\neq\emptyset$ (we keep the indexing of coordinates as in R) and consider $R/_\theta\subseteq\mathbb{A}_1\times\mathbb{A}_2\times\mathbb{A}_3/_\theta$, $R_D\subseteq R$ such that

$$\begin{aligned}R/_\theta &= \{(a,b,c/_\theta)\mid(a,b,c)\in R\},\\ R_D &= \{(a,b,c)\mid(a,b,c)\in R, c\in D\}.\end{aligned}$$

By what is observed above $R_D\cap(C_1\times Q)\neq\emptyset$. Pick $\mathbf{a}\in R_D\cap(C_1\times Q)$ and consider a maximal component S of R_D such that $\mathbf{a}\sqsubseteq\mathbf{b}$ for some $\mathbf{b}\in S$. By Corollary 16(1) $S\subseteq R_D\cap(C_1\times Q)$. Let $R'\subseteq R$ be the algebra generated by S. Clearly, $\mathrm{pr}_1S\cap C_1\neq\emptyset$ and $\mathrm{pr}_{23}S\cap Q\neq\emptyset$. Note that since $\mathbb{A}_1\times\mathbb{A}_3\subseteq\mathrm{pr}_{13}R$, by Corollary 16(1) this implies $\mathrm{pr}_1S\cap C_1=C_1$. Also, obviously, $\mathrm{pr}_{13}R_D=\mathbb{A}_1\times D$. The projection pr_3S is a maximal component of D; denote it E and denote by E' the algebra generated by E. Since $\mathbb{A}_1\times E\subseteq\mathrm{pr}_{13}R$ and $(C_1\times E)\cap\mathrm{pr}_{13}R'\neq\emptyset$, again by Corollary 16(1) we obtain $C_1\times E\subseteq\mathrm{pr}_{13}R'$. Therefore $\mathbb{A}_1\times E'\subseteq\mathrm{pr}_{13}R'$. By Proposition 26, $\mathrm{pr}_{23}R/_\theta$ is either the graph of a mapping, or $\mathbb{A}_2\times\mathbb{A}_3/_\theta$.

Case 1. $\mathrm{pr}_{23}R/_\theta$ is the graph of a mapping $\pi\colon\mathbb{A}_2\to\mathbb{A}_3/_\theta$.

As before, Corollary 16(1) implies that $F=\mathrm{pr}_2S$ is a maximal component of $B=\pi^{-1}(D)$. Then $F'=\mathrm{pr}_2R'$ is generated by F. Since for each $(a,b)\in\mathbb{A}_1\times B\subseteq\mathrm{pr}_{12}R$ there is $c\in D$ with $(a,b,c)\in R$, we have $\mathbb{A}_1\times B\subseteq\mathrm{pr}_{12}R_D$. Also, $\mathrm{pr}_{12}S\cap(C_1\times F)\neq\emptyset$, say, $\mathbf{a}\in\mathrm{pr}_{12}S\cap(C_1\times F)$. As

S is a maximal component in R_D, for any $\mathbf{b} \in C_1 \times F$ such that $\mathbf{a} \sqsubseteq \mathbf{b}$, we have $\mathbf{b} \in \mathrm{pr}_{12}S \cap (C_1 \times F)$. This implies $C_1 \times F \subseteq \mathrm{pr}_{12}S$. Thus, $\mathrm{pr}_{12}R' = \mathbb{A}_1 \times F'$.

Since $|\mathbb{A}_1| + |F'| + |E'| < |\mathbb{A}_1| + |\mathbb{A}_2| + |\mathbb{A}_3|$, and $\mathrm{pr}_{23}R'$ is maximal generated, inductive hypothesis implies $\mathbb{A}_1 \times \mathrm{pr}_{23}R' \subseteq R'$. In particular, there is $(a, b) \in \mathrm{pr}_{23}R' \cap Q \subseteq \mathrm{pr}_{23}R$ such that $\mathbb{A}_1 \times \{(a, b)\} \subseteq R$. To finish the proof we just apply Lemma 25.

Case 2. $\mathrm{pr}_{23}R/_\theta = \mathbb{A}_2 \times \mathbb{A}_3/_\theta$.

Since $|\mathbb{A}_1| + |\mathbb{A}_2| + |\mathbb{A}_3/_\theta| < |\mathbb{A}_1| + |\mathbb{A}_2| + |\mathbb{A}_3|$, $\mathbb{A}_3/_\theta$ is simple, and $\mathrm{pr}_{12}R = \mathbb{A}_1 \times \mathbb{A}_2$, by inductive hypothesis, $R/_\theta = \mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3/_\theta$. Therefore, $\mathrm{pr}_{12}R_D = \mathbb{A}_1 \times \mathbb{A}_2$. Then $\mathrm{pr}_{12}R' = \mathbb{A}_1 \times \mathbb{A}_2$. Indeed, let $C_2 = \mathrm{pr}_2Q$, it is a maximal component of $\mathbb{A}_2$, and $\mathbb{A}_2$ is generated by C_2. Moreover, $\mathbb{A}_1 \times \mathbb{A}_2$ is generated by $C_1 \times C_2$. By the choice of R', there is $(a, b, c) \in R' \cap (C_1 \times C_2 \times E)$. By Corollary 16(1) for any $(a', b') \in C_1 \times C_2$ there is a path from (a, b, c) to (a', b', c') for some $c' \in E$.

Now we argue as in Case 1, except in this case $F' = \mathbb{A}_2$.

Let us assume that the lemma is proved for $n-1$. Then $\mathbb{A}_1 \times \mathrm{pr}_{3,\dots,n}R \subseteq \mathrm{pr}_{1,3,\dots,n}R$. Denoting $\mathrm{pr}_{3,\dots,n}R$ by R' we have $R \subseteq \mathbb{A}_1 \times \mathbb{A}_2 \times R'$, and the conditions of the lemma hold for this subdirect product. Thus $R = \mathbb{A}_1 \times \mathrm{pr}_{2,\dots,n}R$ as required. □

Lemma 36 serves as the base case for the following more general statement.

Lemma 37. *Let R be a subdirect product of algebras $\mathbb{A}_1, \dots, \mathbb{A}_n \in \mathcal{V}$. Let $\mathbb{A}_1$ be generated by a maximal component C_1, $\mathrm{pr}_{2,\dots,n}R$ is maximal generated, say, by a maximal component Q, $R \cap (C_1 \times Q) \neq \emptyset$, and $\mathrm{pr}_{1,i}R = \mathbb{A}_1 \times \mathbb{A}_i$ for $i \in \{2, \dots, n\}$, Then $R = \mathbb{A}_1 \times \mathrm{pr}_{2,\dots,n}R$.*

Proof. For every $i \in \{2, \dots, n\}$ the set $C_i = \mathrm{pr}_iQ$ is a maximal component. Moreover, $\mathbb{A}_i$ is generated by C_i, and therefore is also maximal generated.

We show that for any subalgebra S of $\mathbb{A}_1$ such that (a) $S \cap C_1 \neq \emptyset$, (b) S is maximal generated by its elements from C_1, and (c) $Q \subseteq \mathrm{pr}_{2,\dots,n}(R \cap (S \times C_2 \times \dots \times C_n))$, the following holds: $\{d\} \times Q \subseteq R$ for any $d \in \max(C_1 \cap S)$.

We prove by induction on the size of S. If $|S| = 1$, then its only element belongs to C_1 by (a) and is maximal. Then (c) is equivalent to the claim. Suppose that the result holds for all subalgebras satisfying (a)–(c) smaller than S. If S is simple then the result follows from Lemma 36, since conditions (a)–(c) imply the premises of Lemma 36. (Note that $\mathrm{pr}_{1,i}R = \mathbb{A}_1 \times \mathbb{A}_i$ implies $\max(C_1 \cap S) \times C_i \subseteq \mathrm{pr}_{1i}R$ for $i \in \{2, \dots, n\}$.) Otherwise let θ be a maximal congruence of S, let $R' = \{(c_1, c_2, \dots, c_n) \in R \mid c_1 \in S\}$, and let

$$R^\theta = \{(c_1/_\theta, c_2, \dots, c_n) \mid (c_1, c_2, \dots, c_n) \in R'\}.$$

By Lemma 36 $R^\theta = S/_\theta \times \mathrm{pr}_{2,\dots,n}R$. Take a class S' of θ containing elements from C_1. Observe that S' satisfies condition (c). As is easily seen there is a maximal component B of S' containing elements from C_1. Indeed, take any

$d \in C_1 \cap S'$, then $\mathrm{Ft}^s_{S'}(d) \subseteq C_1$. Let S'' be a subalgebra of S' generated by B, we show it satisfies (a)–(c).

Conditions (a) and (b) are true by the choice of S''. For condition (c) observe first that $Q \subseteq \mathrm{pr}_{2,\dots,n}(R \cap (\max(S') \times Q))$. As for any $d \in C_1 \cap S'$ there is $(a_2, \dots, a_n) \in Q$ with $(d, a_2, \dots, a_n) \in R$ (it follows from the condition $R \cap (C_1 \times Q) \neq \emptyset$ of the lemma and Corollary 16(1)), applying Corollary 16(1) again $Q \subseteq \mathrm{pr}_{2,\dots,n}(R \cap (B \times Q))$, and (c) is also true for S''. By the inductive hypothesis $\{d\} \times Q \subseteq R$ for $d \in B$. Applying Proposition 26 we obtain the result.

Finally, since $\mathbb{A}_1$ contains C_1 and satisfies conditions (a)–(c), it follows that $C_1 \times Q \subseteq R$, and the result is proved. □

Corollary 38. *Let R be a subdirect product of algebras $\mathbb{A}_1, \dots, \mathbb{A}_n \in \mathcal{V}$ such that $\mathrm{pr}_{1i} R$ is linked for any $i \in \{2, \dots, n\}$. Let also $\mathbf{a} \in R$ be such that $\mathbf{a}[1] \in \max(\mathbb{A}_1)$ and $\mathrm{pr}_{2\dots n}\mathbf{a} \in \max(\mathrm{pr}_{2\dots n} R)$. Then $\mathsf{s}(\mathbf{a}[1]) \times \mathsf{s}(\mathrm{pr}_{2\dots n}\mathbf{a}) \subseteq R$.*

Proof. Consider R', the relation generated by $R \cap (C \times D)$, $C = \mathsf{s}(\mathbf{a}[1])$, $D = \mathsf{s}(\mathrm{pr}_{2\dots n}\mathbf{a})$. By Corollary 16 $\mathsf{s}(\mathbf{a}[1]) \subseteq \mathrm{pr}_1 R'$ and $\mathsf{s}(\mathrm{pr}_{2\dots n}\mathbf{a}) \subseteq \mathrm{pr}_{2\dots n} R'$. Moreover, C and D are maximal components in $\mathrm{pr}_1 R'$ and $\mathrm{pr}_{2\dots n} R'$, respectively. Since $\mathrm{pr}_{1i} R$ is linked for $i \in \{2, \dots, n\}$, $C \times \mathrm{pr}_i D \subseteq \mathrm{pr}_{1i} R$ by Proposition 26. As R' is generated by $R \cap (C \times D)$, $\mathrm{pr}_{1i} R'$ is generated by $\mathrm{pr}_{1i} R \cap (C \times \mathrm{pr}_i D)$ implying $C \times \mathrm{pr}_i D \subseteq \mathrm{pr}_{1i} R'$. Subalgebras $\mathrm{pr}_1 R'$ and $\mathrm{pr}_i R'$ are generated by C and $\mathrm{pr}_i D$, respectively, therefore, $\mathrm{pr}_{1i} R' = \mathrm{pr}_1 R' \times \mathrm{pr}_i R'$. Now by Lemma 37 the result follows. □

Declarations

Data availability

Data sharing not applicable to this article as datasets were neither generated nor analyzed.

Compliance with ethical standards

The author is a member of the Editorial Board of Algebra Universalis. Apart from this the author declares that he has no conflict of interest.

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