Graphs of finite algebras: edges, and connectivity

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Abstract. We refine and advance the study of the local structure of idempotent finite algebras started in [A.Bulatov, *The Graph of a Relational Structure and Constraint Satisfaction Problems*, LICS, 2004]. We introduce a graph-like structure on an arbitrary finite idempotent algebra including those admitting type **1**. We show that this graph is connected, its edges can be classified into 4 types corresponding to the local behavior (set, semilattice, majority, or affine) of certain term operations. We also show that if the variety generated by the algebra omits type **1**, then the structure of the algebra can be 'improved' without introducing type **1** by choosing an appropriate reduct of the original algebra. Taylor minimal idempotent algebras introduced recently are a special case of such reducts. Then we refine this structure demonstrating that the edges of the graph of an algebra omitting type **1** can be made 'thin', that is, there are term operations that behave very similar to semilattice, majority, or affine operations on 2-element subsets of the algebra. Finally, we prove certain connectivity properties of the refined structures.

This research is motivated by the study of the Constraint Satisfaction Problem, although the problem itself does not really show up in this paper.

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1. Introduction

The study of the Constraint Satisfaction Problem (CSP) and especially the Dichotomy Conjecture triggered a wave of research in universal algebra, as it turns out that the algebraic approach to the CSP developed in [18, 24] is

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the most prolific one in this area. These developments have led to a number of strong results about the CSP, see, e.g., [1, 4, 5, 9, 11, 13, 17, 23], and ultimately the resolution of the Dichotomy Conjecture in [15, 31, 33]. However, successful application of the algebraic approach also requires new results about the structure of finite algebras. Two ways to describe this structure have been proposed. One is based on absorption properties [2, 3] and has led not only to new results on the CSP, but also to significant developments in universal algebra itself.

In this paper we refine and advance the alternative approach originally developed in [8, 12, 19], which is based on the local structure of finite algebras. This approach identifies subalgebras or factors of an algebra having 'good' term operations, that is, operations of one of the three types: semilattice, majority, or affine. It then explores the graph or hypergraph formed by such subalgebras, and exploits its connectivity properties. In a nutshell, this method stems from the early study of the CSP over so called conservative algebras [11], and has led to a much simpler proof of the Dichotomy Conjecture for conservative algebras [14] and to a characterization of CSPs solvable by consistency algorithms [10]. In spite of these applications the original method suffer from a number of drawbacks that make its use difficult.

In the present paper we refine many of the constructions, generalize them to include algebras admitting type **1**, and fix the deficiencies of the original method. Similar to [8, 19] an edge is a pair of elements a, b such that there is a factor algebra of the subalgebra generated by a, b that has an operation which is semilattice, majority, or affine on the blocks containing a, b. Here we extend the definition of an edge by allowing edges to have the unary type, when the corresponding factor is a set. These operations, or lack thereof, determine the type of edge ab. In this paper we also allow edges to have more than one type if there are several factors witnessing different types.

The main difference from the previous results is the introduction of oriented *thin* majority and affine edges. An edge ab is said to be thin if there is a term operation that is semilattice on $\{a, b\}$, or there is a term operation that satisfies the identities of a majority or affine term on $\{a, b\}$, but only for some choices of values of the variables. In particular, for a thin majority edge we require the existence of a term operation m such that $m(a, b, b) = m(b, a, b) = m(b, b, a) = b$, and for a thin affine edge we require that there is a term operation d with $d(b, a, a) = d(a, a, b) = b$, but m, d do not have to satisfy similar conditions when a, b are swapped. Oriented thin edges allow us to prove a stronger version of the connectivity of the graph related to an algebra. This updated approach makes it possible to give a much simpler proof of the result of [10] (see also [4]), and, eventually, proving the Dichotomy Conjecture [15, 16], see also [31, 32, 33], however, this is a subject of subsequent papers.

2. Preliminaries

In terminology and notation we mainly follow the standard texts on universal algebra [20, 27]. We also assume some familiarity with the basics of the tame congruence theory [22], although its use is restricted to some proofs in the first two sections. All algebras in this paper are assumed to be finite and idempotent.

Algebras will be denoted by $\mathbb{A}, \mathbb{B}$, etc. The subalgebra of an algebra $\mathbb{A}$ generated by a set $B \subseteq \mathbb{A}$ is denoted $\mathsf{Sg}_{\mathbb{A}}(B)$, or if $\mathbb{A}$ is clear from the context simply by $\mathsf{Sg}(B)$. The set of term operations of algebra $\mathbb{A}$ is denoted by $\mathsf{Term}(\mathbb{A})$. Subalgebras of direct products are often considered as relations. An element (a tuple) from $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is denoted in boldface, say, $\mathbf{a}$, and its ith component is referred to as $\mathbf{a}[i]$, that is, $\mathbf{a} = (\mathbf{a}[1], \dots, \mathbf{a}[n])$. The set $\{1, \dots, n\}$ will be denoted by $[n]$. For $I \subseteq [n]$, say, $I = \{i_1, \dots, i_k\}$, $i_1 < \cdots < i_k$, by $\mathrm{pr}_I \mathbf{a}$ we denote the k-tuple $(\mathbf{a}[i_1], \dots, \mathbf{a}[i_k])$, and for $R \subseteq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ by $\mathrm{pr}_I R$ we denote the set $\{\mathrm{pr}_I \mathbf{a} \mid \mathbf{a} \in R\}$. If $I = \{i\}$ or $I = \{i, j\}$ we write $\mathrm{pr}_i, \mathrm{pr}_{ij}$ rather than pr_I. The tuple $\mathrm{pr}_I \mathbf{a}$ and relation $\mathrm{pr}_I R$ are called the *projections* of, respectively, $\mathbf{a}$ and R on I. A subalgebra (a relation) R of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is said to be a *subdirect product* of $\mathbb{A}_1, \dots, \mathbb{A}_n$ if $\mathrm{pr}_i R = \mathbb{A}_i$ for every $i \in [n]$. For a congruence α of $\mathbb{A}$, $a \in \mathbb{A}$, and term operation f of $\mathbb{A}$, by $a/_{\alpha}$ we denote the α-block containing a, by $\mathbb{A}/_{\alpha}$ the factor algebra modulo α, and by $f/_{\alpha}$ the factor operation of f, that is, the operation on $\mathbb{A}/_{\alpha}$ given by $f/_{\alpha}(a_1/_{\alpha}, \dots, a_n/_{\alpha}) = f(a_1, \dots, a_n)/_{\alpha}$. For $B \subseteq \mathbb{A}^2$, the congruence generated by B will be denoted by $\mathsf{Cg}_{\mathbb{A}}(B)$ or just $\mathsf{Cg}(B)$. By $\underline{0}_{\mathbb{A}}, \underline{1}_{\mathbb{A}}$ we denote the least (i.e., the equality relation), and the greatest (i.e., the total relation) congruences of $\mathbb{A}$, respectively. Again, we often simplify this notation to $\underline{0}, \underline{1}$.

An algebra is called a *set* if all of its term operations are projections.

Recall that a *tolerance* of $\mathbb{A}$ is a binary reflexive and symmetric relation compatible with $\mathbb{A}$. The transitive closure of a tolerance is a congruence of $\mathbb{A}$. Let α, β be congruences of $\mathbb{A}$ such that $\alpha \prec \beta$, that is, $\alpha < \beta$ and (α, β) is a prime interval. Then a (α, β)-*minimal set* U is a subset of $\mathbb{A}$, minimal with respect to inclusion among subsets of the form $U = f(\mathbb{A})$, for some unary polynomial f of $\mathbb{A}$ such that $f(\beta) \not\subseteq \alpha$. If this is the case then f can also be chosen to be idempotent. The intersection of U with a β-block is a *trace* if it overlaps with more than one α-block. Any two (α, β)-minimal sets U, V are *polynomially isomorphic*, that is, there are unary polynomials g, h such that $g(U) = V$, $h(V) = U$, and $g^{-1} \circ h, h^{-1} \circ g$ are identity transforations on V and U, respectively. The prime factor (α, β) has type **1**,**2**,**3**,**4**, or **5** if every (α, β)-trace $U/_{\alpha}$ is polynomially equivalent to an algebra with only unary operations, 1-dimensional vector space, 2-element Boolean algebra, lattice, or semilattice, respectively. The type of (α, β) is denoted by $\mathsf{typ}(\alpha, \beta)$.

If α is a congruence of an algebra $\mathbb{A}$ and R is a compatible binary relation, then the α-*closure* of R is defined to be $\alpha \circ R \circ \alpha$. A relation equal to its α-closure is said to be α-*closed*. If (α, β) is a prime factor of $\mathbb{A}$, then the *basic tolerance for* (α, β) (see [22], Chapter 5) is the α-closure of the relation $\alpha \cup \bigcup\{N^2 \mid N \text{ is an } (\alpha, \beta)\text{-trace}\}$ if $\mathsf{typ}(\alpha, \beta) \in \{\mathbf{2}, \mathbf{3}\}$, and it is the α-closure of the compatible relation generated by $\alpha \cup \bigcup\{N^2 \mid N$ is an (α,β)-trace$\}$ if $\mathsf{typ}(\alpha,\beta) \in \{\mathbf{4},\mathbf{5}\}$. The basic tolerance is the smallest α-closed tolerance τ of $\mathbb{A}$ such that $\alpha \neq \tau \subseteq \beta$.

Let (α,β) be a prime factor of $\mathbb{A}$. An (α,β)-*quasi-order* is a compatible reflexive and transitive relation R such that $R \cap R^{-1} = \alpha$, and the transitive closure of $R \cup R^{-1}$ is β. The quotient (α,β) is said to be *orderable* if there exists an (α,β)-quasi-order. By Theorem 5.26 of [22], if $\mathsf{typ}(\alpha,\beta) \neq \mathbf{1}$, then (α,β) is orderable if and only if $\mathsf{typ}(\alpha,\beta) \in \{\mathbf{4},\mathbf{5}\}$.

3. Graph: Thick edges

After recalling several basic facts about idempotent algebras, we start with introducing 'thick' edges, one of the main constructions of this paper.

3.1. Basic facts about idempotent algebras

An algebra $\mathbb{A} = (A;F)$ is *idempotent* if $f(x,\dots,x) = x$ for every $f \in F$. The algebra $\mathbb{A}' = (A,F')$ where F' is the set of all idempotent operations from $\mathsf{Term}(\mathbb{A})$ is said to be the *full idempotent reduct* of $\mathbb{A}$. Let τ be a tolerance. A set $B \subseteq \mathbb{A}$ maximal with respect to inclusion and such that $B^2 \subseteq \tau$ is said to be a *class* of τ. We will need the following simple observation.

Lemma 3.1. *Every class of a tolerance of an idempotent algebra is a subalgebra.*

Proof. Let τ be a tolerance of an idempotent algebra $\mathbb{A}$ and B its class. Then, for any $a \notin B$, there is $b \in B$ such that $(a,b) \notin \tau$. If B is not a subuniverse, then, for a certain term operation f of A and $b_1,\dots,b_n \in B$, we have $a = f(b_1,\dots,b_n) \notin B$. Take $b \in B$ such that $(a,b) \notin \tau$. Since $(b_1,b),\dots,(b_n,b) \in \tau$, we get $(f(b_1,\dots,b_n), f(b,\dots,b)) = (a,b) \in \tau$, a contradiction. □

Let $G = (V,E)$ be a hypergraph. A *path* in G is a sequence $H_1,\dots,H_k$ of hyperedges such that $H_i \cap H_{i+1} \neq \varnothing$, for $1 \le i < k$. The hypergraph G is said to be *connected* if, for any $a,b \in V$, there is a path $H_1,\dots,H_k$ such that $a \in H_1$, $b \in H_k$.

The universe of an algebra $\mathbb{A}$ along with the family of all of its proper subalgebras form a hypergraph denoted by $\mathcal{H}(\mathbb{A})$. Lemma 3.1 implies that, for a simple idempotent algebra $\mathbb{A}$, the hypergraph $\mathcal{H}(\mathbb{A})$ is connected unless $\mathbb{A}$ is tolerance free. In the latter case it can be disconnected.

Recall that an element a of an algebra $\mathbb{A}$ is said to be *absorbing* if whenever $t(x,y_1,\dots,y_n)$ is an $(n+1)$-ary term operation of $\mathbb{A}$ such that t depends on x and $b_1,\dots,b_n \in \mathbb{A}$, then $t(a,b_1,\dots,b_n) = a$. A congruence θ of $\mathbb{A}^2$ is said to be *skew* if it not a product congruence of the form $\theta_1 \times \theta_2 = \{((a_1,a_2),(b_1,b_2)) \mid (a_1,b_1) \in \theta_1, (a_2,b_2) \in \theta_2\}$ for $\theta_1,\theta_2 \in \mathsf{Con}(\mathbb{A})$. In particular, if $\mathbb{A}$ is simple it happens if it is the kernel of no projection mapping of $\mathbb{A}^2$ onto its factors. The following theorem due to Kearnes [25] provides some information about the structure of simple idempotent algebras.

Theorem 3.2 ([25])**.** *Let $\mathbb{A}$ be a simple idempotent algebra. Then exactly one of the following holds:*

(a) $\mathbb{A}$ is abelian and is term equivalent to a subalgebra of a reduct of a module;
(b) $\mathbb{A}$ has an absorbing element;
(c) $\mathbb{A}^2$ has no skew congruence.

Since $\mathbb{A}$ is idempotent, in most cases in item (a) rather than working with a module $\mathbb{M}$ we use its full idempotent reduct. Slightly abusing the terminology we will call such a reduct simply by a module.

Option (a) of Theorem 3.2 can be made more precise. In [30] Valeriote proved that every finite simple abelian idempotent algebra is strictly simple, that is, does not have proper subalgebras containing more than one element. It then follows from the result of Szendrei [29] that every such algebra is either a set or a module.

Proposition 3.3. *Let $\mathbb{A}$ be a simple finite idempotent Abelian algebra. Then $\mathbb{A}$ is either a set, or a module.*

3.2. The four types of edges

Let $\mathbb{A}$ be an algebra with universe A. We introduce graph $\mathcal{G}(\mathbb{A})$ as follows: The vertex set of $\mathcal{G}(\mathbb{A})$ is the set A. A pair ab of vertices is an *edge* if and only if there exists a congruence θ of $\mathsf{Sg}(a,b)$ such that either $\mathsf{Sg}(a,b)/_\theta$ is a set, or there is a term operation f of $\mathbb{A}$ such that either $f/_\theta$ is an affine operation on $\mathsf{Sg}(a,b)/_\theta$, or $f/_\theta$ is a semilattice operation on $\{a/_\theta, b/_\theta\}$, or $f/_\theta$ is a majority operation on $\{a/_\theta, b/_\theta\}$.

If there exists a congruence θ and a term operation f of $\mathbb{A}$ such that $f/_\theta$ is a semilattice operation on $\{a/_\theta, b/_\theta\}$ then ab is said to have the *semilattice type*. Edge ab is of the *majority type* if there are a congruence θ and $f \in \mathsf{Term}(\mathbb{A})$ such that $f/_\theta$ is a majority operation on $\{a/_\theta, b/_\theta\}$, but no operation is semilattice on $\{a/_\theta, b/_\theta\}$. Pair ab has the *affine type* if there are a congruence θ and $f \in \mathsf{Term}(\mathbb{A})$ such that $\mathsf{Sg}(a,b)/_\theta$ is a module and $f/_\theta$ is its affine operation $x-y+z$. Finally, ab is of the *unary type* if $\mathsf{Sg}(a,b)/_\theta$ is a set (in which case $\mathsf{Sg}(a,b)/_\theta = \{a/_\theta, b/_\theta\}$). In all cases we say that congruence θ and operation f *witness* the type of edge ab. As is easily seen, congruence θ can always be chosen to be a maximal congruence of $\mathsf{Sg}(a,b)$ if ab is a unary, semilattice, or affine edge. For majority edges the situation is more complicated as the following example shows. Let $\mathbb{A} = (\{a,b,0\}; f, \wedge)$. The operation $\wedge$ is the semilattice operation which satisfies $a \wedge b = 0$. The operation f is the majority operation on $\{a,b\}$, but if $0 \in \{x,y,z\}$, then $f(x,y,z) = 0$. Clearly, ab is a majority edge with respect to the identity relation, but with respect to any of the two maximal congruences of $\mathbb{A} = \mathsf{Sg}(a,b)$ (partitioning A into $\{a,0\},\{b\}$ and $\{a\},\{b,0\}$, respectively), ab is a semilattice edge.

Note that, for every edge ab of $\mathcal{G}(\mathbb{A})$, there is an associated pair $a/_\theta, b/_\theta$ from the factor subalgebra. We will need both of these types of pairs and will sometimes call $a/_\theta, b/_\theta$ a *thick* edge (see Fig. 1). There may be many congruences certifying the type of the edge ab. In most cases it does not

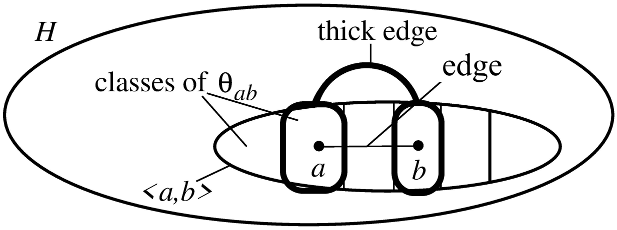

FIGURE 1. Edges and thick edges

make much difference which congruence to choose, so we will use just any congruence certifying the type of ab. Note also that a pair ab may have more than one type witnessed by different congruences θ.

Example 3.4. *Let $A = \{a, b, c\}$ and binary operations f, g on A defined as follows: $f(a, c) = f(c, a) = g(a, c) = g(c, a) = c$, $f(b, c) = f(c, b) = g(b, c) = g(c, b) = c$, $f(a, b) = g(b, a) = c$, and $f(b, a) = b$, $g(a, b) = a$. Let $\mathbb{A} = (A, f, g)$, that is, $\mathbb{A}$ is an 'almost' 3-element semilattice with respect to each of f, g, in which a, b are incomparable, but unlike a 3-element semilattice $\mathbb{A}$ is simple. As is easily seen, each of the pairs $\{a, c\}, \{b, c\}$ generates a 2-element subalgebra term equivalent to a 2-element semilattice, and therefore is an edge of $\mathbb{A}$ of the semilattice type. On the other hand, $\{a, b\}$ generates $\mathbb{A}$, and it can be easily seen that there is no term operation of $\mathbb{A}$ that is semilattice, majority, or affine on $\{a, b\}$. Therefore ab is not an edge.*

3.3. General connectivity

The main result of this section is the connectedness of the graph $\mathcal{G}(\mathbb{A})$ for arbitrary idempotent algebras, and the connection between omitting types **1** and **2** and avoiding edges of certain types. Recall that for a class $\mathfrak{A}$ of algebras $\mathsf{H}(\mathfrak{A})$ and $\mathsf{S}(\mathfrak{A})$ denote the class of all homomorphic images, and the class of all subalgebras of algebras from $\mathfrak{A}$, respectively. More precisely, we prove the following

Theorem 3.5. *Let $\mathbb{A}$ be an idempotent algebra. Then*

(1) $\mathcal{G}(\mathbb{A})$ is connected;
*(2) $\mathsf{HS}(\mathbb{A})$ omits type **1** if and only if $\mathcal{G}(\mathbb{A})$ contains no edges of the unary type;*
*(3) $\mathsf{HS}(\mathbb{A})$ omits types **1** and **2** if and only if $\mathcal{G}(\mathbb{A})$ contains no edges of the unary and affine types.*

First we prove Theorem 3.5 for simple algebras. We will need the following easy observation.

Lemma 3.6. *Let R be an n-ary compatible relation on $\mathbb{A}$ such that, for any $i \in [n]$, $\mathrm{pr}_i R = \mathbb{A}$. Then, for any $i \in [n]$, the relation $\mathsf{tol}_i(R) = \{(a, b) \mid$ there are $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in \mathbb{A}$ such that $(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n)$, $(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \in R\}$ is a tolerance of $\mathbb{A}$.*

Tolerance of the form $\mathsf{tol}_i(R)$ will be called *link tolerance*, or *ith link tolerance*. We will drop mention of R whenever it does not cause confusion.

Proposition 3.7. *Let $\mathbb{A}$ be a simple idempotent algebra. Then one of the following holds:*

(1) $\mathbb{A}$ is a 2-element set;
(2) $\mathbb{A}$ is term equivalent to a module;
(3) $\mathcal{H}(\mathbb{A})$ is connected;
(4) $\mathbb{A} = \mathsf{Sg}(a, b)$ for some $a, b \in \mathbb{A}$, and for any such a, b either a, b are connected in $\mathcal{H}(\mathbb{A})$, or there is a binary term operation f such that f is a semilattice operation on $\{a, b\}$, or there is a ternary term operation g and an automorphism of $\mathbb{A}$ such that g is a majority operation on $\{a, b\}$ and the automorphism swaps a and b.

Proof. We consider the three options given in Theorem 3.2.

(a) Suppose first that $\mathbb{A}$ is abelian. Then by Proposition 3.3 it is either a set or a module. As $\mathbb{A}$ is simple, in the former case it is also 2-element, and we are in case (1) of Proposition 3.7. In the latter case we of course obtain case (2) of the proposition. In the rest of the proof we assume $\mathbb{A}$ non-abelian, in particular the factor $\underline{0}_\mathbb{A} \prec \underline{1}_\mathbb{A}$ has type **3**,**4**, or **5**.

(b) Next assume that $\mathbb{A}$ has an absorbing element, and let a be the absorbing element of $\mathbb{A}$. Then for any $b \in \mathbb{A}$ the set $\{a, b\}$ is a subalgebra. Indeed, let $f(x_1, \dots, x_k)$ be a term operation of $\mathbb{A}$ (for simplicity assume that it depends on all of its arguments). Then for $a_1, \dots, a_k \in \{a, b\}$ we have $f(a_1, \dots, a_k) = a$ if any of the a_i equals a, and $f(a_1, \dots, a_k) = b$ otherwise, as f is idempotent. Therefore if $|\mathbb{A}| \geq 3$ then $\mathcal{H}(\mathbb{A})$ is connected through such 2-element subalgebras, and we are in case (3) of Proposition 3.7. If $|\mathbb{A}| = 2$, say, $\mathbb{A} = \{a, b\}$, then clearly $\mathbb{A} = \mathsf{Sg}(a, b)$. Moreover for any non-unary term operation $f(x_1, \dots, x_k)$ of $\mathbb{A}$ for $g(x, y) = f(x, y, \dots, y)$ we have $g(a, a) = g(a, b) = g(b, a) = a$ and $g(b, b) = b$; that is, g is a semilattice operation on $\mathbb{A}$. We obtain case (4) of Proposition 3.7.

(c) If $\mathsf{typ}(\underline{0}_\mathbb{A}, \underline{1}_\mathbb{A}) \in \{\mathbf{4}, \mathbf{5}\}$, then by Theorem 5.26 of [22], there exists a $(\underline{0}_\mathbb{A}, \underline{1}_\mathbb{A})$-quasi-order $\leq$ on $\mathbb{A}$, which is, clearly, just a compatible partial order. Let $a \leq b \in A$ be such that $a \leq c \leq b$ implies $c = a$ or $c = b$. We claim that $\{a, b\}$ is a subalgebra of $\mathbb{A}$. Indeed, for any term operation $f(x_1, \dots, x_n)$ of $\mathbb{A}$ and any $a_1, \dots, a_n \in \{a, b\}$, we have $a = f(a, \dots, a) \leq f(a_1, \dots, a_n) \leq f(b, \dots, b) = b$. If $|\mathbb{A}| \geq 3$, it follows from Lemma 5.24(3) and Theorem 5.26(2) of [22] that $\leq$ is connected. Therefore we have case (3) of Proposition 3.7. If $|\mathbb{A}| = 2$, then $\mathbb{A}$ has a semilattice or majority term operation, as follows from the description of clones on a 2-element set, [28]. Moreover, if the mapping φ swapping the elements of $\mathbb{A}$ is not an automorphism of $\mathbb{A}$, the algebra $\mathbb{A}$ has a term operation f that is not self-dual (meaning precisely that φ does not commute with f). If this is the case, again, the description from [28] implies that a semilattice operation is a term operation of $\mathbb{A}$.

Suppose that $\mathsf{typ}(\underline{0}_{\mathbb{A}}, \underline{1}_{\mathbb{A}}) = \mathbf{3}$. If $\mathsf{Sg}(a,b) \neq \mathbb{A}$ for all $a,b \in \mathbb{A}$ then $\mathcal{H}(\mathbb{A})$ is connected by subalgebras generated by 2-element sets. So, suppose $\mathbb{A} = \mathsf{Sg}(a,b)$ for some $a,b \in \mathbb{A}$. Note that as $\mathbb{A}$ is simple, the transitive closure of every one of its tolerance different from the equality relation is the total relation. If a tolerance satisfies this condition then we say that it is *connected*. Since if a connected tolerance τ exists, $\mathcal{H}(\mathbb{A})$ is connected by the classes of τ that are subalgebras of $\mathbb{A}$ by Lemma 3.1, we obtain case (3) of Proposition 3.7. Thus, we can assume that $\mathbb{A}$ is tolerance-free.

We consider two cases.

Case 1. There is no automorphism φ of $\mathbb{A}$ such that $\varphi(a) = b$ and $\varphi(b) = a$.

Consider the relation R generated by $(a,b),(b,a)$. By the assumption made, R is not the graph of a bijective mapping. By Lemma 3.6, $\mathsf{tol}_1(R), \mathsf{tol}_2(R)$ are tolerances of $\mathbb{A}$ different from the equality relation. Thus, they both are equal to the total relation. Therefore, there is $c \in \mathbb{A}$ such that $(a,c),(b,c) \in R$. If both $\mathsf{Sg}(a,c), \mathsf{Sg}(b,c)$ are proper subalgebras of $\mathbb{A}$, then a,b are connected in $\mathcal{H}(\mathbb{A})$. Otherwise, let, say, $\mathsf{Sg}(a,c) = \mathbb{A}$. Since $(b,a),(b,c) \in R$ and $\mathbb{A}$ is idempotent, $(b,d) \in R$ for any $d \in \mathbb{A}$. In particular, $(b,b) \in R$. This means that there is a binary term operation f such that $f(a,b) = f(b,a) = b$, that is, f is semilattice on $\{a,b\}$, as required. We are in case (4) of Proposition 3.7.

Case 2. There is an automorphism φ of $\mathbb{A}$ such that $\varphi(a) = b$ and $\varphi(b) = a$.

Consider the ternary relation R generated by $(a,a,b),(a,b,a),(b,a,a)$. As in the previous case, if we show that $(a,a,a) \in R$, then the result follows, as we explain in detail later. Let also $S = \{(c,\varphi(c)) \mid c \in \mathbb{A}\}$ denote the graph of an automorphism φ with $\varphi(a) = b$ and $\varphi(b) = a$.

Claim 1. $\mathrm{pr}_{12}R = \mathbb{A} \times \mathbb{A}$.

Let $Q = \mathrm{pr}_{12}R$ and $Q' = \{(c,\varphi(d)) \mid (c,d) \in Q\}$. As $Q'(x,y) = \exists z(Q(x,z) \wedge S(z,y))$, this relation is compatible. Clearly, $Q = \mathbb{A} \times \mathbb{A}$ if and only if $Q' = \mathbb{A} \times \mathbb{A}$. Notice that $(a,a),(b,b),(a,b) \in Q'$. Since $\mathsf{typ}(\mathbb{A}) = \mathbf{3}$ and $\mathbb{A}$ is tolerance free, every pair $c,d \in \mathbb{A}$ is a trace. Therefore, again as $\mathsf{typ}(\mathbb{A}) = \mathbf{3}$, there is a polynomial operation $g(x)$ with $g(a) = c, g(b) = d$ and, hence, there is a term operation $f(x,y,z)$ such that $f(a,b,x) = g(x)$. For this operation we have

$$f\left(\begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} b \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right) = \begin{pmatrix} c \\ d \end{pmatrix} \in Q'.$$

Since such an operation can be found for any pair $c,d \in \mathbb{A}$, this proves the claim.

Next we show that $\mathsf{tol}_3(R)$ cannot be the equality relation. Suppose for contradiction that it is. Then the relation $\theta = \{((c_1,d_1),(c_2,d_2)) \mid$ there is $e \in \mathbb{A}$ such that $(c_1,d_1,e),(c_2,d_2,e) \in R\}$ is a congruence of $Q = \mathbb{A}^2$. It cannot be a skew congruence, hence, it is the kernel of the projection of $\mathbb{A}^2$ onto one of its factors. Without loss of generality let $\theta = \{((c_1,d_1),(c_2,d_2)) \mid c_1 = c_2\}$. This means that, for any $e \in \mathbb{A}$ and any $(c_1,d_1,e),(c_2,d_2,e) \in R$, we have $c_1 = c_2$. However, $(a,b,a),(b,a,a) \in R$, a contradiction. The same argument applies when $\theta = \{((c_1,d_1),(c_2,d_2)) \mid d_1 = d_2\}$.

Thus, $\mathsf{tol}_3(R)$ is the total relation, and there is $(c,d) \in Q$ such that $(c,d,a),(c,d,b) \in R$ which implies $\{(c,d)\} \times \mathbb{A} \subseteq R$.

CLAIM 2. For any $(c',d') \in \mathrm{pr}_{12}R$, it holds that $(c',d',a) \in R$.

Take a term operation $g(x,y,z)$ such that $g(a,b,c) = c'$ and $g(a,b,\varphi^{-1}(d)) = \varphi^{-1}(d')$. Such an operation exists provided that $c \neq \varphi^{-1}(d)$, because every pair of elements of $\mathbb{A}$ is a trace. Then

$$
\begin{aligned}
g\left(\begin{pmatrix} a\\ b\\ a\end{pmatrix}, \begin{pmatrix} b\\ a\\ a\end{pmatrix}, \begin{pmatrix} c\\ d\\ a\end{pmatrix}\right) &= \begin{pmatrix} g(a,b,c)\\ g(b,a,d)\\ a\end{pmatrix}\\
&= \begin{pmatrix} c'\\ \varphi(g(\varphi^{-1}(b),\varphi^{-1}(a),\varphi^{-1}(d)))\\ a\end{pmatrix}\\
&= \begin{pmatrix} c'\\ \varphi(g(a,b,\varphi^{-1}(d)))\\ a\end{pmatrix} = \begin{pmatrix} c'\\ \varphi(\varphi^{-1}(d'))\\ a\end{pmatrix} = \begin{pmatrix} c'\\ d'\\ a\end{pmatrix} \in R.
\end{aligned}
$$

What is left is to show that there are c', d' such that $(c',d',a) \in R$ and $c' \neq \varphi^{-1}(d')$. Suppose $c = \varphi^{-1}(d)$. If $\mathsf{Sg}(a,c), \mathsf{Sg}(c,b) \neq \mathbb{A}$, then a, b are connected in $\mathcal{H}(\mathbb{A})$. Let $\mathsf{Sg}(a,c) = \mathbb{A}$, and h such that $h(a,c) = b$. Since R is symmetric with respect to any permutation of coordinates, $\{c\} \times \mathbb{A} \times \{d\} \subseteq R$. In particular, $(c,c,d) \in R$. Then

$$
h\left(\begin{pmatrix} a\\ a\\ b\end{pmatrix}, \begin{pmatrix} c\\ c\\ d\end{pmatrix}\right) = \begin{pmatrix} b\\ b\\ a\end{pmatrix},
$$

as $c = \varphi^{-1}(d)$ and $a = \varphi^{-1}(b)$. The tuple (b,b,a) is as required and we can set $c' = d' = b$. The case when $\mathsf{Sg}(c,b) = \mathbb{A}$ is dual by applying φ, c becomes d and we get $\mathsf{Sg}(a,d) = \mathbb{A}$. Then the same argument proceeds with $(c,d,d) \in R$ and $h(a,d) = b$.

Thus, $(a,a,a) \in R$ which means that there is a term operation $f(x,y,z)$ such that $f(a,a,b) = f(a,b,a) = f(b,a,a) = a$. Since φ is an automorphism, we also get $f(b,b,a) = f(b,a,b) = f(a,b,b) = b$, i.e. f is a majority operation on $\{a,b\}$. □

Proof of Theorem 3.5. Suppose for contradiction that $\mathcal{G}(\mathbb{A})$ is disconnected. Let $\mathbb{B}$ be a minimal subalgebra of $\mathbb{A}$ such that $\mathcal{G}(\mathbb{B})$ is disconnected. Let θ be a maximal congruence of $\mathbb{B}$. As is easily seen, if $\mathcal{G}(\mathbb{B}/_\theta)$ is connected then $\mathcal{G}(\mathbb{B})$ is connected. Indeed, if $c/_\theta d/_\theta$ is an edge in $\mathbb{B}/_\theta$ then cd is an edge in $\mathbb{B}$ (see also Lemma 3.8 below). Also, θ-blocks are subalgebras of $\mathbb{B}$ and therefore connected by the choice of $\mathbb{B}$. Take $c, d \in \mathbb{B}$; let $c' = c/_\theta, d' = d/_\theta$. If $\mathsf{Sg}(c,d) \neq \mathbb{B}$ then c, d are connected in $\mathsf{Sg}(c,d)$ by the assumption made. Otherwise $\mathsf{Sg}(c',d') = \mathbb{B}/_\theta$ and we are in the conditions of Proposition 3.7. If $\mathbb{B}/_\theta$ is a set, cd is an edge of the unary type. If $\mathbb{B}/_\theta$ is term equivalent to the full idempotent reduct of a module, then cd is an edge of the affine type. If c', d' are connected in $\mathcal{H}(\mathbb{B}/_\theta)$ as in items (3),(4) of Proposition 3.7, then c, d are connected in $\mathcal{G}(\mathbb{B})$ by the assumption of the minimality of $\mathbb{B}$. In the remaining cases cd is a semilattice or majority edge.

Finally, if $\mathcal{G}(\mathbb{A})$ has an edge ab of the unary type and θ is a maximal congruence of $\mathsf{Sg}(a, b)$ certifying that, then $\{a/_\theta, b/_\theta\}$ is a factor of $\mathbb{A}$ term equivalent to a set, and so $\mathsf{HS}(\mathbb{A})$ admits type **1**. Similarly, if $\mathcal{G}(\mathbb{A})$ has an affine edge then $\mathsf{HS}(\mathbb{A})$ admits type **2**. Conversely, suppose $\mathsf{HS}(\mathbb{A})$ admits type **1**. Then some algebra $\mathbb{B}$ from $\mathsf{HS}(\mathbb{A})$ has a prime interval $\alpha \prec \beta$ in $\mathsf{Con}(\mathbb{B})$ of type **1**. Taking the factor of $\mathbb{B}$ modulo α and restricting $\mathbb{B}$ on a nontrivial β-block, $\mathbb{B}$ can be assumed to be simple and $\underline{0}_\mathbb{B} \prec \underline{1}_\mathbb{B}$ has the unary type. Therefore, by Proposition 3.3 $\mathbb{B}$ is a 2-element set. Let a', b' be the elements of $\mathbb{B}$. Then as $\mathbb{B} \in \mathsf{HS}(\mathbb{A})$ there are elements $a, b \in \mathbb{A}$ such that $a \in a', b \in b'$. As is easily seen, ab is an edge of the unary type. Similarly, if some factor of $\mathbb{A}$ has a prime congruence interval of type **2**, we can find an edge of the affine type in $\mathcal{G}(\mathbb{A})$. $\square$

We complete this section observing several simple properties of edges in connection with subalgebras and factor algebras.

Lemma 3.8. *Let $\mathbb{A}$ be an idempotent algebra and α a congruence of $\mathbb{A}$. Then*

(1) If $a/_\alpha b/_\alpha$, $a, b \in \mathbb{A}$, is an edge in $\mathcal{G}(\mathbb{A}/_\alpha)$ then ab is also an edge in $\mathcal{G}(\mathbb{A})$ of the same type.

(2) If $\mathcal{H}(\mathbb{A}/_\alpha)$ is connected then so is $\mathcal{H}(\mathbb{A})$.

Proof. (1) Let θ be a congruence of $\mathbb{B} = \mathsf{Sg}_{\mathbb{A}/_\alpha}(a/_\alpha, b/_\alpha)$ witnessing the type of the edge $a/_\alpha b/_\alpha$. Let $\theta' \geq \alpha$ be the congruence of $\mathbb{B}' = \{a \in \mathbb{A} \mid a/_\alpha \in \mathbb{B}\}$ such that $\theta'/_\alpha = \theta$, and consider the congruence η which is the restriction of θ' to the algebra $\mathsf{Sg}_\mathbb{A}(a, b)$. Clearly, $\mathsf{Sg}_\mathbb{A}(a, b)/_\eta$ is isomorphic to $\mathbb{B}/_\theta$, thus witnessing that ab is an edge in $\mathcal{G}(\mathbb{A})$ of the same type as $a/_\alpha b/_\alpha$.

(2) This statement follows from the fact that any $a, b \in \mathbb{A}$ such that $(a, b) \in \alpha$ are connected by the subalgebra $a/_\alpha$. $\square$

The following example shows that the converse of Lemma 3.8(1) does not always hold.

Example 3.9. *Let $\mathbb{A}$ be an algebra containing a pair that is not an edge. For example, we may consider the algebra from Example* 3.4*, that is, the universe of $\mathbb{A}$ is $A = \{a, b, c\}$ and the only basic operations are the binary operations f, g acting as described in Example* 3.4*. For a non-edge consider the pair ab. Let $\mathbb{B} = (B, m)$, where $B = \{0, 1\}$ and m is the majority operation on $\{0, 1\}$. We define f, g on B to be the first projections and m on A to be the first projection as well. Let $\mathbb{A}' = (A, f, g, m)$, $\mathbb{B}' = (B, f, g, m)$, and consider $\mathbb{C} = \mathbb{A}' \times \mathbb{B}'$. Let also π_1, π_2 be the projection congruences of $\mathbb{C}$. It is easy to see that $(a, 0)(b, 1)$ is an edge of the majority type as witnessed by the congruence π_2. Indeed, $(a, 0), (b, 1)$ generate $\mathbb{C}$ and $\mathbb{C}/_{\pi_2}$ is isomorphic to $\mathbb{B}'$, which is an edge of the majority type. On the other hand, $\mathbb{C}/_{\pi_1}$ is isomorphic to $\mathbb{A}'$, where*

$(a, 0)/_{\pi_1}, (b, 1)/_{\pi_1}$ *correspond to* a, b*, respectively. Thus,* $(a, 0)/_{\pi_1}(b, 1)/_{\pi_1}$ *is not an edge.*

The next statement follows straightforwardly from definitions.

Lemma 3.10. *Let* $\mathbb{A}$ *be an idempotent algebra and* $\mathbb{B}$ *its subalgebra. Then for any* $a, b \in \mathbb{B}$

(1) the pair ab *is an edge in* $\mathcal{G}(\mathbb{B})$ *if and only if it is an edge in* $\mathcal{G}(\mathbb{A})$*, and it has the same type in both algebras;*

(2) if elements a, b *are connected in* $\mathcal{H}(\mathbb{B})$ *then they are connected in* $\mathcal{H}(\mathbb{A})$*.*

Lemma 3.11. *Let* $\mathbb{A}$ *be an idempotent algebra and* ab *an edge of* $\mathcal{G}(\mathbb{A})$*. Let* θ *be a congruence of* $\mathsf{Sg}(a, b)$ *witnessing that* ab *is an edge. Then for any* $c \in a/_\theta$*,* $d \in b/_\theta$*, the pair* cd *is also an edge of* $\mathcal{G}(\mathbb{A})$ *of the same type.*

Proof. Consider the algebra $\mathbb{B} = \mathsf{Sg}(a, b)/_\theta$. The fact that ab is an edge and its type only depends on the term operations of $\mathbb{B}$. Since $c \in a/_\theta$, $d \in b/_\theta$, the result follows. □

3.4. Adding thick edges

Generally, an edge, or even a thick edge is not a subalgebra. However, it is always possible to find a reduct for which every (thick) edge is a subalgebra. For instance, one can throw away all the term operations of an algebra. Every subset of such a reduct is a subalgebra. More difficult is to find a reduct that keeps the types of edges in $\mathcal{G}(\mathbb{A})$. In this section we show that every idempotent algebra $\mathbb{A}$ omitting type **1** has a reduct $\mathbb{A}'$ such that every one of its (thick) edges of the semilattice or majority type is a subalgebra of $\mathbb{A}'$, and if $\mathbb{A}$ omits certain types then so does $\mathbb{A}'$. More precisely, we prove the following

Theorem 3.12. *Let* $\mathbb{A}$ *be an idempotent algebra. There exists a reduct* $\mathbb{A}'$ *of* $\mathbb{A}$ *such that every thick edge of the semilattice or majority type is a subuniverse of* $\mathbb{A}'$ *and*

(1) if $\mathcal{G}(\mathbb{A})$ *does not contain edges of the unary type, then* $\mathcal{G}(\mathbb{A}')$ *does not contain edges of the unary type;*

(2) if $\mathcal{G}(\mathbb{A})$ *contains no edges of the unary and affine types, then* $\mathcal{G}(\mathbb{A}')$ *contains no edges of the unary and affine types.*

Remark 3.13. If we do not insist on a specific reduct in item (1) of Theorem 3.12, it allows for a simpler proof than we give here. It was suggested by Brady and works for *Taylor-minimal* algebras introduced in [6]. A Taylor-minimal algebra is a finite Taylor algebra, that is, it has a term operation satisfying some Taylor identities, and whose clone of term operations is minimal among clones containing a Taylor operation. First, it can be shown that if $\mathbb{A}$ is an idempotent algebra such that $\mathsf{HS}(\mathbb{A})$ omits type one, there is a reduct of $\mathbb{A}$ that is a Taylor-minimal algebra. It suffices therefore to show that every thick edge of a Taylor-minimal algebra is a subalgebra. Suppose that $\mathbb{A}$ is a Taylor-minimal algebra, then it has a cyclic term f, which can be assumed to be the only basic operation of $\mathbb{A}$. Let also ab be a semilattice or majority edge of $\mathcal{G}(\mathbb{A})$ and congruence θ witnesses that. Then the semilattice or majority operation on $\{a/_\theta, b/_\theta\}$ is a cyclic term on this set, denote it t. The idea now is to compose f and t in such a way that the resulting term is still cyclic on $\mathbb{A}$, but preserves $a/_\theta \cup b/_\theta$. Since $\mathbb{A}$ is Taylor-minimal, this implies that $a/_\theta \cup b/_\theta$ is a subalgebra of $\mathbb{A}$.

It is not clear, however, how to extend the argument above to item (2). Also, item (1) can be proved for a wider class of algebras than Taylor-minimal ones. So, here we give a longer, but more general proof.

Remark 3.14. If an idempotent algebra satisfies the property that every one of its (thick) semilattice or majority edges is a subalgebra, we call it *smooth*.

The main auxiliary statement to prove Theorem 3.12 is Proposition 3.15. We say that an algebra $\mathbb{A}$ is X-*connected*, where $X \subseteq \{$unary, semilattice, majority, affine$\}$, if for any subalgebra $\mathbb{B}$ of $\mathbb{A}$ and any $a, b \in \mathbb{B}$ there is a path in $\mathcal{G}(\mathbb{B})$ connecting a and b and that only contains edges of types from X.

Proposition 3.15. *Let $\mathbb{A}$ be an idempotent algebra, let ab be an edge of $\mathcal{G}(\mathbb{A})$ of semilattice or majority type and θ a congruence witnessing that, and let $R_{ab} = a/_\theta \cup b/_\theta$. Let also F_{ab} denote the set of all term operations of $\mathbb{A}$ preserving R_{ab} and $\mathbb{A}' = (A, F_{ab})$. Then*

(1) if $\mathbb{A}$ is $\{$semilattice, majority, affine$\}$-connected then so is $\mathbb{A}'$;

(2) if $\mathbb{A}$ is $\{$semilattice,majority$\}$-connected, then so is $\mathbb{A}'$.

First, we show how Theorem 3.12 is obtained from Proposition 3.15.

Proof of Theorem 3.12. Let X be one of the sets used in Theorem 3.12, that is, X is $\{$semilattice,majority$\}$, or $\{$semilattice,majority,affine$\}$. We prove that if ab is an edge of $\mathcal{G}(\mathbb{B})$ of the unary or affine type for some algebra $\mathbb{B}$, then there are $c, d \in \mathsf{Sg}(a, b)$ such that any path in $\mathcal{G}(\mathsf{Sg}(c, d))$ from c to d contains an edge of the unary or affine type, respectively. Hence, if $\mathcal{G}(\mathbb{A}')$ contains an edge whose type is not in X, it is not X-connected. By Proposition 3.15, neither is $\mathbb{A}$, which means that if $\mathbb{A}'$ has a unary edge, then $\mathbb{A}$ has a unary edge, while if $\mathbb{A}'$ has no unary edge, but has an affine edge, the same property holds for $\mathbb{A}$. Proceeding inductively, we reach a reduct in which all thick edges of the semilattice and majority type are subuniverses, which has a unary edge iff $\mathbb{A}$ has a unary edge, and the same reduct has no unary edge nor an affine edge iff $\mathbb{A}$ has no unary edge nor an affine edge.

Let θ be a congruence of $\mathbb{B} = \mathsf{Sg}(a, b)$ witnessing that ab is an edge. Choose $c, d \in \mathbb{B}$ such that $c/_\theta \neq d/_\theta$ and $\mathbb{C} = \mathsf{Sg}(c, d)$ is minimal possible with this condition. Suppose also that $c = c_1, c_2, \dots, c_k = d$ is a path in $\mathcal{G}(\mathbb{C})$ connecting c and d. Then for some $i \in [k-1]$ it holds $c_i/_\theta \neq c_{i+1}/_\theta$. We show that $c_i c_{i+1}$ is an edge of $\mathcal{G}(\mathbb{A})$ of the same type as ab.

By the choice of c, d we have $\mathsf{Sg}(c_i, c_{i+1}) = \mathbb{C}$. Let $\eta = \theta|_\mathbb{C}$. Clearly $\mathbb{C}' = \mathbb{C}/_\eta$ is isomorphic to a subalgebra of $\mathbb{B}' = \mathbb{B}/_\theta$. As $\mathbb{B}'$ is a set or a module depending on whether ab has the unary or affine type, so is $\mathbb{C}'$. Therefore η witnesses that c_ic_{i+1} is an edge of the unary or affine type depending on whether ab has the unary or affine type. It remains to show that c_ic_{i+1} does not have any other type. Take a proper congruence χ of $\mathbb{C}$, note that χ separates c_i, c_{i+1} as they generate $\mathbb{C}$. If $\chi \not\leq \eta$ then some χ-block D is not contained in an η-block, a contradiction with the choice of c, d, as D is a subuniverse and a proper subset of $\mathbb{C}$. Finally, if $\chi \leq \eta$, then a semilattice or majority operation on $\{c_i/_\chi, c_{i+1}/_\chi\}$ gives rise to a semilattice or majority operation on $\{c_i/_\eta, c_{i+1}/_\eta\}$ contradicting the assumption that $\mathbb{C}$ is a module or a set. □

Lemma 3.16. *(1) Let ab be a semilattice edge of an algebra $\mathbb{A}$, let θ be a congruence of $\mathbb{A}$ witnessing that, and f a binary term operation which is semilattice on $\{a/_\theta, b/_\theta\}$. Then f can be chosen to satisfy (on $\mathbb{A}$) any one of the two equations:*

$$f(x, f(x, y)) = f(x, y), \qquad f(f(x, y), f(y, x)) = f(x, y).$$

(2) Let ab be a majority edge of an algebra $\mathbb{A}$, let θ be a congruence of $\mathbb{A}$ witnessing that, and m a ternary term operation which is majority on $\{a/_\theta, b/_\theta\}$. Then m can be chosen to satisfy (on $\mathbb{A}$) any one of the two equations:

$$m(x, m(x, y, y), m(x, y, y)) = m(x, y, y),$$
$$m(m(x, y, z), m(y, z, x), m(z, x, y)) = m(x, y, z).$$

Moreover, if for some $c, d \in \mathbb{A}$ and a congruence η of $\mathsf{Sg}_{\mathbb{A}}(c, d)$ there is an operation that is semilattice or majority on $\{a/_\theta, b/_\theta\}$ and the first projection on $\{c/_\eta, d/_\eta\}$, then the operations f, m above can be chosen such that they are the first projection on $\{c/_\eta, d/_\eta\}$.

Proof. (1) To show that f can be chosen to satisfy the equation $f(x, f(x, y)) = f(x, y)$, for every $x, y \in A$, we consider the unary operation $g_x(y) = f(x, y)$. There is a natural number n_x such that $g_x^{n_x}$ is an idempotent transformation of A. Let n be the least common multiple of the n_x, $x \in A$, and

$$h(x, y) = f(\underbrace{x, f(x, \dots f(x}_{n \text{ times}}, y) \dots)).$$

Since $g_x^n(y)$ is idempotent for any $x \in A$, we have $h(x, h(x, y)) = g_x^n(g_x^n(y)) = g_x^n(y) = h(x, y)$. Finally, as is easily seen h equals f on $\{a/_\theta, b/_\theta\}$.

To show that f can be chosen to satisfy the second equation, consider the unary operation g on $\mathbb{A}^2$ given by $(x, y) \mapsto (f(x, y), f(y, x))$. There is n such that g^n is idempotent. Then

$$h(x, y) = \underbrace{f(\dots f(f(f(}_{n \text{ times}}x, y), f(y, x)), f(f(y, x), f(x, y)) \dots)$$

satisfies the required equation and equals f on $\{a/_\theta, b/_\theta\}$.

(2) To show that m can be chosen to satisfy the equation $m(x, m(x, y, y), m(x, y, y)) = m(x, y, y)$, for every $x \in A$, we consider the unary operation

$g_x(y) = m(x, y, y)$. There is a natural number n_x such that $g_x^{n_x}$ is an idempotent transformation of A. Let n be the least common multiple of the n_x, $x \in A$, and

$$h(x, y, z) = g_x^{n-1}(m(x, y, z)) = m(\underbrace{x, m(x, \ldots m(x}_{n \text{ times}}, y, z), m(x, y, z) \ldots)),$$

Observe that $h(x, y, y) = g_x^n(y)$. Since $g_x^n(y)$ is idempotent for any $x \in A$, we have $h(x, h(x, y, y), h(x, y, y)) = g_x^n(g_x^n(y)) = g_x^n(y) = h(x, y, y)$. Finally, as is easily seen h is a majority operation on $\{a/_\theta, b/_\theta\}$.

For the second equation consider the unary operation g on $\mathbb{A}^3$ given by $(x, y, z) \mapsto (m(x, y, z), m(y, z, x), m(z, x, y))$. There is n such that g^n is idempotent. Then

$$h(x, y, z) = \underbrace{m(\ldots m(m(m(}_{n \text{ times}} x, y, z), m(y, z, x), m(z, x, y))) \ldots).$$

satisfies the required equation and equals m on $\{a/_\theta, b/_\theta\}$.

To prove the last statement of the lemma it suffices to observe that if we start with f or m that is the first projection on $\{c/_\eta, d/_\eta\}$, the resulting operation also satisfies this property. $\square$

In the rest of this section we assume the conditions and notation used in Proposition 3.15. Recall that the subalgebra of $\mathbb{A}$ generated by a set $B \subseteq A$ is denoted by $\mathsf{Sg}_{\mathbb{A}}(B)$, while the subalgebra of $\mathbb{A}'$ generated by the same set is denoted by $\mathsf{Sg}_{\mathbb{A}'}(B)$. In general, $\mathsf{Sg}_{\mathbb{A}'}(B) \subseteq \mathsf{Sg}_{\mathbb{A}}(B)$. In what follows, let f [respectively, m] be a term operation of $\mathbb{A}$ that witnesses that ab is a semilattice [respectively, majority] edge, that is, such that $f/_\theta$ [respectively, $m/_\theta$] is a semilattice [respectively, majority] operation on $B' = \{a/_\theta, b/_\theta\}$. We start with two lemmas.

Lemma 3.17. *Let $c, d \in \mathbb{A}$ be such that there exists a congruence η of $\mathbb{C} = \mathsf{Sg}_{\mathbb{A}'}(c, d)$ such that $\mathbb{C}' = \mathsf{Sg}_{\mathbb{A}'}(c, d)/_\eta$ is a subalgebra of a reduct of a module. Then operation f or m can be chosen to be the first projection on $\mathbb{C}'$.*

Proof. As $\mathbb{C}'$ is a reduct of a module over some ring $\mathbb{K}$, the operations f or m (note, they belong to F_{ab}) on $\mathbb{C}'$ have the form

$$f(x, y) = \alpha x + (1 - \alpha)y, \qquad m(x, y, z) = \alpha x + \beta y + \gamma z, \quad \alpha + \beta + \gamma = 1.$$

Let n be such that α^n is an idempotent of $\mathbb{K}$. Then set

$$f'(x, y) = \underbrace{f(f(\ldots f}_{n \text{ times}}(x, y) \ldots, y), y).$$

Since f and f' are idempotent, $f'(x, y) = \alpha^n x + (1 - \alpha^n)y$ on $\mathbb{C}'$ and $f'(x, y) = f(x, y)$ on B'. If $\alpha^n \in \{0, 1\}$, then $f'(x, y)$ or $f'(y, x)$ is the first projection on $\mathbb{C}'$ and we are done. If α^n is a nontrivial idempotent, set

$$f''(x, y) = f'(f'(x, y), x).$$

Again, $f''(x, y) = f(x, y)$ on B'. On the other hand, on $\mathbb{C}'$ we have

$$f''(x, y) = ((\alpha^n)^2 + (1 - \alpha^n))x + \alpha^n(1 - \alpha^n)y = x.$$

For the operation m we need to perform several steps similar to the ones above. The goal is to make sure that α, β, γ can be assumed to be idempotents of $\mathbb{K}$ and such that $\alpha\beta = \alpha\gamma = \beta\gamma = 0$. For such m we can set

$$\begin{aligned} m'(x,y,z) &= m(m(x,y,z), m(z,x,y), m(y,z,x)) \\ &= (\alpha^2+\beta^2+\gamma^2)x + (\alpha\beta+\beta\gamma+\gamma\alpha)y + (\alpha\gamma+\beta\alpha+\gamma\beta)z \\ &= x + \gamma\alpha y + (\beta\alpha+\gamma\beta)z. \end{aligned}$$

As m' is idempotent, $p(x,y) = m'(x,y,y) = x$ on $\mathbb{C}'$ and $p(x,y) = y$ on B'. Then let $m''(x,y,z) = p(x, m(x,y,z))$ which is equal to $m(x,y,z)$ on B', and $m''(x,y,z) = x$ on $\mathbb{C}'$.

It remains to prove that m with the required properties exists. Again, assume that α^n is an idempotent of $\mathbb{K}$ and set

$$m'(x,y,z) = \underbrace{m(m(\dots m}_{n \text{ times}}(x,y,z)\dots,y,z),y,z).$$

We have $m'(x,y,z) = \alpha^n x + \beta' y + \gamma' z$ on $\mathbb{C}'$ for some $\beta', \gamma' \in \mathbb{K}$ with $\alpha^n + \beta' + \gamma' = 1$, and, as is easily seen, m' is a majority operation on B'. Next, set

$$\begin{aligned} m''(x,y,z) &= m'(x, m'(x,y,z), m'(x,y,z)) \\ &= (\alpha^n + (\beta'+\gamma')\alpha^n)x + (\beta'+\gamma')\beta' y + (\beta'+\gamma')\gamma' z. \end{aligned}$$

Since $\beta' + \gamma' = 1 - \alpha^n$ and so $\alpha^n(\beta'+\gamma') = (\beta'+\gamma')\alpha^n = 0$, we get

$$\begin{aligned} \alpha^n + (\beta'+\gamma')\alpha^n &= \alpha^n, \\ \alpha^n(\beta'+\gamma')\beta' &= 0, \\ \alpha^n(\beta'+\gamma')\gamma' &= 0. \end{aligned}$$

Since m'' is still a majority operation on B', we may assume that for m it holds that $\alpha^2 = \alpha$, $\alpha\beta = \alpha\gamma = 0$.

Next, let β^ℓ be an idempotent of $\mathbb{K}$. Repeat the steps above for y:

$$m'(x,y,z) = \underbrace{m(x, m(x, \dots m}_{\ell \text{ times}}(x,y,z)\dots,z),z).$$

As is easily seen, m' is a majority operation on B' and

$$m'(x,y,z) = (1+\beta+\dots+\beta^{\ell-1})\alpha x + \beta^\ell y + \gamma' z$$

for some $\gamma' \in \mathbb{K}$. Since $\alpha\beta = 0$, for the first coefficient we have

$$\begin{aligned} ((1+\beta+\dots+\beta^{\ell-1})\alpha)^2 &= (1+\beta+\dots+\beta^{\ell-1})\alpha + \sum_{i=0,j=1}^{\ell-1,\ell-1} \beta^i\alpha\beta^j\alpha \\ &= (1+\beta+\dots+\beta^{\ell-1})\alpha. \end{aligned}$$

Thus, the first and second coefficients are idempotent. Denote them α' and β', respectively. Note also that $\alpha'\beta' = 0$ and $\alpha'\gamma' = \alpha'(1-\alpha'-\beta') = 0$. Next, set

$$\begin{aligned} m''(x,y,z) &= m'(m'(x,y,z),y,m'(x,y,z)) \\ &= (\alpha'+\gamma')\alpha' x + (\alpha'+1+\gamma')\beta' y + (\alpha'+\gamma')\gamma' z. \end{aligned}$$

We have

$$\begin{aligned} &((\alpha'+\gamma')\alpha')^2 = (\alpha'+\gamma'\alpha')^2 = \alpha' + \alpha'\gamma'\alpha' + \gamma'\alpha'^2 + \gamma'\alpha'\gamma'\alpha' = \alpha' + \gamma'\alpha', \\ &((\alpha'+1+\gamma')\beta')^2 = (\beta' + (1-\beta')\beta')^2 = \beta'^2 = \beta', \\ &(\alpha'+\gamma')\alpha'(\alpha'+1+\gamma')\beta' = (\alpha'+\gamma')\alpha'\beta' = 0, \\ &(\alpha'+\gamma')\alpha'(\alpha'+\gamma')\gamma' = (\alpha'+\gamma')\alpha'\gamma' = 0 \\ &(\alpha'+1+\gamma')\beta'(\alpha'+\gamma')\gamma' = (\alpha'+1+\gamma')\beta'(1-\beta')\gamma' = 0. \end{aligned}$$

Moreover, it is straightforward that m'' is a majority operation on B'. Assuming that the original operation m satisfies these conditions we have $\alpha^2 = \alpha, \beta^2 = \beta, \alpha\beta = \alpha\gamma = \beta\gamma = 0$.

Finally, we repeat the first step again for z by assuming that γ^k is an idempotent of $\mathbb{K}$ and setting

$$m'(x,y,z) = \underbrace{m(x,y,m(x,y,\ldots m}_{k \text{ times}}(x,y,z)\ldots)).$$

As before, it is straightforward to verify that m' is a majority operation on B' and that the coefficients α', β', γ' of m' on $\mathbb{C}'$ satisfy the conditions $\alpha'^2 = \alpha', \beta'^2 = \beta', \gamma'^2 = \gamma'$, $\alpha'\beta' = \alpha'\gamma' = \beta'\gamma' = 0$. $\square$

Notice that if $\mathsf{Sg}_{\mathbb{A}}(c,d)$ is a set or a module, then $\mathsf{Sg}_{\mathbb{A}'}(c,d)$ is a subalgebra of a reduct of a module. Hence we obtain the following corollary.

Corollary 3.18. *If cd is a unary or affine edge witnessed by congruence η of $\mathsf{Sg}_{\mathbb{A}}(c,d)$, then f or m can be chosen to be the first projection on $\mathsf{Sg}_{\mathbb{A}}(c,d)/_\eta$.*

Lemma 3.19. *Let $c,d \in \mathbb{A}$ be such that cd is an edge (of any type including the unary type) in $\mathbb{A}'$. Let also η be a congruence of $\mathsf{Sg}_{\mathbb{A}'}(c,d)$ witnessing that, and assume that $\mathsf{Sg}_{\mathbb{A}}(c',d') = \mathsf{Sg}_{\mathbb{A}}(c,d)$, for any $c',d' \in \mathsf{Sg}_{\mathbb{A}'}(c,d)$ with $c' \in c/_\eta, d' \in d/_\eta$. Then one of the following holds.*

- *(i) cd is a semilattice edge or there is $e \in \mathsf{Sg}_{\mathbb{A}'}(c,d)$ such that ce, de are semilattice edges in $\mathbb{A}'$, or*
- *(ii) $\mathcal{H}(\mathsf{Sg}_{\mathbb{A}'}(c,d))$ is connected, or*
- *(iii) $\mathsf{Sg}_{\mathbb{A}'}(c,d) = \mathsf{Sg}_{\mathbb{A}}(c,d)$ as sets, or*
- *(iv) ab is a majority edge in $\mathbb{A}$ and cd is a semilattice or majority edge in $\mathsf{Sg}_{\mathbb{A}'}(c,d)$.*

Proof. Let $\mathbb{C} = \mathsf{Sg}_{\mathbb{A}'}(c,d)$.

Case 1. ab is a semilattice edge.

Recall that f is a term operation of $\mathbb{A}$ semilattice on $\{a/_\theta, b/_\theta\}$, where θ is a congruence of $\mathsf{Sg}_{\mathbb{A}}(a,b)$ witnessing that ab is a semilattice edge. By Lemma 3.16(1), f can be assumed to satisfy the equation $f(x,f(x,y)) = f(x,y)$. We consider the congruence χ of $\mathbb{C}$ generated by the set

$D = \{(f(c',d'), f(d',c')) \mid c',d' \in \mathbb{C}\}$. Note that if χ is not the total congruence then f is commutative on $\mathbb{C}/_\chi$.

CASE 1.1. χ is not the total congruence.

As f satisfies the equation $f(x, f(x,y)) = f(x,y)$, it is a semilattice operation on $\{c/_\chi, f(c/_\chi, d/_\chi)\}$ and $\{f(c/_\chi, d/_\chi), d/_\chi\}$, implying condition (i) of Lemma 3.19, where $e = f(c,d)$, unless $f(c/_\chi, d/_\chi) \in \{c/_\chi, d/_\chi\}$, in which case cd is a semilattice edge.

CASE 1.2. χ is the total congruence

The congruence χ is the transitive closure of the set $D' = \{(p(c'), p(d')) \mid (c',d') \in D,$ and p is a unary polynomial of $\mathbb{C}\}$. For every such unary polynomial there is a term operation g of $\mathbb{C}$ such that $p(x) = g(c,d,x)$. We first show that if, for every pair $(g(c,d,c'), g(c,d,d'))$, where $(c',d') \in D$, and g is a term operation of $\mathbb{C}$, the subalgebra $\mathsf{Sg}_{\mathbb{A}}(g(c,d,c'), g(c,d,d'))$ of $\mathbb{A}$ is a proper subalgebra of $\mathsf{Sg}_{\mathbb{A}}(c,d)$, then $\mathcal{H}(\mathbb{C})$ is connected implying condition (ii) or (ii). Firstly, since χ is the total congruence, any two elements from $\mathbb{C}$ are connected by a sequence of subalgebras of $\mathbb{C}$ generated by pairs from D'. Secondly, all these subalgebras are proper. Indeed, if $\mathsf{Sg}_{\mathbb{A}'}(g(c,d,c'), g(c,d,d')) = \mathbb{C}$ for some c',d', then $c,d \in \mathsf{Sg}_{\mathbb{A}'}(g(c,d,c'), g(c,d,d')) \subseteq \mathsf{Sg}_{\mathbb{A}}(g(c,d,c'), g(c,d,d'))$. Therefore, $\mathsf{Sg}_{\mathbb{A}}(g(c,d,c'), g(c,d,d')) = \mathsf{Sg}_{\mathbb{A}}(c,d)$, a contradiction.

Suppose now that, for a certain $(c',d') \in D$ and a ternary term operation g of $\mathbb{C}$, we have $\mathsf{Sg}_{\mathbb{A}}(g(c,d,c'), g(c,d,d')) = \mathsf{Sg}_{\mathbb{A}}(c,d)$. Then for any $e \in \mathsf{Sg}_{\mathbb{A}}(c,d)$, there is a term operation h of $\mathbb{A}$ such that $h(g(c,d,c'), g(c,d,d')) = e$. Consider $h'(x,y,z,t) = h(g(x,y,f(z,t)), g(x,y,f(t,z)))$. We have $h'|_{B'}(x,y,z,t) = g(x,y,f(z,t))$, where $B' = \{a/_\theta, b/_\theta\}$. Hence, $h' \in F_{ab}$. On the other hand, $h'(c,d,c'',d'') = e$, where c'',d'' are such that $f(c'',d'') = c'$, $f(d'',c'') = d'$. Thus, $\mathsf{Sg}_{\mathbb{A}'}(c,d) = \mathsf{Sg}_{\mathbb{A}}(c,d)$, and we obtain item (iii).

CASE 2. ab is a majority edge.

Let χ' be the congruence of $\mathbb{C}$ generated by

$$D = \{(m(c',d',d'), m(d',c',d')), (m(c',d',d'), m(d',d',c')), (m(d',d',c'), m(d',c',d')) \mid c',d' \in \mathbb{C}\}.$$

As in Case 1 if χ' is not the total congruence, then $m(x,y,y) = m(y,x,y) = m(y,y,x)$ in $\mathbb{C}/_{\chi'}$. We consider two subcases.

CASE 2.1. χ' is not the total congruence.

Let χ be a maximal congruence of $\mathbb{C}$ containing χ'. If $\eta \not\leq \chi$, then, since $\eta \vee \chi$ is the total congruence, $\mathcal{H}(\mathbb{C})$ is connected by the η- and χ-blocks implying items (ii). So, suppose $\eta \leq \chi$. By the assumption cd is an edge and η witnesses this. The case when cd has the unary type is impossible, because m is not a projection on $\mathbb{C}/_\chi$, and therefore is not a projection on $\mathbb{C}/_\eta$. If cd is of semilattice or majority type, we have case (iv) of the lemma. So, suppose that cd has the affine type, and so $\mathbb{C}/_\eta$ is term equivalent to a module. By Lemma 3.17 there is a term operation $\overline{m}$ such that it is majority on $B' = \{a/_\theta, b/_\theta\}$ and $\overline{m}$ is the first projection on $\mathbb{C}/_\eta$. As is easily seen,

$h(x,y) = \overline{m}(x,y,y)$ is the second projection on B' and the first projection on $\mathbb{C}/_\eta$. We show that $\mathsf{Sg}_{\mathbb{A}'}(c,d) = \mathsf{Sg}_{\mathbb{A}}(c,d)$, thus obtaining item (iii). By the assumptions of the lemma $\mathsf{Sg}_{\mathbb{A}}(h(c,d),d) = \mathsf{Sg}_{\mathbb{A}}(c,d)$, and so it suffices to prove that $\mathsf{Sg}_{\mathbb{A}'}(c,d) = \mathsf{Sg}_{\mathbb{A}}(h(c,d),d)$. Let $e \in \mathsf{Sg}_{\mathbb{A}}(h(c,d),d)$, that is, there is a term operation $g(x,y)$ of $\mathbb{A}$ such that $e = g(h(c,d),d)$. The operation $g'(x,y) = g(h(x,y),y) = y$ on B' and so $g' \in F_{ab}$. On the other hand,

$$e = g'(c,d) = g(h(c,d),d) \in \mathsf{Sg}_{\mathbb{A}'}(c,d).$$

Case 2.2. χ' is the total congruence of $\mathbb{C}$.

Similar to Case 1.2 the congruence generated by D is the transitive closure of the set $D' = \{(g(c,d,c'), g(c,d,d')) \mid (c',d') \in D$, and g is a term operation of $\mathbb{C}\}$. As is shown in Case 1.2, if for every pair $(g(c,d,c'), g(c,d,d')) \in D'$ the subalgebra $\mathsf{Sg}_{\mathbb{A}}(g(c,d,c'), g(c,d,d'))$ of $\mathbb{A}$ is a proper subalgebra of $\mathsf{Sg}_{\mathbb{A}}(c,d)$, then $\mathcal{H}(\mathbb{C})$ is connected, and condition (ii) holds.

Suppose that, for a certain $(c',d') \in D$ and a ternary term operation g of $\mathbb{C}$, we have $\mathsf{Sg}_{\mathbb{A}}(g(c,d,c'), g(c,d,d')) = \mathsf{Sg}_{\mathbb{A}}(c,d)$. Then, for any $e \in \mathsf{Sg}_{\mathbb{A}}(c,d)$, there is a term operation h of $\mathbb{A}$ such that $h(g(c,d,c'), g(c,d,d')) = e$ (here we slightly deviate from the argument in Case 1.2). Without loss of generality we may assume that $c' = m(c'',d'',d'')$, $d' = m(d'',c'',d'')$ for certain $c'',d'' \in \mathbb{C}$. Consider $h'(x,y,z,t) = h(g(x,y,m(z,t,t)), g(x,y,m(t,z,t)))$. We have $h'|_{B'}(x,y,z,t) = g(x,y,m(z,t,t)) = g(x,y,t)$, hence, $h' \in F_{ab}$. On the other hand, $h'(c,d,c'',d'') = e$. Thus, $\mathsf{Sg}_{\mathbb{A}'}(c,d) = \mathsf{Sg}_{\mathbb{A}}(c,d)$. □

Lemma 3.20. *Let $c,d \in \mathbb{A}$ and $\mathbb{C} = \mathsf{Sg}_{\mathbb{A}'}(c,d)$. Suppose that cd is an edge in $\mathbb{A}$, η is a congruence of $\mathsf{Sg}_{\mathbb{A}}(c,d)$ witnessing that, and for any $c' \in c/_\eta, d' \in d/_\eta$, it holds $\mathsf{Sg}_{\mathbb{A}}(c',d') = \mathsf{Sg}_{\mathbb{A}}(c,d)$. Then either c,d are connected in $\mathcal{H}(\mathbb{C})$, or cd is a semilattice edge in $\mathbb{A}'$, or there is $e \in \mathbb{C}$ such that ce, de are semilattice edges of $\mathbb{A}'$, or*

(1) if cd is a majority, affine or unary edge, then cd is an edge of $\mathbb{A}'$ of the same type as it is in $\mathbb{A}$.

(2) if cd is a semilattice edge in $\mathbb{A}$, then cd is a semilattice or majority edge in $\mathbb{A}'$.

Proof. We consider cases when ab is a semilattice and majority edge separately proving that either the conclusion of the items (1) or (2) is true, or that c,d are connected in $\mathcal{H}(\mathbb{C})$, or cd is a semilattice edge of $\mathbb{A}'$, or ce, de are semilattice edges of $\mathbb{A}'$ for some $e \in \mathbb{C}$. First, we give the part of the argument common for both cases.

Let f or m be a term operation of $\mathbb{A}$, semilattice or majority on $B' = \{a/_\theta, b/_\theta\}$, respectively. Let η be a congruence of $\mathsf{Sg}_{\mathbb{A}}(c,d)$ witnessing that cd is an edge and η' a maximal congruence of $\mathsf{Sg}_{\mathbb{A}'}(c,d)$ containing $\eta|_{\mathsf{Sg}_{\mathbb{A}'}(c,d)}$; set $\mathbb{C}' = \mathsf{Sg}_{\mathbb{A}'}(c,d)/_{\eta'}$.

By Proposition 3.7 there are four options for $\mathbb{C}'$: either $\mathbb{C}'$ is a set or a module, or c,d are connected in $\mathcal{H}(\mathbb{C}')$, or there is an operation h of $\mathbb{C}'$ which is either semilattice or majority on $\{c/_{\eta'}, d/_{\eta'}\}$. By Lemmas 3.8(2) and 3.10(2) if c,d are connected in $\mathcal{H}(\mathbb{C}')$, then they are also connected in

$\mathcal{H}(\mathbb{C})$ and there is nothing to prove. Otherwise cd is an edge in $\mathbb{C}'$, and hence, by Lemmas 3.8(1) and 3.10(1) it is also an edge in $\mathbb{A}'$. Note that if there are $c' \in c/_{\eta'}, d' \in d/_{\eta'}$ such that $\mathsf{Sg}_{\mathbb{A}'}(c', d') \subsetneq \mathbb{C}$, then c and d are connected in $\mathcal{H}(\mathbb{C})$ by the η'-blocks $c/_{\eta'}, d/_{\eta'}$ and the subalgebra $\mathsf{Sg}_{\mathbb{A}'}(c', d')$. Hence we assume that for any $c' \in c/_{\eta'}, d' \in d/_{\eta'}$ it holds that $\mathsf{Sg}_{\mathbb{A}'}(c', d') = \mathbb{C}$.

Suppose first that $\mathbb{C}'$ has a term operation that is semilattice or majority on $\{c/_{\eta'}, d/_{\eta'}\}$. Note that cd can only be a semilattice or majority edge of $\mathbb{A}$. Indeed, if cd is a unary or affine edge of $\mathbb{A}$, then $\mathsf{Sg}_{\mathbb{A}}(c, d)/_{\eta}$ is a set or a module, and therefore so are $\mathbb{C}$ and $\mathbb{C}'$, implying that this case is impossible. If cd is a semilattice edge of $\mathbb{A}$, there is nothing to prove. Suppose that cd is a majority edge of $\mathbb{A}$, which is witnessed by η. As we want to show that cd remains a majority edge of $\mathbb{A}'$, it suffices to show that there is no term operation that is semilattice on $\{c/_{\eta'}, d/_{\eta'}\}$. Suppose the contrary, and let ℓ be a binary operation that is semilattice on $\{c/_{\eta'}, d/_{\eta'}\}$, say, $\ell(c/_{\eta'}, d/_{\eta'}) = d/_{\eta'}$. Then $\ell(c, d) = d'$ for some $d' \in d/_{\eta'}$. By the assumption $\mathsf{Sg}_{\mathbb{A}'}(c, d') = \mathbb{C}$. Therefore there is a binary operation ℓ' of $\mathbb{C}$ such that $\ell'(c, d') = d$. Set $\ell''(x, y) = \ell'(x, \ell(x, y))$, for this operation we have $\ell''(c, d) = d$, and $\ell''(d, c) = d'' \in d/_{\eta'}$. By the same argument there exists a binary operation r such that $r(c, d'') = d$. Now consider $n(x, y) = r(y, \ell''(x, y))$. We have $n(c, d) = r(d, \ell''(c, d)) = r(d, d) = d$ and $n(d, c) = r(c, \ell''(d, c)) = r(c, d'') = d$. Thus, cd has to be a semilattice edge of $\mathbb{A}$, a contradiction.

Next, suppose that $\mathbb{C}'$ is either a set or a module. By Lemma 3.17 m can be chosen to be the first projection on $\mathbb{C}'$. Also, by Lemmas 3.16 and 3.17 f can be chosen to be the first projection on $\mathbb{C}'$ and additionally to satisfy the identity $f(f(x, y), f(y, x)) = f(x, y)$ on $\mathbb{A}'$. Then by Lemma 3.19 either $\mathcal{H}(\mathbb{C})$ is connected, or cd is a semilattice edge of $\mathbb{A}'$, or there is $e \in \mathbb{C}$ such that ce, de are semilattice edges of $\mathbb{A}'$, or $\mathsf{Sg}_{\mathbb{A}'}(c, d) = \mathsf{Sg}_{\mathbb{A}}(c, d)$. There is nothing to prove in the first three cases, so suppose that $\mathsf{Sg}_{\mathbb{A}'}(c, d) = \mathsf{Sg}_{\mathbb{A}}(c, d)$.

Case 1. ab is a semilattice edge.

Let $c' = f(c, d), d' = f(d, c)$, and note that $f(c', d') = c', f(d', c') = d'$ and $c' \stackrel{\eta'}{\equiv} c, d' \stackrel{\eta'}{\equiv} d$. Choose a maximal congruence η'' of $\mathsf{Sg}_{\mathbb{A}}(c', d')$ with $\eta \leq \eta''$. By the assumption above $\mathbb{C} = \mathsf{Sg}_{\mathbb{A}'}(c, d) = \mathsf{Sg}_{\mathbb{A}}(c, d)$, which means that every subalgebra of $\mathsf{Sg}_{\mathbb{A}}(c, d)$ is a subalgebra of $\mathbb{C}$, and η'' is a congruence of $\mathbb{C}$. By Proposition 3.7 applied to $\mathsf{Sg}_{\mathbb{A}}(c', d')/_{\eta''}$ either c', d', and therefore c, d are connected in $\mathcal{H}(\mathsf{Sg}_{\mathbb{A}}(c', d'))$ and hence in $\mathcal{H}(\mathbb{C})$ or $c'd'$ is an edge in $\mathbb{A}$ witnessed by η''. If $\eta'' \not\leq \eta'$, then c, d are connected in $\mathcal{H}(\mathbb{C})$ by η'- and η''-blocks. So, suppose that $c'd'$ is an edge in $\mathbb{A}$ and $\eta'' \leq \eta'$. As by the assumptions of Proposition 3.15 $\mathbb{A}$ has no unary edges, let g be a semilattice, majority, or affine operation witnessing that $c'd'$ is an edge of $\mathbb{A}$. Then the operation $g'(x, y) = g(f(x, y), f(y, x))$ in the first case and the operation

$$g'(x, y, z) = g(f(x, f(y, z)), f(y, f(z, x)), f(z, f(x, y)))$$

in the two latter cases belongs to F_{ab} and is a semilattice, majority or affine operation on $\{c'/_{\eta''}, d'/_{\eta''}\}$ (or on $\mathsf{Sg}_{\mathbb{A}'}(c', d')/_{\eta''}$ if g is affine), respectively, and hence on $\{c'/_{\eta'}, d'/_{\eta'}\}$, as well. This shows that $c'd'$ cannot be a semilattice or majority edge of $\mathbb{A}$ in this case, and if it is an affine edge of $\mathbb{A}$, then it is also an affine edge of $\mathbb{A}'$. It remains to show that the same holds for cd. If some operation h witnesses that cd is a semilattice or majority edge, since $\eta \le \eta''$, the operation h also witnesses that $\mathsf{Sg}_{\mathbb{A}}(c,d)/_{\eta''} = \mathsf{Sg}_{\mathbb{A}}(c',d')/_{\eta''}$ is not a module or a set. Finally, if cd is affine and $\mathsf{Sg}_{\mathbb{A}}(c,d)/_{\eta}$ is a module, then so is $\mathsf{Sg}_{\mathbb{A}}(c',d')/_{\eta''}$.

Case 2. ab is a majority edge.

We start with an auxiliary statement.

Claim. For any $c_1, c_2 \in c/_{\eta'}$ and $d_1, d_2, d_3, d_4 \in d/_{\eta'}$ there is a term operation p of $\mathbb{C}$ with $p(d_1, c_1, d_3) = p(d_2, d_4, c_2) = d_3$ and $p(c/_{\eta'}, d/_{\eta'}, d/_{\eta'}) = c/_{\eta'}$ on $\mathbb{C}'$.

Let R be the subalgebra of $\mathbb{C}^2$ generated by $\{(d_1,d_2),(c_1,d_4),(d_3,c_2)\}$. Since $\mathsf{Sg}_{\mathbb{A}'}(d_1,c_1) = \mathbb{C}$, there is $d' \in \mathsf{Sg}_{\mathbb{A}'}(d_2,d_4) \subseteq d/_{\eta'}$ such that $(d_3,d') \in R$. Again, as $\mathsf{Sg}_{\mathbb{A}'}(c_2,d') = \mathbb{C}$, we have $\{d_3\} \times \mathbb{C} \subseteq R$, which means in particular that an operation p satisfying the first two equalities exists.

To show the last condition we consider two cases. First, if $\mathbb{C}'$ is a set then p is a projection. Also, as we proved $p(d/_{\eta'}, d/_{\eta'}, c/_{\eta'}) = p(d/_{\eta'}, c/_{\eta'}, d/_{\eta'}) = d/_{\eta'}$, it can only be the first projection. Second, if $\mathbb{C}'$ is a module over a ring $\mathbb{K}$ then $p(x,y,z) = \alpha x + \beta y + \gamma z$ for some coefficients satisfying $\alpha+\beta+\gamma = 1$, where 1 is the unity of $\mathbb{K}$. We have

$$\alpha d/_{\eta'} + \beta d/_{\eta'} + \gamma c/_{\eta'} = d/_{\eta'}, \qquad \alpha d/_{\eta'} + \beta c/_{\eta'} + \gamma d/_{\eta'} = d/_{\eta'}.$$

These equalities imply $\gamma c/_{\eta'} = \gamma d/_{\eta'}$ and $\beta c/_{\eta'} = \beta d/_{\eta'}$. Therefore

$$p(c/_{\eta'}, d/_{\eta'}, d/_{\eta'}) = \alpha c/_{\eta'} + \beta d/_{\eta'} + \gamma d/_{\eta'} = \alpha c/_{\eta'} + \beta c/_{\eta'} + \gamma c/_{\eta'} = c/_{\eta'}.$$

The Claim is proved.

Let $c_1 = m(c,d,d), d_2 = m(d,d,c), d_3 = m(d,c,d)$, where $c_1 \in c/_{\eta'}$ and $d_2, d_3 \in d/_{\eta'}$. Using the Claim we show that m can be chosen such that $d_2 = d_3$. There is a term operation p of $\mathbb{C}$ such that $p(d_2,d_3,c_1) = p(d_3,c_1,d_2) = d_2$ and $p(c_1,d_2,d_3) \stackrel{\eta'}{\equiv} c$. Then for

$$m'(x,y,z) = p(m(x,y,z), m(y,z,x), m(z,x,y))$$

we obtain

$$m'(c,d,d) = p(c_1,d_2,d_3) \stackrel{\eta'}{\equiv} c, \quad m'(d,d,c) = p(d_2,d_3,c_1) = d_2,$$
$$m'(d,c,d) = p(d_3,c_1,d_2) = d_2.$$

Moreover, as is easily seen, $m' \in F_{ab}$. Thus, we assume $c_1 = m(c,d,d), d_2 = m(d,d,c) = m(d,c,d)$. Note that at this point we can no longer assume that m is the first projection on $\mathbb{C}'$, although it still satisfies the conditions $m(c,d,d) \stackrel{\eta'}{\equiv} c, m(d,c,d) \stackrel{\eta'}{\equiv} m(d,d,c) \stackrel{\eta'}{\equiv} d$.

By our assumption $\mathsf{Sg}_{\mathbb{A}'}(c_1,d_2) = \mathsf{Sg}_{\mathbb{A}'}(c,d)$. Choose a maximal congruence η'' of $\mathsf{Sg}_{\mathbb{A}}(c,d)$ with $\eta \le \eta''$. By Proposition 3.7 applied to $\mathsf{Sg}_{\mathbb{A}}(c,d)/_{\eta''}$ either c_1, d_2, and therefore c, d, are connected in $\mathcal{H}(\mathsf{Sg}_{\mathbb{A}}(c,d))$ and therefore in $\mathcal{H}(\mathbb{C})$, or c_1d_2 is an edge in $\mathbb{A}$ witnessed by η''. If $\eta'' \not\leq \eta'$, then c, d are connected in $\mathcal{H}(\mathbb{C})$ by η'- and η''-blocks. So, suppose that c_1d_2 is an edge in $\mathbb{A}$ and $\eta'' \leq \eta'$.

Let g be a semilattice, majority, or affine operation witnessing that c_1d_2 is an edge of $\mathbb{A}$. Then the operation

$$g'(x, y, z) = g(m(x, y, z), m(y, z, x))$$

in the first case and the operation

$$g'(x, y, z) = g(m(x, y, z), m(y, z, x), m(z, x, y))$$

in the two latter cases belongs to F_{ab} and satisfies the following conditions. If g is a semilattice operation and $g(c_1, d_2) \stackrel{\eta''}{\equiv} d_2$, then

$$\begin{aligned} g'(c, d, d) &= g(m(c, d, d), m(d, d, c)) = g(c_1, d_2) \stackrel{\eta''}{\equiv} d_2, \\ g'(d, d, c) &\stackrel{\eta''}{\equiv} g'(d, c, d) \stackrel{\eta''}{\equiv} d_2. \end{aligned}$$

If $g(c_1, d_2) \stackrel{\eta''}{\equiv} c_1$ then

$$\begin{aligned} g'(c, d, d) &= g(m(c, d, d), m(d, d, c)) = g(c_1, d_2) \stackrel{\eta''}{\equiv} c_1, \\ g'(d, d, c) &= g(d_2, d_2) \stackrel{\eta''}{\equiv} d_2, \\ g'(d, c, d) &= g(d_2, c_1) \stackrel{\eta''}{\equiv} c_1. \end{aligned}$$

If g is a majority or affine operation, then

$$\begin{aligned} g'(c, d, d) &= g(m(c, d, d), m(d, d, c), m(d, c, d)) = g(c_1, d_2, d_2) \stackrel{\eta''}{\equiv} d_2, \\ g'(d, d, c) &\stackrel{\eta''}{\equiv} g'(d, c, d) \stackrel{\eta''}{\equiv} d_2, \end{aligned}$$

and

$$\begin{aligned} g'(c, d, d) &= g(m(c, d, d), m(d, d, c), m(d, c, d)) = g(c_1, d_2, d_2) \stackrel{\eta''}{\equiv} c_1, \\ g'(d, d, c) &\stackrel{\eta''}{\equiv} c_1, \end{aligned}$$

respectively. Since $\eta'' \leq \eta'$, the same equalities hold modulo η'. If g is semilattice or majority, the equalities above show that g' is not a projection or affine operation on $\mathbb{C}'$. Thus, c_1d_2 cannot be a semilattice or majority edge of $\mathbb{A}$ in this case, and if it is an affine edge of $\mathbb{A}$, then it is also an affine edge of $\mathbb{A}'$. Then we complete the proof in the same way as in Case 1 using c_1, d_2 instead of c', d'. Case 2 and the proof of the lemma are completed. □

Proof of Proposition 3.15. We prove items (1) and (2) simultaneously. We need to show that any $c, d \in A$ are connected by a path containing only edges of the correct types. We proceed by induction on the cardinality of subuniverses of $\mathbb{A}'$ to show that for any $\mathbb{B} \in \mathsf{Sub}(\mathbb{A}')$, any $c, d \in \mathbb{B}$ are connected. For the base case of induction we use $|\mathbb{B}| = 1$.

Let $\mathbb{B} \in \mathsf{Sub}(\mathbb{A}')$ and let the statement be proved for all $\mathbb{B}' \in \mathsf{Sub}(\mathbb{A}')$ with $|\mathbb{B}'| < |\mathbb{B}|$. Observe that if all subalgebras of $\mathbb{B}$ generated by 2 elements are connected, so is $\mathbb{B}$. Therefore, it suffices to assume that $\mathbb{B}$ is generated by two elements, say, $c, d \in A$. Moreover, by Theorem 3.5 we can assume that cd is an edge of $\mathbb{A}'$; let congruence η of $\mathbb{C} = \mathsf{Sg}_{\mathbb{A}'}(c,d)$ witness that. Assume that all the proper subalgebras of $\mathbb{C}$ satisfy the X-connectedness properties given in the proposition. We may also assume that for any $c' \in c/_\eta, d' \in d/_\eta$, it holds $\mathsf{Sg}_{\mathbb{A}}(c',d') = \mathsf{Sg}_{\mathbb{A}}(c,d)$. Indeed, if $\mathsf{Sg}_{\mathbb{A}}(c',d') \neq \mathsf{Sg}_{\mathbb{A}}(c,d)$, then choose c', d' so that $\mathsf{Sg}_{\mathbb{A}}(c',d')$ is minimal possible. If $\mathsf{Sg}_{\mathbb{A}'}(c',d') \neq \mathsf{Sg}_{\mathbb{A}'}(c,d)$, then by Lemma 3.11 $c'd'$ is an edge in $\mathbb{A}'$ of the same type as cd, and by the induction hypothesis satisfies the required conditions. Hence c and d are connected by $\mathsf{Sg}_{\mathbb{A}'}(c',d')$ and the congruence blocks $c/_\eta$, $d/_\eta$. Otherwise as is easily seen, $c'd'$ is an edge of $\mathbb{A}'$ of the same type as cd. Therefore, we can replace c, d with c', d'.

Let θ, and f or m be the congruence of $\mathsf{Sg}_{\mathbb{A}}(a,b)$, and a semilattice or majority operation witnessing the type of ab. We now consider the options given in Lemma 3.19. In case (i) elements c and d are connected by a semilattice path. In case (ii) $\mathcal{H}(\mathbb{C})$ is connected and we use the inductive hypothesis. In case (iv) there is nothing to prove, because cd is an edge of one of the required types. Therefore, assume $\mathsf{Sg}_{\mathbb{A}'}(c,d) = \mathsf{Sg}_{\mathbb{A}}(c,d)$.

The elements c and d are connected by a path $c = e_1, e_2, \dots, e_k = d$ in $\mathcal{G}(\mathsf{Sg}_{\mathbb{A}}(c,d))$, where e_ie_{i+1} is an edge of $\mathbb{A}$. Therefore we need to show connectedness for every pair e_ie_{i+1} which is an edge in $\mathbb{A}$. Observe that the conditions of Lemma 3.20 can be assumed. Indeed, if η is the congruence of $\mathsf{Sg}_{\mathbb{A}}(e_i, e_{i+1})$ witnessing that e_ie_{i+1} is an edge, and $c' \in e_i/_\eta, d' \in e_{i+1}/_\eta$ such that $\mathsf{Sg}_{\mathbb{A}}(c',d') \subsetneq \mathsf{Sg}_{\mathbb{A}}(e_i, e_{i+1})$, then e_i, e_{i+1} are connected in $\mathcal{H}(\mathsf{Sg}_{\mathbb{A}'}(c,d))$ and the result follows by the induction hypothesis. Now, by Lemma 3.20 e_i, e_{i+1} are either connected by semilattice edges in $\mathcal{G}(\mathsf{Sg}_{\mathbb{A}'}(c,d))$, or they are connected in $\mathcal{H}(\mathsf{Sg}_{\mathbb{A}'}(c,d))$, or e_ie_{i+1} is an edge in $\mathbb{A}'$ of the same type as e_ie_{i+1} if it is unary, affine, or majority in $\mathbb{A}$, and e_ie_{i+1} is semilattice or majority, if it is semilattice in $\mathbb{A}$. Therefore all the connectedness conditions are preserved. □

3.5. Unified operations

To conclude this section we prove that the term operations certifying the types of edges of smooth algebras can be significantly unified (cf. Proposition 2 from [7]).

Theorem 3.21. *Let $\mathbb{A}$ be an idempotent algebra and $E_1, \dots, E_s$ its thick edges such that every E_i of the semilattice or majority type is a subuniverse of $\mathbb{A}$. There are term operations f, g, h of $\mathbb{A}$ such that for any $E_i = \{a/_\theta, b/_\theta\}$, $i \in [s]$, where θ is a congruence witnessing that ab is an edge*

(i) $f|_{E_i}$ is a semilattice operation if ab is a semilattice edge; it is the first projection if ab is a majority or affine edge;

(ii) $g|_{E_i}$ *is a majority operation if ab is a majority edge; it is the first projection if ab is an affine edge, and* $g|_{E_i}(x,y,z) = f|_{E_i}(x, f|_{E_i}(y,z))$ *if ab is semilattice;*

(iii) $h|_{\mathsf{Sg}(a,b)/\theta}$ *is an affine operation if ab is an affine edge; it is the first projection if ab is a majority edge, and* $h|_{E_i}(x,y,z) = f|_{E_i}(x, f|_{E_i}(y,z))$ *if ab is semilattice.*

Operations f, g, h are projections on every edge of the unary type.

Proof. First, we show that there is an operation f that is semilattice on each semilattice edge from $E_1, \dots, E_s$. Let $B_1, \dots, B_n$ be a list of all thick semilattice edges from $E_1, \dots, E_s$. That is, each B_i is a 2-element set. Let also $f_1, \dots, f_n$ be a list of term operations of the algebra such that $f_i|_{B_i}$ is a semilattice operation. Notice that every binary idempotent operation on a 2-element set is either a projection or a semilattice operation, and every binary operation of a module can be represented in the form $\alpha x + (1-\alpha)y$. Since each f_i is idempotent, for any i,j, $f_i|_{B_j}$ is either a projection, or a semilattice operation. We prove by induction, that the operation f^i constructed via the following rules is a semilattice operation on $B_1, \dots, B_i$:

- $f^1 = f_1$;
- $f^i(x,y) = f_i(f^{i-1}(x,y), f^{i-1}(y,x))$.

The base case of induction, $i = 1$ holds by the choice of f_1. Since $f^{i-1}(x,y)|_{B_j} = f^{i-1}(y,x)|_{B_j}$, for $j \in [i-1]$, we have

$$f^i(x,y)|_{B_j} = f^{i-1}(x,y)|_{B_j}.$$

Suppose that f^{i-1} satisfies the required conditions. If $f^{i-1}|_{B_i}$ is a projection, say, $f^{i-1}|_{B_i}(x,y) = x$, then

$$f^i(x,y)|_{B_i} = f_i(f^{i-1}(x,y), f^{i-1}(y,x))|_{B_i} = f_i(x,y)|_{B_i},$$

that is, a semilattice operation on B_i. Otherwise $f^{i-1}(x,y) = f^{i-1}(y,x)$ on B_i, and we have $f^i(x,y)|_{B_i} = f^{i-1}(y,x))|_{B_i}$, which is a semilattice operation.

Thus, for each thick edge E_j, $j \in [s]$, $f^n|_{E_j}$ is a semilattice operation if E_j is semilattice and either a semilattice operation or a projection if E_j is majority or unary, and f^n on $\mathsf{Sg}(E_j)$ is $\alpha x + (1-\alpha)y$ if E_j is affine. However, if E_j is not semilattice, then the subalgebra with the universe E_j (or $\mathsf{Sg}(E_j)$) has no semilattice operation. Therefore, if E_j is majority or unary edge, then $f^n|_{\mathsf{Sg}(E_j)}$ is a projection, and if E_j is an affine edge, then $f^n|_{\mathsf{Sg}(E_j)}$ can be one of the two types, either a projection, or $\alpha x + (1-\alpha)y$ for some module $\mathbb{M}$ over a ring $\mathbb{K}$ and some $\alpha \in \mathbb{K}$.

Let $D_1, \dots, D_\ell$ be a list of all thick affine edges. Set $f_0^\dagger = f^n$ and inductively define $f_1^\dagger, \dots, f_\ell^\dagger$ as follows. For $i \in [\ell]$ if $f_{i-1}^\dagger|_{\mathsf{Sg}(D_i)}$ is a projection, set $f_i^\dagger = f_{i-1}^\dagger$. If $f_{i-1}^\dagger|_{\mathsf{Sg}(D_i)} = \alpha x + (1-\alpha)y$ for some $\mathbb{M}, \mathbb{K}$, and $\alpha \in \mathbb{K}$, then by Lemma 3.17 applied to the algebra $\mathbb{A}' = (A, f_{i-1}^\dagger)$ one can transform $f_{i-1}^\dagger$ into a projection on $\mathsf{Sg}(D_i)$ without changing its action on semilattice and majority edges. Let $f_i^\dagger$ be this transformed operation. As is easily seen, $f_i^\dagger$ is a projection on $\mathsf{Sg}(D_j)$ for each $j < i$. Therefore f^n can be assumed to be a projection on every non-semilattice edge E_j. Finally, it is easy to check that $f(x, y) = f^n(f^n(x, y), x)$ satisfies the conditions of the theorem, regardless on whether $f^n|_{\mathsf{Sg}(E_j)}$ is the first or second projection.

Now let $C_1, \dots, C_k$, $D_1, \dots, D_\ell$ be lists of all thick majority and all thick affine edges respectively, from $E_1, \dots, E_s$, and $g_1, \dots, g_k$, $h_1, \dots, h_\ell$ lists of term operations of the algebra $\mathbb{A}$ such that $g_i|_{C_i}$ is a majority operation, and $h_i|_{D'_i}$ is the affine operation, where $D'_i = \mathsf{Sg}(D_i)$. Let F be the set of term operations of $\mathbb{A}$. Notice first, that since neither $\mathbb{C}_i = (C_i; F|_{C_i})$ nor $\mathbb{D}'_i = (D'_i; F|_{D'_i})$ has a term semilattice operation, every one of their binary term operation is either a projection or an operation of the form $\alpha x + (1 - \alpha)y$. Therefore, for any i, j, $g_i|_{C_j}(x, y, y), g_i|_{C_j}(y, x, y), g_i|_{C_j}(y, y, x)$, $h_i|_{C_j}(x, y, y), h_i|_{C_j}(y, x, y)$, $h_i|_{C_j}(y, y, x) \in \{x, y\}$, and also $g_i|_{D'_j}(x, y, y), g_i|_{D'_j}(y, x, y)$, $g_i|_{D'_j}(y, y, x)$, $h_i|_{D'_j}(x, y, y)$, $h_i|_{D'_j}(y, x, y), h_i|_{D'_j}(y, y, x) \in \{x, y, \alpha x + (1 - \alpha)y\}$. This means in particular that each of the operations $g_i|_{C_j}, h_i|_{C_j}$ is of one of the following types: a projection, the minority operation, the majority operation, or a 2/3-minority operation.

First we show that for any $1 \le i \le \ell$ there is h^i such that $h^i|_{D'_j}$ is an affine operation for $j \le i$. As before, $h^1 = h_1$ gives the base case of induction. If h^{i-1} is constructed, then if $h^{i-1}|_{D'_i}$ is an affine operation then set $h^i = h^{i-1}$. Otherwise, $h^{i-1}|_{D'_i} = \alpha x + \beta y + \gamma z$ with $\alpha + \beta + \gamma = 1$. Then set $h'(x, y) = h^{i-1}(x, y, y)$ and observe that $h'(x, y)|_{D'_i} = \alpha x + (1 - \alpha)y$ and $h'(x, y)|_{D'_j} = x$ for $j < i$. Then let

$$h''_1(x, y, z) = h'(x, h_i(x, y, z)).$$

Then

$$\begin{aligned} h''_1(x, y, z)|_{D'_i} &= \alpha x + (1 - \alpha)z - (1 - \alpha)y + (1 - \alpha)x \\ &= x - (1 - \alpha)y + (1 - \alpha)z, \quad \text{and} \\ h''_1(x, y, z)|_{D'_j} &= x, \qquad \text{for } j < i. \end{aligned}$$

Similarly, we can obtain $h''_3(x, y, z)$ with the property that $h''_3(x, y, z)|_{D'_i} = x - (1 - \gamma)y + (1 - \gamma)z$ and $h''_3(x, y, z)|_{D'_j} = x$. Furthermore, set

$$h''_2(x, y, z) = h''_1(h^{i-1}(x, y, z), z, x).$$

As is easily seen, for this operation we have

$$\begin{aligned} h''_2(x, y, z)|_{D'_i} &= \alpha x + \beta y + \gamma z - (1 - \alpha)z + (1 - \alpha)x = x + \beta y - \beta z, \quad \text{and} \\ h''_2(x, y, z)|_{D'_j} &= x - y + z, \qquad \text{for } j < i. \end{aligned}$$

Finally, we set

$$h^i(x, y, z) = h''_1(h''_2(h''_3(x, y, z), y, z), y, z).$$

Again, we have

$$\begin{aligned} h^i(x,y,z)|_{D'_i} &= x-(1-\gamma)y+(1-\gamma)z+\beta y-\beta z-(1-\alpha)y+(1-\alpha)z\\ &= x-y+z, \quad \text{and}\\ h^i(x,y,z)|_{D'_j} &= x-y+z, \qquad \text{for } j<i. \end{aligned}$$

Next, we prove that every g_i, $i\in[k]$, can be chosen such that $g_i(x,y,z)=x$ on D'_j for all $j\in[\ell]$. As in the proof of Lemma 3.17, it suffices to find a term operation $p(x,y)$ such that $p(x,y)=x$ on C_i and $p(x,y)=y$ on D'_j for $j\in[\ell]$. As C_i is a majority but not a semilattice edge, the operation h^ℓ from the previous paragraph on C_i can be one of the following operations: a projection, a majority operation, a 2/3-minority operation, or a minority operation. If h^ℓ is a projection on C_i, say, $h^\ell(x,y,z)=x$, or majority, then set $p(x,y)=h^\ell(x,x,y)$. If h^ℓ is 2/3 minority satisfying $h^\ell(x,y,y)=y$ or $h^\ell(y,y,x)=y$, then set $p(x,y)=h^\ell(y,x,x)$ in the former case, and $p(x,y)=h^\ell(x,x,y)$ in the latter case. If h^ℓ is 2/3-minority satisfying $h^\ell(x,y,x)=x$, then set $p(x,y)=h^\ell(x,h^\ell(x,y,x),x)$; again it is easy to check that p satisfies the required conditions. Finally, if h^ℓ on C_i is the minority operation, suppose $g_i(x,y,z)=\alpha_jx+\beta_jy+\gamma_jz$ on D'_j, $j\in[\ell]$. Then set $s_1(x,y)=h^\ell(g_i(x,y,y),y,g_i(y,y,x))$ and $s_2(x,y)=g_i(x,y,x)$. As is easily seen, $s_1(x,y)=y, s_2(x,y)=x$ on C_i and $s_1(x,y)=s_2(x,y)=(1-\beta_j)x+\beta_jy$ on D'_j. Then set $p(x,y)=h^\ell(s_1(x,y),s_2(x,y),y)$. We have $p(x,y)=h^\ell(y,x,y)=x$ on C_i and $p(x,y)=y$ on each D'_j, $j\in[\ell]$, as required.

Now, we prove by induction that for every $1\le i\le k$ there is an operation $g^i(x,y,z)$ which is majority on C_j for $j\le i$ and is the first projection on D'_r, $r\in[\ell]$. The operation $g^1=g_1$ gives the base case of induction. Let us assume that g^{i-1} is already found. If $g^{i-1}|_{C_j}$ is the majority operation, set $g^i=g^{i-1}$. Otherwise, it is either a projection, or a 2/3-minority operation, or the minority operation. In all these case its variables can be permuted such that $g^{i-1}|_{C_i}(x,y,y)=x$. Then the operation $p(x,y)=g^{i-1}(x,y,y)$ satisfies the conditions $p|_{C_i}(x,y)=x$, and $p|_{C_j}(x,y)=y$ for all $j\in[i-1]$. Therefore, the operation

$$g^i(x,y,z)=p(g_i(x,y,z),g^{i-1}(x,y,z))$$

satisfies the required conditions. Note that g^i is a projection (the same one) on every D'_r, $r\in[\ell]$, but not necessarily the first projection. So, we may need to permute the variables of g^i to obtain the desired result.

Operation g^k acts correctly on the majority and affine edges. To make g^k act correctly on the semilattice edges we set

$$g(x,y,z)=g^k(f(x,f(y,z)),f(y,f(z,x)),f(z,f(x,y))).$$

Finally, set

$$\begin{aligned} p(x,y)&=g(x,y,y),\\ \overline{h}(x,y,z)&=p(h^\ell(x,y,z),x), \end{aligned}$$

and

$$h(x,y,z) = \overline{h}(f(x,f(y,z)), f(y,f(z,x)), f(z,f(x,y))).$$

Since all the operations are projections on thick edges of the unary type, as is easily seen h satisfies the conditions required. □

Theorem 3.21 implies that for a smooth algebra $\mathbb{A}$ there are operations f, g, h that act as the theorem prescribes on all thick edges of $\mathbb{A}$. Moreover, it can be easily extended to finite classes of algebras.

Corollary 3.22. *Let $\mathcal{K}$ be a finite class of finite smooth algebras. Then there are term operations f, g, h of $\mathcal{K}$ such that conditions (i)–(iii) of Theorem* 3.21 *are true for any edge ab of any $\mathbb{B} \in \mathcal{K}$.*

Proof. Let $\mathcal{K} = \{\mathbb{A}_1, \ldots, \mathbb{A}_n\}$ and $\mathbb{A} = \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$. Observe that for any $i \in [n]$ and any edge ab of $\mathbb{A}_i$, any pair of the form $\mathbf{a}, \mathbf{b} \in \mathbb{A}$, where $\mathbf{a}[j] = \mathbf{b}[j]$ for $j \neq i$ and $\mathbf{a}[i] = a$, $\mathbf{b}[i] = b$, is an edge of $\mathbb{A}$ of the same type as ab. We apply Theorem 3.21 to the list of thick edges of $\mathbb{A}$ of the form $\{\mathbf{a}/_{\overline{\theta}}, \mathbf{b}/_{\overline{\theta}}\}$ where $\mathbf{a}, \mathbf{b}$ are as above for some $i \in [n]$ and edge ab of $\mathbb{A}_i$, θ is a congruence of $\mathsf{Sg}_{\mathbb{A}_i}(a,b)$ and $\overline{\theta}$ is the congruence of $\mathsf{Sg}_{\mathbb{A}}(\mathbf{a}, \mathbf{b})$ given by $(\mathbf{c}, \mathbf{d}) \in \overline{\theta}$ iff $(\mathbf{c}[i], \mathbf{d}[i]) \in \theta$. Then by Theorem 3.21 there are term operations f, g, h of $\mathbb{A}$, that is, of $\mathcal{K}$, satisfying the conditions (i)–(iii) of the theorem on those thick edges. It remains to observe that f, g, h satisfy the conditions (i)–(iii) for each $\mathbb{A}_i$. □

4. Thin edges

Although edges and thick edges as they have been introduced so far reflect some aspects of the structure of idempotent algebras, they are not very useful from the technical perspective, the way we are going to use them. To improve the construction of (thick) edges we introduce thin edges, that are always just a pair of elements, without any congruences or quotient algebras involved. Later we show that the graph of an algebra $\mathbb{A}$ based on these thin edges retains many of the useful properties of $\mathcal{G}(\mathbb{A})$, most importantly, connectivity. In a certain sense the connectivity of this new graph is even improved relative to $\mathcal{G}(\mathbb{A})$. This will come at a price: thin edges are inevitably directed even when it does not seem natural or necessary, and are not always subalgebras. Our first goal is to introduce thin edges, prove their existence, and show that operations f, g, h from Theorem 3.21 can be assumed to have a number of additional properties.

Fix a finite class $\mathcal{K}$ of similar smooth algebras. The definitions of majority and affine thin edges depend on this class.

We start with an observation that operations f, g, h identified in Corollary 3.22 can be assumed to satisfy certain identities.

Lemma 4.1. *Operations f, g, h identified in Corollary* 3.22 *can be chosen such that*

(1) $f(x, f(x,y)) = f(x,y)$ for all $x, y \in \mathbb{A}$ and all $\mathbb{A} \in \mathcal{K}$;

(2) $g(x, g(x, y, y), g(x, y, y)) = g(x, y, y)$ *for all* $x, y \in \mathbb{A}$ *and all* $\mathbb{A} \in \mathcal{K}$*;*
(3) $h(h(x, y, y), y, y) = h(x, y, y)$ *for all* $x, y \in \mathbb{A}$ *and all* $\mathbb{A} \in \mathcal{K}$*.*

Proof. Items (1) and (2) follow from Lemma 3.16.

(3) Let $h_b(x) = h(x, b, b)$ for $b \in \mathbb{A}$. The goal is to find h such that $h_b(h_b(x)) = h_b(x)$ for all b and all x. Let $h'_0(x, y, z) = h(x, y, z)$ and $h'_{i+1}(x, y, z) = h'_i(h(x, y, y), y, z)$ for $i \geq 0$. Then $h'_i(x, b, b) = h^i_b(x)$. Clearly, $h_b^{|A|!}$ is idempotent for every $b \in \mathbb{A}$, and thus $h'_{|A|!}(h'_{|A|!}(x, y, y), y, y) = h'_{|A|!}(x, y, y)$. It remains to show that every function $h_i(x, y, z)$ is a replacement for h. That is, for any affine edge ab,

$$h_{i+1}(a, b, b) \stackrel{\theta}{\equiv} h_{i+1}(b, b, a) \stackrel{\theta}{\equiv} a,$$

where $\theta \in \mathsf{Con}(\mathsf{Sg}(a, b))$ witnesses that ab is an affine edge. By induction we have

$$h_{i+1}(a, b, b) = h_i(h(a, b, b), b, b) \stackrel{\theta}{\equiv} h_i(a, b, b) \stackrel{\theta}{\equiv} a,$$

$$h_{i+1}(b, b, a) = h_i(h(b, b, b), b, a) = h_i(b, b, a) \stackrel{\theta}{\equiv} a.$$

As is easily seen, the resulting operation h acts on semilattice and majority edges as prescribed by Theorem 3.21. □

4.1. Semilattice edges

In this section we focus on semilattice edges of the graph $\mathcal{G}(\mathbb{A})$. Note first that if one fixes a term operation f such that f is a semilattice operation on every thick semilattice edge of $\mathcal{G}(\mathbb{A})$, then it is possible to define an orientation of every semilattice edge. A semilattice edge ab is oriented from a to b if $f(a/_\theta, b/_\theta) = f(b/_\theta, a/_\theta) = b/_\theta$, where θ is a congruence witnessing that ab is a semilattice edge. Clearly, this orientation strongly depends on the choice of the term operation f.

We shall now improve the choice of operation f and restrict the kind of semilattice edges we will use later. A semilattice edge ab such that the equality relation witnesses that it is a semilattice edge will be called a *thin semilattice edge.* A binary operation $\ell(x, y)$ is said to satisfy the *Semilattice Shift Condition* (SLS condition for short), if

$$\text{for any } a, b \in \mathbb{A}, \text{ either } a = \ell(a, b) \text{ or the pair } (a, \ell(a, b)) \text{ is a thin semilattice edge.} \qquad \text{(SLS)}$$

Proposition 4.2. *Let* $\mathbb{A}$ *be a smooth idempotent algebra. There is an SLS binary term operation* f *of* $\mathbb{A}$ *such that* f *is a semilattice operation on every thick semilattice edge of* $\mathcal{G}(\mathbb{A})$*, and* f *is the first projection on every majority or affine thick edge.*

Proof. Let f be a binary term operation such that f is semilattice on every semilattice edge and $f(x, f(x, y)) = f(x, y)$. Let $a, b \in \mathbb{A}$ be such that $f(a, b) \neq a$, and set $b_0 = f(a, b)$ and $b_{i+1} = f(a, f(b_i, a))$ for $i > 0$.

Claim 1. For any i, $f(a, b_i) = b_i$.

Indeed, $f(a,b_0)=f(a,f(a,b))=f(a,b)=b_0$, and for any $i>0$

$$f(a,b_i)=f(a,f(a,f(b_{i-1},a)))=f(a,f(b_{i-1},a))=b_i.$$

Let $\mathbb{B}_i=\mathsf{Sg}(a,b_i)$. Then $\mathbb{B}_0\supseteq\mathbb{B}_1\supseteq\ldots$, and there is k with $\mathbb{B}_{k+1}=\mathbb{B}_k$.

Claim 2. $f(a,b_k)=f(b_k,a)=b_k$.

Since $b_k\in\mathbb{B}_{k+1}=\mathsf{Sg}(a,b_{k+1})$, there is a term operation t such that $b_k=t(a,b_{k+1})$. Let $s(x,y)=t(x,f(x,f(y,x)))$. For this operation we have

$$s(a,b_k)=t(a,f(a,f(b_k,a)))=t(a,b_{k+1})=b_k$$
$$s(b_k,a)=t(b_k,f(b_k,f(a,b_k)))=t(b_k,f(b_k,b_k))=b_k.$$

This means that ab_k is a semilattice edge, and the congruence witnessing it is the equality relation. Since $\mathbb{A}$ is smooth, every one of its thick semilattice edge is a subalgebra, and by the choice of f, it is a semilattice operation on every such pair. Moreover, by Claim 1 $f(a,b_k)=f(b_k,a)=b_k$.

Let ℓ be the maximal among the numbers chosen as before Claim 2 for all pairs a,b with $f(a,b)\neq a$. Let $f_0=f$, and $f_{i+1}(x,y)=f(x,f(f_i(x,y),x))$ for $i\geq 0$. Let also $f'=f_\ell$.

Claim 3. For any $a,b\in A$, either $f'(a,b)=a$, or the pair ac, where $c=f'(a,b)$ is a thin semilattice edge.

If $f(a,b)=a$ then it is straightforward that $f'(a,b)=a$. Suppose $f(a,b)\neq a$. Note that in the notation introduced before Claim 1 $b_i=f_i(a,b)$. Indeed, we have $b_0=f(a,b)=f_0(a,b)$, and then

$$f_{i+1}(a,b)=f(a,f(f_i(a,b),a))=f(a,f(b_i,a))=b_{i+1}.$$

We prove by induction on i that $f_i(a,c)=f_i(c,a)=c$. Since $\mathbb{B}_\ell=\mathbb{B}_{\ell+1}$, where the $\mathbb{B}_i$ are constructed as before, by Claim 2 $f(a,c)=f(c,a)=c$. This gives the base case of induction. Suppose $f_i(a,c)=f_i(c,a)=c$. Then

$$f_{i+1}(a,c)=f(a,f(f_i(a,c),a)=f(a,f(c,a))=c$$
$$f_{i+1}(c,a)=f(c,f(f_i(c,a),c)=f(c,f(c,c))=c.$$

Claim 3 is proved.

To complete the proof it suffices to check that f' is a semilattice operation on every (thick) semilattice edge of $\mathcal{G}(\mathbb{A})$. However, this is straightforward from the construction of f'. Also, as f is the first projection on every thick edge of every non-semilattice type, so is f'. $\Box$

It will be convenient for us to denote binary operation f that satisfies the conditions of Theorem 3.21, Lemma 4.1(1), and Proposition 4.2 by $\cdot$, that is, to write $x\cdot y$ or just xy for $f(x,y)$, whenever it does not cause a confusion. The fact that ab is a thin semilattice edge we will also denote by $a\leq b$. In other words, $a\leq b$ if and only if $a\cdot b=b\cdot a=b$.

Next we show that those semilattice edges ab, for which the equality relation does not witness that they are semilattice edges, can be thrown out of the graph $\mathcal{G}(\mathbb{A})$ such that the graph remains connected. Therefore, we can assume that every semilattice edge is thin. Let $\mathbb{A}\in\mathcal{K}$ be an algebra, $a,b\in\mathbb{A}$,

$\mathbb{B} = \mathsf{Sg}(a,b)$, and θ a congruence of $\mathbb{B}$. Pair ab is said to be *minimal* with respect to θ if for any $b' \in b/_\theta$, $b \in \mathsf{Sg}(a,b')$.

Corollary 4.3. *Let ab be a semilattice edge, θ a congruence of $\mathsf{Sg}(a,b)$ that witnesses this, and $c \in a/_\theta$. Then there is $d \in b/_\theta$ such that cd is a thin semilattice edge. Moreover, cd is a thin semilattice edge for every $d \in b/_\theta$ such that cd is minimal with respect to d.*

Proof. By Proposition 4.2 $cb = c$ or $c \le cb$. Since $d = cb \in b/_\theta$ the former option is impossible. Therefore cd is a thin semilattice edge. To prove the last claim let $d \in b/_\theta$ be such that cd is a minimal pair with respect to θ. Let also $d' = c \cdot d$; we have $c \le d'$. Then there is a term operation p such that $p(c,d') = d$. Let $f'(x,y) = p(x, x \cdot y)$ and $d'' = f'(d,c)$. Then $f'(c,d) = d$ and $d'' \in b/_\theta$. Again, there is a term operation r such that $r(c,d'') = d$. Then for $f''(x,y) = r(x, f'(y,x))$ we have $f''(c,d) = r(c, f'(d,c)) = r(c,d'') = d$ and $f''(d,c) = r(d, f'(c,d)) = r(d,d) = d$. □

4.2. Thin majority edges

Here we introduce thin majority edges in a way similar to thin semilattice edges, although in a weaker sense.

A ternary term operation g' is said to satisfy the *majority condition* (with respect to $\mathcal{K}$) if it satisfies the identity from Lemma 4.1(2) and g' is a majority operation on every thick majority edge of every algebra from $\mathcal{K}$. By Corollary 3.22 an operation satisfying the majority condition exists.

A pair ab is called a *thin majority edge* (with respect to $\mathcal{K}$) if

(*) for any term operation g' satisfying the majority condition, the subalgebras $\mathsf{Sg}(a, g'(a,b,b)), \mathsf{Sg}(a, g'(b,a,b)), \mathsf{Sg}(a, g'(b,b,a))$ contain b.

The operation g from Corollary 3.22 does not have to satisfy any specific conditions on a thin majority edge, except what follows from its definition. Also, thin majority edges are directed, since a, b in the definition occur asymmetrically.

If in addition to the condition above ab is also a majority edge, a congruence θ witnesses that, and ab is a minimal pair with respect to θ, we say that ab is a *special majority edge.* We now show that thin majority edges have term operations on them that are similar to majority operations.

Next we prove a property, Lemma 4.5, of thin majority edges that will be useful in the future. We start with an auxiliary lemma.

Lemma 4.4. *Let $\mathbb{A}$ be a smooth algebra and g' a ternary term operation that is a majority operation on every thick majority edge.*

(1) For every binary term operation t, the operations $g'(t(x, g'(x,y,z)), y, z)$, $g'(x, t(y, g'(x,y,z)), z)$, $g'(x, y, t(z, g'(x,y,z)))$ are majority on every thick majority edge.

(2) If for some $a, b \in \mathbb{A}$, $g'(a,b,b) = b$ [or $g'(b,a,b) = b$, or $g'(b,b,a) = b$], then there is a term operation g'' satisfying the majority condition and such that $g''(a,b,b) = b$ [or $g''(b,a,b) = b$, or $g''(b,b,a) = b$, respectively].

(3) If for some $a, b \in \mathbb{A}$, $g'(a,a,b) = b$, then there is a term operation g'' satisfying the majority condition and such that $g''(a,a,b) = b$.

Proof. (1) Let ab be a majority edge of $\mathbb{A}$ and θ a congruence of $\mathsf{Sg}(a,b)$ that witnesses that. Then $\mathbb{B} = \mathsf{Sg}(a,b)/_\theta$ is a 2-element algebra that has a majority operation but does not have a semilattice term operation. This means that t is a projection on $\mathbb{B}$. If $t(x,y) = x$ on $\mathbb{B}$, then

$$g'(t(x, g'(x,y,z)), y, z) = g'(x,y,z)$$

on $\mathbb{B}$, as required. If $t(x,y) = y$ on $\mathbb{B}$, then

$$g''(x,y,z) = g'(t(x, g'(x,y,z)), y, z) = g'(g'(x,y,z), y, z).$$

It is now straightforward to verify that g'' is majority on $\mathbb{B}$. A proof for the other two operations is quite similar.

(2) To show the existence of an operation g'' satisfying the equation $g''(x, g''(x,y,y), g''(x,y,y)) = g''(x,y,y)$ we use the construction from the proof of Lemma 3.16. We consider the unary operation $g_x(y) = g'(x,y,y)$ and $g''(x,y,z) = g_x^{n-1}(g'(x,y,z))$. By Lemma 3.16 g'' is a majority operation on every thick majority edge. We show that for any $i \in [n-1]$ it holds that $g^\dagger(a,b,b) = b$ [or $g^\dagger(b,a,b) = b$, or $g^\dagger(b,b,a) = b$], where $g^\dagger(x,y,z) = g_x^i(g'(x,y,z))$. We proceed by induction on i. Assuming $g_x^0(g'(x,y,z)) = g'(x,y,z)$, the base case follows from the conditions of the lemma. Suppose $g^\dagger(x,y,z) = g_x^i(g'(x,y,z))$ and $g^\dagger(a,b,b) = b$. We need to check that $g_a(g^\dagger(a,b,b)) = b$ [$g_b(g^\dagger(b,a,b)) = b$,$g_b(g^\dagger(b,b,a)) = b$], which can be easily verified

$$\begin{aligned}
g_a(g^\dagger(a,b,b)) &= g'(a, g^\dagger(a,b,b), g^\dagger(a,b,b)) = g'(a,b,b) = b,\\
g_b(g^\dagger(b,a,b)) &= g'(b, g^\dagger(b,a,b), g^\dagger(b,a,b)) = g'(b,b,b) = b,\\
g_b(g^\dagger(b,b,a)) &= g'(b, g^\dagger(b,b,a), g^\dagger(b,b,a)) = g'(b,b,b) = b.
\end{aligned}$$

(3) Here we use a slightly different construction. Define operations g_i inductively by setting

$$\begin{aligned}
g_1(x,y,z) &= g'(x,y,z) \quad \text{and}\\
g_{i+1} &= g'(x, g_i(x,y,y), g_i(x,y,z)).
\end{aligned}$$

If $g_i(a,a,b) = b$, then $g_{i+1}(a,a,b) = g'(a, g_i(a,a,a), g_i(a,a,b)) = g'(a,a,b) = b$. Also, if g_i is a majority operation on some thick majority edge, it is straightforward to verify that so is g_{i+1} on that edge. Therefore by the assumptions on g', for every i the operation g_i is a majority operation on every thick majority edge of any $\mathbb{B} \in \mathcal{K}$, and $g_i(a,a,b) = b$. We now find n such that g_n satisfies the identity from Lemma 3.16(2). Let $g_x(y) = g'(x,y,y)$, n_x the idempotent power of g_x, and n the least common multiple of n_x for all $x \in \mathbb{B} \in \mathcal{K}$. Then we have

$$g_n(x,y,y) = \underbrace{g_x \circ g_x \circ \cdots \circ g_x(y)}_{n \text{ times}} = g_x^n(y),$$

and so

$$g_n(x, g_n(x, y, y), g_n(x, y, y)) = g_x^n(g_x^n(y)) = g_x^n(y) = g_n(x, y, y),$$

as required. □

Lemma 4.5. *Let $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3 \in \mathcal{K}$, and let a_1b_1, a_2b_2, and a_3b_3 be thin majority edges in $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$. Then there is an operation g' satisfying the majority condition such that $g'(a_1, b_1, b_1) = b_1$, $g'(b_2, a_2, b_2) = b_2$, $g'(b_3, b_3, a_3) = b_3$. In particular, for any thin majority edge ab there is an operation g' satisfying the majority condition such that $g'(a, b, b) = g'(b, a, b) = g'(b, b, a) = b$.*

Proof. Let g be an operation satisfying the conditions of Corollary 3.22; in particular, it satisfies the majority condition. Let R be the subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$ generated by $\mathbf{a}_1 = (a_1, b_2, b_3), \mathbf{a}_2 = (b_1, a_2, b_3), \mathbf{a}_3 = (b_1, b_2, a_3)$. Let $b_1^1 = g(a_1, b_1, b_1)$. By the definition of thin majority edges there is a term operation t such that $b_1 = t(a_1, b_1^1)$. Consider $g_1'(x, y, z) = g(t(x, g(x, y, z)), y, z)$, by Lemma 4.4(1) g_1' is a majority operation on every thick majority edge and

$$g_1'(a_1, b_1, b_1) = g(t(a_1, g(a_1, b_1, b_1)), b_1, b_1) = g(b_1, b_1, b_1) = b_1.$$

By Lemma 4.4(2) g_1' can be converted into g' satisfying the majority condition and such that $g'(a_1, b_1, b_1) = b_1$. Also let $g'(b_2, a_2, b_2) = b_2^2$ and $g'(b_3, b_3, a_3) = b_3^2$, so that $g'(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = (b_1, b_2^2, b_3^2)$.

Again, as g' satisfies the majority condition, by the definition of thin majority edges there is a term operation s such that $b_2 = s(a_2, b_2^2)$. Consider $g_1''(x, y, z) = g'(x, s(y, g'(x, y, z)), z)$, by Lemma 4.4(1) g_1'' is a majority operation on every thick majority edge and

$$g_1''(b_2, a_2, b_2) = g'(b_2, s(a_2, g'(b_2, a_2, b_2)), b_2) = g'(b_2, b_2, b_2) = b_2.$$

Also

$$g_1''(a_1, b_1, b_1) = g'(a_1, s(b_1, g'(a_1, b_1, b_1)), b_1) = g'(a_1, s(b_1, b_1), b_1) = b_1.$$

Operation g_1'' can be transformed into g'' satisfying the majority condition and such that $g''(a_1, b_1, b_1) = b_1, g''(b_2, a_2, b_2) = b_2$. Let $g''(b_3, b_3, a_3) = b_3^3$, so that $g''(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = (b_1, b_2, b_3^3)$.

Now we use the same construction once more. As g'' satisfies the majority condition, there is a binary term operation q such that $q(a_3, b_3^3) = b_3$. Consider operation $g_1'''(x, y, z) = g''(x, y, q(z, g''(x, y, z)))$. As before, we obtain

$$g_1'''(b_3, b_3, a_3) = g''(b_3, b_3, q(a_3, g''(b_3, b_3, a_3))) = g''(b_3, b_3, b_3) = b_3.$$

Also

$$g_1'''(a_1, b_1, b_1) = g''(a_1, b_1, q(b_1, g''(a_1, b_1, b_1))) = g''(a_1, b_1, q(b_1, b_1)) = b_1,$$

and

$$g_1'''(b_2, a_2, b_2) = g''(b_2, a_2, q(b_2, g''(b_2, a_2, b_2))) = g''(b_2, a_2, q(b_2, b_2)) = b_2.$$

Thus, $g_1'''(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = (b_1, b_2, b_3)$. We now apply Lemma 4.4(2) to obtain g''' that satisfies the conditions of the lemma. □

Next we show that every majority edge has a thin majority edge associated with it.

Lemma 4.6. *Let $\mathbb{A} \in \mathcal{K}$ be a smooth idempotent algebra, ab a majority edge in $\mathbb{A}$, and θ a congruence of $\mathsf{Sg}(a,b)$ witnessing that. Then for any $c \in a/_\theta$ and $d \in b/_\theta$ such that cd is a minimal pair with respect to the restriction θ' of θ on $\mathsf{Sg}(c,d)$, the pair cd is a special thin majority edge.*

Proof. Since $g'(c,d,d) \stackrel{\theta}{\equiv} g'(d,c,d) \stackrel{\theta}{\equiv} g'(d,d,c) \stackrel{\theta}{\equiv} d$ for any g' satisfying the majority condition and cd is a minimal pair with respect to θ', we have $d \in \mathsf{Sg}(c, g'(c,d,d)), \mathsf{Sg}(c, g'(d,c,d)), \mathsf{Sg}(c, g'(d,d,c))$. □

Since for any $c \in a/_\theta, d \in b/_\theta$ such that $\mathsf{Sg}(c,d)$ is minimal among the subalgebras of the form $\mathsf{Sg}(c,d')$, $d' \in b/_\theta$, the pair cd is minimal with respect to θ, we obtain the following

Corollary 4.7. *For any majority edge ab, where θ is a witnessing congruence, and any $c \in a/_\theta$ there is $d \in b/_\theta$ such that cd is a special thin majority edge.*

A natural question is, of course, whether it is also possible to find a 'symmetric' thin majority edge in any thick majority edge. In other words, is it true that for any majority edge ab (witnessed by congruence θ of $\mathsf{Sg}(a,b)$) there are $c \in a/_\theta, d \in b/_\theta$ such that cd is a majority edge witnessed by the equality relation? Unfortunately, the following example derived from one suggested by M.Kozik [26] shows that this is not true in general.

Example 4.8. Let $A = \{0,1,2,3\}$, and let θ be the equivalence relation on A with blocks $\{0,2\}$ and $\{1,3\}$. Define two ternary operations maj and min on A as follows: maj is majority and min is the first projection on $A/_\theta$. On each of the θ-blocks $\{0,2\},\{1,3\}$, the operation maj is the third projection, and min is the minority operation. Finally, for any $a,b,c \in A$ such that $(b,c) \in \theta$, but $(a,b) \notin \theta$ we set $\mathrm{maj}(a,b,c) = \mathrm{maj}(b,a,c) = c$, $\mathrm{maj}(b,c,a) = \mathrm{maj}(b,c,a-1)$, $\min(a,b,c) = \min(a+2,b-1,c-1)$, $\min(b,a,c) = \min(b,a-1,c)$, $\min(b,c,a) = \min(b,c,a-1)$. Here $+$ and $-$ are modulo 4. Also, note that in the given conditions $\mathrm{maj}(b,c,a-1), \min(a+2,b-1,c-1), \min(b,a-1,c), \min(b,c,a-1)$ are defined by the action of $\min, \mathrm{maj}$ on θ-blocks. Let $\mathbb{A} = (A, \mathrm{maj}, \min)$. As is easily seen, $\mathbb{A}$ omits type **1**, and any pair ab, $a \in \{0,2\}, b \in \{1,3\}$, is a majority edge, as witnessed by the congruence θ. It can be verified by straightforward computation (use Universal Algebra Calculator [21] for that) that for no such pair there is a term operation of $\mathbb{A}$ that is majority on $\{a,b\}$.

4.3. Thin affine edges

In this section we introduce thin affine edges in a similar fashion as thin majority and semilattice edges.

We say that a term operation h' of a smooth algebra $\mathbb{A} \in \mathcal{K}$ satisfies the *minority condition* (with respect to $\mathcal{K}$) if it satisfies the identity from Lemma 4.1(3), and for any $\mathbb{B} \in \mathcal{K}$ and every affine edge ab of $\mathbb{B}$ witnessed by a congruence θ of $\mathsf{Sg}_{\mathbb{B}}(a,b)$, the operation h' is a Mal'tsev operation on

$\mathsf{Sg}_{\mathbb{B}}(a,b)/_\theta$. By Corollary 3.22 and Lemma 4.1(3) an operation satisfying the minority condition exists.

A pair ab is called a *thin affine edge* (with respect to $\mathcal{K}$) if

(**) $h(b,a,a) = b$, where h is the operation identified in Corollary 3.22, and $b \in \mathsf{Sg}(a, h'(a,a,b))$ for every term operation h' satisfying the minority condition.

The operation h from Corollary 3.22 does not have to satisfy any specific conditions on thin minority edges, except what follows from its definition. Also, thin affine edges are directed, since a, b in the definition occur asymmetrically.

Lemma 4.9. *Let $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{K}$, and let ab and cd be thin affine edges in $\mathbb{A}_1, \mathbb{A}_2$. Then there is an operation h' and such that $h'(b,a,a) = b$ and $h'(c,c,d) = d$. In particular, for any thin affine edge ab there is an operation h' such that $h'(b,a,a) = h'(a,a,b) = b$.*

Proof. Let R be the subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2$ generated by $(b,c), (a,c), (a,d)$. By the definition of thin affine edges,

$$\begin{pmatrix} b \\ d' \end{pmatrix} = h\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}\right) \in R,$$

where h is an operation satisfying the minority condition that exists by Corollary 3.22 and Lemma 4.1(3). Then as $d \in \mathsf{Sg}_{\mathbb{A}_2}(c, h(c,c,d))$, there is a term operation $r(x,y)$ such that $d = r(c,d')$. Therefore

$$\begin{pmatrix} b \\ d \end{pmatrix} = r\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} b \\ d' \end{pmatrix}\right) \in R.$$

The result follows. $\square$

Lemma 4.10. *Let $\mathbb{A} \in \mathcal{K}$ be an algebra, ab an affine edge in it, and θ the congruence of $\mathsf{Sg}(a,b)$ witnessing that. Then there exists $b' \in b/_\theta$ such that ab' is a minimal pair and $h(b',a,a) = b'$. Moreover, for any such b', the pair ab' is a thin affine edge.*

Proof. Suppose a, b satisfy the conditions of the lemma. Choose b'' such that ab'' is a minimal pair. Let $b' = h(b'',a,a) \in \mathsf{Sg}(a,b'')$, since h satisfies the conditions of Lemma 4.1(3) $h(b',a,a) = b'$. Note that by the choice of b'' the pair ab' is minimal with respect to the restriction θ' of θ on $\mathsf{Sg}(a,b')$. Since $h'(a,a,b') \in b/_\theta$ for any h' satisfying the minority condition, this means that $b' \in \mathsf{Sg}(a, h'(a,a,b'))$. Condition (**) follows. $\square$

Corollary 4.11. *For any affine edge ab, where θ is a witnessing congruence, there is $b' \in b/_\theta$ such that ab' is a thin affine edge.*

4.4. Some useful terms

First, we make a useful observation concerning the presence of thick and thin edges of certain types.

Proposition 4.12. *Let $\mathcal{K}$ be a finite class of smooth algebras without edges of the unary type. Then there exists an algebra from $\mathcal{K}$ containing a thin semilattice (majority, affine) edge if and only if there is an algebra in $\mathcal{K}$ containing a thick edge of the same type.*

Proof. Corollaries 4.3, 4.7, and 4.11 imply that if an algebra $\mathbb{A}$ contains a thick edge of a certain type, it also contains a thin edge of the same type. Conversely, every thin semilattice edge is also a thick one by definition. Suppose that $\mathcal{K}$ does not contain algebras with majority (affine) edges. Then any operation satisfying the conditions of Lemma 4.1 also satisfies the majority (minority) condition, including the first projection $p(x, y, z)$. Let $a, b \in \mathbb{A} \in \mathcal{K}$, $a \neq b$. Then $p(a, b, b) = a$ ($p(a, a, b) = a$), and so $\mathsf{Sg}_{\mathbb{A}}(a, p(a, b, b)) = \{a\}$ ($\mathsf{Sg}_{\mathbb{A}}(a, p(a, a, b)) = \{a\}$). Therefore, ab cannot be a thin edge of the majority (affine) type. □

The next two lemmas show that smooth algebras always have a range of term operations that behave in a predictable way on thin or thick edges. These properties will be very helpful later when proving various statements about the structure of smooth algebras.

Lemma 4.13. *Let $\mathbb{A}_1, \mathbb{A}_2 \in \mathcal{K}$, and let ab and cd be thin edges in $\mathbb{A}_1, \mathbb{A}_2$. If they have different types, then there is a binary term operation p such that $p(b, a) = b$, $p(c, d) = d$.*

Proof. Let R be the subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2$ generated by $(b, c), (a, d)$. Let also g, h be the operations from Corollary 3.22.

If ab is majority and $c \leq d$, then by the definition of thin majority edges there is a binary term operation r such that $b = r(a, g(a, b, b))$. Then

$$\begin{pmatrix} b \\ d \end{pmatrix} = r\left(\begin{pmatrix} a \\ d \end{pmatrix}, g\left(\begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}\right)\right) \in R,$$

as $g(x, y, z) = xyz$ on semilattice edges. Therefore $p(x, y) = r(y, g(y, x, x))$ satisfies the conditions.

If ab is affine and $c \leq d$, then by the definition of thin affine edges,

$$\begin{pmatrix} b \\ d \end{pmatrix} = h\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}\right) \in R,$$

as $h(x, y, z) = xyz$ on semilattice edges. Therefore $p(x, y) = h(x, y, y)$ satisfies the conditions.

If ab is affine and cd is majority, then set

$$\begin{pmatrix} b' \\ d' \end{pmatrix} = h\left(\begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}\right) \in R.$$

By the definition of thin affine edges there is a binary term operation r such that $b = r(a, h(a, a, b))$. Consider operation

$$g'(x, y, z) = g(r(x, h(x, y, z)), r(y, h(y, x, z)), z).$$

It satisfies the following condition.

$$\begin{aligned} g'(a,a,b) &= g(r(a,h(a,a,b)), r(a,h(a,a,b)), b) \\ &= g(r(a,b'), r(a,b'), b) = g(b,b,b) = b. \end{aligned}$$

Also, on any thick majority edge $\{c'/_\theta, d'/_\theta\}$ of an algebra from $\mathcal{K}$, where θ witnesses that $c'd'$ is a majority edge, $h(x,y,z) = x$, therefore

$$\begin{aligned} g'(x,y,z) &= g(r(x,h(x,y,z)), r(y,h(y,x,z)), z) \\ &= g(r(x,x), r(y,y), z) = g(x,y,z) \end{aligned}$$

on $\mathsf{Sg}_{\mathbb{A}_2}(c',d')/_\theta$.

By Lemma 4.4(3) there is an the operation g'' that satisfies the majority condition and such that $g''(a,a,b) = b$. As cd is a thin majority edge, there is a binary term operation s such that $s(c, g''(d,d,c)) = d$. Thus,

$$\begin{pmatrix} b \\ d \end{pmatrix} = s\left(\begin{pmatrix} b \\ c \end{pmatrix}, \begin{pmatrix} b \\ g''(d,d,c) \end{pmatrix}\right) = s\left(\begin{pmatrix} b \\ c \end{pmatrix}, g''\left(\begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ c \end{pmatrix}\right)\right) \in R.$$

The result follows. □

Lemma 4.14. *(1) Let ab be a thin majority edge of an algebra $\mathbb{A} \in \mathcal{K}$. There is a term operation t_{ab} such that $t_{ab}(a,b) = b$ and $t_{ab}(c,d) \stackrel{\eta}{\equiv} c$ for any affine edge cd of any $\mathbb{A}' \in \mathcal{K}$, where the type of cd is witnessed by congruence η.*

(2) The operation $h'(x,y,z) = h(z,y,x)$ satisfies the following conditions: $h'(a,a,b) = b$ for any thin affine edge ab of any $\mathbb{A}' \in \mathcal{K}$, and $h'(d,c,c) \stackrel{\eta}{\equiv} d$ for any affine edge cd of any $\mathbb{A}' \in \mathcal{K}$, where the type of cd is witnessed by congruence η. Moreover, $h'(x,c',d')$ is a permutation of $\mathsf{Sg}(c,d)/_\eta$ for any $c', d' \in \mathsf{Sg}(c,d)$.

Proof. Let g,h be operations satisfying the conditions of Corollary 3.22.

(1) Let $c_1d_1, \dots, c_\ell d_\ell$ be a list of all affine edges of algebras in $\mathcal{K}$, $c_i, d_i \in \mathbb{A}_i$ and θ_i the corresponding congruences. Let $b' = g(a,b,b)$. By the definition of thin majority edges $b \in \mathsf{Sg}_{\mathbb{A}}(a,b')$ and there is a binary term operation r such that $b = r(a,b')$. By Corollary 3.22 g is the first projection on $\mathsf{Sg}(c_i,d_i)/_{\theta_i}$. Let $t_{ab}(x,y) = r(x, g(x,y,y))$. Then

$$\begin{aligned} t_{ab}(a,b) &= r(a, g(a,b,b)) = b, \\ t_{ab}(c_i,d_i) &= r(c_i, g(c_i,d_i,d_i)) \stackrel{\theta_i}{\equiv} c_i, \quad \text{for } i \in [\ell]. \end{aligned}$$

This means that t_{ab} satisfies the required conditions.

(2) The first result follows from the definition of thin affine edges and the fact that h is a Mal'tsev operation on $\mathsf{Sg}(c,d)/_\eta$ for every affine edge cd and congruence η witnessing that.

Let $c', d' \in \mathsf{Sg}(c,d)$. Since $\mathbb{B} = \mathsf{Sg}(c,d)/_\eta$ is a module, in particular, it is an Abelian algebra and $h'(x,c^*,c^*) = x$ for all $c^* \in \mathbb{B}$, the second result follows. □

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Declarations

Data availability

Data sharing not applicable to this article as datasets were neither generated nor analyzed.

Compliance with ethical standards

The author is a member of the Editorial Board of Algebra Universalis. Apart from this the author declares that he has no conflict of interest.

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