# A dichotomy theorem for nonuniform CSPs

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**Abstract**

In this paper we prove the Dichotomy Conjecture on the complexity of nonuniform constraint satisfaction problems posed by Feder and Vardi[1]

# Contents



[1]Apart from correcting numerous typos and inaccuracies, this version of the paper is different from the first *Arxiv* version in the following ways:
– A self-contained high level presentation of the main results is added.
– An inconsistency between the definition and the use of the chaining condition is fixed.
– The proof of Lemma 45 is expanded for improved readability.
– Section “Auxiliary Lemmas” is split into two sections, Sections 7 and 8, and moved forward.

# 1 Introduction

In a Constraint Satisfaction Problem (CSP) the question is, given two similar relational structures, decide whether or not it is possible to satisfy a given set of constraints. Constraints are often represented by specifying a relation, which is a set of allowed combinations of values some variables can take simultaneously. If the constraints allowed in a problem have to come from some set $\Gamma$ of relations; such a restricted problem is referred to as the *nonuniform CSP* and denoted $\mathrm{CSP}(\Gamma)$. The set $\Gamma$ is then called a constraint language. A systematic study of the complexity of nonuniform CSPs was started by Schaefer in [47] who showed that for every constraint language $\Gamma$ over a 2-element set the problem $\mathrm{CSP}(\Gamma)$ is either solvable in polynomial time or is NP-complete. Schaefer also asked about the complexity of $\mathrm{CSP}(\Gamma)$ for languages over larger sets. The next step in the study of nonuniform CSPs was made in the seminal paper by Feder and Vardi [31, 32], who apart from considering numerous aspects of the problem, posed the *Dichotomy Conjecture* that states that for every finite constraint language $\Gamma$ over a finite set the problem $\mathrm{CSP}(\Gamma)$ is either solvable in polynomial time or is NP-complete. This conjecture has become a focal point of the CSP research and most of the effort in this area revolves to some extent around the Dichotomy Conjecture.

The Dichotomy Conjecture was approached with different methods; however, the most effective one turned out to be the *algebraic approach* that associates every constraint language with its (universal) algebra of polymorphisms. The method was first developed in a series of papers by Jeavons and coauthors [36, 37, 38] and then refined by Bulatov, Krokhin, Barto, Kozik, Maroti, Zhuk and others [5, 8, 6, 3, 25, 15, 27, 42, 43, 48, 50, 49]. While the complexity of $\mathrm{CSP}(\Gamma)$ has been already solved for some interesting classes of structures such as graphs [33], the algebraic approach allowed the researchers to confirm the Dichotomy Conjecture in a number of more general cases: for languages over a set of size up to 7 [13, 16, 41, 50, 49], so called conservative languages [14, 17, 18, 2], and some classes of digraphs [7]. It also allowed to design the main classes of CSP algorithms [6, 23, 20, 11, 35], and refine the exact complexity of the CSP [1, 8, 29, 40].

In this paper we confirm the Dichotomy Conjecture for arbitrary finite structures. More precisely we prove the following

**Theorem 1** *For any finite constraint language $\Gamma$ over a finite set the problem $\mathrm{CSP}(\Gamma)$ is either solvable in polynomial time or is NP-complete.*

The proved criterion matches the algebraic form of the Dichotomy Conjecture suggested in [25]. The hardness part of the conjecture has been known for long time. Therefore the main achievement of this paper is a polynomial time algorithm for problems satisfying the tractability condition. More specifically, we suggest such an algorithm for languages that contain all the constant relations of the form $\{(a)\}$, and this implies a general dichotomy due to the results of [25].

Using the algebraic language we can state the result in a stronger form. Let $\mathbb{A}$ be a finite idempotent algebra and let $\mathrm{CSP}(\mathbb{A})$ denote the union of problems $\mathrm{CSP}(\Gamma)$ such that every term operation of $\mathbb{A}$ is a polymorphism of $\Gamma$. Problem $\mathrm{CSP}(\mathbb{A})$ is no longer a nonuniform CSP, and Theorem 1 allows for problems

$\mathrm{CSP}(\Gamma) \subseteq \mathrm{CSP}(\mathbb{A})$ to have different solution algorithms even when $\mathbb{A}$ meets the tractability condition. We show that the solution algorithm only depends on the algebra $\mathbb{A}$.

**Theorem 2** *For a finite idempotent algebra that satisfies the conditions of the Dichotomy Conjecture there is a uniform solution algorithm for* $\mathrm{CSP}(\mathbb{A})$.

The paper consists of two parts. The first part of the paper aims at a self-contained introduction into the main ideas of the solution algorithm. The second part mostly concerns with further development of the algebraic approach and technical proofs of the results, and is significantly more involved. We start with introducing the terminology and notation for CSPs that is used throughout the paper and reminding the basics of the algebraic approach. Then in Section 3.1 we introduce the key ingredients used in the algorithm: separating congruences and quasi-centralizer. Then in Section 3.2 we apply these concepts to CSPs to, first, demonstrate how quasi-centralizers help to decompose an instance into smaller subinstances, and, second, to introduce a new kind of minimality condition for CSPs, *block minimality*. After that we state the main results used by the algorithm and describe the algorithm itself. We complete the first part by introducing the main technical construction to give an idea of why the algorithm works.

# Part I
# Outline of the algorithm

## 2 Introduction to CSP

For a detailed introduction to CSP and the algebraic approach to its structure the reader is referred to a very recent and very nice survey by Barto et al. [9]. In preliminaries to this paper we therefore focus on what is needed for our result.

### 2.1 CSP, universal algebra and the Dichotomy conjecture

The 'combinatorial' formulation of the CSP best fits our purpose. Fix a finite set $A$ and let $\Gamma$ be a *constraint language* over $A$, that is, a set — not necessarily finite — of relations over $A$. The (*nonuniform*) *Constraint Satisfaction Problem* (*CSP*) associated with language $\Gamma$ is the problem $\mathrm{CSP}(\Gamma)$, in which, an *instance* is a pair $(V, \mathcal{C})$, where $V$ is a set of variables; and $\mathcal{C}$ is a set of *constraints*, i.e. pairs $\langle \mathbf{s}, R \rangle$, where $\mathbf{s} = (v_1, \ldots, v_k)$ is a tuple of variables from $V$, the *constraint scope*, and

$R \in \Gamma$, the $k$-ary *constraint relation*. We always assume that relations are given explicitly by a list of tuples. The way constraints are represented does not matter if $\Gamma$ is finite, of course, but it may change the complexity of the problems for infinite languages. The goal is to find a *solution*, that is a mapping $\varphi : V \to A$ such that for every constraint $\langle \mathbf{s}, R\rangle$, $\varphi(\mathbf{s}) \in R$.

We will often use the set of solutions of a CSP instance $\mathcal{P} = (V, \mathcal{C})$ or its subproblems (to be defined later), viewed either as a $|V|$-ary relation or as a set of mappings $\varphi : V \to A$. It will be denoted by $\mathcal{S}_{\mathcal{P}}$, or just $\mathcal{S}$ if $\mathcal{P}$ is clear from the context.

Jeavons et al. in [36, 37] were first to observe that higher order symmetries of constraint languages called polymorphisms play a significant role in the study of the complexity of the CSP. A *polymorphism* of a relation $R$ over $A$ is an operation $f(x_1, \dots, x_k)$ on $A$ such that for any choice of $\mathbf{a}_1, \dots, \mathbf{a}_k \in R$ we have $f(\mathbf{a}_1, \dots, \mathbf{a}_k) \in R$. If this is the case we also say that $f$ *preserves* $R$, or that $R$ is *invariant* with respect to $f$. A polymorphism of a constraint language $\Gamma$ is an operation that is a polymorphism of every $R \in \Gamma$.

**Theorem 3 ([36, 37])** *For constraint languages* $\Gamma, \Delta$, *where* $\Gamma$ *is finite, if every polymorphism of* $\Delta$ *is also a polymorphism of* $\Gamma$, *then* $\mathrm{CSP}(\Gamma)$ *is polynomial time reducible to* $\mathrm{CSP}(\Delta)$. *(In fact, this can be improved to a log-space reduction.)*

Listed below are the several types of polymorphisms that occur frequently throughout the paper. The presence of each of these polymorphisms imposes restrictions on the structure of invariant relations that can be used in designing a solution algorithm. Some of such results we will mention later.

– *Semilattice* operation is a binary operation $f(x, y)$ such that $f(x, x) = x$, $f(x, y) = f(y, x)$, and $f(x, f(y, z)) = f(f(x, y), z)$ for all $x, y, z \in A$;

– $k$-ary *near-unanimity* operation is a $k$-ary operation $u(x_1, \dots, x_k)$ such that $u(y, x, \dots, x) = u(x, y, x, \dots, x) = \dots = u(x, \dots, x, y) = x$ for all $x, y \in A$; a ternary near-unanimity operation $m$ is said to be a *majority* operation, it satisfies the equations $m(y, x, x) = m(x, y, x) = m(x, x, y) = x$;

– *Mal'tsev* operation is a ternary operation $h(x, y, z)$ satisfying the equations $h(x, y, y) = h(y, y, x) = x$ for all $x, y \in A$; the *affine* operation $x - y + z$ of an Abelian group is a special case of Mal'tsev operations;

– $k$-ary *weak near-unanimity* operation is a $k$-ary operation $w$ that satisfies the same equations as a near-unanimity operations $w(y, x, \dots, x) = w(x, y, x, \dots, x) = \dots = w(x, \dots, x, y)$, except the last one.

The next step in discovering more structure behind nonuniform CSPs has been made in paper [25], in which universal algebras were brought into the picture. A *(universal) algebra* is a pair $\mathbb{A} = (A, F)$ consisting of a set $A$, the *universe* of $\mathbb{A}$, and a set $F$ of operations on $A$. Operations from $F$ together with operations that can be obtained from them by means of composition are called the *term* operations of $\mathbb{A}$.

Algebras allow for a more general definition of CSPs that is used here. Let $\mathrm{CSP}(\mathbb{A})$ denote the class of nonuniform CSPs $\{\mathrm{CSP}(\Gamma) \mid \Gamma \subseteq \mathsf{Inv}(F)\}$, where $\mathsf{Inv}(F)$ denotes the set of all relations invariant with respect to all operations from $F$. Note that the tractability of $\mathrm{CSP}(\mathbb{A})$ can be understood in two ways: as the existence of a polynomial-time algorithm for every $\mathrm{CSP}(\Gamma)$ from this class, or as the existence of a uniform polynomial-time algorithm for all such problems. One of the implications of our results is that these two types of tractability are equivalent. From the formal standpoint we will use the stronger one.

The main structural elements are subalgebras, congruences, and quotient algebras. For $B \subseteq A$ and an operation $f$ on $A$ by $f|_B$ we denote the restriction of $f$ on $B$. Algebra $\mathbb{B} = (B, \{f|_B \mid f \in F\})$ is called a *subalgebra* of $\mathbb{A}$ if $f(b_1, \ldots, b_k) \in B$ for any $b_1, \ldots, b_k \in B$ and any $f \in F$.

Congruences play a very significant role in our algorithm, and we discuss them more carefully. A *congruence* is an equivalence relation $\theta \in \mathsf{Inv}(F)$. This means that for any operation $f \in F$ and any $(a_1, b_1), \ldots, (a_k, b_k) \in \theta$ it holds $(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k)) \in \theta$. Therefore it is possible to define an algebra on $A/_\theta$, the set of $\theta$-blocks, by setting $f/_\theta(a_1^\theta, \ldots, a_k^\theta) = (f(a_1, \ldots, a_k))/_\theta$ for $a_1, \ldots, a_k \in A$, where $a^\theta$ denotes the $\theta$-block containing $a$. The resulting algebra $\mathbb{A}/_\theta$ is called the *quotient algebra modulo* $\theta$.

The following are examples of congruences and quotient algebras.

– Let $\mathbb{A}$ be any algebra. Then the equality relation $\underline{0}_\mathbb{A}$ and the full binary relation $\underline{1}_\mathbb{A}$ on $\mathbb{A}$ are congruences of $\mathbb{A}$. The quotient algebra $\mathbb{A}/_{\underline{0}_\mathbb{A}}$ is $\mathbb{A}$ itself, while $\mathbb{A}/_{\underline{1}_\mathbb{A}}$ is a 1-element algebra.

– Let $\mathbb{S}_n$ be the permutation group on an $n$-element set and binary relation $\theta$ is given by: $(a, b) \in \theta$ for $a, b \in \mathbb{S}_n$ if and only if $a$ and $b$ have the same parity as permutations. Then $\theta$ is a congruence of $\mathbb{S}_n$ and $\mathbb{S}_n/_\theta$ is the 2-element group.

– Let $\mathbb{L}_n$ be an $n$-dimensional vector space and $\mathbb{L}'$ its $k$-dimensional subspace. The binary relation $\pi$ given by: $(\overline{a}, \overline{b}) \in \pi$ if and only if $\overline{a}, \overline{b}$ have the same orthogonal projection on $\mathbb{L}'$, is a congruence of $\mathbb{L}_n$ and $\mathbb{L}_n/_\pi$ is $\mathbb{L}'$.

The (ordered) set of all congruences of $\mathbb{A}$ is denoted by $\mathsf{Con}(\mathbb{A})$. This set is actually a lattice. By $\mathsf{HS}(\mathbb{A})$ we denote the set of all quotient algebras of all subalgebras of $\mathbb{A}$.

The results of [25] reduce the dichotomy conjecture to idempotent algebras. An algebra $\mathbb{A} = (A, F)$ is said to be *idempotent* if every operation $f \in F$ satisfies the equation $f(x, \ldots, x) = x$. If $\mathbb{A}$ is idempotent, then all the constant relations

$\{(a)\}$ are invariant under $F$. Therefore studying CSPs over idempotent algebras is the same as studying the CSPs that allow all constant relations. Another useful property of idempotent algebras is that every block of every its congruence is a subalgebra. We now can state the algebraic version of the dichotomy theorem.

**Theorem 4** *For a finite idempotent algebra* $\mathbb{A}$ *the following are equivalent:*

*(1)* $\mathrm{CSP}(\mathbb{A})$ *is solvable in polynomial time;*

*(2)* $\mathbb{A}$ *has a weak near-unanimity term operation;*

*(3) every algebra from* $\mathsf{HS}(\mathbb{A})$ *has a nontrivial term operation (that is not a* projection*, an operation of the form* $f(x_1, \ldots, x_k) = x_i$*);*

*Otherwise* $\mathrm{CSP}(\mathbb{A})$ *is NP-complete.*

The hardness part of this theorem is proved in [25]; the equivalence of (2) and (3) was proved in [24] and [44]. The equivalence of (1) to (2) and (3) is the main result of this paper. In the rest of the paper we assume all algebras to satisfy the conditions (2),(3).

### 2.2 Bounded width and the few subalgebras algorithm

Leaving aside occasional combinations thereof, there are only two standard types of algorithms solving the CSP. In this section we give a brief introduction into them.

**CSPs of bounded width.** Algorithms of the first kind are based on the idea of local propagation, that is formally described below. By $[n]$ we denote the set $\{1, \ldots, n\}$. For sets $A_1, \ldots, A_n$ tuples from $A_1 \times \cdots \times A_n$ are denoted in boldface, say, $\mathbf{a}$; the $i$th component of $\mathbf{a}$ is referred to as $\mathbf{a}[i]$. An $n$-ary relation $R$ over sets $A_1, \ldots, A_n$ is any subset of $A_1 \times \cdots \times A_n$. For $I = \{i_1, \ldots, i_k\} \subseteq [n]$ by $\mathrm{pr}_I\mathbf{a}, \mathrm{pr}_I R$ we denote the *projections* $\mathrm{pr}_I\mathbf{a} = (\mathbf{a}[i_1], \ldots, \mathbf{a}[i_k])$, $\mathrm{pr}_I R = \{\mathrm{pr}_I\mathbf{a} \mid \mathbf{a} \in R\}$ of tuple $\mathbf{a}$ and relation $R$. If $\mathrm{pr}_i R = A_i$ for each $i \in [n]$, relation $R$ is said to be a *subdirect product* of $A_1 \times \cdots \times A_n$.

Let $\mathcal{P} = (V, \mathcal{C})$ be a CSP instance. For $W \subseteq V$ by $\mathcal{P}_W$ we denote the *restriction* of $\mathcal{P}$ onto $W$, that is, the instance $(W, \mathcal{C}_W)$, where for each $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, the set $\mathcal{C}_W$ includes the constraint $C_W = \langle \mathbf{s} \cap W, \mathrm{pr}_{\mathbf{s}\cap W} R\rangle$. The set of solutions of $\mathcal{P}_W$ will be denoted by $\mathcal{S}_W$.

Unary solutions, that is, when $|W| = 1$ play a special role. One may always assume that allowed values for a variable $v \in V$ is the set $\mathcal{S}_v$. We call this set the *domain* of $v$ and assume that CSP instances may have different domains, which nevertheless are always subalgebras or quotient algebras of the original algebra $\mathbb{A}$.

It will be convenient to denote the domain of $v$ by $\mathbb{A}_v$. The domain $\mathbb{A}_v$ may change as a result of transformations of the instance. Instance $\mathcal{P}$ is said to be *1-minimal* if for every $v \in V$ and every constraint $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ such that $v \in \mathbf{s}$, it holds $\mathrm{pr}_v R = \mathbb{A}_v$.

Instance $\mathcal{P}$ is said to be *(2,3)-consistent* if it has a *(2,3)-strategy*, that is, a collection of relations $R^X$, $X \subseteq V$, $|X| = 2$ satisfying the following conditions:
– for every $X \subseteq V$ with $|X| \leq 2$ and every $C = \langle \mathbf{s}, R \rangle$, $\mathrm{pr}_{\mathbf{s}\cap X} R^X \subseteq \mathrm{pr}_{\mathbf{s}\cap X} R$;
– for every $X = \{u, v\} \subseteq V$, any $w \in V - X$, and any $(a, b) \in R^X$, there is $c \in \mathbb{A}_w$ such that $(a, c) \in R^{\{u,w\}}$ and $(b, c) \in R^{\{v,w\}}$.

We will always assume that a (2,3)-consistent instance has a constraint $C^X = \langle X, \mathcal{S}_X \rangle$ for every $X \subseteq V$, $|X| = 2$. Then clearly $R^X \subseteq \mathcal{S}_X$. Let the collection of relations $R^X$ be denoted by $\mathcal{R}$. A tuple $\mathbf{a}$ whose entries are indexed with elements of $W \subseteq V$ and such that $\mathrm{pr}_X \mathbf{a} \in R^X$ for any $X \subseteq W$, $|X| = 2$, will be called *$\mathcal{R}$-compatible*. If a (2,3)-consistent instance $\mathcal{P}$ with a (2,3)-strategy $\mathcal{R}$ satisfies the additional condition
– for every constraint $C = \langle \mathbf{s}, R \rangle$ of $\mathcal{P}$ every tuple $\mathbf{a} \in R$ is $\mathcal{R}$-compatible,

it is called *(2,3)-minimal*. For $k \in \mathbb{N}$, $(k, k+1)$-strategies, $(k, k+1)$-consistency, and $(k, k+1)$-minimality are defined in a similar way replacing 2,3 with $k, k+1$.

Instance $\mathcal{P}$ is said to be *minimal* (or *globally minimal*) if for every $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ and every $\mathbf{a} \in R$ there is a solution $\varphi \in \mathcal{S}$ such that $\varphi(\mathbf{s}) = \mathbf{a}$.

Any instance can be transformed to a 1-minimal, (2,3)-consistent, or (2,3)-minimal instance in polynomial time using the standard constraint propagation algorithms (see, e.g. [30]). These algorithms work by changing the constraint relations and the domains of the variables eliminating some tuples and elements from them. We call such a process *tightening* the instance. It is important to notice that if the original instance belongs to CSP($\mathbb{A}$) for some algebra $\mathbb{A}$, that is, all its constraint relations are invariant under the term operations of $\mathbb{A}$, the constraint relations obtained by propagation algorithms are also invariant under term operations of $\mathbb{A}$, and so the resulting instance also belongs to CSP($\mathbb{A}$). Establishing minimality amounts to solving the problem and therefore not always can be easily done.

If a constraint propagation algorithm solves a CSP, the problem is said to be of bounded width. More precisely, $\mathrm{CSP}(\Gamma)$ (or $\mathrm{CSP}(\mathbb{A})$) is said to have *bounded width* if for some $k$ every $(k, k+1)$-minimal instance from $\mathrm{CSP}(\Gamma)$ (or $\mathrm{CSP}(\mathbb{A})$) has a solution. Problems of bounded width are very well studied, see the older survey [26] and a more recent paper [4].

**Theorem 5 ([4, 20, 39])** *For an idempotent algebra $\mathbb{A}$ the following are equivalent:*

*(1)* $\mathrm{CSP}(\mathbb{A})$ *has bounded width;*

*(2) every (2,3)-minimal instance from* $\mathrm{CSP}(\mathbb{A})$ *has a solution;*

*(3)* $\mathbb{A}$ *has a weak near-unanimity term of arity* $k$ *for every* $k \geq 3$*;*

*(4) every algebra* $\mathsf{HS}(\mathbb{A})$ *has a nontrivial operation, and none of them is equivalent to a module.*

**Omitting semilattice edges and the few subpowers property.** The second type of CSP algorithms can be viewed as a generalization of Gaussian elimination, although, it utilizes just one property also used by Gaussian elimination: the set of solutions of a system of linear equations or a CSP has a set of generators of size linear in the number of variables. The property that for every instance $\mathcal{P}$ of $\mathrm{CSP}(\mathbb{A})$ its solution space $\mathcal{S}_\mathcal{P}$ has a set of generators of polynomial size is nontrivial, because there are only exponentially many such sets, while, as is easily seen CSPs with $n$ variables may have up to double exponentially many different sets of solutions. Formally, an algebra $\mathbb{A} = (A, F)$ has *few subpowers* if for every $n$ there are only exponentially many $n$-ary relations in $\mathsf{Inv}(F)$.

Algebras with few subpowers are well studied, completely characterized, and the CSP over such an algebra has a polynomial-time solution algorithm, see, [11, 35]. In particular, such algebras admit a characterization in terms of the existence of a term operation with special properties, an *edge* term. We however need only a subclass of algebras with few subpowers that appeared in [20] and is defined as follows.

A pair of elements $a, b \in \mathbb{A}$ is said to be a *semilattice edge* if there is a binary term operation $f$ of $\mathbb{A}$ such that $f(a,a) = a$ and $f(a,b) = f(b,a) = f(b,b) = b$, that is, $f$ is a semilattice operation on $\{a, b\}$.

**Proposition 6 ([20])** *If an idempotent algebra* $\mathbb{A}$ *has no semilattice edges, it has few subpowers, and therefore* $\mathrm{CSP}(\mathbb{A})$ *is solvable in polynomial time.*

Semilattice edges have other useful properties including the following one that we use for reducing a CSP to smaller problems.

**Lemma 7 ([19])** *For any idempotent algebra* $\mathbb{A}$ *there is a binary term operation* $xy$ *of* $\mathbb{A}$ *(think multiplication) such that* $xy$ *is a semilattice operation on any semilattice edge and for any* $a, b \in \mathbb{A}$ *either* $ab = a$ *or* $\{a, ab\}$ *is a semilattice edge.*

# 3 Solving CSPs

## 3.1 Congruence separation and centralizers

In this section we introduce two of the key ingredients of the algorithm.

**Separating congruences** Unlike the vast majority of the literature on the algebraic approach to the CSP we use not only term operations, but also polynomial operations of an algebra. It should be noted however that the first to use polynomials for CSP algorithms was Maroti in [43]. We make use of some ideas from that paper in the next section.

Let $f(x_1, \dots, x_k, y_1, \dots, y_\ell)$ be a $k+\ell$-ary term operation of an algebra $\mathbb{A}$ and $b_1, \dots, b_\ell \in \mathbb{A}$. The operation $g(x_1, \dots, x_k) = f(x_1, \dots, x_k, b_1, \dots, b_\ell)$ is called a *polynomial* of $\mathbb{A}$. The name 'polynomial' refers to usual polynomials. Indeed, if $\mathbb{A}$ is a ring, its polynomials as just defined are the same as polynomials in the regular sense. A polynomial that depends on only one variable is said to be a *unary* polynomial.

While polynomials of $\mathbb{A}$ do not have to be polymorphisms of relations from $\mathsf{Inv}(F)$, congruences and unary polynomials are in a special relationship. More precisely, an equivalence relation over $\mathbb{A}$ is a congruence if and only if it is preserved by all the unary polynomials of $\mathbb{A}$.

Let $\mathbb{A}$ be an algebra. For $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ we write $\alpha \prec \beta$ if $\alpha < \beta$ and $\alpha \le \gamma \le \beta$ in $\mathsf{Con}(\mathbb{A})$ implies $\gamma = \alpha$ or $\gamma = \beta$; if this is the case we call $(\alpha, \beta)$ a *prime interval* in $\mathsf{Con}(\mathbb{A})$. Let $\alpha \prec \beta$ and $\gamma \prec \delta$ be prime intervals in $\mathsf{Con}(\mathbb{A})$. We say that $\alpha \prec \beta$ can be *separated* from $\gamma \prec \delta$ if there is a unary polynomial $f$ of $\mathbb{A}$ such that $f(\beta) \not\subseteq \alpha$, but $f(\delta) \subseteq \gamma$. The polynomial $f$ in this case is said to *separate* $\alpha \prec \beta$ from $\gamma \prec \delta$.

In a similar way separation can be defined for prime intervals in different coordinate positions of a relation. Let $R$ be a subdirect product of $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$. Then $R$ is also an algebra and its polynomials can be defined in the same way. Let $i, j \in [n]$ and let $\alpha \prec \beta, \gamma \prec \delta$ be prime intervals in $\mathsf{Con}(\mathbb{A}_i)$ and $\mathsf{Con}(\mathbb{A}_j)$, respectively. Interval $\alpha \prec \beta$ can be separated from $\gamma \prec \delta$ if there is a unary polynomial $f$ of $R$ such that $f(\beta) \not\subseteq \alpha$ but $f(\delta) \subseteq \gamma$ (note that the actions of $f$ on $\mathbb{A}_i, \mathbb{A}_j$ are polynomials of those algebras).

The binary relation 'cannot be separated' on the set of prime intervals of an algebra or factors of a relation is easily seen to be reflexive and transitive. Under certain mild conditions it can also be shown to be symmetric in a certain sense (Lemma 45), and so for the purpose of our algorithm it can be treated as an equivalence relation.

**Quasi-Centralizers.** The second ingredient introduced here is the notion of quasi-centralizer of a prime interval of congruences. It is similar to the centralizer as it is defined in commutator theory, albeit the exact relationship between the two concepts is not quite clear, and so we name differently for safety.

For an algebra $\mathbb{A}$, a term operation $f(x, y_1, \dots, y_k)$, and $\mathbf{a} \in \mathbb{A}^k$, let $f^{\mathbf{a}}(x) =$

$f(x, \mathbf{a})$. Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\alpha \leq \beta$, and let $\zeta(\alpha, \beta) \subseteq \mathbb{A}^2$ denote the following binary relation: $(a, b) \in \zeta(\alpha, \beta)$ if an only if, for any term operation $f(x, y_1, \ldots, y_k)$, any $i \in [k]$, and any $\mathbf{a}, \mathbf{b} \in \mathbb{A}^k$ such that $\mathbf{a}[i] = a$, $\mathbf{b}[i] = b$, and $\mathbf{a}[j] = \mathbf{b}[j]$ for $j \neq i$, it holds $f^{\mathbf{a}}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{b}}(\beta) \subseteq \alpha$. The relation $\zeta(\alpha, \beta)$ is always a congruence of $\mathbb{A}$ (Lemma 50) and its effect on the structure of algebra $\mathbb{A}$ is illustrated by the following statement.

**Lemma 8** *Let $\zeta(\alpha, \beta) = \underline{1}_{\mathbb{A}}$, $a, b, c \in \mathbb{A}$ and $(b, c) \in \beta$. Then $(ab, ac) \in \alpha$.*

Fig. 1(a),(b) shows the effect of large quasi-centralizers on the structure of algebra $\mathbb{A}$. Dots there represent $\alpha$-blocks (assume $\alpha$ is the equality relation), ovals represent $\beta$-blocks, let they be $B$ and $C$, and such that there is at least one semilattice edge between $B$ and $C$. If $\zeta(\alpha, \beta)$ is the full relation, Lemmas 7 and 8 imply that for any $a \in B$ and any $b, c \in C$ we have $ab = ac$, and so $ab$ is the only element of $C$ such that $\{a, ab\}$ is a semilattice edge. In other words, we have a mapping that can also be shown injective from $B$ to $C$. We will use this mapping to lift any solution with a value from $B$ to a solution with a value from $C$.

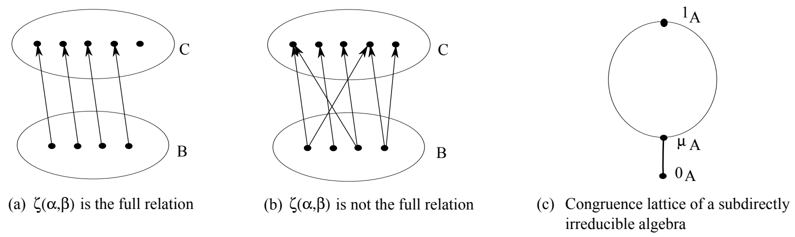


Figure 1:

## 3.2 The algorithm

We have seen in the previous section that big centralizers impose strong restrictions on the structure of an algebra. We start this section showing that small centralizers restrict the structure of CSPs.

**Decomposition of CSPs.** Let $R$ be a binary relation, a subdirect product of $\mathbb{A} \times \mathbb{B}$, and $\alpha \in \mathsf{Con}(\mathbb{A})$, $\gamma \in \mathsf{Con}(\mathbb{B})$. Relation $R$ is said to be $\alpha\gamma$*-aligned* if, for any $(a, c), (b, d) \in R$, $(a, b) \in \alpha$ if and only if $(c, d) \in \gamma$. This means that if $A_1, \ldots, A_k$ are the $\alpha$-blocks of $\mathbb{A}$, then there are also $k$ $\gamma$-blocks of $\mathbb{B}$ and they can be labeled $B_1, \ldots, B_k$ in such a way that

$$R = (R \cap (A_1 \times B_1)) \cup \cdots \cup (R \cap (A_k \times B_k)).$$

**Lemma 9** *Let* $R, \mathbb{A}, \mathbb{B}$ *be as above and* $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$*,* $\gamma, \delta \in \mathsf{Con}(\mathbb{B})$*, with* $\alpha \prec \beta$*,* $\gamma \prec \delta$*. If* $(\alpha, \beta)$ *and* $(\gamma, \delta)$ *cannot be separated, then* $R$ *is* $\zeta(\alpha, \beta)\zeta(\gamma, \delta)$*-aligned.*

Lemma 9 provides a way to decompose CSP instances. Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance from CSP($\mathbb{A}$), in particular, $\mathcal{C}$ contains a constraint $C^{\{v,w\}} = \langle (v, w), R^{\{v,w\}} \rangle$ for every $v, w \in V$, and these relations form a (2,3)-strategy for $\mathcal{P}$. Due to (2,3)-minimality the domain of variables from $V$ do not have to be $\mathbb{A}$ itself, but can be subalgebras of $\mathbb{A}$. Recall that $\mathbb{A}_v$ denotes the domain of $v \in V$. Also, let $W \subseteq V$ and congruences $\alpha_v, \beta_v \in \mathsf{Con}(\mathbb{A}_v)$ for $v \in W$ be such that $\alpha_v \prec \beta_v$, and for any $v, w \in W$ the intervals $(\alpha_v, \beta_v)$ and $(\alpha_w, \beta_w)$ cannot be separated in $R^{\{v,w\}}$.

Denoting $\zeta_v = \zeta(\alpha_v, \beta_v)$ we see that there is a one-to-one correspondence between $\zeta_v$ and $\zeta_w$ blocks of $\mathbb{A}_v$ and $\mathbb{A}_w$. Moreover, by (2,3)-minimality these correspondences are consistent, that is, if $u, v, w \in W$ and $B_u, B_v, B_w$ are $\zeta_u$-, $\zeta_v$- and $\zeta_w$-blocks, respectively, such that $R^{\{u,v\}} \cap (B_u \times B_v) \neq \varnothing$ and $R^{\{v,w\}} \cap (B_v \times B_w) \neq \varnothing$, then $R^{\{u,w\}} \cap (B_u \times B_w) \neq \varnothing$. This means that $\mathcal{P}_W$ can be split into several instances, whose domains are $\zeta_v$-blocks.

**Lemma 10** *Let* $\mathcal{P}, W, \alpha_v, \beta_v$ *be as above. Then* $\mathcal{P}_W$ *can be decomposed into a collection of instances* $\mathcal{P}_1, \dots, \mathcal{P}_k$*,* $\mathcal{P}_i = (W, \mathcal{C}_i)$ *such that every solution of* $\mathcal{P}_W$ *is a solution of one of the* $\mathcal{P}_i$ *and for every* $v \in V$ *its domain in* $\mathcal{P}_i$ *is a* $\zeta_v$*-block.*

**Irreducibility.** In order to formulate the algorithm properly we need one more transformation of algebras. An algebra $\mathbb{A}$ is said to be *subdirectly irreducible* if the intersection of all its nontrivial (different from the equality relation) congruences is nontrivial. This smallest nontrivial congruence $\mu_{\mathbb{A}}$ is called the *monolith* of $\mathbb{A}$, see Fig. 1(c). It is a folklore observation that any CSP instance can be transformed in polynomial time to an instance, in which the domain of every variable is a subdirectly irreducible algebra. We will assume this property of all the instances we consider.

**Block-minimality.** Lemma 10 allows one to establish a much stronger version of local consistency, block-minimality; in fact, it is not local anymore. The definitions below are designed in such a way that to allow for an efficient procedure to establish block-minimality. This is achieved either by allowing for decomposing a subinstance into instances over smaller domains as in Lemma 10, or by replacing large domains with their quotient algebras.

Let $\alpha_v$ be a congruence of $\mathbb{A}_v$ for each $v \in V$. By $\mathcal{P}/_{\overline{\alpha}}$ we denote the instance $(V, \mathcal{C}^{\overline{\alpha}})$ constructed as follows: the domain of $v \in V$ is $\mathbb{A}_v/_{\alpha_v}$; for every constraint $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, the set $\mathcal{C}^{\overline{\alpha}}$ includes the constraint $\langle \mathbf{s}, R/_{\overline{\alpha}_\mathbf{s}}\rangle$, where $\mathbf{s} = (v_1, \dots, v_k)$ and $R/_{\overline{\alpha}} = \{(\mathbf{a}[v_1]^{\alpha_{v_1}}, \dots, \mathbf{a}[v_k]^{\alpha_{v_k}}) \mid \mathbf{a} \in R\}$.

Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance and $\{R^X \mid X \subseteq V, |X| = 2\}$ is its (2,3)-strategy. Let $\overline{\beta} = (\beta_v)_{v\in V}$, $\beta_v \in \mathsf{Con}(\mathbb{A}_v)$, $v \in V$, be a collection of congruences. Let $\mathcal{W}^{\mathcal{P}}(\overline{\beta})$ denote the set of triples $(v, \alpha, \beta)$ such that $v \in V$, $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$, and $\alpha \prec \beta \le \beta_v$. Also, $\mathcal{W}^{\mathcal{P}}$ denotes $\mathcal{W}^{\mathcal{P}}(\overline{\beta})$ when $\beta_v$ is the full relation for all $v \in V$. We will omit the superscript $\mathcal{P}$ whenever it is clear from the context.

For every $(v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})$, let $W_{v,\alpha\beta,\overline{\beta}}$ denote the set of variables $w \in V$ such that $(\alpha, \beta)$ and $(\gamma, \delta)$ cannot be separated in $R^{vw}$ for some $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_w)$ with $(w, \gamma, \delta) \in \mathcal{W}^{\mathcal{P}}(\overline{\beta})$. Let $\mathcal{W}'(\overline{\beta})$ (and respectively $\mathcal{W}'$) denote the set of triples $(v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})$ (respectively, from $\mathcal{W}$), for which $\zeta(\alpha, \beta)$ is the full relation.

We say that algebra $\mathbb{A}_v$ is *semilattice free* if it does not contain semilattice edges. Let $\mathsf{size}(\mathcal{P})$ denote the maximal size of domains of $\mathcal{P}$ that are not semilattice free and $\mathsf{MAX}(\mathcal{P})$ be the set of variables $v \in V$ such that $|\mathbb{A}_v| = \mathsf{size}(\mathcal{P})$ and $\mathbb{A}_v$ is not semilattice free. For an instance $\mathcal{P}$ we say that an instance $\mathcal{P}'$ is *strictly smaller* than instance $\mathcal{P}$ if $\mathsf{size}(\mathcal{P}') < \mathsf{size}(\mathcal{P})$. For $Y \subseteq V$ let $\mu_v^Y = \mu_v$ if $v \in Y$ and $\mu_v^Y = \underline{0}_v$ otherwise.

Instance $\mathcal{P}$ is said to be *block-minimal* if for every $(v, \alpha, \beta) \in \mathcal{W}$ (here $\beta_v = \underline{1}_v$, $v \in V$) if the following conditions hold:

(BM1) for every $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ if $(v, \alpha, \beta) \notin \mathcal{W}'$, and the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}/_{\overline{\mu}^Y}$ otherwise, where $Y = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$, is minimal;

(BM2) if $(v, \alpha, \beta) \in \mathcal{W}'$, then for every $(w, \gamma, \delta) \in \mathcal{W} - \mathcal{W}'$ the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}/_{\overline{\mu}^Y}$, where $Y = \mathsf{MAX}(\mathcal{P}) - (W_{v,\alpha\beta,\overline{\beta}} \cap W_{w,\gamma\delta,\overline{\beta}})$ is minimal.

Observe that $W_{v,\alpha\beta,\overline{\beta}}$ can be large, even equal to $V$. However if $(v, \alpha, \beta) \notin \mathcal{W}'$ the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ splits into a union of disjoint problems over smaller domains, and so its minimality can be established by recursing to strictly smaller problems. On the other hand, if $(v, \alpha, \beta) \in \mathcal{W}'$ then $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ may not be decomposable. Since we need an efficient procedure of establishing block-minimality, this explains the complications introduced in (BM1),(BM2). In the first case $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}/_{\overline{\mu}^Y}$ can be solved for each tuple $\mathbf{a} \in R$. Taking the quotient algebras of the domains guarantees that we recurse to strictly smaller instances. In the second case $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}} \cap W_{w,\gamma\delta,\overline{\beta}}}/_{\overline{\mu}^Y}$ is decomposable, and we branch on those strictly smaller subproblems.

**Lemma 11** *Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance. Then $\mathcal{P}$ can be transformed to an equivalent block-minimal instance $\mathcal{P}'$ by solving a quadratic number of strictly smaller CSPs.*

**The algorithm** In the algorithm we distinguish three cases depending on semilattice edges and quasi-centralizers of the domains of variables. In each case we employ different methods of solving or reducing the instance to a strictly smaller one.

Let $\mathcal{P} = (V, \mathcal{C})$ be a subdirectly irreducible (2,3)-minimal instance. Let $\mathsf{Center}(\mathcal{P})$ denote the set of variables $v \in V$ such that $\zeta(\underline{0}_v, \mu_v) = \underline{1}_v$. Let $\mu^*_v = \mu_v$ if $v \in \mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P})$ and $\mu^*_v = \underline{0}_v$ otherwise.

*Semilattice free domains.* If no domain of $\mathcal{P}$ contains a semilattice edge then by Proposition 6 $\mathcal{P}$ can be solved in polynomial time, using the few subalgebras algorithm, as shown in [35, 20].

*Trivial centralizers.* If $\mu^*_v = \underline{0}_v$ for all $v \in V$, block-minimality guarantees the existence of a solution, and we can use Lemma 11 to solve the instance.

**Theorem 12** *If $\mathcal{P}$ is subdirectly irreducible, (2,3)-minimal, block-minimal, and $\mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P}) = \varnothing$, then $\mathcal{P}$ has a solution.*

*Nontrivial centralizers.* Suppose that $\mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P}) \neq \varnothing$. In this case we consider the problem $\mathcal{P}/_{\overline{\mu}^*}$. For this problem either $\mathsf{size}(\mathcal{P}/_{\overline{\mu}^*}) < \mathsf{size}(\mathcal{P})$, or $\mathsf{MAX}(\mathcal{P}/_{\overline{\mu}^*}) \cap \mathsf{Center}(\mathcal{P}/_{\overline{\mu}^*}) = \varnothing$; in either case it can be solved by the previous case or by recursion to a strictly smaller problem. We find a solution $\varphi$ of $\mathcal{P}/_{\overline{\mu}^*}$ satisfying the following conditions. For every $v \in \mathsf{Center}(\mathcal{P})$ there is $a \in \mathbb{A}_v$ such that $\{a, \varphi(v)\}$ is a semilattice edge if $\mu^*_v = \underline{0}_v$, or, if $\mu^*_v = \mu_v$, there is $b \in \varphi(v)$ such that $\{a, b\}$ is a semilattice edge. Then we apply the transformation of $\mathcal{P}$ suggested by Maroti in [43]. By $\mathcal{P} \cdot \varphi$ we denote the instance $(V, \mathcal{C}_\varphi)$ given by the rule: for every $\mathbf{C} = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ the set $\mathcal{C}_\varphi$ contains a constraint $\langle \mathbf{s}, R \cdot \varphi \rangle$. To construct $R \cdot \varphi$ choose a tuple $\mathbf{b} \in R$ such that $\mathbf{b}[v]^{\mu^*_v} = \varphi(v)$ for all $v \in \mathbf{s}$; this is possible because $\varphi$ is a solution of $\mathcal{P}/_{\overline{\mu}^*}$. Then set $R \cdot \varphi = \{\mathbf{a} \cdot \mathbf{b} \mid \mathbf{a} \in R\}$. By the results of [43] and Lemma 8 the instance $\mathcal{P} \cdot \varphi$ has a solution if and only if $\mathcal{P}$ does and $\mathsf{size}(\mathcal{P} \cdot \varphi) < \mathsf{size}(\mathcal{P})$.

**Theorem 13** *If $\mathcal{P}/_{\overline{\mu}^*}$ is 1-minimal, then $\mathcal{P}$ can be reduced in polynomial time to a strictly smaller instance.*

**Comments on the algorithm** Using Lemma 11 and Theorems 12,13 it is not difficult to see that the algorithm runs in polynomial time. Indeed, every time it makes a recursive call it calls on a problem whose non-semilattice free domains have strictly smaller size, and therefore the depth of recursion is bounded by $|\mathbb{A}|$ if we are dealing with $\mathrm{CSP}(\mathbb{A})$.

**Algorithm 1** Procedure SolveCSP

**Require:** A CSP instance $\mathcal{P} = (V, \mathcal{C})$ from $\mathrm{CSP}(\mathbb{A})$

**Ensure:** A solution of $\mathcal{P}$ if one exists, 'NO' otherwise

1: **if** all the domains are semilattice free **then**

2: Solve $\mathcal{P}$ using the few subpowers algorithm and RETURN the answer

3: **end if**

4: Transform $\mathcal{P}$ to a subdirectly irreducible, block-minimal and (2,3)-minimal instance

5: $\mu_v^* = \mu_v$ for $v \in \mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P})$ and $\mu_v^* = \underline{0}_v$ otherwise

6: $\mathcal{P}^* = \mathcal{P}/_{\overline{\mu}^*}$

7: */* Check the 1-minimality of $\mathcal{P}^*$*

8: **for** every $v \in V$ and $a \in \mathbb{A}_v/_{\mu_v^*}$ **do**

9: $\mathcal{P}' = \mathcal{P}^*_{(v,a)}$ */* Add a constraint $\langle (v), \{a\} \rangle$ fixing the value of $v$ to $a$*

10: Transform $\mathcal{P}'$ to a subdirectly irreducible, (2,3)-minimal instance $\mathcal{P}''$

11: If $\mathsf{size}(\mathcal{P}'') < \mathsf{size}(\mathcal{P})$ call SolveCSP on $\mathcal{P}''$ and flag $a$ if $\mathcal{P}''$ has no solution

12: Establish block-minimality of $\mathcal{P}''$; if the problem changes, return to Step 10

13: If the resulting instance is empty, flag the element $a$

14: **end for**

15: If there are flagged values, tighten the instance by removing the flagged elements and start over

16: Use Theorem 13 to reduce $\mathcal{P}$ to an instance $\mathcal{P}'$ with $\mathsf{size}(\mathcal{P}') < \mathsf{size}(\mathcal{P})$

17: Call SolveCSP on $\mathcal{P}'$ and RETURN the answer

In order to prove Theorem 12 we introduce $\overline{\beta}$-strategies that are somewhat similar to (2,3)-strategies in the sense that they are also collections of relations defined through some sort of minimality condition and are consistent. We show how such constructions can be used to prove Theorem 12.

Let $\mathcal{P} = (V, \mathcal{C})$ be a subdirectly irreducible, (2,3)-minimal and block-minimal instance. Let $\mathbb{A}_v$ denote the domain of $v \in V$. Also, let $\beta_v \in \mathsf{Con}(\mathbb{A}_v)$ and $B_v$ a $\beta_v$-block. Let $\mathcal{R}$ be a collection of relations $R_{C,v,\alpha\beta}$ for every $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$, $(v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})$ and such that $S(C, v, \alpha\beta) = \mathbf{s} \cap W_{v,\alpha\beta,\overline{\beta}}$ is its set of coordinate positions. Similar to (2,3)-minimality a tuple $\mathbf{a} \in \prod_{w \in X} \mathbb{A}_x$ for some $X \subseteq V$, is called *$\mathcal{R}$-compatible* if for any $C \in \mathcal{C}$ and $(v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})$ it holds $\mathrm{pr}_T \mathbf{a} \in \mathrm{pr}_T R_{C,v,\alpha\beta}$, where $T = X \cap S(C, v, \alpha\beta)$. Collection $\mathcal{R}$ is said to be a *$\overline{\beta}$-strategy* with respect to $(B_V)_{v \in V}$ if the following conditions hold for every $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ and $(v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})$ (let $W = W_{v,\alpha\beta,\overline{\beta}}$)[2].

(S1) the relations $R^{X,\mathcal{R}}$, where $R^{X,\mathcal{R}}$ consists of $\mathcal{R}$-compatible tuples from $R^X$ for $X \subseteq V$, $|X| \leq 2$, form a nonempty $(2,3)$-strategy for $\mathcal{P}^{\mathcal{R}}$;

(S2) for every $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$ (let $U = W_{w,\gamma\delta}$) and every $\mathbf{a} \in \mathrm{pr}_{\mathbf{s} \cap W \cap U} R_{C,v,\alpha\beta}$ it holds: if $(w, \gamma, \delta) \notin \mathcal{W}'$ then $\mathbf{a}$ extends to an $\mathcal{R}$-compatible solution $\varphi$ of $\mathcal{P}_U$; otherwise if $(v, \alpha, \beta) \notin \mathcal{W}'$ then $\mathbf{a}$ extends to an $\mathcal{R}$-compatible solution of $\mathcal{P}_U/_{\overline{\mu}^{Y_1}}$ with $Y_1 = \mathsf{MAX}(\mathcal{P}) - (W \cap U)$; and if $(v, \alpha, \beta) \in \mathcal{W}'$ then $\mathbf{a}$ extends to an $\mathcal{R}$-compatible solution of $\mathcal{P}_U/_{\overline{\mu}^{Y_2}}$, where $Y_2 = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$;

(S3) $R \cap \prod_{w \in \mathbf{s}} B_w \neq \varnothing$ and for any $I \subseteq \mathbf{s}$ any $\mathcal{R}$-compatible tuple $\mathbf{a} \in \mathrm{pr}_I R$ extends to an $\mathcal{R}$-compatible tuple $\mathbf{b} \in R$.

Let $\mathcal{P}$ be a block-minimal instance, $\beta_v = \underline{1}_v$ and $B_v = \mathbb{A}_v$ for $v \in V$. Then as it not hard to see the collection of relations $\mathcal{R} = \{R_{C,v,\alpha\beta} \mid (v, \alpha, \beta) \in \mathcal{W}(\overline{\beta}), C \in \mathcal{C}\}$ given by $R_{C,v,\alpha\beta} = \mathrm{pr}_{\mathbf{s} \cap W_{v,\alpha\beta,\overline{\beta}}} R$ for $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ is a $\overline{\beta}$-strategy with respect to $\overline{B}$. Also, by (S3) a $\overline{\gamma}$-strategy with $\gamma_v = \underline{0}_v$ gives a solution of $\mathcal{P}$. Our goal is therefore to show that a $\overline{\beta}$-strategy for any $\overline{\beta}$ can be 'reduced', that is, transformed to a $\overline{\beta}'$-strategy for some $\overline{\beta}' < \overline{\beta}$. Note that this reduction of strategies is where the condition $\mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P}) = \varnothing$ is used. Indeed, suppose that $\beta_v = \mu_v^*$. Then by conditions (S1)–(S3) we only have information about solutions to problems of the form $\mathcal{P}_W/_{\overline{\mu}^*}$ or something very close to that. Therefore this barrier cannot be penetrated.

[2]This is a 'light' version of the conditions that a $\overline{\beta}$-strategy has to satisfy. A complete and precise list of conditions can be found in Section 9.3 or [22].

# Part II
# Technicalities

We start this part with preliminaries, where apart from additional definitions and notation, we remind some of the results of [15, 19, 20] related to colored graphs of algebras and relational structures and also some of their properties. In Sections 4.4–4.6 we advance these results a little further. Then in Section 5 we introduce in a more detailed way the method of separating factors in congruence lattices using polynomial operations of the algebra. This method constitutes the basis for our algorithm. Some preliminary versions of this approach can be found in [12, 21]. In Section 6 we introduce again and study the quasi-centralizer operator on congruence lattices that is similar to the well studied centralizer operator, although the precise relationship between the two is not quite clear. In particular, it allows to split certain CSPs into smaller ones. In Sections 7 and 8 we prove two important technical results

In Section 9 we give a description of the algorithm, and prove its running time and partially soundness. In very broad strokes the algorithm works as follows. If none of the domains of $\mathcal{P}$ contains a semilattice edge in the sense of colored graphs of algebras, then $\mathcal{P}$ can be solved by the few subpowers algorithm [11, 35], as shown in [20]. Otherwise in most cases the problem can be solved by establishing *block-minimality* similar to that in [21]. The problematic case when block-minimality does not provide a solution, or rather when it cannot be established is roughly speaking when the domains of the instance have nontrivial centers in the sense of the commutator theory. In this case we show in Section 9.2 that a solution of a problem $\mathcal{P}'$ obtained from $\mathcal{P}$ by replacing some of its domains with quotient algebras modulo their centers allows one to reduce the number of semilattice edges in those domains, and we can recurse to an instance with smaller domains.

The key ingredient of our result is presented in Section 9.3. There for block-minimal instances we introduce strategies that are in certain aspects similar to strategies used to solve problems of bounded width, but allow us to approach general CSPs. Then in Section 10 we show, Theorem 67, that if for a CSP instance $\mathcal{P}$ satisfying the block-minimality conditions such a strategy exists, one can improve (tighten) the strategy to obtain a solution of the quotient problem $\mathcal{P}'$ needed to reduce semilattice edges. This theorem is the most difficult and technically involved part of the proof. Tightening of a strategy works by (effectively) reducing domains of the CSP to a class of a maximal congruence, and then repeating the process as long as possible. The main cases of tightening considered are: when the interval formed by the maximal congruence used and the full congruence is Abelian, and when it is non-Abelian, Sections 10.1 and 10.2, respectively. In the two cases we use quite different transformations of the strategy. In the Abelian case the argument is based on the rectangularity of relations understood in a general sense, while in the non-Abelian case the transformation is similar to that used for bounded width CSPs in [20].

# 4 Preliminaries

We expand upon many of the definitions and notation given in Part I. For the sake of convenience we also repeat some of the definitions given in Part I.

## 4.1 Universal algebra and CSP: notation and agreements

We assume familiarity with the basics of universal algebra and the algebraic approach to the CSP. For reference on universal algebra please use [28, 45]; for the algebraic approach see the recent survey [9] and earlier papers [5, 3, 25, 27, 26, 19].

By $[n]$ we denote the set $\{1, \dots, n\}$. For sets $A_1, \dots, A_n$ tuples from $A_1 \times \dots \times A_n$ are denoted in boldface, say, $\mathbf{a}$; the $i$th component of $\mathbf{a}$ is referred to as $\mathbf{a}[i]$. An $n$-ary relation $R$ over sets $A_1, \dots, A_n$ is any subset of $A_1 \times \dots \times A_n$. For $I = \{i_1, \dots, i_k\} \subseteq [n]$ by $\mathrm{pr}_I \mathbf{a}, \mathrm{pr}_I R$ we denote the *projections* $\mathrm{pr}_I \mathbf{a} = (\mathbf{a}[i_1], \dots, \mathbf{a}[i_k])$, $\mathrm{pr}_I R = \{\mathrm{pr}_I \mathbf{a} \mid \mathbf{a} \in R\}$ of tuple $\mathbf{a}$ and relation $R$. If $\mathrm{pr}_i R = A_i$ for each $i \in [n]$, relation $R$ is said to be a *subdirect product* of $A_1 \times \dots \times A_n$. It will be convenient to use $\overline{A}$ for $A_1 \times \dots \times A_n$, or for $\prod_{v \in V} A_v$ if the sets $V$ and $A_v$ are clear from the context. For $I \subseteq [n]$ or $I \subseteq V$ we will use $\overline{A}_I$, for $\prod_{i \in I} A_i$, or if $I$ is clear from the context just $\overline{A}$.

Algebras will be denoted by $\mathbb{A}, \mathbb{B}$ etc.; we often do not distinguish between subuniverses and subalgebras. For $B \subseteq \mathbb{A}$ the subalgebra generated by $B$ is denoted $\mathsf{Sg}(B)$. For $C \subseteq \mathbb{A}^2$ the congruence generated by $C$ is denoted $\mathsf{Cg}(C)$. The equality relation and the full congruence of algebra $\mathbb{A}$ are denoted $\underline{0}_{\mathbb{A}}$ and $\underline{1}_{\mathbb{A}}$, respectively. Often when we need to use one of these trivial congruences of an algebra indexed in some way, say, $\mathbb{A}_i$, we write $\underline{0}_i, \underline{1}_i$ for $\underline{0}_{\mathbb{A}_i}, \underline{1}_{\mathbb{A}_i}$. The set of all polynomials (unary polynomials) of $\mathbb{A}$ is denoted by $\mathsf{Pol}(\mathbb{A})$ and $\mathsf{Pol}_1(\mathbb{A})$, respectively. We frequently use operations on subalgebras of direct products of algebras, say, $R \subseteq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$. If $f$ is such an operation (say, $k$-ary) then we denote its component-wise action also by $f$, e.g. $f(a_1, \dots, a_k)$ for $a_1, \dots, a_k \in \mathbb{A}_i$. In the same way we denote the action of $f$ on projections of $R$, e.g. $f(\mathbf{a}_1, \dots, \mathbf{a}_k)$ for $I \subseteq [n]$ and $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathrm{pr}_I R$. What we mean will always be clear from the context. We use similar agreements for collections of congruences. If $\alpha_i \in \mathsf{Con}(\mathbb{A}_i)$ then $\overline{\alpha}$ denotes the congruence $\alpha_1 \times \dots \times \alpha_n$ of $R$. If $I \subseteq [n]$ we use $\overline{\alpha}_I$ to denote $\prod_{i \in I} \alpha_i$. If it does not lead to a confusion we write $\overline{\alpha}$ for $\overline{\alpha}_I$. Sometimes $\alpha_i$

are specified for $i$ from a certain set $I \subseteq [n]$, then by $\overline{\alpha}$ we mean the congruence $\prod_{i\in[n]} \alpha'_i$ where $\alpha'_i = \alpha_i$ if $i \in I$ and $\alpha'_i$ is the equality relation otherwise. For example, if $\alpha \in \mathsf{Con}(\mathbb{A}_1)$ then $R/_\alpha$ means the factor of $R$ modulo $\alpha \times \underline{0}_2 \times \cdots \times \underline{0}_n$. For $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ we write $\alpha \prec \beta$ if $\alpha < \beta$ and $\alpha \le \gamma \le \beta$ in $\mathsf{Con}(\mathbb{A})$ implies $\gamma = \alpha$ or $\gamma = \beta$. In this paper all algebras are finite, idempotent and omit type **1**.

The (*nonuniform*) *Constraint Satisfaction Problem* (*CSP*) associated with a relational structure $\mathbf{B}$ is the problem $\mathrm{CSP}(\mathbf{B})$, in which, given a structure $\mathbf{A}$ of the same signature as $\mathbf{B}$, the goal is to decide whether or not there is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. For a class of similar algebras $\mathcal{A} = \{\mathbb{A}_i \mid i \in I\}$ for some set $I$ an *instance* of $\mathrm{CSP}(\mathcal{A})$ is a triple $(V, \delta, \mathcal{C})$, where $V$ is a set of variables; $\delta : V \to \mathcal{A}$ is a *type function* that associates every variable with a *domain* in $\mathcal{A}$. Finally, $\mathcal{C}$ is a set of *constraints*, i.e. pairs $\langle \mathbf{s}, R\rangle$, where $\mathbf{s} = (v_1, \dots, v_k)$ is a tuple of variables from $V$, the *constraint scope*, and $R \in \mathsf{Inv}(\mathcal{A})$, a subset of $\mathbb{A}_{\delta(v_1)} \times \cdots \times \mathbb{A}_{\delta(v_k)}$, the *constraint relation*. The goal is to find a *solution*, that is a mapping $\varphi : V \to \bigcup \mathcal{A}$ such that $\varphi(v) \in \mathbb{A}_{\delta(v)}$ and for every constraint $\langle \mathbf{s}, R\rangle$, $\varphi(\mathbf{s}) \in R$. It is easy to see that if $\mathcal{A}$ is a class containing just one algebra $\mathbb{A}$, then $\mathrm{CSP}(\mathcal{A})$ can be viewed as the union of $\mathrm{CSP}(\mathbf{A})$ for all relational structures $\mathbf{A}$ invariant under the operations of $\mathbb{A}$. To simplify the notation we always write $\mathbb{A}_v$ rather than $\mathbb{A}_{\delta(v)}$, because the mapping $\delta$ is always clear from the context. This also allows us to simplify the notation for instances to $\mathcal{P} = (V, \mathcal{C})$. To allow for transformations of CSP described below we assume that $\mathcal{A}$ is closed under taking subalgebras and quotient algebras.

The set of solutions of a CSP instance $\mathcal{P} = (V, \mathcal{C})$ will be denoted by $\mathcal{S}_\mathcal{P}$, or just $\mathcal{S}$ if $\mathcal{P}$ is clear from the context. For $W \subseteq V$ by $\mathcal{P}_W$ we denote the *restriction* of $\mathcal{P}$ onto $W$, that is, the instance $(W, \mathcal{C}_W)$, where for each $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, the set $\mathcal{C}_W$ includes the constraint $C_W = \langle \mathbf{s} \cap W, \mathrm{pr}_{\mathbf{s}\cap W} R\rangle$. The set of solutions of $\mathcal{P}_W$ will be denoted by $\mathcal{S}_W$. For $v \in V$ and a subalgebra $\mathbb{B}$ of $\mathbb{A}_v$ by $\mathcal{P}_{(v,\mathbb{B})}$ we denote the instance $\mathcal{P}$ with an extra constraint $\langle \{v\}, \mathbb{B}\rangle$; note that this is essentially equivalent to reducing the domain of $v$, and this is how we usually consider this construction. For $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ let $R'$ be a subalgebra of $R$ and $C' = \langle \mathbf{s}, R'\rangle$. The instance obtained from $\mathcal{P}$ replacing $C$ with $C'$ is denoted by $\mathcal{P}_{C\to C'}$. The transformation of $\mathcal{P}$ by reducing the domain of a variable $v \in V$ or reducing a constraint $C \in \mathcal{C}$, that is, transforming $\mathcal{P}$ into $\mathcal{P}_{(v,\mathbb{B})}$ or $\mathcal{P}_{C\to C'}$ in such a way that the new instance has a solution if and only if $\mathcal{P}$ does, will be called *tightening* of $\mathcal{P}$. Let $\alpha_v$ be a congruence of $\mathbb{A}_v$ for each $v \in V$. By $\mathcal{P}/_{\overline{\alpha}}$ we denote the instance $(V, \mathcal{C}^{\overline{\alpha}})$ constructed as follows: the domain of $v \in V$ is $\mathbb{A}_v/_{\alpha_v}$; for every constraint $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, the set $\mathcal{C}^{\overline{\alpha}}$ includes the constraint $\langle \mathbf{s}, R/_{\overline{\alpha}_\mathbf{s}}\rangle$. Note that if $\mathcal{A}$ is closed under taking subalgebras and quotient algebras, then applying a transformation of one of these kinds to an instance from $\mathrm{CSP}(\mathcal{A})$ results again in an instance from $\mathrm{CSP}(\mathcal{A})$.

Instance $\mathcal{P}$ is said to be *minimal* (or *globally minimal*) if for every $C = \langle \mathbf{s}, R\rangle \in$

$\mathcal{C}$ and every $\mathbf{a} \in R$ there is a solution $\varphi \in \mathcal{S}$ such that $\varphi(\mathbf{s}) = \mathbf{a}$. Instance $\mathcal{P}$ is said to be *1-minimal* if for every $v \in V$ and every constraint $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ such that $v \in \mathbf{s}$, $\mathrm{pr}_v R = \mathcal{S}_v$. Instance $\mathcal{P}$ is said to be *(2,3)-consistent* if it has a *(2,3)-strategy*, that is, a collection of relations $R^X$, $X \subseteq V$, $|X| = 2$ satisfying the following conditions:

– for every $X \subseteq V$ with $|X| \leq 2$ and any constraint $C = \langle \mathbf{s}, R \rangle$, $\mathrm{pr}_{\mathbf{s} \cap X} R^X \subseteq \mathrm{pr}_{\mathbf{s} \cap X} R$;

– for every $X = \{u, v\} \subseteq V$, any $w \in V - X$ and any $(a, b) \in R^X$, there is $c \in \mathbb{A}_w$ such that $(a, c) \in R^{\{u,w\}}$ and $(b, c) \in R^{\{v,w\}}$.

We will always assume that a (2,3)-consistent instance has a constraint $C^X = \langle X, \mathcal{S}_X \rangle$ for every $X \subseteq V$, $|X| = 2$. Then clearly $R^X \subseteq \mathcal{S}_X$. Let the collection of relations $R^X$ be denoted by $\mathcal{R}$. A tuple $\mathbf{a}$ whose entries are indexed with elements of $W \subseteq V$ such that $\mathrm{pr}_X \mathbf{a} \in R^X$ for any $X \subseteq W$, $|X| = 2$, will be called *$\mathcal{R}$-compatible*. If a (2,3)-consistent instance $\mathcal{P}$ with a (2,3)-strategy $\mathcal{R}$ satisfies the additional condition

– for every constraint $C = \langle \mathbf{s}, R \rangle$ of $\mathcal{P}$ every tuple $\mathbf{a} \in R$ is $\mathcal{R}$-compatible,

it is called *(2,3)-minimal*. Any instance can be transformed to a 1-minimal, (2,3)-consistent, or (2,3)-minimal instance in polynomial time using the standard constraint propagation algorithms (see, e.g. [30] or [26]). These algorithms tighten the instance.

## 4.2 Minimal sets and polynomials

We will use the following basic facts from the tame congruence theory [34], often without further notice.

Let $\mathbb{A}$ be a finite algebra and $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ with $\alpha \prec \beta$. An $(\alpha, \beta)$*-minimal set* is a set minimal with respect to inclusion among the sets of the form $f(\mathbb{A})$, where $f \in \mathsf{Pol}_1(\mathbb{A})$ is such that $f(\beta) \not\subseteq \alpha$. Sets $B, C$ are said to be *polynomially isomorphic* in $\mathbb{A}$ if there are $f, g \in \mathsf{Pol}_1(\mathbb{A})$ such that $f(B) = C$, $g(C) = B$, and $f \circ g, g \circ f$ are identity mappings on $C$ and $B$, respectively.

**Lemma 14 (Theorem 2.8, [34])** *Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\alpha \prec \beta$. Then the following hold.*

*(1) Any $(\alpha, \beta)$-minimal sets $U, V$ are polynomially isomorphic.*

*(2) For any $(\alpha, \beta)$-minimal set $U$ and any $f \in \mathsf{Pol}_1(\mathbb{A})$, if $f(\beta|_U) \not\subseteq \alpha$ then $f(U)$ is an $(\alpha, \beta)$-minimal set, $U$ and $f(U)$ are polynomially isomorphic, and $f$ witnesses this fact.*

*(3) For any $(\alpha, \beta)$-minimal set $U$ there is $f \in \mathsf{Pol}_1(\mathbb{A})$ such that $f(\mathbb{A}) = U$, $f(\beta) \not\subseteq \alpha$, and $f$ is idempotent, in particular, $f$ is the identity mapping on $U$.*

*(4) For any $(a,b) \in \beta - \alpha$ and an $(\alpha,\beta)$-minimal set $U$ there is $f \in \mathsf{Pol}_1(\mathbb{A})$ such that $f(\mathbb{A}) = U$ and $(f(a), f(b)) \in \beta|_U - \alpha|_U$. Moreover, $f$ can be chosen to satisfy the conditions of item (3).*

*(5) For any $(\alpha,\beta)$-minimal set $U$, $\beta$ is the transitive closure of*

$$\alpha \cup \{(f(a), f(b)) \mid (a,b) \in \beta|_U, f \in \mathsf{Pol}_1(\mathbb{A})\}.$$

*In fact, as $\alpha \prec \beta$ this claim can be strengthen to the following. For any $(a,b) \in \beta - \alpha$, $\beta$ is the transitive closure of*

$$\alpha \cup \{(f(a), f(b)) \mid f \in \mathsf{Pol}_1(\mathbb{A})\}.$$

*(6) For any $f \in \mathsf{Pol}_1(\mathbb{A})$ such that $f(\beta) \not\subseteq \alpha$ there is an $(\alpha,\beta)$-minimal set $U$ such that $f$ witnesses that $U$ and $f(U)$ are polynomially isomorphic.*

For an $(\alpha,\beta)$-minimal set $U$ and a $\beta$-block $B$ such that $\beta|_{U\cap B} \neq \alpha|_{U\cap B}$, the set $U \cap B$ is said to be an $(\alpha,\beta)$-*trace*. A 2-element set $\{a,b\} \subseteq U \cap B$ such that $(a,b) \in \beta - \alpha$, is called an $(\alpha,\beta)$-*subtrace*. The union $Q$ of the traces from $U$ is called the *body* of $U$, and $U - Q$ is called the *tail* of $U$. Depending on the structure of its minimal sets the interval $(\alpha,\beta)$ can be of one of the five types, **1**–**5**. Since we assume the tractability conditions of the Dichotomy Conjecture, type **1** does not occur in algebras we deal with.

**Lemma 15 (Section 4 of [34])** *Let $\alpha,\beta \in \mathsf{Con}(\mathbb{A})$ and $\alpha \prec \beta$. Then the following hold.*

*(1) If $\mathsf{typ}(\alpha,\beta) = \mathbf{2}$ then every $(\alpha,\beta)$-trace is polynomially equivalent to a 1-dimensional vector space.*

*(2) If $\mathsf{typ}(\alpha,\beta) \in \{\mathbf{3},\mathbf{4},\mathbf{5}\}$ then every $(\alpha,\beta)$-minimal set $U$ contains exactly one trace $T$, and if $\mathsf{typ}(\alpha,\beta) \in \{\mathbf{3},\mathbf{4}\}$, $T$ contains only 2 elements. Also, $T/_\alpha$ is polynomially equivalent to a Boolean algebra, 2-element lattice, or 2-element semilattice, respectively.*

Intervals $(\alpha,\beta),(\gamma,\delta)$, $\alpha,\beta,\gamma,\delta \in \mathsf{Con}(\mathbb{A})$ and $\alpha \prec \beta, \gamma \prec \delta$ are said to be *perspective* if $\beta = \alpha \vee \delta, \gamma = \alpha \wedge \delta$, or $\delta = \beta \vee \gamma, \alpha = \beta \wedge \gamma$.

**Lemma 16 (Lemma 6.2, [34])** *Let $\alpha,\beta,\gamma,\delta \in \mathsf{Con}(\mathbb{A})$ be such that $\alpha \prec \beta, \gamma \prec \delta$ and intervals $(\alpha,\beta),(\gamma,\delta)$ are perspective. Then $\mathsf{typ}(\alpha,\beta) = \mathsf{typ}(\gamma,\delta)$ and a set $U$ is $(\alpha,\beta)$-minimal if and only if it is $(\gamma,\delta)$-minimal.*

We will also use polynomials that behave on a minimal set in a particular way.

**Lemma 17 (Lemmas 4.16, 4.17, [34])** *Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\alpha \prec \beta$, and $\mathsf{typ}(\alpha, \beta) \in \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$. Let $U$ be an $(\alpha, \beta)$-minimal set and $T$ its only trace. Then there is element $1 \in T$ and a binary polynomial $g$ of $\mathbb{A}$ such that*

*(1) $(1, a) \notin \alpha$ for any $a \in U - \{1\}$;*

*(2) for all $a \in U - \{1\}$, the algebra $(\{a, 1\}, g)$ is a semilattice with neutral element $1$, that is, $g(1, 1) = 1$ and $g(1, a) = g(a, 1) = g(a, a) = a$.*

*(3) for any $a \in U - \{1\}$ and any $b \in T - \{1\}$, $g(a, b) \stackrel{\alpha}{\equiv} g(b, a) \stackrel{\alpha}{\equiv} a$;*

*(4) for all $a, b \in U$, $g(a, g(a, b)) = g(a, b)$.*

*Polynomial $g$ is said to be a* pseudo-meet *on $U$.*

## 4.3 Coloured graphs

In [15, 27] we introduced a local approach to the structure of finite algebras. As we use this approach throughout the paper, we present it here in some details, see also [19]. For the sake of the definitions below we slightly abuse terminology and by a module mean the full idempotent reduct of a module.

For an algebra $\mathbb{A}$ graph $\mathcal{G}(\mathbb{A})$ is defined as follows. The vertex set is the universe $A$ of $\mathbb{A}$. A pair $ab$ of vertices is an *edge* if and only if there exists a congruence $\theta$ of $\mathsf{Sg}(a, b)$, other than the full congruence and a term operation $f$ of $\mathbb{A}$ such that either $\mathsf{Sg}(a, b)/_\theta$ is a module and $f$ is an affine operation on it, or $f$ is a semilattice operation on $\{a^\theta, b^\theta\}$, or $f$ is a majority operation on $\{a^\theta, b^\theta\}$. (Note that we use the same operation symbol in this case.) Usually, $\theta$ is chosen to be a maximal congruence of $\mathsf{Sg}(a, b)$.

If there are a congruence $\theta$ and a term operation $f$ of $\mathbb{A}$ such that $f$ is a semilattice operation on $\{a^\theta, b^\theta\}$ then $ab$ is said to have the *semilattice type*. An edge $ab$ is of *majority type* if there are a congruence $\theta$ and a term operation $f$ such that $f$ is a majority operation on $\{a^\theta, b^\theta\}$ and there is no semilattice term operation on $\{a^\theta, b^\theta\}$. Finally, $ab$ has the *affine type* if there are $\theta$ and $f$ such that $f$ is an affine operation on $\mathsf{Sg}(a, b)/_\theta$ and $\mathsf{Sg}(a, b)/_\theta$ is a module; in particular it implies that there is no semilattice or majority operation on $\{a^\theta, b^\theta\}$. In all cases we say that congruence $\theta$ *witnesses* the type of edge $ab$. Observe that a pair $ab$ can still be an edge of more than one type as witnessed by different congruences, although this has consequences in this paper.

Omitting type **1** can be characterized as follows.

**Theorem 18 ([15, 19])** *An idempotent algebra $\mathbb{A}$ omits type **1** (that is, the variety generated by $\mathbb{A}$ omits type if and only if $\mathcal{G}(\mathbb{B})$ is connected for every subalgebra $\mathbb{B}$ of $\mathbb{A}$.*

*Moreover, a finite class $\mathcal{A}$ of similar idempotent algebras closed under subalgebras and quotient algebras omit type* **1** *if and only if $\mathcal{G}(\mathbb{A})$ is connected for any $\mathbb{A} \in \mathcal{A}$.*

For the sake of the dichotomy conjecture, it suffices to consider *reducts* of an algebra $\mathbb{A}$ omitting type **1**, that is, algebras with the same universe but reduced set of term operations, as long as reducts also omit type **1**. In particular, we are interested in reducts of $\mathbb{A}$, in which semilattice and majority edges are subalgebras.

**Theorem 19 ([15, 19])** *Let $\mathbb{A}$ be an algebra such that $\mathcal{G}(\mathbb{B})$ is connected for all subalgebras of $\mathbb{B}$ of $\mathbb{A}$, and let $ab$ be an edge of $\mathcal{G}(\mathbb{A})$ of the semilattice or majority type witnessed by congruence $\theta$, and $R_{ab} = a^\theta \cup b^\theta$. Let also $F_{ab}$ denote set of term operations of $\mathbb{A}$ preserving $R_{ab}$, and $\mathbb{A}' = (A, F_{ab})$. Then $\mathcal{G}(\mathcal{B}')$ is connected for all subalgebras $\mathbb{B}'$ of $\mathbb{A}'$.*

An algebra $\mathbb{A}$ such that $a^\theta \cup b^\theta$ is a subuniverse of $\mathbb{A}$ for every semilattice or majority edge $ab$ of $\mathbb{A}$ is called *sm-smooth*. If $\mathcal{A}$ is the class of all quotient algebras of subalgebras of an sm-smooth algebra $\mathbb{A}$, it is easy to see that that every $\mathbb{B} \in \mathcal{A}$ is sm-smooth. Although it is not needed in this paper, for any finite class $\mathcal{A}$ omitting type **1** there is a class $\mathcal{A}'$ of sm-smooth algebras which are reducts of algebras from $\mathcal{A}$, and such that $\mathcal{A}'$ omits type **1**, as well. In the rest of the paper all algebras are assumed to be sm-smooth.

The next statement uniformizes the operations witnessing the type of edges.

**Theorem 20 ([15, 19])** *Let $\mathcal{A}$ be a class of similar idempotent algebra closed under taking subalgebras and quotient algebras. There are term operations $f, g, h$ of $\mathcal{A}$ such that for any $\mathbb{A} \in \mathcal{A}$ and any $a, b \in \mathbb{A}$ operation $f$ is a semilattice operation on $\{a^\theta, b^\theta\}$ if $ab$ is a semilattice edge; $g$ is a majority operation on $\{a^\theta, b^\theta\}$ if $ab$ is a majority edge; $h$ is an affine operation on $\mathsf{Sg}(a, b)/_\theta$ if $ab$ is an affine edge, where $\theta$ witnesses the type of the edge. Moreover, $f, g, h$ can be chosen such that*

*(1) $f(x, f(x, y)) = f(x, y)$ for all $x, y \in \mathbb{A}$, $\mathbb{A} \in \mathcal{A}$;*

*(2) $g(x, g(x, y, y), g(x, y, y)) = g(x, y, y)$ for all $x, y \in \mathbb{A}$, $\mathbb{A} \in \mathcal{A}$;*

*(3) $h(h(x, y, y), y, y) = h(x, y, y)$ for all $x, y \in \mathbb{A}$, $\mathbb{A} \in \mathcal{A}$.*

*There is a term operation $t$ such that for any affine edge $ab$ and a majority, edge $cd$ witnessed by congruences $\eta$ and $\theta$, respectively, $t(a, b) \stackrel{\eta}{\equiv} a$ and $t(c, d) \stackrel{\theta}{\equiv} d$.*

Unlike majority and affine operations, for a semilattice edge $ab$ and a congruence $\theta$ of $\mathsf{Sg}(a, b)$ witnessing that, there can be semilattice operations acting differently on $\{a^\theta, b^\theta\}$, which corresponds to the two possible orientations of $ab$. In every such case by fixing operation $f$ from Theorem 20 we effectively choose one of the two orientations. In this paper we do not really care about what orientation is preferable.

In [19] we introduced a stronger notion of edge. A pair $ab$ of elements of algebra $\mathbb{A}$ is called a *thin semilattice edge* if $ab$ is a semilattice edge, and the congruence witnessing that is the equality relation. In other words, $f(a,a) = a$ and $f(a,b) = f(b,a) = f(b,b) = b$. We denote the fact that $ab$ is a thin semilattice edge by $a \le b$. Thin semilattice edges allow us to introduce a directed graph $\mathcal{G}_s(\mathbb{A})$, whose vertices are the elements of $\mathbb{A}$, and the arcs are the thin semilattice edges. We then can define *semilattice-connected* and *strongly semilattice-connected* components of $\mathcal{G}_s(\mathbb{A})$. We will also use the natural order on the set of strongly semilattice-connected components of $\mathcal{G}_s(\mathbb{A})$: for components $A, B$, we write $A \le B$ if there is a directed path in $\mathcal{G}_s(\mathbb{A})$ connecting a vertex from $A$ with a vertex from $B$. Elements from the maximal strongly connected components (or simply *maximal components*) of $\mathcal{G}_s(\mathbb{A})$ are called *maximal* elements of $\mathbb{A}$ and the set of all such elements is denoted by $\max(\mathbb{A})$. A directed path in $\mathcal{G}_s(\mathbb{A})$ is called a *semilattice path* or *s-path*. If there is an s-path from $a$ to $b$ we write $a \sqsubseteq b$.

**Proposition 21 ([15, 19])** *Let $\mathcal{A}$ be a finite class of similar idempotent algebras closed under taking subalgebras and quotient algebras. There is a binary term operation $f$ of $\mathcal{A}$ such that $f$ is a semilattice operation on $\{a^\theta, b^\theta\}$ for every semilattice edge $ab$ of any $\mathbb{A} \in \mathcal{A}$, where congruence $\theta$ witnesses the type of $ab$, and, for any $a, b \in \mathbb{A}$, either $a = f(a,b)$ or the pair $(a, f(a,b))$ is a thin semilattice edge of $\mathbb{A}$. Operation $f$ with this property will be denoted by a dot (think multiplication).*

Let operations $g, h$ be as in Theorem 20. A pair $ab$ from $\mathbb{A} \in \mathcal{A}$ is called a *thin majority edge* if (a) it is a majority edge, let congruence $\theta$ witness this, (b) for any $c \in b^\theta$, $b \in \mathsf{Sg}(a,c)$, (c) $g(a,b,b) = b$, and (d) there exists a ternary term operation $g'$ such that $g'(a,b,b) = g'(b,a,b) = g'(b,b,a) = b$. Finally, a pair $ab$ is called a *thin affine edge* if (a) it is an affine edge, let congruence $\theta$ witness this, (b) for any $c \in b^\theta$, $b \in \mathsf{Sg}(a,c)$, (c) $h(b,a,a) = b$, (d) there exists a ternary term operation $h'$ such that $h'(b,a,a) = h'(a,a,b) = b$, and (e) $a$ is maximal in $\mathsf{Sg}(a,b)$. Note that the operations $h, g$ from Theorem 20 do not have to be majority or affine operations on thin edges; thin edges do not have to be even closed under $g, h$. Thin edges of all types are oriented. We therefore can define yet another directed graph, $\mathcal{G}'(\mathbb{A})$, in which the arcs are the thin edges of all types.

**Lemma 22 ([19])** *Let $\mathbb{A}$ be an algebra.*

*(1) Let $ab$ be a semilattice or majority edge in $\mathbb{A}$, and $\theta$ the congruence of $\mathsf{Sg}(a,b)$ witnessing that. Then there is $b' \in b^\theta$ such that $ab'$ is a thin semilattice or majority*

*edge, respectively.*

*(2) Let $ab$ be an affine edge, and $\theta$ the congruence of $\mathsf{Sg}(a,b)$ witnessing that. Then there are $a' \in a^\theta$ and $b' \in b^\theta$ such that $a \sqsubseteq a'$ in $a^\theta$ and $a'b'$ is a thin affine edge.*

The following simple properties of thin edges will be useful. Note that a subdirect product of algebras (a relation) is also an algebra, and so edges and thin edges can be defined for relations as well.

**Lemma 23 ([19])** *(1) Let $\mathbb{A}$ be an algebra and $ab$ a thin edge. Then $ab$ is a thin edge in any subalgebra of $\mathbb{A}$ containing $a, b$, and $a^\theta b^\theta$ is a thin edge in $\mathbb{A}/_\theta$ for any congruence $\theta$.*

*(2) Let $R$ be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$, $I \subseteq [n]$, and $\mathbf{ab}$ a thin edge in $R$. Then $\mathrm{pr}_I\mathbf{a}\mathrm{pr}_I\mathbf{b}$ is a thin edge in $\mathrm{pr}_I R$ of the same type as $\mathbf{ab}$.*

We will need stronger versions of Lemmas 18 and 20 of [19]. Let $\mathcal{A}$ be a finite class of similar idempotent algebras closed under taking subalgebras and quotient algebras.

**Lemma 24** *(1) Let $ab$ be a thin majority edge of algebra $\mathbb{A} \in \mathcal{A}$. There is a term operation $t_{ab}$ such that $t_{ab}(a,b) = b$ and $t_{ab}(c,d) \stackrel{\theta_{cd}}{\equiv} c$ for all affine edges $cd$ of all $\mathbb{A}' \in \mathcal{A}$, where the type of $cd$ is witnessed by congruence $\theta_{cd}$.*

*(2) Let $ab$ be a thin affine edge of algebra $\mathbb{A} \in \mathcal{A}$. There is a term operation $h_{ab}$ such that $h_{ab}(a,a,b) = b$ and $h_{ab}(d,c,c) \stackrel{\theta_{cd}}{\equiv} d$ for all affine edges $cd$ of all $\mathbb{A}' \in \mathcal{A}$, where the type of $cd$ is witnessed by congruence $\theta_{cd}$. Moreover, $h_{ab}(x, c', d')$ is a permutation of $\mathsf{Sg}(c,d)/_{\theta_{cd}}$ for any $c', d' \in \mathsf{Sg}(c,d)$.*

*(3) Let $ab$ and $cd$ be thin edges in algebras $\mathbb{A}, \mathbb{A}' \in \mathcal{A}$, respectively. If they have different types there is a binary term operation $p$ such that $p(a,b) = a$, $p(c,d) = d$. If both edges are affine then there is a term operation $h'$ such that $h'(a,a,b) = b$ and $h'(d,c,c) = d$.*

**Proof:** (1) Let $c_1d_1, \dots, c_\ell d_\ell$ be a list of all affine edges of algebras in $\mathcal{A}$, $c_i, d_i \in \mathbb{A}_i$ and $\theta_{c_id_i}$ the corresponding congruences. Set $\mathbf{c} = (c_1, \dots, c_\ell), \mathbf{d} = (d_1, \dots, d_\ell)$. Let $R$ be the subalgebra of $\mathbb{A} \times \prod_{i=1}^{\ell} \mathbb{A}_i$ generated by $(a, \mathbf{c}), (b, \mathbf{d})$. Pair $ab$ is also a majority edge, let it be witnessed by a congruence $\theta$. By Theorem 20

$$\begin{pmatrix} b' \\ \mathbf{c}' \end{pmatrix} = t\left(\begin{pmatrix} a \\ \mathbf{c} \end{pmatrix}, \begin{pmatrix} b \\ \mathbf{d} \end{pmatrix}\right) \in R,$$

where $b' \in b^\theta$ and $\mathbf{c}'[i] \stackrel{\theta_{c_id_i}}{\equiv} \mathbf{c}[i]$, as $t$ is the first projection on $\mathsf{Sg}(c_i,d_i)/_{\theta_{c_id_i}}$ and a second projection on $\mathsf{Sg}(a,b)/_\theta$. Then as $b \in \mathsf{Sg}(a,b')$, we get $(b, \mathbf{c}'') \in R$ for some $\mathbf{c}''$ such that $\mathbf{c}''[i] \stackrel{\theta_{c_i d_i}}{\equiv} \mathbf{c}[i]$. This means there is a binary term operation $t_{ab}$ such that

$$t_{ab}\left(\begin{pmatrix} a \\ \mathbf{c} \end{pmatrix}, \begin{pmatrix} b \\ \mathbf{d} \end{pmatrix}\right) = \begin{pmatrix} b \\ \mathbf{c}'' \end{pmatrix}.$$

The result follows.

(2) We use the notation from item (1) except $ab$ now is a thin affine edge and $R$ is generated by $(a, \mathbf{d}), (a, \mathbf{c}), (b, \mathbf{c})$. By condition (a) of the definition of thin affine edges,

$$\begin{pmatrix} b' \\ \mathbf{d}' \end{pmatrix} = h\left(\begin{pmatrix} a \\ \mathbf{d} \end{pmatrix}, \begin{pmatrix} a \\ \mathbf{c} \end{pmatrix}, \begin{pmatrix} b \\ \mathbf{c} \end{pmatrix}\right) \in R,$$

where $b' \in b^\theta$ and $\mathbf{d}'[i] \stackrel{\theta_{c_i d_i}}{\equiv} \mathbf{d}[i]$, as $h$ is a Mal'tsev operation on $\mathsf{Sg}(a, b)/_\theta$ and on $\mathsf{Sg}(c_i, d_i)/_{\theta_{c_i d_i}}$. Then as $b \in \mathsf{Sg}(a, b')$, by condition (b) we get $(b, \mathbf{d}'') \in R$ for some $\mathbf{d}''$ such that $\mathbf{d}''[i] \stackrel{\theta_{c_i d_i}}{\equiv} \mathbf{d}[i]$. The first result follows.

Let now $h_{ab}$ be the term operation we constructed and $c', d' \in \mathsf{Sg}(c_i, d_i)$, $i \in [\ell]$. Since $\mathbb{B} = \mathsf{Sg}(c_i, d_i)/_{\theta_{c_i d_i}}$ is a module, in particular, it is an Abelian algebra and $h_{ab}(x, c^*, c^*) = x$ for all $c^* \in \mathbb{B}$, the second result follows.

(3) Follows from [19], Lemmas 15,18,19,20. □

### 4.4 Maximality

A directed path in $\mathcal{G}'(\mathbb{A})$ is called an *asm-path*, if there is an asm-path from $a$ to $b$ we write $a \sqsubseteq_{asm} b$. If all edges of this path are semilattice or affine, it is called an *affine-semilattice path* or an *as-path*, if there is an as-path from $a$ to $b$ we write $a \sqsubseteq_{as} b$. Similar to maximal components, we consider strongly connected components of $\mathcal{G}'(\mathbb{A})$ with majority edges removed, and the natural partial order on such components. The maximal components will be called *as-components*, and the elements from as-components are called *as-maximal*; the set of all as-maximal elements of $\mathbb{A}$ is denoted by $\mathsf{amax}(\mathbb{A})$. If $a$ is an as-maximal element, the as-component containing $a$ is denoted $\mathsf{as}(a)$. An alternative way to define as-maximal elements is as follows: $a$ is as-maximal if for every $b \in \mathbb{A}$ such that $a \sqsubseteq_{as} b$ it also holds that $b \sqsubseteq_{as} a$. Finally, element $a \in \mathbb{A}$ is said to be *universally maximal* (or *u-maximal* for short) if for every $b \in \mathbb{A}$ such that $a \sqsubseteq_{asm} b$ it also holds that $b \sqsubseteq_{asm} a$. The set of all u-maximal elements of $\mathbb{A}$ is denoted $\mathsf{umax}(\mathbb{A})$.

**Proposition 25 ([19])** *Let $\mathbb{A}$ be an algebra. Then*

*(1) any $a, b \in \mathbb{A}$ are connected in $\mathcal{G}'(\mathbb{A})$ with an undirected path;*

*(2) any $a, b \in \max(\mathbb{A})$ (or $a, b \in \mathsf{amax}(\mathbb{A})$, or $a, b \in \mathsf{umax}(\mathbb{A})$) are connected in $\mathcal{G}'(\mathbb{A})$ with a directed path.*

**Proof:** Item (2) is only proved in [19] for maximal and as-maximal elements; so we prove it here for u-maximal elements as well. Let $a', b' \in \mathbb{A}$ be maximal elements of $\mathbb{A}$ such that $a \sqsubseteq a'$ and $b \sqsubseteq b'$. Then by Proposition 25 for maximal elements $a' \sqsubseteq_{asm} b'$, and, as $b$ is u-maximal, $b' \sqsubseteq_{asm} b$. □

Since for every $a \in \mathbb{A}$ there is a maximal $a' \in \mathbb{A}$ such that $a \sqsubseteq a'$, Proposition 25 implies that there is only one u-maximal component. U-maximality has an additional useful property, it is somewhat hereditary, as it made precise in the following

**Lemma 26** *Let $\mathbb{B}$ be a subalgebra of $\mathbb{A}$ containing a u-maximal element of $\mathbb{A}$. Then every element u-maximal in $\mathbb{B}$ is also u-maximal in $\mathbb{A}$. In particular, if $\alpha$ is a congruence of $\mathbb{A}$ and $\mathbb{B}$ is a u-maximal $\alpha$-block, that is $\mathbb{B}$ is a u-maximal element in $\mathbb{A}/_\alpha$, then* $\mathsf{umax}(B) \subseteq \mathsf{umax}(\mathbb{A})$.

**Proof:** Let $a \in \mathbb{B}$ be an element u-maximal in $\mathbb{A}$, let $b \in \mathsf{umax}(\mathbb{B})$. For any $c \in \mathbb{A}$ with $b \sqsubseteq_{asm} c$ we also have $c \sqsubseteq_{asm} a$. Finally, since $b \in \mathsf{umax}(\mathbb{B})$ and $a \in \mathbb{B}$, we have $a \sqsubseteq_{asm} b$. For the second part of the lemma we need to find a u-maximal element in $\mathbb{B}$. Let $b \in \mathsf{umax}(\mathbb{A})$. Then as $\mathbb{B}$ is u-maximal in $\mathbb{A}/_\alpha$ applying Lemma 22 we get that there is $a' \in \mathbb{B}$ such that $b \sqsubseteq_{asm} a'$. Clearly, $a' \in \mathsf{umax}(\mathbb{A})$. □

Let $\mathbb{A}$ be an algebra and $a \in \mathbb{A}$. By $\mathrm{Ft}_{\mathbb{A}}(a)$ we denote the set of elements $a$ is connected to (in terms of semilattice paths); similarly, by $\mathrm{Ft}^{as}_{\mathbb{A}}(a)$ and $\mathrm{Ft}^{asm}_{\mathbb{A}}(a)$ we denote the set of elements $a$ is as-connected and asm-connected to. Also, $\mathrm{Ft}_{\mathbb{A}}(C) = \bigcup_{a\in C} \mathrm{Ft}_{\mathbb{A}}(a)$ ($\mathrm{Ft}^{as}_{\mathbb{A}}(C) = \bigcup_{a\in C} \mathrm{Ft}^{as}_{\mathbb{A}}(a)$, $\mathrm{Ft}^{asm}_{\mathbb{A}}(C) = \bigcup_{a\in C} \mathrm{Ft}^{asm}_{\mathbb{A}}(a)$, respectively) for $C \subseteq \mathbb{A}$. Note that if $a$ is an as-maximal element then $\mathsf{as}(a) = \mathrm{Ft}^{as}_{\mathbb{A}}(a)$, and $a \in \mathrm{Ft}^{asm}_{\mathbb{A}}(b)$ for any $b \in \mathbb{A}$. We will need the following statements.

**Lemma 27 (The Maximality Lemma,[20])** *Let $R$ be a subdirect product of $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$, $I \subseteq [n]$.*

*(1) For any $\mathbf{a} \in R$, $\mathbf{b} \in \mathrm{pr}_I R$ with $\mathrm{pr}_I \mathbf{a} \leq \mathbf{b}$, there is $\mathbf{b}' \in R$ such that $\mathbf{a} \leq \mathbf{b}'$ and $\mathrm{pr}_I \mathbf{b}' = \mathbf{b}$.*

*(2) For any $\mathbf{a} \in R$, $\mathbf{b} \in \mathrm{pr}_I R$ such that $(\mathrm{pr}_I \mathbf{a})\mathbf{b}$ is a thin majority edge there is $\mathbf{b}' \in R$ such that $\mathbf{a}\mathbf{b}'$ is a thin majority edge, and $\mathrm{pr}_I \mathbf{b}' = \mathbf{b}$.*

*(3) For any $\mathbf{a} \in R$, $\mathbf{b} \in \mathrm{pr}_I R$ such that $(\mathrm{pr}_I \mathbf{a})\mathbf{b}$ is a thin affine edge there are $\mathbf{a}', \mathbf{b}' \in R$ such that $\mathbf{a} \sqsubseteq \mathbf{a}'$, $\mathbf{a}'\mathbf{b}'$ is a thin affine edge, and $\mathrm{pr}_I \mathbf{a}' = \mathrm{pr}_I \mathbf{a}$, $\mathrm{pr}_I \mathbf{b}' = \mathbf{b}$.*

*(4) For any $\mathbf{a} \in R$, and an s-path (as-path, asm-path) $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathrm{pr}_I R$ with $\mathrm{pr}_I \mathbf{a} = \mathbf{b}_1$, there is an s-path (as-path,asm-path, respectively) $\mathbf{b}'_1, \dots, \mathbf{b}'_\ell \in R$ such that $\mathrm{pr}_I \mathbf{b}'_\ell = \mathbf{b}_\ell$.*

*(5) For any* $\mathbf{b} \in \max(\mathrm{pr}_I R)$ *(*$\mathbf{b} \in \mathsf{amax}(\mathrm{pr}_I R)$*,* $\mathbf{b} \in \mathsf{umax}(\mathrm{pr}_I R)$*) there is* $\mathbf{b}' \in \max(R)$ *(*$\mathbf{b}' \in \mathsf{amax}(R)$*,* $\mathbf{b}' \in \mathsf{umax}(R)$*, respectively), such that* $\mathrm{pr}_I \mathbf{b}' = \mathbf{b}$. *In particular,* $\mathrm{pr}_{[n]-I}\mathbf{b}' \in \max(\mathrm{pr}_{[n]-I} R)$ *(*$\mathrm{pr}_{[n]-I}\mathbf{b}' \in \mathsf{amax}(\mathrm{pr}_{[n]-I} R)$*,* $\mathrm{pr}_{[n]-I}\mathbf{b}' \in \mathsf{umax}(\mathrm{pr}_{[n]-I} R)$*, respectively).*

**Proof:** Items (1) and (3) are proved in [20], and items of (4) and (5) are only proved for s- and as-paths, and, respectively, for maximal and as-maximal elements. Items (4) and (5) for asm-paths and u-maximal elements follow from (1)–(3).

(2) Observe that it suffices to consider binary relations $R$. Indeed, $R$ can be viewed as a subdirect product of $\mathrm{pr}_I R \times \mathrm{pr}_{[n]-I} R$. So, suppose $n = 2$ and $I = \{1\}$. We have $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = b_1$. Let $\theta$ be a maximal congruence of $\mathsf{Sg}(a_1, b_1)$ witnessing that $a_1 b_1$ is an majority edge. Choose $\mathbf{b}'' = (b_1', b_2) \in R$ such that $b_1' \in b_1^\theta$ and $R' = \mathsf{Sg}(\mathbf{a}, \mathbf{b}'')$ is minimal possible with this condition. It suffices to prove the lemma for $R'$, since $b_1 \in \mathsf{Sg}(a_1, b_1')$ and so $b_1 \in \mathrm{pr}_1 R'$, and $a_1 b_1$ is still a thin majority edge. This means that $(b_1', b_2)$ can be chosen such that $b_1' = b_1$. Also, by taking $\begin{pmatrix} b_1 \\ b_2' \end{pmatrix} = g\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right)$ we may assume by Theorem 20 that $g(a_2, b_2, b_2) = b_2$. As is easily seen, the pair $(a_1, a_2)(b_1, b_2)$ is a majority edge as witnessed by congruence $\theta' = \theta \times \underline{1}_{\mathbb{A}_2'}$ where $\mathbb{A}_2' = \mathsf{Sg}(a_2, b_2)$. By the choice of $\mathbf{b}''$ the pair $(b_1, b_2)$ belongs to $\mathsf{Sg}((a_1, a_2), (c_1, c_2))$ for any $(c_1, c_2) \in (b_1, b_2)^{\theta'}$, and it only remains to prove condition (d) of the definition of thin majority edges.

Let $g'$ be the operation from condition (d) for $a_1 b_1$. Then

$$g'\left(\begin{pmatrix} (a_1, a_2) \\ (b_1, b_2) \\ (b_1, b_2) \end{pmatrix}, \begin{pmatrix} (b_1, b_2) \\ (a_1, a_2) \\ (b_1, b_2) \end{pmatrix}, \begin{pmatrix} (b_1, b_2) \\ (b_1, b_2) \\ (a_1, a_2) \end{pmatrix}\right) = \begin{pmatrix} (b_1, b_2') \\ (b_1, b_2'') \\ (b_1, b_2''') \end{pmatrix}.$$

Since $(b_1, b_2) \in \mathsf{Sg}((a_1, a_2), (b_1, b_2'))$ by the choice of $(b_1, b_2)$, there is a term operation $r_1$ such that

$$r_1\left(\begin{pmatrix} (a_1, a_2) \\ (b_1, b_2) \\ (b_1, b_2) \end{pmatrix}, \begin{pmatrix} (b_1, b_2') \\ (b_1, b_2'') \\ (b_1, b_2''') \end{pmatrix}\right) = \begin{pmatrix} (b_1, b_2) \\ (b_1, b_2^*) \\ (b_1, b_2^{**}) \end{pmatrix}.$$

Repeating this for the second and third coordinate positions by finding $r_2, r_3$ with

$$r_2\left(\begin{pmatrix} (b_1, b_2) \\ (a_1, a_2) \\ (b_1, b_2) \end{pmatrix}, \begin{pmatrix} (b_1, b_2) \\ (b_1, b_2^*) \\ (b_1, b_2^{**}) \end{pmatrix}\right) = \begin{pmatrix} (b_1, b_2) \\ (b_1, b_2) \\ (b_1, b_2^\dagger) \end{pmatrix}, \quad r_3\left(\begin{pmatrix} (b_1, b_2) \\ (b_1, b_2) \\ (a_1, a_2) \end{pmatrix}, \begin{pmatrix} (b_1, b_2) \\ (b_1, b_2) \\ (b_1, b_2^\dagger) \end{pmatrix}\right) = \begin{pmatrix} (b_1, b_2) \\ (b_1, b_2) \\ (b_1, b_2) \end{pmatrix},$$

we obtain a ternary operation $g''$ such that

$$\begin{aligned} g''\left(\begin{pmatrix}a_1\\a_2\end{pmatrix},\begin{pmatrix}b_1\\b_2\end{pmatrix},\begin{pmatrix}b_1\\b_2\end{pmatrix}\right) &= g''\left(\begin{pmatrix}b_1\\b_2\end{pmatrix},\begin{pmatrix}a_1\\a_2\end{pmatrix},\begin{pmatrix}b_1\\b_2\end{pmatrix}\right)\\ &= g''\left(\begin{pmatrix}b_1\\b_2\end{pmatrix},\begin{pmatrix}b_1\\b_2\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}\right) = \begin{pmatrix}b_1\\b_2\end{pmatrix},\end{aligned}$$

confirming property (d). □

The following lemma considers a special case of as-components in subdirect products, and is straightforward.

**Lemma 28** *Let $R$ be a subdirect product of $\mathbb{A}_1 \times \mathbb{A}_2$, $B, C$ as components of $\mathbb{A}_1, \mathbb{A}_2$, respectively, and $B \times C \subseteq R$. Then $B \times C$ is an as-component of $R$.*

We complete this section with an auxiliary statement that will be needed later.

**Lemma 29** *Let $\alpha \prec \beta$, $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, let $B$ be a $\beta$-block and $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$. Then $B/_\alpha$ is term equivalent to a module. In particular, every pair of elements of $B/_\alpha$ is a thin affine edge in $\mathbb{A}/_\alpha$.*

**Proof:** As $\mathbb{A}$ is an idempotent algebra that generates a variety omitting type **1**, and $(\alpha, \beta)$ is a simple interval in $\mathsf{Con}(\mathbb{A})$ of type **2**, by Theorem 7.11 of [34] there is a term operation of $\mathbb{A}$ that is Mal'tsev on $B/_\alpha$. Since $\beta$ is Abelian on $B/_\alpha$, we get the result. □

## 4.5 Quasi-decomposition and quasi-majority

We make use of the property of quasi-2-decomposability proved in [20].

**Theorem 30 (The 2-Decomposition Theorem, [20])** *If $R$ is an $n$-ary relation, $X \subseteq [n]$, tuple $\mathbf{a}$ is such that $\mathrm{pr}_J\mathbf{a} \in \mathrm{pr}_J R$ for any $J \subseteq [n]$, $|J| = 2$, and $\mathrm{pr}_X\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_X R)$, there is a tuple $\mathbf{b} \in R$ with $\mathrm{pr}_J\mathbf{b} \in \mathrm{Ft}^{as}_{\mathrm{pr}_J R}(\mathrm{pr}_J\mathbf{a})$ for any $J \subseteq [n]$, $|J| = 2$, and $\mathrm{pr}_X\mathbf{b} = \mathrm{pr}_X\mathbf{a}$.*

One useful implication of the 2-Decomposition Theorem 30 is the existence of term operation resembling a majority function. We state this theorem for finite classes of algebras rather than a single algebra, because it concerns as-components that in subalgebras of products may have complicated structure.

**Theorem 31** *Let $\mathcal{A}$ be a finite class of finite similar sm-smooth algebras omitting type $\mathbf{1}$. There is a term operation $\mathsf{maj}$ of $\mathcal{A}$ such that for any $\mathbb{A} \in \mathcal{A}$ and any $a, b \in \mathbb{A}$, $\mathsf{maj}(a, a, b), \mathsf{maj}(a, b, a), \mathsf{maj}(b, a, a) \in \mathrm{Ft}^{as}_{\mathbb{A}}(a)$.*

*In particular, if $a$ is as-maximal, then $\mathsf{maj}(a, a, b), \mathsf{maj}(a, b, a), \mathsf{maj}(b, a, a)$ belong to the as-component of $\mathbb{A}$ containing $a$.*

**Proof:** Let $\{a_1, b_1\}, \ldots, \{a_n, b_n\}$ be a list of all pairs of elements from algebras of $\mathcal{A}$, let $a_i, b_i \in \mathbb{A}_i$. Define relation $R$ to be a subdirect product of $\mathbb{A}_1^3 \times \cdots \times \mathbb{A}_n^3$ generated by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, where for every $i \in [n]$, $\mathrm{pr}_{3i-2,3i-1,3i}\mathbf{a}_1 = (a_i, a_i, b_i)$, $\mathrm{pr}_{3i-2,3i-1,3i}\mathbf{a}_2 = (a_i, b_i, a_i)$, $\mathrm{pr}_{3i-2,3i-1,3i}\mathbf{a}_3 = (b_i, a_i, a_i)$. In other words the triples $(\mathbf{a}_1[3i-2], \mathbf{a}_2[3i-2], \mathbf{a}_3[3i-2])$, $(\mathbf{a}_1[3i-1], \mathbf{a}_2[3i-1], \mathbf{a}_3[3i-1])$, $(\mathbf{a}_1[3i], \mathbf{a}_2[3i], \mathbf{a}_3[3i])$ have the form $(a_i, a_i, b_i), (a_i, b_i, a_i), (b_i, a_i, a_i)$, respectively. Therefore it suffices to show that $R$ contains a tuple $\mathbf{b}$ such that $a_i \sqsubseteq_{as} \mathbf{b}[j]$, where $j \in \{3i, 3i-1, 3i-2\}$. However, since $(a_{i_1}, a_{i_2}) \in \mathrm{pr}_{j_1 j_2} R$ for any $i_1, i_2 \in [n]$ and $j_1 \in \{3i_1, 3i_1 - 1, 3i_1 - 2\}$, $j_2 \in \{3i_2, 3i_2 - 1, 3i_2 - 2\}$, this follows from the 2-Decomposition Theorem 30. □

A function $\mathsf{maj}$ satisfying the properties from Theorem 31 will be called a *quasi-majority function*.

## 4.6 Rectangularity

Let $R$ be a subdirect product of $\mathbb{A}_1, \mathbb{A}_2$. By $R[c], R^{-1}[c']$ for $c \in \mathbb{A}_1, c' \in \mathbb{A}_2$ we denote the sets $\{b \mid (c, b) \in R\}, \{a \mid (a, c') \in R\}$, respectively, and for $C \subseteq \mathbb{A}_1, C' \subseteq \mathbb{A}_2$ we use $R[C] = \bigcup_{c \in C} R[c]$, $R^{-1}[C'] = \bigcup_{c' \in C'} R^{-1}[c']$, respectively. Binary relations $\mathsf{tol}_1, \mathsf{tol}_2$ on $\mathbb{A}_1, \mathbb{A}_2$ given by $\mathsf{tol}_1(R) = \{(a, b) \mid R[a] \cap R[b] \neq \varnothing\}$ and $\mathsf{tol}_2(R) = \{(a, b) \mid R^{-1}[a] \cap R^{-1}[b] \neq \varnothing\}$, respectively, are called *link tolerances* of $R$. They are tolerances of $\mathbb{A}_1$, $\mathbb{A}_2$, respectively, that is invariant reflexive and symmetric relations. The transitive closures $\mathsf{lk}_1, \mathsf{lk}_2$ of $\mathsf{tol}_1(R), \mathsf{tol}_2(R)$ are called *link congruences*, and they are, indeed, congruences. Relation $R$ is said to be *linked* if the link congruences are full congruences.

**Lemma 32 ([20])** *Let $R$ be a subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2$ and let $a \in \mathbb{A}_1$ and $B = R[a]$. For any $b \in \mathbb{A}_1$ such that $ab$ is thin edge, and any $c \in R[b] \cap B$, $\mathrm{Ft}^{as}_B(c) \subseteq R[b]$.*

**Proof:** The case when $a \leq b$ or $ab$ is affine is considered in [20], so suppose that $ab$ is majority. Let $D = \mathrm{Ft}^{as}_B(c) \cap R[b]$. Set $D$ is nonempty, as $c \in D$. If $D \neq \mathrm{Ft}^{as}_B(c)$, there are $b_1 \in D$ and $b_2 \in \mathrm{Ft}^{as}_B(c) - D$ such that $b_1 b_2$ is a thin edge. By Lemma 24(3) there is a term operation $p$ such that $p(a, b) = b$ and $p(b_2, b_1) = b_2$. Then $\begin{pmatrix} b \\ b_2 \end{pmatrix} = p\left(\begin{pmatrix} a \\ b_2 \end{pmatrix}, \begin{pmatrix} b \\ b_1 \end{pmatrix}\right) \in R$. The result follows. □

**Proposition 33 ([20])** *Let $R \leq \mathbb{A}_1 \times \mathbb{A}_2$ be a linked subdirect product and let $B_1, B_2$ be as-components of $\mathbb{A}_1, \mathbb{A}_2$, respectively, such that $R \cap (B_1 \times B_2) \neq \varnothing$. Then $B_1 \times B_2 \subseteq R$.*

**Corollary 34 (The Rectangularity Corollary)** *Let $R$ be a subdirect product of $\mathbb{A}_1$ and $\mathbb{A}_2$, $\mathsf{lk}_1, \mathsf{lk}_2$ the link congruences, and let $B_1, B_2$ be as-components of a*

$\mathsf{lk}_1$*-block and a* $\mathsf{lk}_2$*-block, respectively, such that* $R \cap (B_1 \times B_2) \neq \varnothing$*. Then* $B_1 \times B_2 \subseteq R$*.*

**Proposition 35** *Let* $R$ *be a subdirect product of* $\mathbb{A}_1$ *and* $\mathbb{A}_2$*,* $\mathsf{lk}_1, \mathsf{lk}_2$ *the link congruences, and let* $B_1$ *be an as-component of a* $\mathsf{lk}_1$*-block and* $B_2' = R[B_1]$*; let* $B_2 = \mathsf{umax}(B_2')$*. Then* $B_1 \times B_2 \subseteq R$*.*

**Proof:** Let $B_2'$ be a subset of a $\mathsf{lk}_2$-block $C$. By the Maximality Lemma 27(5) $B_2'$ contains an as-maximal element $a$ of $C$. By the Rectangularity Corollary 34 $B_1 \times \{a\} \subseteq R$. It then suffices to show that $B_1 \times \mathrm{Ft}^{asm}_{B_2'}(a) \subseteq R$.

Suppose for $D \subseteq \mathrm{Ft}^{asm}_{B_2'}(a)$ it holds $B_1 \times D \subseteq R$. If $D \neq \mathrm{Ft}^{asm}_{B_2'}(a)$, there are $b_1 \in D$ and $b_2 \in \mathrm{Ft}^{asm}_{B_2'}(a) - D$ such that $b_1 b_2$ is a thin edge. By Lemma 32 $B_1 \times \{b_2\} \subseteq R$; the result follows. $\square$

# 5 Separating congruences

In this section we introduce and study the relationship between prime factors in the congruence lattice of an algebra, or in congruence lattices of factors in a subdirect products. It was first introduced in [12] and used in the CSP research in [21].

## 5.1 Special polynomials, mapping pairs

We start with several technical results. They demonstrate the connection between minimal sets of an algebra $\mathbb{A}$ and the structure of its graph $\mathcal{G}'(\mathbb{A})$. Let $\mathbb{A}$ be an algebra and let $Q^{\mathbb{A}}_{ab}$, $a, b \in \mathbb{A}$, denote the subdirect product of $\mathbb{A}^2$ generated by $\{(x, x) \mid x \in \mathbb{A}\} \cup \{(a, b)\}$.

**Lemma 36** *(1)* $Q^{\mathbb{A}}_{ab} = \{(f(a), f(b)) \mid f \in \mathsf{Pol}_1(\mathbb{A})\}$*.*
*(2) For any* $f \in \mathsf{Pol}_1(\mathbb{A})$*,* $(f(a), f(b)) \in \mathsf{tol}_1(Q^{\mathbb{A}}_{ab})$*. In particular,* $\mathsf{lk}_1(Q^{\mathbb{A}}_{ab}) = \mathsf{Cg}(a, b)$*; denote this congruence by* $\alpha$*.*
*(3)* $Q^{\mathbb{A}}_{ab} \subseteq \mathsf{Cg}(a, b)$*.*
*(4) Let* $B_1, B_2$ *be* $\alpha$*-blocks, and* $C_1, C_2$ *as-components of* $B_1, B_2$*, respectively, such that* $f(a) \in C_1$ *and* $f(b) \in C_2$ *for a polynomial* $f$*. Then* $C_1 \times C_2 \subseteq Q^{\mathbb{A}}_{ab}$*.*

**Proof:** (1) follows directly from the definitions.

(2) Take $f \in \mathsf{Pol}_1(\mathbb{A})$ and let $f(x) = g(x, a_1, \ldots, a_k)$ for a term operation $g$ of $\mathbb{A}$. Then $\begin{pmatrix} f(a) \\ f(b) \end{pmatrix} = g\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_k \\ a_k \end{pmatrix}\right) \in R$ and $\begin{pmatrix} f(b) \\ f(b) \end{pmatrix} = g\left(\begin{pmatrix} b \\ b \end{pmatrix}, \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_k \\ a_k \end{pmatrix}\right) \in R$. Thus $(f(a), f(b)) \in \mathsf{tol}_1(Q^{\mathbb{A}}_{ab})$.

(3) follows from (1), and (4) follows from (2),(3), and the Rectangularity Corollary 34. □

We say that $a$ is *$\alpha$-maximal* for a congruence $\alpha \in \mathsf{Con}(\mathbb{A})$ if $a$ is as-maximal in the subalgebra $a^\alpha$.

**Corollary 37** *Let $\alpha \in \mathsf{Con}(\mathbb{A})$ and $\underline{0} \prec \alpha$. Then for any $a, b \in \mathbb{A}$ with $a \stackrel{\alpha}{\equiv} b$ and any $\alpha$-maximal $c, d \in \mathbb{A}$, $c \neq d$, with $c \stackrel{\alpha}{\equiv} d$, belonging to the same as-component of $c^\alpha$, there is $f \in \mathsf{Pol}_1(\mathbb{A})$ such that $c = f(a)$, $d = f(b)$.*

**Proof:** The result follows from Lemma 36(4). □

Recall that for $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ with $\alpha \prec \beta$ a pair $\{a, b\}$ is called an $(\alpha, \beta)$-*subtrace* if $(a, b) \in \beta - \alpha$ and $a, b \in U$ for some $(\alpha, \beta)$-minimal set $U$.

**Corollary 38** *Let $\alpha \in \mathsf{Con}(\mathbb{A})$ and $\underline{0} \prec \alpha$, and let $c, d \in \mathbb{A}$, $c \stackrel{\alpha}{\equiv} d$, be $\alpha$-maximal.*

*(1) If $c, d$ belong to the same as-component of $c^\alpha$, then $\{c, d\}$ is a $(\underline{0}, \alpha)$-subtrace.*

*(2) If there is a $(\underline{0}, \alpha)$-subtrace $\{c', d'\}$ such that $c' \in \mathsf{as}(c)$ and $d' \in \mathsf{as}(d)$ then $\{c, d\}$ is a $(\underline{0}, \alpha)$-subtrace as well.*

**Proof:** (1) Take any $(\underline{0}, \alpha)$-minimal set $U$, and $a, b \in U$ with $a \stackrel{\alpha}{\equiv} b$. By Corollary 37 there is $f \in \mathsf{Pol}_1(\mathbb{A})$ with $c = f(a)$, $d = f(b)$. By Lemma 14(3) $U' = f(U)$ is a $(\underline{0}, \alpha)$-minimal set.

(2) As in (1) one can argue that $(c', d') \in Q^{\mathbb{A}}_{ab}$, that is, $Q^{\mathbb{A}}_{ab} \cap (\mathsf{as}(c) \times \mathsf{as}(d)) \neq \varnothing$. We then complete by Lemma 36(4). □

**Lemma 39** *For any $\alpha \in \mathsf{Con}(\mathbb{A})$ with $\underline{0} \prec \alpha$ such that $|D| > 1$ for some as-component $D$ of an $\alpha$-block, the prime factor $\underline{0} \prec \alpha$ has type* **2** *or* **3**.

**Proof:** Let $a, b \in D$ for an as-component $D$ of an $\alpha$-block. Then by Corollary 37 there is a polynomial $f$ such that $f(a) = b$ and $f(b) = a$. Also, $a, b$ belong to some $(\underline{0}, \alpha)$-minimal set. This rules out types **4** and **5**. Since $\mathbb{A}$ omits type **1**, this only leaves types **2** and **3**. □

**Lemma 40** *Let $\alpha \in \mathsf{Con}(\mathbb{A})$ with $\underline{0} \prec \alpha$ be such that some $\alpha$-block contains a semilattice or majority edge. Then the prime factor $(\underline{0}, \alpha)$ has type* **3**, **4** *or* **5**.

**Proof:** We need to show that $(\underline{0}, \alpha)$ does not have type **2**. Let $B$ the $\alpha$-block containing a semilattice or majority edge. Then $B$ contains a non-Abelian subalgebra, which implies $(\underline{0}, \alpha)$ is also non-Abelian. □

## 5.2 Separation

Let $\mathbb{A}$ be an algebra, and let $\alpha \prec \beta$ and $\gamma \prec \delta$ be prime intervals in $\mathsf{Con}(\mathbb{A})$. We say that $\alpha \prec \beta$ can be *separated* from $\gamma \prec \delta$ if there is a unary polynomial $f \in \mathsf{Pol}_1(\mathbb{A})$ such that $f(\beta) \not\subseteq \alpha$, but $f(\delta) \subseteq \gamma$. The polynomial $f$ in this case is said to *separate* $\alpha \prec \beta$ from $\gamma \prec \delta$.

Since we often consider relations rather than single algebras, we also introduce separability in a slightly different way. Let $R$ be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$. Let $i, j \in [n]$ and let $\alpha_i \prec \beta_i$, $\alpha_j \prec \beta_j$ be prime intervals in $\mathsf{Con}(\mathbb{A}_i)$ and $\mathsf{Con}(\mathbb{A}_j)$, respectively. Interval $\alpha_i \prec \beta_i$ can be separated from $\alpha_j \prec \beta_j$ if there is a unary polynomial $f$ of $R$ such that $f(\beta_i) \not\subseteq \alpha_i$ but $f(\beta_j) \subseteq \alpha_j$. Similarly, the polynomial $f$ in this case is said to *separate* $\alpha_i \prec \beta_i$ from $\alpha_j \prec \beta_j$

First, we observe a connection between separation in a single algebra and in relations.

**Lemma 41** *Let $R$ be the binary equality relation on $\mathbb{A}$. Let $\alpha_1 = \alpha, \beta_1 = \beta$ be viewed as congruences of the first factor of $R$, and $\alpha_2 = \gamma, \beta_2 = \delta$ as congruences of the second factor of $R$. Prime interval $\alpha \prec \beta$ can be separated from $\gamma \prec \delta$ as intervals in $\mathsf{Con}(\mathbb{A})$ if and only if $\alpha_1 \prec \beta_1$ can be separated from $\alpha_2 \prec \beta_2$ in $R$.*

**Proof:** Note that for any polynomial $f$ its action on the first and second projections of $R$ are the same polynomial of $\mathbb{A}$. Therefore $\alpha \prec \beta$ can be separated from $\gamma \prec \delta$ in $\mathsf{Con}(\mathbb{A})$ if and only if, there is $f \in \mathsf{Pol}_1(\mathbb{A})$, $f(\beta) \not\subseteq \alpha$ while $f(\delta) \subseteq \gamma$. This condition can be expressed as follows: there is $f \in \mathsf{Pol}_1(R)$, $f(\beta_1) \not\subseteq \alpha_1$ while $f(\beta_2) \subseteq \alpha_2$, which precisely means that $\alpha_1 \prec \beta_1$ can be separated from $\alpha_2 \prec \beta_2$ in $R$. □

In what follows when proving results about separation we will always assume that we deal with a relation — a subdirect product — and that the prime intervals in question are from congruence lattices of different factors of the subdirect product. If this is not the case, one can duplicate the factor containing the prime intervals and apply Lemma 41.

Let $R$ be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $I \subseteq [n]$, and let $f$ be a polynomial of $\mathrm{pr}_I R$, that is, there are a term operation $g$ of $R$ and $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathrm{pr}_I R$ such that $f(x_1, \ldots, x_\ell) = g(x_1, \ldots, x_\ell, \mathbf{a}_1, \ldots, \mathbf{a}_k)$. The tuples $\mathbf{a}_i$ can be extended to tuples $\mathbf{a}'_i \in R$. Then the polynomial of $R$ given by $f(x_1, \ldots, x_\ell) = g(x_1, \ldots, x_\ell, \mathbf{a}'_1, \ldots, \mathbf{a}'_k)$ is said to be an *extension* of $f$ to a polynomial of $R$.

**Lemma 42** *Let $R$ be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $i, j \in [n]$, and $\alpha_i \prec \beta_i$, $\alpha_j \prec \beta_j$ for $\alpha_i, \beta_i \in \mathsf{Con}(\mathbb{A}_i)$, $\alpha_j, \beta_j \in \mathsf{Con}(\mathbb{A}_j)$. Let also a unary polynomial $f$ of $R$ separate $\alpha_i \prec \beta_i$ from $\alpha_j \prec \beta_j$. Then $f$ can be chosen idempotent and such that $f(\mathbb{A}_i)$ is a $(\alpha_i, \beta_i)$-minimal set.*

**Proof:** Let $g$ be a polynomial separating $(\alpha_i,\beta_i)$ from $(\alpha_j,\beta_j)$. Since $g(\beta_i)\not\subseteq\alpha_i$, by Lemma 14(6) there is an $(\alpha_i,\beta_i)$-minimal set $U$ such that $g(\beta_i|_U)\not\subseteq\alpha_i$. Let $V=g(U)$, by Lemma 14(2) $V$ is a $(\alpha_i,\beta_i)$-minimal set. Let $h$ be a unary polynomial such that $h$ maps $V$ onto $U$ and $h\circ g|_U$ is the identity mapping. Let also $h'$ be an extension of $h$ to a polynomial of $R$. Then $h'\circ g$ separates $i$ from $j$. Now $f$ can be chosen to be an appropriate power of $h'\circ g$. □

For a subdirect product $R\subseteq\mathbb{A}_1\times\cdots\times\mathbb{A}_n$ the relation 'cannot be separated' on prime factors of the $\mathbb{A}_i$s is clearly reflexive and transitive. If the algebras $\mathbb{A}_i$ are Mal'tsev, it is also symmetric (for partial results see [12, 21]). Moreover, it can be shown that it remains 'almost' symmetric when the $\mathbb{A}_i$s contain no majority edges. In the general case however the situation is more complicated. Next we introduce conditions that make the 'cannot be separated' relation to some extent symmetric, at least in what concerns our algorithm, as it will be demonstrated in Lemma 45.

Let $\beta_i\in\mathsf{Con}(\mathbb{A}_i)$, let $B_i$ be a $\beta_i$-block for $i\in[n]$, and let $B'_i=\mathrm{pr}_i(R\cap\overline{B})$. Let also $\mathcal{U}$ be a set of unary polynomials of $R$, $i$ an element from $[n]$, and $\alpha,\beta\in\mathsf{Con}(\mathbb{A}_i)$ with $\alpha\prec\beta\le\beta_i$. Let $T_{\mathbb{A}_i}(a',b';\alpha,\beta,\mathcal{U})\subseteq\beta/_\alpha\subseteq(\mathbb{A}_i/_\alpha)^2$ for $a',b'\in B'_i/_\alpha$, $(a',b')\in\beta-\alpha$, denote the set of pairs $(a,b)\in\beta/_\alpha$ such that there is a polynomial $g\in\mathcal{U}$ satisfying the following conditions: $g(\{a',b'\})=\{a,b\}$ and $g(\mathbb{A}_i)$ is a $(\alpha,\beta)$-minimal set. Note that these conditions imply that $\{a,b\}$ is a $(\alpha,\beta)$-subtrace. We say that $\alpha$ and $\beta$ are *$\mathcal{U}$-chained* in $R$ with respect to $\overline{\beta},\overline{B}$ if for any $a',b'\in B'_i/_\alpha$ with $(a',b')\in\beta-\alpha$, the following conditions hold:

(G1) For a $\beta/_\alpha$-block $E$ such that $E\cap\mathsf{umax}(B'_i)\ne\varnothing$ let $E'=E\cap B'_i/_\alpha$. Then $(a,b)\in T_{\mathbb{A}_i}(a',b';\alpha,\beta,\mathcal{U})$ for any $a,b$ from the same as-component of $E'$.

(G2) For any $\beta/_\alpha$-block $E$ such that $E\cap\mathsf{umax}(B'_i)\ne\varnothing$, and any $a,b\in\mathsf{umax}(E')$, where $E'=E\cap B'_i/_\alpha$, there is a sequence $a=a_1,\dots,a_k=b$ in $E'$ such that $\{a_i,a_{i+1}\}\in T_{\mathbb{A}_i}(a',b';\alpha,\beta,\mathcal{U})$ for any $i\in[k-1]$.

Also, if for elements $a,b$ and any $a',b'$ there is a sequence of elements satisfying (G2), we say that $a$ and $b$ are *subtrace connected*; congruences $\alpha,\beta,\overline{\beta}$, congruence classes $\overline{B}$, and set of polynomials $\mathcal{U}$ will always be clear from the context in this case. Observe that $\mathcal{U}$-chaining amounts to saying that polynomials from $\mathcal{U}$ do not allow any congruences of $\beta$-blocks viewed as subalgebras between $\alpha$ and $\beta$, at least where u-maximal elements are concerned.

A unary polynomial $f$ is said to be *$\overline{B}$-preserving* if $f(\overline{B})\subseteq\overline{B}$. We call relation $R$ *chained* with respect to $\overline{\beta},\overline{B}$ if

(Q1) for any $\alpha,\beta\in\mathsf{Con}(\mathbb{A}_i)$, $i\in[n]$, such that $\alpha\prec\beta\le\beta_i$, congruences $\alpha$ and $\beta$ are $\mathcal{U}_B$-chained in $R$, where $\mathcal{U}_B$ is the set of all $\overline{B}$-preserving polynomials of $R$

(Q2) for any $\alpha,\beta\in\mathsf{Con}(\mathbb{A}_i)$, $\gamma,\delta\in\mathsf{Con}(\mathbb{A}_j)$, $i,j\in[n]$, such that $\alpha\prec\beta\le\beta_i$, $\gamma\prec\delta\le\beta_j$, and $(\alpha,\beta)$ can be separated from $(\gamma,\delta)$, congruences $\alpha$ and $\beta$ are

$\mathcal{U}^*$-chained in $R$, where $\mathcal{U}^*$ is the set of all $\overline{B}$-preserving polynomials of $R$ such that $g(\delta) \subseteq \gamma$.

Polynomials from $\mathcal{U}^*$ in condition (Q2) will be called $(\gamma, \delta, \overline{B})$-*good*.

**Lemma 43** *(1) Any constant polynomial from $\overline{B} \cap R$ is $(\gamma, \delta, \overline{B})$-good.*

*(2) If $f$ is a $k$-ary term function of $R$ and $g_1, \ldots, g_k$ are $(\gamma, \delta, \overline{B})$-good polynomials, then $f(g_1(x), \ldots, g_k(x))$ is $(\gamma, \delta, \overline{B})$-good.*

*(3) Let $T(a', b')$ denote $T_{\mathbb{A}_i}(a', b'; \alpha, \beta, \mathcal{U})$ for $\mathcal{U} \in \{\mathcal{U}_{\overline{B}}, \mathcal{U}^*\}$. If $\{a, b\} \in T(a', b')$ then $T(a, b) \subseteq T(a', b')$.*

*(4) If there is a $\beta/_\alpha$-block $E$ such that $E \cap \mathsf{umax}(B'_i) \neq \varnothing$, let $E' = E \cap B'_i/_\alpha$, and if $E'$ contains a nontrivial as-component, then there is a set $T \subseteq \beta/_\alpha$ such that $T \subseteq T(a', b')$ for any $a', b' \in B'_i/_\alpha$, $a' \overset{\beta/\alpha}{\equiv} b'$ and $T$ satisfies conditions (G1),(G2) for $T(a', b')$.*

*(5) Let $a', b' \in B'_i/_\alpha$, $a' \overset{\beta/\alpha}{\equiv} b'$ be such that $T(a', b')$ is minimal among sets of this form. Then for any $(a, b) \in T(a', b')$ there is $h \in \mathcal{U}$ such that $h$ is idempotent and $h(a) \overset{\alpha}{\equiv} a, h(b) \overset{\alpha}{\equiv} b$.*

**Proof:** Items (1),(2) are straightforward.

(3) Let $\{a'', b''\} \in T(a, b)$. Then there are polynomials $f, g \in \mathcal{U}$ with $\{a, b\} = f(\{a', b'\})$ and $\{a'', b''\} = g(\{a, b\})$. Then $g \circ f \in \mathcal{U}$ by item (2) or definition and $g \circ f(\{a', b'\}) = \{a'', b''\}$.

(4) Take $a, b \in C$ where $C$ is a nontrivial as-component in $E'$. By (G2) $\{a, b\} \in T(a', b')$ for any appropriate $a', b'$. Therefore by (3) $T = T(a, b) \subseteq T(a', b')$.

(5) Let $\{a, b\} \in T(a', b')$. Then by (3) $T(a, b) \subseteq T(a', b')$, and therefore by the minimality of $T(a', b')$ we get $T(a, b) = T(a', b')$. The result follows by definition of $T(a', b')$. □

The next lemma shows how we will use the property of being chained.

**Lemma 44** *Let $R$ be a subdirect product of $\mathbb{A}_1, \ldots, \mathbb{A}_n$ chained with respect to $\overline{\beta}, \overline{B}$, where $\beta_i \in \mathsf{Con}(\mathbb{A}_i)$ and $B_i$ is a $\beta_i$-block, and $R' = R \cap (B_1 \times \cdots \times B_n)$, $B'_i = \mathrm{pr}_i R'$. Let also $\mathsf{lk}$ be the link congruence of $B'_i$ with respect to $\mathrm{pr}_{ij} R'$ for some $i, j \in [n]$, and $\delta = \mathsf{Cg}(\mathsf{lk})$ the congruence of $\mathbb{A}_i$ generated by $\mathsf{lk}$. Then for any $\gamma \in \mathsf{Con}(\mathbb{A}_i)$ with $\gamma \prec \delta$ it holds $(\delta/_\gamma)|_{\mathsf{umax}(E)} = (\mathsf{lk}/_\gamma)|_{\mathsf{umax}(E)}$ for every $\delta|_{B'_i}$-block $E$ u-maximal in $B'_i/_\delta$.*

**Proof:** If $\gamma \prec \delta$ then by the choice of $\delta$ there are $a, b \in B'_i$ with $(a, b) \in \delta - \gamma$. Let $E$ be the intersection of $B'_i$ with a $\delta$-block. We apply condition (Q1) to $\alpha = \gamma \wedge \beta_i$ and $\beta = \delta \wedge \beta_i$. By condition (Q1) for any $a', b' \in \mathsf{umax}(E)$ there is a sequence $a' = a_1, \dots, a_k = b'$ such that $\{a_\ell, a_{\ell+1}\} = f_\ell(\{a, b\})$ for some $\overline{B}$-preserving polynomial $f_\ell$ for each $\ell \in [k-1]$. This means that $(a_\ell, a_{\ell+1}) \in \mathsf{lk}$, and so $(a', b') \in \mathsf{lk}$. $\square$

The following lemma establishes the weak symmetricity of separability relation mentioned before.

**Lemma 45** *Let $R$ be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $\beta_i \in \mathsf{Con}(\mathbb{A}_i)$, $B_i$ a $\beta_i$-block such that $R$ is chained with respect to $\overline{\beta}, \overline{B}$; $R' = R \cap \overline{B}$, $B'_i = \mathrm{pr}_i R'$. Let also $\alpha \prec \beta \leq \beta_1$, $\gamma \prec \delta = \beta_2$, where $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_1)$, $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_2)$. If $B'_2/_\gamma$ has a nontrivial as-component $C'$ and $(\alpha, \beta)$ can be separated from $(\gamma, \delta)$, then there is a $\overline{B}$-preserving polynomial $g$ such that $g(\beta|_{B'_1}) \subseteq \alpha$ and $g(\delta) \not\subseteq \gamma$. Moreover, for any $c, d \in C'$ polynomial $f$ can be chosen such that $f(c) = c, f(d) = d$.*

**Proof:** As is easily seen, we can assume that $\alpha, \gamma$ are equality relations. We need to show that there is $g$ such that $g$ collapses $\beta$ but does not collapse $\beta_2 = \delta$.

First we show that there are $c, d \in B'_2$ such that for any $(a, b) \in \beta|_{B'_1}$ there is a polynomial $h^{ab}$ of $R$ such that

(1) $h^{ab}$ is idempotent;

(2) $h^{ab}(a) = h^{ab}(b)$;

(3) $h^{ab}(c) = c$, $h^{ab}(d) = d$.

We consider two cases.

CASE 1. There is an element $c$ from a nontrivial as-component of $B'_2$ such that $(a, c, \mathbf{b}) \in R'$ for some $a \in B'$, a $\beta$-block such that $B' \cap \mathsf{umax}(B'_1) \neq \varnothing$ and $|\mathsf{umax}(B' \cap B'_1)| > 1$.

First, we choose $d$ to be any element other than $c$ of the nontrivial as-component $C'$ of $B'_2$ containing $c$. Let $T_1$ be the minimal set of $(\alpha, \beta)$-subtraces as in Lemma 43(4) for $\mathcal{U}^*$, the set of $(\gamma, \delta, \overline{B})$-good polynomials. We start with the case when $(a, b) \in T_1$. Even more specifically, as $c$ is as-maximal in $B'_2$, by the Maximality Lemma 27(5) $a$ can be chosen from $\mathsf{umax}(B' \cap B'_1)$. Take an $(\alpha, \beta)$-subtrace $\{a, b\} \in T_1$, such a subtrace exists by condition (Q1).

By $Q^* \subseteq \mathbb{A}_1^2 \times \mathbb{A}_2^2 \times R$ we denote the relation generated by $\{(a, b, c, d, \mathbf{a})\} \cup \{(x, x, y, y, \mathbf{z}) \mid \mathbf{z} \in R, \mathbf{z}[1] = x, \mathbf{z}[2] = y\}$, where $\mathbf{a}$ is an arbitrary element from $R'$. Let $Q = \mathrm{pr}_{1234} Q^*$ and $Q' = \mathrm{pr}_{1234}(Q^* \cap (B'_1 \times B'_1 \times B'_2 \times B'_2 \times \overline{B}))$. Observe that $Q$ is exactly the set of quadruples $(f(a), f(b), f(c), f(d))$ for unary polynomials $f$ of $R$ and $Q'$ is exactly the set of quadruples $(f(a), f(b), f(c), f(d))$ for $\overline{B}$-preserving unary polynomials $f$ of $R$. We prove that $Q'$ contains a quadruple of the form $(a', a', c, d)$; the result then follows.

Let also $Q_1 = \mathrm{pr}_{1,2}Q = Q_{ab}^{\mathbb{A}_1}$, $Q_2 = \mathrm{pr}_{3,4}Q = Q_{cd}^{\mathbb{A}_2}$; set $Q'_1 = \mathrm{pr}_{1,2}Q'$, $Q'_2 = \mathrm{pr}_{3,4}Q'$. Note that $\mathrm{pr}_1 Q' = \mathrm{pr}_2 Q' = B'_1$ and $\mathrm{pr}_3 Q' = \mathrm{pr}_4 Q' = B'_2$, because $\mathrm{pr}_{12} R' \subseteq \mathrm{pr}_{13} Q', \mathrm{pr}_{24} Q'$. Let $\mathsf{lk}_1, \mathsf{lk}_2$ denote the link congruences of $Q'$ viewed as a subdirect product of $Q'_1$ and $Q'_2$. Note that these congruences may be different from the link congruences of $Q$ restricted to $Q_1 \cap (B'_1 \times B'_1)$, $Q_2 \cap (B'_2 \times B'_2)$, respectively. We show that $(a', a')$ for some $a' \in B'_1$ is as-maximal in a $\mathsf{lk}_1$-block, $(c, d)$ is as-maximal in a $\mathsf{lk}_2$-block, and $Q' \cap (\mathsf{as}(a', a') \times \mathsf{as}(c, d)) \neq \varnothing$. By the Rectangularity Corollary 34 this implies the result.

Claim 1. $(\alpha \times \beta)|_{Q'_1} \subseteq \mathsf{lk}_1$ and $(\gamma \times \delta)|_{Q'_2} \subseteq \mathsf{lk}_2$.

We first prove that $(\alpha \times \beta)|_{Q'_1} \subseteq \mathsf{lk}_1$. Relation $Q'$ contains tuples $(a, b, c, d)$, $(a, b, c', c')$, $(a, a, c', c')$, $(a, a, c, c)$ for some $c' \in B'_2$. Indeed, $(a, b, c, d) \in Q'$ by definition, $(a, a, c, c) \in Q$ because $(a, c, \mathbf{b}) \in R$, and $(a, b, c', c'), (a, a, c', c')$ can be chosen to be the images of $(a, b, c, d)$ and $(a, a, c, c)$, respectively, under a $\overline{B}$-preserving polynomial $g^{ab}$ such that $g^{ab}(a) = a$, $g^{ab}(b) = b$ and $g^{ab}(\delta) \subseteq \gamma$. Such a polynomial exists because $R$ is chained and because $(\alpha, \beta)$ can be separated from $(\gamma, \delta)$. This implies that $(c, d) \stackrel{\mathsf{lk}_2}{\equiv} (c, c)$. Let $\eta_1, \eta_2$ be congruences of $Q_1, Q_2$ generated by $((a, b), (a, a))$ and $((c, d), (c, c))$, respectively. Then

$$\eta_1|_{Q'_1} = (\alpha \times \beta)|_{Q'_1}, \quad \text{and} \quad \eta_2|_{Q'_2} = (\gamma \times \delta)|_{Q'_2}.$$

Indeed, in the case of, say, $\alpha \times \beta$, relation $Q'_1$ consists of pairs $(g(a), g(b))$ for a $\overline{B}$-preserving unary polynomial $g$ of $\mathbb{A}_1$. Since $(a, b) \stackrel{\alpha \times \beta}{\equiv} (a, a)$, for any $(a', b') \in Q'_1$ it holds that

$$(a', b') = (g(a), g(b)) \stackrel{\eta_1}{\equiv} (g(a), g(a)) = (a', a').$$

For $Q'_2$ and $\gamma \times \delta$ the argument is similar. Since $(a, b), (a, a)$ are in the same $\mathsf{lk}_1$-block, $(\alpha \times \beta)|_{Q'_1} \subseteq \mathsf{lk}_1$.

Observing from the same tuples as before that $(c, d) \stackrel{\mathsf{lk}_2}{\equiv} (c, c)$, we prove $(\gamma \times \delta)|_{Q'_2} \subseteq \mathsf{lk}_2$ by a similar argument. Claim 1 is proved.

Claim 2. Let $E = B' \cap B'_1$, where $B'$ is the $\beta$-block containing $a, b$. Then $(\beta \times \beta)|_{\mathsf{umax}(E) \times \mathsf{umax}(E)} \subseteq \mathsf{lk}_1$.

By the assumption for any $(\alpha, \beta)$-subtrace $(a', b') \in T_1 \cap E^2$ there is a $\overline{B}$-preserving polynomial $g^{a'b'}$ satisfying $g^{a'b'}(a') = a'$, $g^{a'b'}(b') = b'$, and $g^{a'b'}(B'_2) = \{c'\} \subseteq B'_2$. Applying $g^{a'b'}$ to tuples $(a, b, c, d)$, $(a, a, c, c)$, and $(b, b, d', d')$ for any

$d'$ such that $(b,d) \in \mathrm{pr}_{12}R'$, we obtain $(a',b',c',c'), (a',a',c',c'), (b',b',c',c') \in Q'$. The second two tuples imply that $(a',a') \stackrel{\mathsf{lk}_1}{\equiv} (b',b')$, and therefore $(a'',a'') \stackrel{\mathsf{lk}_1}{\equiv} (b'',b'')$ for any $a'',b'' \in \mathsf{umax}(E)$. Along with Claim 1 this proves the result.

CLAIM 3. $(c,d)$ is as-maximal in a $\mathsf{lk}_2$-block.

If for some $e,e' \in B'_2$ we have $(e,e) \stackrel{\mathsf{lk}_2}{\equiv} (e',e')$, then, as $(e,e')$ generates $\delta$, for any $(\gamma,\delta)$-subtrace $\{e'',e'''\} \in T_{\mathbb{A}_2}(e,e') = T_{\mathbb{A}_2}(e,e';\gamma,\delta,\mathcal{U}_{\overline{B}})$ there is a $\overline{B}$-preserving polynomial $f'$ with $f'(\{e,e'\}) = \{e'',e'''\}$. Applying this polynomial to the tuples witnessing that $(e,e) \stackrel{\mathsf{lk}_2}{\equiv} (e',e')$ we get $(e'',e'') \stackrel{\mathsf{lk}_2}{\equiv} (e''',e''')$. Therefore by condition (Q1) all tuples of the form $(x,x)$, $x \in \mathsf{umax}(B'_2)$, are $\mathsf{lk}_2$-related. Since by condition (G1) $\{c,d\}$ is a $(\gamma,\delta)$-subtrace from $T_{\mathbb{A}_2}(c,d) \subseteq T_{\mathbb{A}_2}(e,e')$, using Claim 1 this implies that $\mathsf{lk}_2|_{Q''} = (\delta \times \delta)|_{Q''}$, where $Q'' = Q'_2 \cap (\mathsf{umax}(B'_2) \times \mathsf{umax}(B'_2))$. In particular, $C' \times C'$, where $C'$ is the as-component of $B'_2$ containing $c,d$, is contained in $Q'_2$, and is contained in a $\mathsf{lk}_2$-block. All elements of $C' \times C'$ are as-maximal in $Q''$.

Otherwise, since the inclusion $(\gamma \times \delta)|_{Q'_2} \subseteq \mathsf{lk}_2$ implies that if $(c_1,d_1) \stackrel{\mathsf{lk}_2}{\equiv} (c_2,d_2)$ then $(c_1,c_1) \stackrel{\mathsf{lk}_2}{\equiv} (c_2,c_2)$, by Claim 1 we have $\mathsf{lk}_2|_{Q''} = (\gamma \times \delta)|_{Q''}$. In particular, $\{c\} \times C'$ is contained in a $\mathsf{lk}_2$-block. Since $c,d$ are as-maximal, $(c,d)$ is as-maximal in this $\mathsf{lk}_2$-block. Claim 3 is proved.

By the Maximality Lemma 27(5) there is an element $(a',b')$ as-maximal in a $\mathsf{lk}_1$-block $D$ such that $(a',b',c,d) \in Q'$. If $a' = b'$ then we are done. Otherwise by condition (G1) and Lemma 43(3) $\{a',b'\}$ is an $(\alpha,\beta)$-subtrace from $T_1$, also $(a',c) \in R$ because $\mathrm{pr}_{1,3}Q \subseteq R$, and we can replace $a,b$ with $a',b'$. Observe that if we show the existence of a polynomial $g$ such that $g(a') = g(b')$ and $g(c) = c$, $g(d) = d$, this will witness the existence of $g'$ with $g'(a) = g'(b)$ and $g'(c) = c$, $g'(d) = d$. Let $E = a'^{\beta} \cap B'_1$. Note that by Claim 2 $(\beta \times \beta)|_{\mathsf{umax}(E) \times \mathsf{umax}(E)} \subseteq \mathsf{lk}_1$ by the Rectangularity Corollary 34 $\mathsf{umax}(D) \times \mathsf{as}(c,d) \subseteq Q'$. Therefore, again $a',b'$ can be chosen as-maximal in $E$. We use $a,b$ for $a',b'$ from now on.

CLAIM 4. $(a,a)$ is as-maximal in $Q''_1 = Q'_1 \cap (E \times E)$.

Let $\eta_1,\eta_2$ be the link congruences of $B'_1, B'_2$, respectively, with respect to $Q'_1$; as $Q'_1 \subseteq Q^{\mathbb{A}_1}_{ab}$ we have $\eta_1,\eta_2 \le \beta$. On the other hand, since $Q'_1$ consists of pairs of the form $(x,x)$ and $(\alpha,\beta)$-subtraces, and since $\mathsf{umax}(E)$ belongs to a block of the transitive closure of $T_1$, it is easy to see that $\mathsf{umax}(E)$ is a subset of both a $\eta_1$- and $\eta_2$-blocks. Indeed, let $e,e' \in \mathsf{umax}(E)$ and $e = e_1,\dots,e_k = e'$ are such that $\{e_i,e_{i+1}\} \in T_1$. This means that either $(e_i,e_{i+1}) \in Q'_1$ or $(e_{i+1},e_i) \in Q'_1$. Since $(e_i,e_i),(e_{i+1},e_{i+1}) \in Q'_1$ by construction, in either case we have $(e_i,e_{i+1}) \in \eta_1,\eta_2$.

Let $E'$ be the as-component of $E$ containing $a$; such an as-component exists by the choice of $a, b$. As $(a,a) \in Q'_1 \cap (E' \times E') \neq \varnothing$, by the Rectangularity Corollary 34 $E' \times E' \subseteq Q'_1$. Since $E'$ is an as-component in $E$, by Lemma 28 $E' \times E'$ is an as-component in $Q''_1$. In particular $(a,a)$ is as-maximal in $Q''_1$. Claim 4 is proved.

CLAIM 5. $(a,a,c,d) \in Q'$.

To prove this claim we find a subalgebra $Q''$ of $Q'$ such that it is linked enough and both $(a,a)$ and $(c,d)$ belong to as-components of $\mathrm{pr}_{12}Q'', \mathrm{pr}_{34}Q''$, respectively, and then apply the Rectangularity Corollary 34.

Let $F$ be the as-component of the $\mathsf{lk}_2$-block containing $(c,d)$. By Claim 3 it is either $\{c\} \times C'$ or $C' \times C'$. Since $Q''_1$ belongs to a $\mathsf{lk}_1$-block and $(a,a,c,c) \in Q'$, by the Maximality Lemma 27(4) for any $(a',b') \in E' \times E'$ there are $(c',d') \in F$ such that $(a',b',c'd') \in Q'$. Now consider $Q'' = Q' \cap (E \times E \times B'_2 \times B'_2)$. Clearly, $Q'_1 \subseteq \mathrm{pr}_{12}Q''$. Also, since $(a,b)$ is as-maximal in a $\mathsf{lk}_1$-block, by the Rectangularity Corollary 34 $\{(a,b)\} \times F \subseteq Q''$, implying $F \subseteq \mathrm{pr}_{34}Q''$. If $\theta_1, \theta_2$ denote the link congruences of $\mathrm{pr}_{12}Q'', \mathrm{pr}_{34}Q''$ with respect to $Q''$, the observation above implies that the as-components of $\mathrm{pr}_{12}Q''$ containing $(a,a)$ and $(a,b)$ belong to the same $\theta_1$-block, and $F$ belongs to a $\theta_2$-block. Therefore again by the Rectangularity Corollary 34 we get $(E' \times E') \times (\{c\} \times C') \subseteq Q'$, in particular $(a,a,c,d) \in Q$. Thus, there is a polynomial $h$ such that $h(a) = h(b) = a$ and $h(c) = c, h(d) = d$.

So far we have proved that for any subtrace $(a,b) \in T_1$, where $a$ is a fixed element such that $(a,c) \in \mathrm{pr}_{12}R'$, there is a polynomial $h^{ab}$ satisfying the conditions stated in the beginning of the proof.

CLAIM 6. For every $(\alpha,\beta)$-subtrace $\{a',b'\}$ from $T_1 \cap (E \times E)$ (recall that $E = a^{\beta} \cap B'_1$) there is a polynomial $h$ such that $h(a) = h(b)$ and $h(c) = c, h(d) = d$.

Let us consider $T_1$ as a graph; we can introduce the distance $r(x)$ of element $x$ from $a$. In particular, all elements from $\mathsf{umax}(E)$ belong to the connected component containing $a$. Let $D(i) \subseteq E$ denote the set of elements at distance at most $i$ from $a$. By what is proved above there is a composition $h^*$ of polynomials $h^{ab}$ for $b \in D(1)$ such that $h^*(D(1)) \subseteq \{a\}$. By Lemma 14(2) every $\overline{B}$-preserving polynomial maps every $(\alpha,\beta)$-subtrace either to a singleton, or to a $(\alpha,\beta)$-subtrace from $T_1$. Therefore by induction we also get $h^*(D(i+1)) \subseteq D(i)$. Therefore composing several copies of $h^*$ collapses $a'$ and $b'$ and leaves $c, d$ unchanged.

We now can prove the result in Case 1. Thus, we have proved that for any $(\alpha,\beta)$-subtrace $(a',b') \in T_1$ from a $\beta$-block $E$ such that $\mathrm{pr}_{12}R' \cap (E \times \{c\}) \neq \varnothing$, a polynomial $h^{a'b'}$ with the required properties exists. Suppose now that $a',b' \in B'_1$ be any such that $(a',b') \in \beta$. Take any $c',d' \in B'_2$ such that $(a,c') \in Q'$.

By Lemmas 36 and 43(5) there is an idempotent $\overline{B}$-preserving polynomial $g$ such that $g(c') = c, g(d') = d$. If $g(a') = g(b')$, we are done, as $g$ may serve as $h^{a'b'}$. Otherwise, as before consider the relation $Q^\dagger \subseteq \mathbb{A}_1^2 \times \mathbb{A}_2^2 \times R$ generated by $\{(g(a'), g(b'), c, d, \mathbf{a})\} \cup \{(x, x, y, y, \mathbf{z}) \mid \mathbf{z} \in R, \mathbf{z}[1] = x, \mathbf{z}[2] = y\}$, where $\mathbf{a}$ is an arbitrary element from $R'$, $Q^\ddagger = \mathrm{pr}_{1234}(Q^\dagger \cap (B'_1 \times B'_1 \times B'_2 \times B'_2 \times \overline{B}))$, $Q''_1 = \mathrm{pr}_{12}Q^\ddagger$, $Q''_2 = \mathrm{pr}_{34}Q^\ddagger$, and the link congruences $\mathsf{lk}''_1, \mathsf{lk}''_2$ of $Q''_1, Q''_2$ with respect to $Q^\ddagger$. By Claim 1 $(c, d)$ is as-maximal in a $\mathsf{lk}''_2$-block. By the Maximality Lemma 27(4) there is u-maximal $(a'', b'') \in Q''_1$ such that $(a'', b'', c, d) \in Q'$ and $(g(a'), (b')) \sqsubseteq_{as} (a'', b'')$. In particular, $a'', b''$ can be chosen u-maximal in a $\beta$-block $E$ such that $\mathrm{pr}_{12}R' \cap (E \times \{c\}) \neq \varnothing$. By what was proved there is a polynomial $h^{a''b''}$ with $h^{a''b''}(a'') = h^{a''b''}(b'')$ and $h^{a''b''}(c) = c, h^{a''b''}(d) = d$. Also, as $(a'', b'', c, d) \in Q^\ddagger$, there is a polynomial $h'$ with $h'(a') = a''$, $h'(b') = b''$, and $h'(c) = c$, $h'(d) = d$. Then set $h^{a'b'} = h^{a''b''} \circ h' \circ g$.

CASE 2. For every element $c$ from a nontrivial as-component of $B'_2$ and any $a \in B'_1$ such that $(a, c) \in R$ element $a$ belongs to a $\beta$-block $B'$ such that $B' \cap \mathsf{umax}(B'_1) = \varnothing$ or $|B\mathsf{umax}(B' \cap B'_1)| = 1$.

We use the same elements $c, d \in C'$, an as-component of $B'_2$. For any $(a, b) \in \beta|_{B'_1}$ choose $c', d' \in B'_2$ such that $(a, c'), (b, d') \in \mathrm{pr}_{12}R'$. (Recall that we are assuming $\alpha$ and $\gamma$ to be equality relations.) If $c' = d'$, that is, $(b, c') \in R$, choose $d'$ to be an arbitrary element from $B'_2$. By Lemmas 36, 43(5) and because $R$ is chained there is an idempotent $\overline{B}$-preserving polynomial $g$ such that $g(c') = c, g(d') = d$. Let $g(a) = a'$, $g(b) = b'$. Then $(a', c) \in R$ and $b' \stackrel{\beta}{\equiv} a'$. Since $g$ is $\overline{B}$-preserving, $b' \in B'_1$. We again as in Case 1 consider the relation $Q^* \subseteq \mathbb{A}_1^2 \times \mathbb{A}_2^2 \times R$ generated by $\{(a', b', c, d, \mathbf{a})\} \cup \{(x, x, y, y, \mathbf{z}) \mid \mathbf{z} \in R, \mathbf{z}[1] = x, \mathbf{z}[2] = y\}$, where $\mathbf{a}$ is an arbitrary element from $R'$, $Q^{**} = \mathrm{pr}_{1234}(Q^* \cap (B'_1 \times B'_1 \times B'_2 \times B'_2 \times \overline{B}))$, $Q^*_1 = \mathrm{pr}_{12}Q^{**}$, $Q^*_2 = \mathrm{pr}_{34}Q^{**}$, and the link congruences $\mathsf{lk}^*_1, \mathsf{lk}^*_2$ of $Q^*_1, Q^*_2$ with respect to $Q'$. By Claim 1 $(c, d)$ is as-maximal in a $\mathsf{lk}^*_2$-block. By the Maximality Lemma 27(4) there is u-maximal $(a'', b'') \in Q^*_1$ such that $(a'', b'', c, d) \in Q^{**}$ and $(a', b') \sqsubseteq_{as} (a'', b'')$. Since $Q^*_1 \subseteq Q^{\mathbb{A}_1}_{a'b'}$, $a'' \stackrel{\beta}{\equiv} b''$. Therefore, $a'', b''$ can be chosen u-maximal in their $\beta$-block, and by the assumption of Case 2 $a'' = b''$. This means there is a $\overline{B}$-preserving polynomial $h$ with $h(a') = h(b')$ and $h(c) = c, h(d) = d$. Polynomial $h^{ab}$ can now be chosen to be $h \circ g$.

Finally, we use polynomials $h^{ab}$ to construct a single polynomial that collapses $\beta$ on $\mathsf{umax}(E)$, where $E = B' \cap B'_1$ for every $\beta$-block $B'$. To this end it suffices that such a polynomial collapses every $(\alpha, \beta)$-subtrace from $T_1$. Fix $c, d$ and $h^{ab}$ for all subtraces $\{a, b\} \in T_1$. Let $T_1 = \{V_1, \dots, V_k\}$, and if $V_\ell = \{a, b\}$ is the subtrace number $\ell$, $h^\ell$ denotes $h^{ab}$. Take a sequence $1 = \ell_1, \ell_2, \dots$ such that $h^{(1)} = h_{\ell_1}$, $V_{\ell_2}$ is a subset of $h^{(1)}(\mathbb{A})$, and, for $s > 2$, $V_{\ell_s}$ is a subset of the range of $h^{(s-1)} =$

$h^{\ell_{s-1}} \circ \ldots \circ h^{\ell_1}$. Since $|\mathsf{Im}\ (h^{(s)})| < |\mathsf{Im}\ (h^{(s-1)})|$, there is $r$ such that $\mathsf{Im}\ (h^{(r)})$ contains no subtrace $V_\ell$ for any $\ell$. Therefore setting $h(x) = h_{\ell_r} \circ \ldots \circ h_{\ell_1}(x)$ we have that $h$ collapses all the pairs $V_\ell$, and $h$ acts identically on $\{c, d\}$. Since $T_1$ connects all elements in $\mathsf{umax}(B'_1)$, the result follows. □

## 5.3 Collapsing polynomials

We say that prime factors $(\alpha, \beta)$ and $(\gamma, \delta)$ *cannot be separated* if $(\alpha, \beta)$ cannot be separated from $(\gamma, \delta)$ and $(\gamma, \delta)$ cannot be separated from $(\alpha, \beta)$. In this section we introduce and prove the existence of polynomials that collapse all prime intervals in congruence lattices of factors of a subproduct, except for a set of factors that cannot be separated from each other.

**Lemma 46** *(1) Let $\mathbb{A}$ be an algebra. If prime factors $\alpha \prec \beta$ and $\gamma \prec \delta$ in $\mathsf{Con}(\mathbb{A})$ are perspective, then they cannot be separated.*
*(2) If $\alpha \prec \beta$ and $\gamma \prec \delta$ from $\mathsf{Con}(\mathbb{A})$ cannot be separated, then a set $U$ is a $(\alpha, \beta)$-minimal set if and only if it is a $(\gamma, \delta)$-minimal set.*
*(3) Let $R$ be a subdirect product of $\mathbb{A}$ and $\mathbb{B}$, $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\gamma, \delta \in \mathbb{B}$ such that $\alpha \prec \beta$, $\gamma \prec \delta$, and let $\alpha \prec \beta$ and $\gamma \prec \delta$ cannot be separated. Then for any $(\alpha, \beta)$-minimal set $U$ there is a unary idempotent polynomial $f$ such that $f(\mathbb{A}) = U$ and $f(\mathbb{B})$ is a $(\gamma, \delta)$-minimal set.*

**Proof:** (1) Follows from Lemma 16.

(2) Let $f$ be a polynomial of $\mathbb{A}$ such that $f(\mathbb{A}) = U$ and $f(\beta) \not\subseteq \alpha$. Since $(\alpha, \beta)$ cannot be separated from $(\gamma, \delta)$, we have $f(\delta) \not\subseteq \gamma$ and therefore $U$ contains a $(\gamma, \delta)$-minimal set $U'$. If $U' \neq U$, there is a polynomial $g$ with $g \circ f(\delta) \not\subseteq \gamma$ and $g \circ f(\mathbb{A}) = U'$. In particular, $|g(U)| < |U|$, and so $g \circ f(\beta) \subseteq \alpha$; a contradiction with the assumption that $(\gamma, \delta)$ cannot be separated from $(\alpha, \beta)$.

(3) Take an idempotent polynomial $g$ of $R$ such that $g(\mathbb{B})$ is a $(\gamma, \delta)$-minimal set. Then, as $(\gamma, \delta)$ cannot be separated from $(\alpha, \beta)$, $g(\beta) \not\subseteq \alpha$. By Lemma 14(6) there is an $(\alpha, \beta)$-minimal set $U' \subseteq g(\mathbb{A})$. Let $g', h$ be polynomials of $R$ such that $g'(U) = U'$, $h(U') = U$ and $h(\mathbb{A}) = U$, which exist by Lemma 14(1). Then $h' = h \circ g \circ g'$ is such that $h'(\mathbb{A}) = h'(U) = U$, $h'(\beta) \not\subseteq \alpha$ and therefore $h'(\delta) \not\subseteq \gamma$. Then iterating $h'$ sufficiently many times we get an idempotent polynomial $f$ satisfying the same properties. □

Let $R$ be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, and choose $\beta_j \in \mathsf{Con}(\mathbb{A}_j)$, $j \in [n]$. Let also $i \in [n]$, and $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_i)$ be such that $\alpha \prec \beta \leq \beta_i$; let also $B_j$ be a $\beta_j$-block. We call an idempotent unary polynomial $f$ of $R$ *$\alpha\beta$-collapsing for $\overline{\beta}, \overline{B}$* if $f$ is $\overline{B}$-preserving, $f(\beta) \not\subseteq \alpha$, $f(\delta|_{B_j}) \subseteq \gamma|_{B_j}$ for every $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_j)$, $j \in [n]$, with $\gamma \prec \delta \leq \beta_j$, and such that $(\alpha, \beta)$ can be separated from $(\gamma, \delta)$ or $(\gamma, \delta)$ can be separated from $(\alpha, \beta)$, and $|f(R)|$ is minimal possible.

**Lemma 47** *Let $R$, $\alpha, \beta$, and $\beta_j$, $j \in [n]$, be as above and $R$ chained with respect to $\overline{\beta}, \overline{B}$. Let also $R' = R \cap \overline{B}$. Then if $\beta = \beta_i$ and $\mathrm{pr}_i R'/_\alpha$ contains a nontrivial as-component, then there exists an $\alpha\beta$-collapsing polynomial for $\overline{\beta}, \overline{B}$.*

**Proof:** Suppose $i = 1$, let $B_1' = \mathrm{pr}_1 R'$ and $C$ be a nontrivial as-component of $B_1'/_\alpha$. Take a $(\alpha, \beta)$-subtrace $\{a, b\} \subseteq B_1'$ such that $a^\alpha, b^\alpha \in C$. Since $R$ is chained with respect to $\overline{\beta}, \overline{B}$, by (Q1) and Lemma 43(5) there is a $\overline{B}$-preserving idempotent polynomial $f$ of $R$ such that $f(\mathbb{A}_1)$ is an $(\alpha, \beta)$-minimal set and $a^\alpha, b^\alpha \in f(\mathbb{A}_1)/_\alpha$. Let polynomial $f$ be such that $f(R)$ is minimal possible. We show that $f$ is $\alpha\beta$-collapsible.

Let $j \in [n]$ and $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_j)$ be such that $\gamma \prec \delta \leq \beta_j$, and $(\alpha, \beta), (\gamma, \delta)$ can be separated. Since $R$ is chained, by Lemma 45 there is an idempotent unary $\overline{B}$-preserving polynomial $f_{j\gamma\delta}$ of $R$ such that $f_{j\gamma\delta}(\mathbb{A}_1)$ is an $(\alpha, \beta)$-minimal set with $a^\alpha, b^\alpha \in f_{j\gamma\delta}(\mathbb{A}_1)/_\alpha$ and $f_{j\gamma\delta}(\delta|_{B_j}) \subseteq \gamma|_{B_j}$. Then if $f(\delta|_{B_j}) \not\subseteq \gamma$, then let $g = f_{j\gamma\delta} \circ f$. We have $g(\beta) \not\subseteq \alpha$, but $g(\delta|_{B_j}) \subseteq \gamma$ implying $|g(R)| < |f(R)|$, a contradiction with minimality of $f(R)$. $\square$

## 5.4 Separation and minimal sets

In this section we show a connection between the fact that two prime intervals cannot be separated, their types, and link congruences.

**Lemma 48** *Let $R$ be a subdirect product of $\mathbb{A}$ and $\mathbb{B}$ and let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\gamma, \delta \in \mathsf{Con}(\mathbb{B})$ be such that $\alpha \prec \beta$, $\gamma \prec \delta$, and $(\alpha, \beta), (\gamma, \delta)$ cannot be separated. Let also $\mathsf{lk}_1, \mathsf{lk}_2$ be the link congruences of $\mathbb{A}, \mathbb{B}$, respectively. If $\mathsf{typ}(\alpha, \beta) \neq \mathbf{2}$ then $\mathsf{lk}_1 \wedge \beta \leq \alpha$, $\mathsf{lk}_2 \wedge \delta \leq \gamma$.*

**Proof:** Assume as usual $\alpha = \underline{0}_{\mathbb{A}}$, $\gamma = \underline{0}_{\mathbb{B}}$. By Lemma 46(3) there is a unary polynomial $f$ such that $f(\mathbb{A}_1) = U_1$, $f(\mathbb{A}_2) = U_2$ are $(\underline{0}_1, \alpha_1)$- and $(\underline{0}_2, \alpha_2)$-minimal sets, respectively. We first study the structure of $R \cap (U_1 \times U_2)$ and then show how it can be used to prove the lemma. By $N_1 = \{0, 1\}$ we denote the only trace of $U_1$; by $T_1$ we denote the tail of $U_1$. By Lemma 17 there is a polynomial $p(x, y)$ with $p(\mathbb{A}_1, \mathbb{A}_1) = U$ and such that $p$ is a semilattice operation on $N$, say, $p(0, 1) = 0$, and $p$ is a semilattice operation on $\{0, a\}, \{1, a\}$ with $p(a, 0) = p(a, 1) = a$ for any $a \in T_1$. There are two cases.

CASE 1. $\mathsf{typ}(\gamma, \delta) \neq \mathbf{2}$.

Let $N_2 = \{0', 1'\}$ be the trace of $U_2$ and $T_2$ the tail of $U_2$. We may assume $p(x, p(x, y)) = p(x, y)$. Observe first that $p$ preserves $N_2$. Indeed, otherwise $p(x, x)$ is not a permutation, as $p(0', 0'), p(0', 1'), p(1', 0'), p(1', 1')$ belong to the same $\beta_2$-block, and if they do not belong to $N_2$ then they are all equal, a contradiction with the assumption that $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ cannot be separated. A binary operation on a 2-element set is either a projection, or a semilattice operation.

Suppose first that $p$ is a projection, say, the first projection on $N_2$. If $(\{0\} \times N_2) \cap R \neq \varnothing$, say, $(0, a) \in R$, then $f'(x) = p\left(x, \begin{pmatrix} 0 \\ a \end{pmatrix}\right)$ satisfies the conditions: $f'(N_1) = \{0\}$, that is, $f'(\alpha_1) \subseteq \underline{0}_1$, and $f'(x) = x$ on $N_2$; a contradiction that $(\gamma, \delta)$ cannot be separated from $(\alpha, \beta)$. If $(\{1\} \times N_2) \cap R \neq \varnothing$, say, $(1, a) \in R$, then $f'(x) = p\left(\begin{pmatrix} 1 \\ a \end{pmatrix}, x\right)$ satisfies the conditions: $f'(x) = x$ on $N_1$, that is, $f'(\beta_1) \not\subseteq \underline{0}_1$, and $f'(N_2) = \{a\}$ on $N_2$; a contradiction that $(\alpha, \beta)$ cannot be separated from $(\gamma, \delta)$. Therefore, for some $a \in T_1$, $(a, 1') \in R$. The operation $f' = p\left(x, \begin{pmatrix} a \\ 1' \end{pmatrix}\right)$ is the projection on $N_2$ and $f'(N_1) = \{a\}$; a contradiction again.

Suppose now that $p$ is a semilattice operation on $N_2$. Let $1'$ be the neutral element of $p$. If $(a, 1') \in R$ for some $a \in T_1$, then $f'(x) = p\left(x, \begin{pmatrix} a \\ 1' \end{pmatrix}\right)$ is the projection on $N_2$, and $f'(N_1) = \{a\}$. If $(a, 1') \in R$ for no $a \in T_1$, then we continue as follows. If $(0, 1')$ or $(1, 0')$ belong to $R$, then one of the operations $p\left(x, \begin{pmatrix} 0 \\ 1' \end{pmatrix}\right)$ and $p\left(x, \begin{pmatrix} 1 \\ 0' \end{pmatrix}\right)$ contradicts the assumption that $i, j$ cannot be separated. Therefore $R \cap (U_1 \times U_2) \subseteq \{(1, 1')\} \cup ((\{0\} \cup T_1) \times (\{0'\} \cup T_2))$.

Suppose that either $\mathsf{lk}_1 \cap \beta \neq \underline{0}_{\mathbb{A}}$ or $\mathsf{lk}_2 \cap \delta \neq \underline{0}_{\mathbb{B}}$, where $\mathsf{lk}_1, \mathsf{lk}_2$ are the link congruences of $\mathbb{A}, \mathbb{B}$ with respect to $R$. Assume the latter. Then, as $\underline{0}_{\mathbb{B}} \leq \mathsf{lk}_2$, there is a $(\underline{0}_{\mathbb{B}}, \delta)$-trace $N$ such that $(a, b) \in \mathsf{lk}_2$. This means there are $a_1, \ldots, a_k \in \mathbb{A}$ and $b_1, \ldots, b_{k+1} \in \mathbb{B}$ with $a = b_1, b = b_{k+1}$, and $(a_i, b_i), (a_i, b_{i+1}) \in R$. Take a polynomial $f$ of $R$ such that $U_1 = f(\mathbb{A})$, $U_2 = f(\mathbb{B})$ are $(\underline{0}_{\mathbb{A}}, \beta)$-, and $(\underline{0}_{\mathbb{B}}, \delta)$-minimal sets, respectively, and such that $U_2$ is a $(\underline{0}_{\mathbb{B}}, \delta)$-minimal set containing $N$ as a trace and $f(a) = a, f(b) = b$. Then, as is easily seen, $R \cap (U_1 \times U_2)$ does not have the form described above. Thus, $\mathsf{lk}_1 \wedge \beta = \underline{0}_{\mathbb{A}}$ and $\mathsf{lk}_2 \wedge \delta = \underline{0}_{\mathbb{B}}$.

CASE 2. $\mathsf{typ}(\underline{0}_{\mathbb{B}}, \delta) = \mathbf{2}$.

As in Case 1, since $p(x, x) = x$ on $U_2$, operation $p$ preserves every trace of $U_2$. Let $N_2$ be a trace in $U_2$. Then $N_2$ is polynomially equivalent to a one-dimensional vector space over $\mathrm{GF}(q)$ where $q$ is a prime power. Since $p$ is idempotent, it can be represented in the form $\gamma x + (1 - \gamma) y$, $\gamma \in \mathrm{GF}(q)$. We may assume that $\gamma = 1$. Indeed, if $\gamma = 0$ then consider $p(y, x)$ instead of $p(x, y)$. Otherwise, the operation

$$\underbrace{p(p \ldots p}_{q-1 \text{ times}}(x, y), y \ldots, y)$$

satisfies the required conditions. Now we can complete the proof as in Case 1. □

The proof of Lemma 48 also implies

**Corollary 49** *Let $R$ be a subdirect product of $\mathbb{A}$ and $\mathbb{B}$ and let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\gamma, \delta \in \mathsf{Con}(\mathbb{B})$ be such that $\alpha \prec \beta$, $\gamma \prec \delta$, and $(\alpha, \beta), (\gamma, \delta)$ cannot be separated. Then* $\mathsf{typ}(\alpha, \beta) = \mathsf{typ}(\gamma, \delta)$.

# 6 Centralizers and decomposition of CSPs

In this section we introduce an operator on congruence lattices similar to the centralizer in commutator theory, and study its properties and its connection to decompositions of CSPs.

## 6.1 Quasi-Centralizer

For an algebra $\mathbb{A}$, a term operation $f(x, y_1, \ldots, y_k)$, and $\mathbf{a} \in \mathbb{A}^k$, let $f^{\mathbf{a}}(x) = f(x, \mathbf{a})$. Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\alpha \leq \beta$, and let $\zeta(\alpha, \beta) \subseteq \mathbb{A}^2$ denote the following binary relation: $(a, b) \in \zeta(\alpha, \beta)$ if an only if, for any term operation $f(x, y_1, \ldots, y_k)$, any $i \in [k]$, and any $\mathbf{a}, \mathbf{b} \in \mathbb{A}^k$ such that $\mathbf{a}[i] = a$, $\mathbf{b}[i] = b$, and $\mathbf{a}[j] = \mathbf{b}[j]$ for $j \neq i$, it holds $f^{\mathbf{a}}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{b}}(\beta) \subseteq \alpha$.

**Lemma 50** *For any $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\alpha \leq \beta$:*

*(1) $\zeta(\alpha, \beta)$ is an equivalence relation.*

*(2) $\zeta(\alpha, \beta)$ is the greatest binary relation $\delta$ satisfying the condition: for any term operation $f(x, y_1, \ldots, y_k)$, and any $\mathbf{a}, \mathbf{b} \in \mathbb{A}^k$ such that $(\mathbf{a}[j], \mathbf{b}[j]) \in \delta$ for $j \in [k]$, it holds $f^{\mathbf{a}}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{b}}(\beta) \subseteq \alpha$.*

*(3) $\zeta(\alpha, \beta)$ is a congruence of $\mathbb{A}$.*[3]

**Proof:** (1) $\zeta(\alpha, \beta)$ is clearly reflexive and symmetric. Suppose $(a, b), (b, c) \in \zeta(\alpha, \beta)$. Let $f(x, y_1, \ldots, y_k)$ be a term operation, $i \in [k]$, and $\mathbf{a}, \mathbf{c} \in \mathbb{A}^k$ such that $\mathbf{a}[i] = a, \mathbf{c}[i] = c$ and $\mathbf{a}[j] = \mathbf{c}[j]$ for $j \neq i$. Let $\mathbf{b} \in \mathbb{A}^k$ be such that $\mathbf{b}[i] = b$ and $\mathbf{b}[j] = \mathbf{a}[j]$ for $j \neq i$. Then $f^{\mathbf{a}}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{b}}(\beta) \subseteq \alpha$, which is if and only if $f^{\mathbf{c}}(\beta) \subseteq \alpha$.

(2) As is easily seen, $\delta$ is reflexive. Choosing $\mathbf{a}, \mathbf{b}$ that differ in only one position, we show that $\delta \subseteq \zeta(\alpha, \beta)$.

Let us show the reverse inclusion. Let $f, \mathbf{a}, \mathbf{b}$ be as in item (2) of the lemma, except $(\mathbf{a}[i], \mathbf{b}[i]) \in \zeta(\alpha, \beta)$, rather than $\delta$. Set $\mathbf{a}_i \in \mathbb{A}^k$, $i \in \{0, \ldots, k\}$, as

[3] Congruence $\zeta(\alpha, \beta)$ appeared in [34], but completely inconsequentially, they did not study it at all. It is easy to see, thanks to K.Kearnes, that $\zeta(\alpha, \beta)$ is greater than the centralizer of $\alpha$ and $\beta$, but the reverse inclusion is unclear.

follows: $\mathbf{a}_i[j] = \mathbf{a}[j]$ for $j \le i$ and $\mathbf{a}_i[j] = \mathbf{b}[j]$ for $j > i$. Then $f^{\mathbf{a}_i}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{a}_{i+1}}(\beta) \subseteq \alpha$. Thus, $f^{\mathbf{a}}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{b}}(\beta) \subseteq \alpha$.

(3) By (1) $\zeta(\alpha, \beta)$ is an equivalence relation, so, we only need to show it is preserved by term operations. Let $g(z_1, \dots, z_m)$ be a term operation and $\mathbf{a}, \mathbf{b} \in \mathbb{A}^m$ such that $(\mathbf{a}[i], \mathbf{b}[i]) \in \zeta(\alpha, \beta)$ for $i \in [m]$. Let also $a = g(\mathbf{a})$ and $b = g(\mathbf{b})$. We show that $(a, b) \in \zeta(\alpha, \beta)$. Take a term operation $f(x, y_1, \dots, y_k)$, $i \in [k]$, and $\mathbf{a}', \mathbf{b}' \in \mathbb{A}^k$ such that $\mathbf{a}'[i] = a, \mathbf{b}'[i] = b$, and $(\mathbf{a}'[j], \mathbf{b}'[j]) \in \zeta(\alpha, \beta)$ for $j \ne i$. Without loss of generality, $i = k$. Let also

$$h(x, y_1, \dots, y_{k-1}, z_1, \dots, z_m) = f(x, y_1, \dots, y_{k-1}, g(z_1, \dots, z_m)),$$

and $\mathbf{a}'' = (\mathbf{a}'[1], \dots, \mathbf{a}'[k-1], \mathbf{a}[1], \dots, \mathbf{a}[m]), \mathbf{b}'' = (\mathbf{b}'[1], \dots, \mathbf{b}'[k-1], \mathbf{b}[1], \dots, \mathbf{b}[m])$. Then $(\mathbf{a}''[j], \mathbf{b}''[j]) \in \zeta(\alpha, \beta)$ for all $j \in [k + m - 1]$. Therefore $f^{\mathbf{a}'}(\beta) = h^{\mathbf{a}''}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{b}'}(\beta) = h^{\mathbf{b}''}(\beta) \subseteq \alpha$. □

The congruence $\zeta(\alpha, \beta)$ will be called the *quasi-centralizer* of $\alpha, \beta$. Next we prove several properties of quasi-centralizer similar to some extent to the properties of the regular centralizer. The following statement is one of the key ingredients of the algorithm.

**Proposition 51** *If $\zeta(\alpha, \beta) \ge \beta$, then $(\alpha, \beta)$ has type* **2***, and for any $\beta$-blocks $B, C$ such that $B \le C$ in $\mathbb{A}/_\beta$ (that is $BC$ is a thin semilattice edge in $\mathbb{A}/_\beta$) and they belong to the same $\zeta(\alpha, \beta)$-block, there is an injective mapping $\sigma\colon B/_\alpha \to C/_\alpha$ such that for any $a \in B/_\alpha$, $a \le \sigma(a)$ and $a \not\le b$ for any other $b \in C$.*

**Proof:** Clearly, we may assume $\alpha = \underline{0}$. Suppose that $\zeta(\alpha, \beta) \ge \beta$, and suppose first that $\mathsf{typ}(\underline{0}, \beta) \ne \mathbf{2}$. Take any $(\underline{0}, \beta)$-minimal set $U$, its only trace $N$, and a pseudo-meet operation $p$ on $U$. Then the polynomial $p(x, 0)$ does not collapse $\beta$, as $f(0, 0) = 0$, $f(1, 0) = 1$, while the polynomial $p(x, 1)$ does, a contradiction with the assumption $\zeta(\alpha, \beta) \ge \beta$.

Suppose now that $\mathsf{typ}(\underline{0}, \beta) = \mathbf{2}$. Then by Corollary 38(1) for any $a, b \in \mathbb{A}$ with $a \overset{\beta}{\equiv} b$, there is a $(\underline{0}, \beta)$-minimal set $U$ such that $a, b \in U$.

Let $B, C$ be $\beta$-blocks and $B \le C$ in $\mathbb{A}/_\beta$. By Lemma 22(1) for any $a \in \mathbb{A}$ there is $b \in C$ with $a \le b$. Suppose the statement of the proposition is not true. Then there are two possibilities.

1. For some $a \in B$ and $b, c \in C$, $a \le b, a \le c$. Let $f$ be a polynomial such that $U = f(\mathbb{A})$ is a $(\underline{0}, \beta)$-minimal set and $b, c \in U$. Consider $g_1(x) = a \cdot f(x)$ and $g_2(x) = b \cdot f(x)$. Clearly, $g_1(b) = b, g_1(c) = c$, so $g_1(\beta) \not\subseteq \underline{0}$. On the other hand, $g_2(b) = b = g_2(c)$, that is, $g_2(\beta) \subseteq \underline{0}$, as $|g_2(\mathbb{A})| \le |U|$, a contradiction with the assumption $(a, b) \in \zeta(\underline{0}, \beta)$.

2. For some $a \in C$ and $b, c \in B$, $b \le a, c \le a$. Let $f$ be a polynomial such that $U = f(\mathbb{A})$ is a $(\underline{0}, \beta)$-minimal set and $b, c \in U$. Consider $g_1(x) = f(x) \cdot a$

and $g_2(x) = f(x) \cdot b$. Clearly, $g_1(b) = g_1(c) = a$, so $g_1(\beta) \subseteq \underline{0}$, as $|g_1(\mathbb{A})| \leq |U|$. On the other hand, $g_2(b) = b, g_2(c) = c$, that is, $g_2(\beta) \not\subseteq \underline{0}$ a contradiction again. □

**Corollary 52** *Let $\zeta(\alpha, \beta) = \underline{1}_{\mathbb{A}}$, $a, b, c \in \mathbb{A}$ and $b \stackrel{\beta}{\equiv} c$. Then $ab \stackrel{\alpha}{\equiv} ac$.*

**Proof:** We have $ab \stackrel{\beta}{\equiv} ac$ and $a \leq ab, a \leq ac$. By Proposition 51 $ab \stackrel{\alpha}{\equiv} ac$. □

**Remark 53** *Recently, Payne [46] designed a polynomial time algorithm for the following class of algebras: Every algebra $\mathbb{A}$ from this class has a congruence $\alpha$ such that $\mathbb{A}/_\alpha$ is a semilattice, and the interactions between $\alpha$-blocks satisfy a certain condition. It seems that Lemma 51 is similar to what this condition can provide.*

An injective mapping between $\beta$-blocks $B, C$ inside a $\zeta(\alpha, \beta)$-block can also be established whenever $BC$ is any thin edge in $\mathbb{A}/_\beta$, as the following lemma shows.

**Lemma 54** *Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ such that $\alpha \prec \beta \leq \zeta = \zeta(\alpha, \beta)$ and let $B, C$ be $\beta$-blocks from the same $\zeta$-block such that $BC$ is a thin edge in $\mathbb{A}/_\beta$. For any $b \in B, c \in C$ such that $bc$ is a thin edge the polynomial $f(x) = x \cdot c$ if $b \leq c$, $f(x) = t_{bc}(x, c)$ if $bc$ is majority, and $f(x) = h_{bc}(x, b, c)$ if $bc$ is affine, where $t_{ab}, h_{ab}$ are the operations from Lemma 24, is an injective mapping from $B/_\alpha$ to $C/_\alpha$.*

**Proof:** We can assume that $\alpha$ is the equality relation. Suppose $f(a_1) = f(a_2)$ for some $a_1, a_2 \in B$. Since $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$, by Corollary 38(1) every pair of elements of $B$ is an $(\alpha, \beta)$-subtrace. Let $f'$ be an idempotent unary polynomial such that $f'(a_1) = a_1$, $f'(a_2) = a_2$, and $f'(\mathbb{A})$ is an $(\alpha, \beta)$-minimal set.

If $b \leq c$, let $g(x, y) = f'(x) \cdot y$. Then $g^c = g(x, c) = f(x)$ on $\{a_1, a_2\}$, that is, $g^c(a_1) = g^c(a_2)$ implying $g^c(\beta) \subseteq \alpha$. On the other hand, $g^b(x) = f'(x)$ on $\{a_1, a_2\}$ implying $g^b(\beta) \not\subseteq \alpha$, a contradiction with the assumption $b \stackrel{\zeta}{\equiv} c$.

If $bc$ is a thin majority edge, set $g(x, y) = t_{bc}(f'(x), y)$. Then $g^c(a_1) = f(a_1) = f(a_2) = g^c(a_2)$, and so $g^c(\beta) \subseteq \alpha$. On the other hand, since $B/_\alpha$ is a module, $a_1b, a_2b$ are affine edges and $\alpha$ witnesses that. Therefore $g^b(a_1) = a_1$ and $g^b(a_2) = a_2$, implying $g^b(\beta) \not\subseteq \alpha$, and we have a contradiction again.

Finally, if $bc$ is a thin affine edge, we consider the polynomials $g(x, y, z) = h_{bc}(f'(x), y, z)$ and $g^{bc}(x) = g(x, b, c), g^{a_1a_1}(x) = g(x, a_1, a_1)$. Again, $g^{bc}(a_1) = f(a_1) = f(a_2) = g^{bc}(a_2)$, while

$$g^{a_1a_1}(a_1) = h_{bc}(f'(a_1), a_1, a_1) = a_1 \neq h_{bc}(f'(a_2), a_1, a_1) = g^{a_1a_1}(a_2),$$

since by Lemma 24 $h_{bc}(x, a_1, a_1)$ is a permutation. This implies that $g^{bc}(\beta) \subseteq \alpha$ and $g^{a_1 a_1}(\beta) \not\subseteq \alpha$, a contradiction. $\Box$

**Lemma 55** *Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ be such that $\alpha \prec \beta$ and $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$, and $\zeta = \zeta(\alpha, \beta)$. Then for any $\beta$-blocks $B_1, B_2$ that belong to the same $\zeta$-block $C$ and such that $B_1 \sqsubseteq_{asm} B_2$ and $B_2 \sqsubseteq_{asm} B_1$ in $C/_\beta$, $|B_1/_\alpha| = |B_2/_\alpha|$.*

**Proof:** Since there is an asm-path from $B_1$ to $B_2$ and back, the result follows from Lemma 54. $\Box$

Let $\mathbb{A}$ be an algebra and $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\alpha \prec \beta$. Element $a \in \mathbb{A}$ is said to be *$\alpha\beta$-minimal* if it belongs to an $(\alpha, \beta)$-trace. Let $\mathsf{Z}_{\mathbb{A}}(\alpha, \beta)$ denote the set of all $\alpha\beta$-minimal elements of $\mathbb{A}$. By Lemma 14(5) $\mathsf{Z}_{\mathbb{A}}(\alpha, \beta)$ intersects every $\alpha$-block from a nontrivial $\beta$-block. The following lemma shows that $\mathsf{Z}_{\mathbb{A}}(\alpha, \beta)$ can be much larger than that. Due to the way we will use it in the future the statement of the lemma is not quite straightforward.

**Lemma 56** *Let $\alpha, \beta, \gamma, \delta \in \mathsf{Con}(\mathbb{A})$ be such that $\gamma \prec \delta \le \beta$, $\alpha \prec \beta$, intervals $(\alpha, \beta), (\gamma, \delta)$ cannot be separated, and $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$; let $B$ be a $\beta$-block. If $a \in \mathsf{Z}_{\mathbb{A}}(\gamma, \delta) \cap B$ then for any $b \in B$ such that $a \sqsubseteq_{asm} b$ in $B$, $b \in \mathsf{Z}_{\mathbb{A}}(\gamma, \delta)$.*

*In particular, $\mathsf{umax}(B) \subseteq \mathsf{Z}_{\mathbb{A}}(\gamma, \delta)$.*

*Moreover, if $a \stackrel{\alpha}{\equiv} b$, $a \sqsubseteq_{asm} b$ in $a^\alpha$, and $f$ is a polynomial such that $f(a) = a$, $f(\mathbb{A})$ is an $(\alpha, \beta)$-minimal set, and $N$ its trace with $a \in N$, then there is a polynomial $g$ such that $g(b) = b$, $g(\mathbb{A})$ is an $(\alpha, \beta)$-minimal set, $N'$ is its trace containing $b$ and $N'/_\alpha = N/_\alpha$.*

**Proof:** Let $f$ be an idempotent unary polynomial of $\mathbb{A}$ such that $a \in N$, a trace in $U = f(\mathbb{A})$, a $(\gamma, \delta)$-minimal set. Note that $f(B) \subseteq B$ and $f(\beta) \not\subseteq \alpha$. It suffices to consider the case when $ab$ is a thin edge.

Depending on the type of the edge $ab$ we set $f'(x) = f(x) \cdot b$, $f'(x) = t_{ab}(f(x), b)$, or $f'(x) = h_{ab}(f(x), a, b)$, if $ab$ is semilattice, majority or affine, respectively. Note also that by Lemma 24 $f'(a) = b$, and therefore if $f'(\delta) \not\subseteq \gamma$ we have $f'(\mathbb{A})$ is a $(\gamma, \delta)$-minimal set, and $b$ belongs to it.

Since $(\alpha, \beta)$ and $(\gamma, \delta)$ cannot be separated, by Lemma 46(2) $U$ is an $(\alpha, \beta)$-minimal set. Hence, there are $a_1, a_2 \in B/_\alpha$ such that $a_1 \ne a_2$ and $f(a_1) = a_1, f(a_2) = a_2$. Since $a_1 a_2$ is an affine edge in $\mathbb{B}/_\alpha$, depending on the type of $ab$ we have:

– if $a \le b$, then $f'(a_i) = a_i \cdot b^\alpha = a_i$ for $i = 1, 2$;

– if $ab$ is majority, then $f'(a_i) = t_{ab}(a_i, b^\alpha) = a_i$, as $a_i \stackrel{\beta/\alpha}{\equiv} b^\alpha$ for $i = 1, 2$ and by Lemma 24(1);

– if $ab$ is affine, then by Lemma 24(2) $h_{ab}(x, a^\alpha, b^\alpha)$ is a permutation on $B/_\alpha$, in

particular, $f'(a_1) \neq f'(a_2)$.
In either case we obtain $f'(\beta) \not\subseteq \alpha$, implying $f'(\delta) \not\subseteq \gamma$.

For the last claim of the lemma it suffices to notice that if $a \stackrel{\alpha}{\equiv} b$ we have $f'(x) \stackrel{\alpha}{\equiv} f(x)$ for $x \in B$. □

## 6.2 Decomposition of CSPs

In this section we show that if intervals in congruence lattices of domains in a CSP instance cannot be separated, they induce certain decomposition of the instance or its subinstances. The components of this decomposition are instances over smaller domains, which are, actually, blocks of the corresponding quasi-centralizers.

Let $R$ be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $i, j \in [n]$, and $\alpha_i \in \mathsf{Con}(\mathbb{A}_i)$, $\alpha_j \in \mathsf{Con}(\mathbb{A}_j)$. The coordinate positions $i, j$ are said to be *$\alpha_i\alpha_j$-aligned in $R$* if, for any $(a, c), (b, d) \in \mathrm{pr}_{ij}R$, $(a, b) \in \alpha_i$ if and only if $(c, d) \in \alpha_j$. Or in other words, the link congruences of $\mathbb{A}_i, \mathbb{A}_j$ with respect to $\mathrm{pr}_{ij}R$ are no greater than $\alpha_i, \alpha_j$, respectively.

**Lemma 57** *Let $R$ be a subdirect product of $\mathbb{A}_1 \times \mathbb{A}_2$, $\alpha_i, \beta_i \in \mathsf{Con}(\mathbb{A}_i)$, $\alpha_i \prec \beta_i$, for $i = 1, 2$. If $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ cannot be separated from each other, then the coordinate positions 1,2 are $\zeta(\alpha_1, \beta_1)\zeta(\alpha_2, \beta_2)$-aligned in $R$.*

**Proof:** Let us assume the contrary, that is, without loss of generality there are $a, b \in \mathbb{A}_1$ and $c, d \in \mathbb{A}_2$ with $(a, c), (b, d) \in R$, $(a, b) \in \zeta(\alpha_1, \beta_1)$, but $(c, d) \notin \zeta(\alpha_2, \beta_2)$. Therefore there is $g(x, y_1, \ldots, y_k)$, a term operation of $\mathbb{A}_2$, $i \in [k]$, and $\mathbf{c}, \mathbf{d} \in \mathbb{A}_2^k$ with $\mathbf{c}[i] = c$, $\mathbf{d}[i] = d$ and $\mathbf{c}[j] = \mathbf{d}[j]$ for $j \neq i$, such that $g^{\mathbf{c}}(\beta_2) \subseteq \alpha_2$ but $g^{\mathbf{d}}(\beta_2) \not\subseteq \alpha_2$, or the other way round. Extend $g$ to a term operation $g$ of $R$, and choose $\mathbf{a}, \mathbf{b} \in \mathbb{A}_1^k$ such that $\mathbf{a}[i] = a, \mathbf{b}[i] = b$, $\mathbf{a}[j] = \mathbf{b}[j]$ for $j \neq i$, and $(\mathbf{a}[j], \mathbf{c}[j]), (\mathbf{b}[j], \mathbf{d}[j]) \in R$ for $j \in [k]$. Then $g^{\mathbf{a}}(\beta_1) \subseteq \alpha_1$ if and only if $g^{\mathbf{b}}(\beta_1) \subseteq \alpha_1$. Therefore, there is a polynomial of $R$ that separates $(\alpha_1, \beta_1)$ from $(\alpha_2, \beta_2)$ or the other way round, a contradiction. □

Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance, in particular, for every $X \subseteq V$, $|X| = 2$, it contains a constraint $C^X = \langle X, R^X \rangle$. Let $w_1, w_2, \in V$. We say that $w_1, w_2$ are *$\alpha_1\alpha_2$-aligned* in $\mathcal{P}$, where $\alpha_1 \in \mathsf{Con}(\mathbb{A}_{w_1})$, $\alpha_2 \in \mathsf{Con}(\mathbb{A}_{w_2})$, if they are $\alpha_1\alpha_2$-aligned in $R^{w_1w_2}$. For $\alpha_v \in \mathsf{Con}(\mathbb{A}_v)$, $v \in V$, instance $\mathcal{P}$ is said to be *$\overline{\alpha}$-aligned* if every $w_1, w_2$ are $\alpha_{w_1}\alpha_{w_2}$-aligned. This means that there are one-to-one mappings $\varphi_{w_1w_2} : \mathbb{A}_{w_1}/_{\alpha_{w_1}} \to \mathbb{A}_{w_1}/_{\alpha_{w_1}}$ such that whenever $(a, b) \in R^{w_1w_2}$, $b^{\alpha_{w_2}} = \varphi_{w_1w_2}(a^{w_1})$. Observe that since $\mathcal{P}$ is (2,3)-minimal, these mappings are consistent, that is, for any $u, v, w \in V$, $\varphi_{vw} \circ \varphi_{uv} = \varphi_{uw}$. Therefore $\mathcal{P}$ can be represented as a disjoint union of instances $\mathcal{P}_1, \ldots, \mathcal{P}_k$, where $k$ is the number of $\alpha_v$-blocks for any $v \in V$ and the domain of $v \in V$ of $\mathcal{P}_i$ is the $i$-th $\alpha_v$-block.

Let again $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance and let $\overline{\beta}$, $\beta_v \in \mathsf{Con}(\mathbb{A}_v)$, $v \in V$, be a collection of congruences. Let $\mathcal{W}^{\mathcal{P}}(\overline{\beta})$ denote the set of triples $(v, \alpha, \beta)$ such that $v \in V$, $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$, and $\alpha \prec \beta \le \beta_v$. Also, $\mathcal{W}^{\mathcal{P}}$ denotes $\mathcal{W}^{\mathcal{P}}(\overline{\beta})$ when $\beta_v = \underline{1}_v$ for all $v \in V$. We will omit the superscript $\mathcal{P}$ whenever it is clear from the context. For every $(v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})$, let $Z$ denote the set of triples $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$ such that $(\alpha, \beta)$ and $(\gamma, \delta)$ cannot be separated in $R^{vw}$. Slightly abusing the terminology we will also say that $(\alpha, \beta)$ and $(\gamma, \delta)$ cannot be separated in $\mathcal{P}$. Then let $W_{v,\alpha\beta,\overline{\beta}} = \{w \in V \mid (w, \gamma, \delta) \in Z \text{ for some } \gamma, \delta \in \mathsf{Con}(\mathbb{A}_w)\}$. We will omit the subscript $\overline{\beta}$ whenever possible. The following statement is an easy corollary of Lemma 57.

**Theorem 58** *Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance and $(v, \alpha, \beta) \in \mathcal{W}$. For $w \in W_{v,\alpha\beta,\overline{\beta}}$, where $\beta_v = \underline{1}_v$ for $v \in V$, let $(w, \gamma, \delta) \in \mathcal{W}$ be such that $(\alpha, \beta)$ and $(\gamma, \delta)$ cannot be separated and $\zeta_w = \zeta(\gamma, \delta)$. Then $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ is $\overline{\zeta}$-aligned.*

## 7 The Congruence Lemma

This section contains several technical results that will be used in the proof of soundness of the algorithm. The main result is the Congruence Lemma 63. Lemmas 61 and 62 are auxiliary and only used in the proof of the Congruence Lemma 63. We start with introducing two closure properties of algebras and their subdirect products, this time under certain polynomials.

We say that a set $A$ is *as-closed* in algebra $\mathbb{B}$, $A \subseteq \mathbb{B}$, if $A \cap \mathsf{umax}(\mathbb{B}) \neq \varnothing$ and, for every $a, b \in \mathbb{B}$ such that $a \sqsubseteq_{as} b$ in $\mathbb{B}$ and $a \in A \cap \mathsf{umax}(\mathbb{B})$, element $b$ also belongs to $A$.

Let $R$ be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$ and $Q$ a subalgebra of $R$. We say that $Q$ is *polynomially closed* in $R$ if for any $i \in [n]$ and any polynomial $f$ of $R$ the following condition holds: for any $\mathbf{a}, \mathbf{b} \in \mathsf{umax}(Q)$ such that $f(\mathbf{a}) = \mathbf{a}$ and for any $\mathbf{c} \in \mathsf{Sg}(\mathbf{a}, f(\mathbf{b}))$ such that $\mathbf{a} \sqsubseteq_{as} \mathbf{c}$ in $\mathsf{Sg}(\mathbf{a}, f(\mathbf{b}))$, the tuple $\mathbf{c}$ belongs to $Q$.

**Remark 59** *Polynomially closed subalgebras of Mal'tsev algebras are congruence blocks. In the general case the structure of polynomially closed subalgebras is more intricate. The intuition (although not entirely correct) is that if for some block $B$ of a congruence $\beta$ and a congruence $\alpha$ with $\alpha \prec \beta$ the set $B/_{\alpha}$ contains several as-components, a polynomially closed subalgebra contains some of them and has empty intersection with the rest. However, since this is true only for factor sets, and we do not even consider non-as-maximal elements, the actual structure is more 'fractal'.*

The following lemma follows from the definitions, Lemma 22, and the fact that congruences are invariant under polynomials.

**Lemma 60** *(1) Let $R$ be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$ and $Q_1, Q_2$ relations polynomially closed in $R$, and $Q_1 \cap Q_2 \cap \mathsf{umax}(Q) \neq \varnothing$. Then $Q_1 \cap Q_2$ is polynomially closed in $R$.*

*In particular, let $\beta_i \in \mathsf{Con}(\mathbb{A}_i)$ and $B_i$ a u-maximal $\beta_i$-block. Then $Q_1 \cap \overline{B}$ is polynomially closed in $R$.*

*If $Q_1, Q_2$ are as-closed in $R$, then $Q_1 \cap Q_2$ is as-closed in $R$.*

*(2) Let $Q_i$ be polynomially closed in $R_i$, $i \in [k]$, and let $R, Q$ be pp-defined through $R_1, \dots, R_k$ and $Q_1, \dots, Q_k$, respectively, by the same pp-formula $\Phi$; that is, $R = \Phi(R_1, \dots, R_k)$ and $Q = \Phi(Q_1, \dots, Q_k)$. Let also $\mathsf{umax}(Q) \cap \mathsf{umax}(R) \neq \varnothing$. Then $Q$ is polynomially closed in $R$.*

*If $Q_i$ is as-closed in $R_i$ then $Q$ is as-closed in $R$.*

*(3) Let $R$ be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$, $\beta_i \in \mathsf{Con}(\mathbb{A}_i)$, $i \in [n]$, and let $Q$ be polynomially closed in $R$. Then $Q/_{\overline{\beta}}$ is polynomially closed in $R$.*

*If $Q$ is as-closed in $R$ then $Q/_{\overline{\beta}}$ is as-closed in $R/_{\overline{\beta}}$.*

The first two lemmas in the rest of this section study the structure of binary relations that have in their domains a pair of prime factors of type **2** that cannot be separated. They show that if we restrict ourselves to blocks of the link congruences then this structure is very uniform. The third lemma, Lemma 63 (the Congruence Lemma), is an important technical result. To explain what it amounts to saying consider this: let $Q \subseteq \mathbb{A}' \times \mathbb{B}'$ be a subdirect product and the link congruence of $\mathbb{A}'$ is the equality relation. Then, clearly, $Q$ is the graph of a mapping $\sigma : \mathbb{B}' \to \mathbb{A}'$, and the kernel of this mapping is the link congruence $\eta$ of $\mathbb{B}'$ with respect to $Q$. Suppose now that $Q$ is a subalgebra of $R$, a subdirect product of $\mathbb{A} \times \mathbb{B}$ such that $\mathbb{A}'$ is a subalgebra of $\mathbb{A}$ and $\mathbb{B}'$ is a subalgebra of $\mathbb{B}$. Then the restriction of the link congruence of $\mathbb{A}$ with respect to $R$ to $\mathbb{A}'$ does not have to be the equality relation, and similarly the restriction of the link congruence of $\mathbb{B}$ to $\mathbb{B}'$ does not have to be $\eta$. Most importantly, the restriction of $\mathsf{Cg}(\eta)$, the congruence of $\mathbb{B}$ generated by $\eta$, to $\mathbb{B}'$ does not have to be $\eta$. The Congruence Lemma 63 shows, however, that this is exactly what happens when $Q$ and $\mathbb{A}', \mathbb{B}'$ satisfy some additional conditions, such as being chained and polynomially closed.

In the next two lemmas let $R$ be a subdirect product of $\mathbb{A}_1 \times \mathbb{A}_2$, $\beta_1, \beta_2$ congruences of $\mathbb{A}_1, \mathbb{A}_2$ and $B_1, B_2$ $\beta_1$- and $\beta_2$-blocks, respectively; $R$ is chained with respect to $(\beta_1, \beta_2), (B_1, B_2)$ and $R^* = R \cap (B_1 \times B_2)$, $B_1^* = \mathrm{pr}_1 R^*, B_2^* = \mathrm{pr}_2 R^*$. Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_1)$, $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_2)$ be such that $\alpha \prec \beta \leq \beta_1$, $\gamma \prec \delta \leq \beta_2$, $\mathsf{typ}(\alpha, \beta) = \mathsf{typ}(\gamma, \delta) = \mathbf{2}$, and $(\alpha, \beta), (\gamma, \delta)$ cannot be separated. Let also $\zeta_1 = \zeta(\alpha, \beta)|_{B_1^*}$, $\zeta_2 = \zeta(\gamma, \delta)|_{B_2^*}$ and $\mathsf{lk}_1^*, \mathsf{lk}_2^*$ the link congruences of $B_1^*, B_2^*$, respectively, with respect to $R^*$. Let $F, G$ be $\zeta_1$-, $\zeta_2$-blocks such that $R^* \cap (F \times G) \neq \varnothing$ and $F, G$ contain nontrivial $\beta$- and $\delta$-blocks $A, B$, respectively (that is, $|A/_\alpha|, |B/_\gamma| > 1$).

By Lemma 55 all the $\beta$-blocks $A' \in F/_\beta$, $A \sqsubseteq_{asm} A'$ in $F$ (respectively, all $\delta$-blocks $B' \in G/_\delta$, $B \sqsubseteq_{asm} B'$ in $G$) are also nontrivial. Note that by Lemma 57 $\mathsf{lk}_1^* \leq \zeta_1$ and $\mathsf{lk}_2^* \leq \zeta_2$. Let also $D \subseteq F, E \subseteq G$ be blocks of $\mathsf{lk}_1^*, \mathsf{lk}_2^*$ such that $R^* \cap (D \times E) \neq \varnothing$.

The first lemma claims that for the link congruence $\mathsf{lk}_2^*$ there are only two options: either it is a subset of $\gamma$ on the $\zeta_2$-block $G$, or it contains $\delta/_\gamma$ on $\mathsf{umax}(G)$.

**Lemma 61** *Suppose that $B \cap E \neq \varnothing$ and that $\mathsf{lk}_2^*$ is nontrivial on the $\delta$-block $B$, that is, there are distinct $a, b \in (B \cap B_2^*)/_\gamma$ with $(a, b) \in \mathsf{lk}_2^*$, or equivalently $\mathsf{lk}_2^* \wedge \delta$ is not a subset of $\gamma$ on $B \cap B_2^*$. Then*

*(1) if $G \cap \mathsf{umax}(B_2^*) \neq \varnothing$ then $\delta|_{\mathsf{umax}(G)} \leq \mathsf{lk}_2^* \vee \gamma|_{B_2^*}$; and*

*(2) any $B' \in G/_\delta$ with $B \sqsubseteq_{asm} B'$ in $G/_\delta$ is nontrivial, that is, $|B'/_\gamma| > 1$. In particular, $\mathsf{umax}(D), \mathsf{umax}(E)$ and $\mathsf{umax}(F), \mathsf{umax}(G)$ do not intersect any trivial $\beta$- and $\delta$-blocks, respectively.*

**Proof:** Since $\mathsf{lk}_1^* \leq \zeta_1$ and $\mathsf{lk}_2^* \leq \zeta_2$, (2) follows by Lemma 55. Also, as $\mathsf{lk}_2^*$ is nontrivial on a $\delta$-block, we obtain (1) by Lemma 44. $\square$

The second lemma amounts to saying that if $\mathsf{lk}_2^*$ is nontrivial on $G$ then it not only contains $\delta$ (modulo $\gamma$), but also that if an element of $F$ is related by $R$ to some element of a $\delta$-block $B$, it is also related to the entire $B$ (again, modulo $\gamma$).

**Lemma 62** *Suppose $\delta|_{\mathsf{umax}(G)} \leq \mathsf{lk}_2^* \vee \gamma|_{B_2^*}$ and sets $A, B$ satisfy one of the following two conditions:*

*(1) let $A \subseteq F, B \subseteq G$ be $\beta$- and $\delta$-blocks, respectively, such that $(A, B) \in \mathsf{umax}((R \cap (F \times G))/_{\beta\times\delta})$.*

*(2) let $A, B$ be $\beta$-, and $\delta$-blocks, respectively, and such that $A' = A \cap D \neq \varnothing$ and $(A', B)\mathsf{umax}((R \cap (D \times G))/_{\beta\times\delta})$.*

*Then either $R \cap (A \times B) = \varnothing$, or for any $c \in A$ with $B \cap R[c] \neq \varnothing$ we have $\{c\} \times B/_\gamma \subseteq R/_\gamma$.*

Note that in condition (2) $D$ does not have to contain u-maximal elements of $F$, and similarly $E$ does not have to contain u-maximal elements of $G$. Thus, (1) is not necessarily a more general condition.

**Proof:** We prove the lemma for condition (2), that is, when $(A', B)\mathsf{umax}((R \cap (D \times G))/_{\beta\times\delta})$. It will be clear that case (1) follows from the same argument.

We assume $\gamma = \underline{0}_2$. Since $B$ is a module, it is as-connected. Therefore if some element of $B$ belongs to an as-component of $E$, the whole set $B$ is contained in that as-component. By the Rectangularity Corollary 34, this means that if $R \cap (\{c\} \times B) \neq \varnothing$ for $c \in \mathsf{amax}(D)$, then $\{c\} \times B \subseteq R$, and the result for $B$ follows.

Now we show that if $\{c\} \times B \subseteq R$ for some $c \in D$ and $B \in E/_\delta$ then $\{d\} \times b'^\delta \subseteq R$ for any $d \in D$ and $b' \in E$ such that $(c,b)(d,b')$ is a thin edge in $R \cap (D \times E)$ for some $b \in B$. As is easily seen this implies the result. There are 3 possible cases.

CASE 1. $b'^\delta = B$, that is $cd$ is a thin edge and $(d,b') \in R$, $b' \in B$. Then $\{d\} \times B \subseteq R$.

This case follows from Lemma 32.

CASE 2. $c = d$, that is, $BB'$ is a thin edge in $\mathbb{A}_2/_\delta$ where $B' = b'^\delta$ and $(c,b') \in R$.

Let $f(x)$ be the unary polynomial of $R$ constructed as in Lemma 54, that is, $f(x) = x \cdot \begin{pmatrix} c \\ b' \end{pmatrix}$, $f(x) = t_{bb'}\left(x, \begin{pmatrix} c \\ b' \end{pmatrix}\right)$, or $f(x) = h_{bb'}\left(x, \begin{pmatrix} c \\ b \end{pmatrix}, \begin{pmatrix} c \\ b' \end{pmatrix}\right)$, depending on the type of $bb'$. Then by Lemma 54 $f : B \to B'$ is a bijection, and therefore maps $\{c\} \times B \subseteq R$ onto $\{c\} \times B'$, implying $\{c\} \times B' \subseteq R$.

CASE 3. $cd$ and $bb'$ are thin edges of the same type.

Let $B' = b'^\delta$. Similar to Case 2 depending on the type of $cd$ we consider polynomial $f(x) = x \cdot \begin{pmatrix} d \\ b' \end{pmatrix}$, $f(x) = t_{cd}\left(x, \begin{pmatrix} d \\ b' \end{pmatrix}\right)$, or $f(x) = h_{cd}\left(x, \begin{pmatrix} c \\ b \end{pmatrix}, \begin{pmatrix} d \\ b' \end{pmatrix}\right)$ for some $b \in B$. We have $f(c) = d$ and $f : B \to B'$ is a bijection by Lemma 54, thus proving that $\{d\} \times B' \subseteq R$.

If condition (1) holds the prove is essentially the same, except we need to use the same starting point as above, and consider pairs from $\mathsf{umax}(R \cap (F \times G))$. $\Box$

We are now in a position to state and prove the main result of the section. Let again $R$ be a subdirect product of $\mathbb{A}_1 \times \mathbb{A}_2$, $\beta_1, \beta_2$ congruences of $\mathbb{A}_1, \mathbb{A}_2$ and $B_1, B_2$ $\beta_1$- and $\beta_2$-blocks, respectively. Also, let $R$ be chained with respect to $(\beta_1, \beta_2), (B_1, B_2)$ and $R^* = R \cap (B_1 \times B_2)$, $B_1^* = \mathrm{pr}_1 R^*, B_2^* = \mathrm{pr}_2 R^*$. Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_1)$ be such that $\alpha \prec \beta \leq \beta_1$. This time we do not assume that $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$.

**Lemma 63 (The Congruence Lemma)** *Suppose $\alpha = \underline{0}_1$ and let $R'$ be a polynomially closed subalgebra of $R^*$ and such that $B_1' = \mathrm{pr}_1 R'$ contains an as-component $C$ of $B_1^*$ and $R' \cap \mathsf{umax}(R^*) \neq \varnothing$. Then either*

*(1) $C \times \mathsf{umax}(B_2'') \subseteq R'$, where $B_2'' = R'[C]$, or*

*(2) there are $\eta, \theta \in \mathsf{Con}(\mathbb{A}_2)$ with $\eta \prec \theta \leq \beta_2$ such that these intervals cannot be separated.*

*Moreover, in case (2) $R' \cap (C \times B_2'')$ is the graph of a mapping $\varphi : B_2'' \to C$ such that the kernel of $\varphi$ is the restriction of $\eta$ on $B_2''$.*

**Proof:** Note that if $|C| = 1$, the lemma is trivially true. Let $B_2' = \mathrm{pr}_2 R'$. We assume $\beta_2|_{B_2'} \neq \lambda|_{B_2'}$ for any congruence $\lambda \leq \beta_2$; otherwise replace $\beta_2$ with $\lambda$. Let $\mathsf{lk}_1', \mathsf{lk}_2'$ be the link congruences of $B_1', B_2'$ with respect to $R'$. Let $\eta \leq \beta_2$ be such that $\eta|_{\mathsf{umax}(B_2')} \subseteq \mathsf{lk}_2'$ and $\eta$ is maximal among congruences of $\mathbb{A}_2$ with this property. We show that either $\eta = \beta_2$ or it is one of the congruences in item (2) of the lemma. If $\eta$ is the total relation on $\mathsf{umax}(B_2')$, we are done by Proposition 35; otherwise there are two cases.

CASE 1. For some $\theta \in \mathsf{Con}(\mathbb{A}_2)$ with $\eta \prec \theta \leq \beta_2$ the intervals $(\underline{0}_1, \beta_1), (\eta, \theta)$ can be separated.

In this case we prove that $\eta$ has to be $\beta_2$ and we have option (1) of the lemma. Since $R$ is chained, by Lemmas 43(4) and 47 there is a set $T \subseteq B_1^* \times B_1^*$ of $(\underline{0}_1, \beta_1)$-subtraces such that any pair of elements from $\mathsf{umax}(B_1^*)$ belongs to the transitive closure of $T$, and for any $(a, b) \in T$ there is a $(B_1, B_2)$-preserving polynomial $f$ such that $f(a) = a, f(b) = b$, and $f(\theta|_{B_2^*}) \subseteq \eta$. This means that $C$ belongs to the $\mathsf{lk}_1^*$-block of $B_1^*$, where $\mathsf{lk}_1^*$ is the link congruence with respect to $R^*/_\eta$. Therefore $C \times \mathsf{umax}(R^*[C])/_\eta \subseteq R^*/_\eta$. Observe that as $R' \subseteq R^*$, the link congruence of $B_1^*$ with respect to $R^*$ restricted to $C$ contains $\mathsf{lk}_1'|_C$. Therefore, we also have $C \times \mathsf{umax}(R^*[C]) \subseteq R^*$. Note that by the assumption $R' \cap \mathsf{umax}(R^*) \neq \varnothing$ of the lemma both $R^*[C]$ and $B_2'$ contain a u-maximal element from $B_2^*$. Since $B_2'' \subseteq R^*[C]$, by Lemma 26 we have $\mathsf{umax}(B_2'') \subseteq \mathsf{umax}(R^*[C])$. Therefore $C \times \mathsf{umax}(B_2'') \subseteq R^*$.

We are going to argue that the same inclusion holds for $R'$. But first we show that for any thin semilattice or affine edge $ab$ of $C$ and any $c \in \mathsf{umax}(R^*[C])$ there is a polynomial $g$ such that $g(a) = a, g(b) = b$, $f(\theta|_{B_2^*}) \subseteq \eta$, and $g(c) = c$. Note that since $R$ is chained, all such pairs $\{a, b\}$ belong to $T$. Since every pair of elements of $C$ is a $(\underline{0}_1, \beta_1)$-subtrace, again, as $R$ is chained, and by Lemma 43(5) this is true for some $c \in R^*[C]$. Suppose $cc'$ is a thin edge in $R^*[C]$; by Lemma 32 this implies that $(a, c), (b, c), (a, c'), (b, c') \in R$. Then as in Lemma 56 we find a polynomial satisfying the required properties for $c'$. Specifically, $g'(x) = g(x) \cdot \begin{pmatrix} a \\ c' \end{pmatrix}$, $g'(x) = t\left(g(x), \begin{pmatrix} a \\ c' \end{pmatrix}\right)$, and $g'(x) = h'\left(g(x), \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} a \\ c' \end{pmatrix}\right)$, where $t$ and $h$ are the operations from Lemma 24(3), depending on the type of $cc'$ and $ab$.

Now we are back to proving that $C \times \mathsf{umax}(B_2'') \subseteq R'$. Observe that $R'/_{\mathsf{lk}_1' \times \mathsf{lk}_2'}$ is the graph of a bijective mapping $\varphi : B_1'/_{\mathsf{lk}_1'} \to B_2'/_{\mathsf{lk}_2'}$. Take $a, b \in C$ and $c \in \mathsf{umax}(B_2'')$ such that $(a, c) \in R'$, $ab$ is a thin semilattice or affine edge and $(a, b) \notin \mathsf{lk}_1'$. Let also $(b, d) \in R'$. By what is proved there is a polynomial $f$

of $R$ such that $f(a) = a, f(b) = b$, $f(c) = c \stackrel{\eta}{\equiv} d' = f(d)$, and $f(\theta|_{B_2^*}) \subseteq \eta$. In particular, $(a, d'), (b, d') \in R$. Since $(a, c) \in R'$ and $(a, c) \leq (b, c')$ or $(a, c)(b, c')$ is an affine edge for some $c' \in \mathsf{Sg}(c, d')$, we obtain $(b, c') \in R'$, as $R'$ is polynomially closed. Since $c' \stackrel{\eta}{\equiv} c$ and $\eta \leq \mathsf{lk}'_2$, we get a contradiction with $(a, b) \notin \mathsf{lk}'_1$.

CASE 2. For all $\theta \in \mathsf{Con}(\mathbb{A}_2)$ with $\eta \prec \theta \leq \beta_2$ the intervals $(\underline{0}_1, \beta_1), (\eta, \theta)$ cannot be separated.

Suppose $\mathsf{lk}'_2|_{B''_2} \not\subseteq \eta|_{B''_2}$. Without loss of generality let $\eta = \underline{0}_2$. Then there are $a, b \in B''_2$ and $c \in C$ such that $(c, a), (c, b) \in R'$. Let $\theta$ any congruence with $\eta \prec \theta \leq \mathsf{Cg}(\eta \cup \{(a, b)\}) \leq \beta_2$. If $\mathsf{typ}(\underline{0}_1, \beta) = \mathbf{3}$ then by Lemma 48 such $a, b$ do not exists, as long as $(\underline{0}_1, \beta_1), (\eta, \theta)$ cannot be separated. Finally, if $\mathsf{typ}(\underline{0}_1, \beta) \in \{\mathbf{4}, \mathbf{5}\}$, $C$ is a singleton by Lemma 39, and the result is trivial.

Suppose now that $\mathsf{typ}(\underline{0}_1, \beta_1) = \mathsf{typ}(\underline{0}_2, \theta) = \mathbf{2}$. Since $R$ is chained $a, b$ can be assumed to be from $\mathsf{umax}(B''_2)$, and so $\eta|_{\mathsf{umax}(B''_2)} < \mathsf{lk}'_2|_{(B''_2)}$. Also, this implies by Proposition 35 that for any $\mathsf{lk}'_1$-block $E$ and the corresponding $\mathsf{lk}'_2$-block $E'$ it holds $E \times \mathsf{umax}(E') \subseteq R'$. Since by the choice of $\eta$, $\mathsf{lk}'_2 \wedge \theta \not\leq \eta$, pairs $(c, a), (c, b)$ can be chosen such that $a \stackrel{\theta}{\equiv} b$. We prove that $\theta|_{\mathsf{umax}(B''_2)} \subseteq \mathsf{lk}'_2$ producing a contradiction with the choice of $\eta$.

In this case $B_1^*$ is a module, $C = B_1^*$ implying $B'_2 = B''_2$, and by Lemma 57 $R[B_1^*]$ is a subset of a $\zeta(\underline{0}_2, \theta)$-block. Then $\theta \leq \mathsf{lk}''_2 \vee \eta$, where $\mathsf{lk}''_2$ is the link congruence of $\mathbb{A}_2$ with respect to $R$, and as $R$ is chained, by Lemma 44 $\theta|_{\mathsf{umax}(E)} \leq \mathsf{lk}^*_2|_{B_2^*}$ for any $\mathsf{lk}^*_2$-block $E \subseteq B_2^*$, since $B_2^*/_{\mathsf{lk}^*_2}$ is a module, and therefore $E$ is u-maximal in this set. Thus we are in the conditions of Lemma 62. Therefore if $(c, d) \in R^*$ then $(c, e) \in R^*$ for any $e \stackrel{\theta}{\equiv} d$ for any $c \in C$ and $d, e \in \mathsf{umax}(R[C])$.

Again, we now extend this property to $R'$ using the assumption that $R$ is polynomially closed. Since any pair $\{a', b'\} \subseteq B_2^*$ with $a' \stackrel{\theta}{\equiv} b'$ is a $(\eta, \theta)$-subtrace, as $R$ is chained, there is a $(B_1, B_2)$-preserving polynomial $f$ such that $f(a) = a'$ and $f(b) = b'$. Now, use the pairs $(c, a), (c, b) \in R'$. For any $b' \in \mathsf{umax}(B''_2)$ with $b \stackrel{\theta}{\equiv} b'$, let $b'' \in b^\theta$ be such that $h(b, b'', a) = b'$, where $h$ is the function from Theorem 20; such $b''$ exists because $h(b, x, a)$ is a permutation on every $\theta$-block (recall that a $\theta$-block is a module in this case). Since $R$ is chained, there is a polynomial $f$ such that $f(a) = a, f(b) = b''$ and $f(c) = d$ for some $d \in B_1^* = C$. The mapping $g(x) = h\left(x, f(x), \begin{pmatrix} d \\ a \end{pmatrix}\right)$ is such that $g\begin{pmatrix} c \\ a \end{pmatrix} = \begin{pmatrix} c \\ a \end{pmatrix}$ and $g\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} c \\ b' \end{pmatrix}$, because, again, $B_1^*$ is a module. Since $R'$ is polynomially closed and $(c, b) \sqsubseteq_{as} (c, b')$ we have $(c, b') \in R'$; and as $b'$ is arbitrary from $a^\theta$, we have $\{c\} \times a^\theta \subseteq R'$. Thus, we have proved the property for a specific $\theta$-block; next we extend it to other $\theta$-

blocks.

Suppose $\{c\} \times E \subseteq R'$ for some $\theta$-block $E$ and a $\theta$-block $E'$ is such that for some $a \in E, b \in E' \cap B_2''$, $ab$ is a thin edge and $(d, b) \in R'$ for some $d \in C$. Then by Lemma 54 mapping $g(x)$ that is defined as $x \cdot \begin{pmatrix} d \\ b \end{pmatrix}$, $t_{ab}\left(x, \begin{pmatrix} d \\ b \end{pmatrix}\right)$, $h_{ab}\left(x, \begin{pmatrix} c \\ a \end{pmatrix}, \begin{pmatrix} d \\ b \end{pmatrix}\right)$ depending on the type of $ab$ is injective on $E$. In particular, if $ab$ is semilattice or majority then $g$ maps $\{c\} \times E$ to $\{c\} \times E'$, $g(c, a) = (c, b), g(c, a') = (c, b')$ and $b \neq b'$ whenever $a \neq a'$; and since $t_{ab}, h_{ab}$ are term operations and all the tuples involved belong to $R'$, $(c, b), (c, b') \in R'$. If $ab$ is affine then $g$ maps $\{c\} \times E$ to $\{d\} \times E'$, and $g(c, a) = (d, b), g(c, a') = (d, b')$ and $b \neq b'$ whenever $a \neq a'$, and $(d, b), (d, b') \in R'$. In either case, $\mathsf{lk}_2'$ is nontrivial on $E'$, and applying the argument from the previous paragraph we obtain $\{c\} \times E' \subseteq R'$ or $\{d\} \times E' \subseteq R'$. Therefore $\theta|_{\mathsf{umax}(B_2'')} \subseteq \mathsf{lk}_2'|_{\mathsf{umax}(B_2'')}$, a contradiction with the choice of $\eta$. □

# 8 Chaining

In this section we first introduce a property of relations which is slightly stronger than chaining; this is the property that will be used in further proofs. Then we show in Lemma 64 that this property is preserved under certain transformations of the relation.

We call relation $R$ *strongly chained* with respect to $\overline{\beta}, \overline{B}$, where $\beta_i \in \mathsf{Con}(\mathbb{A}_i)$ and $B_i$ is a $\beta_i$-block for $i \in [n]$, if

(Q1s) for any $I \subseteq [n]$ and $\alpha, \beta \in \mathsf{Con}(\mathrm{pr}_I R)$ such that $\alpha \prec \beta \leq \overline{\beta}_I$, $\alpha$ and $\beta$ are $\mathcal{U}_B$-chained in $R$, where $\mathcal{U}_B$ is the set of all $\overline{B}$-preserving polynomials of $R$

(Q2s) for any $\alpha, \beta \in \mathsf{Con}(\mathrm{pr}_I R)$, $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_j)$, $j \in [n]$, such that $\alpha \prec \beta \leq \overline{\beta}_I$, $\gamma \prec \delta \leq \beta_j$, and $(\alpha, \beta)$ can be separated from $(\gamma, \delta)$, the congruences $\alpha$ and $\beta$ are $\mathcal{U}^*$-chained in $R$, where $\mathcal{U}^*$ is the set of all $\overline{B}$-preserving polynomials $g$ of $R$ such that $g(\delta) \subseteq \gamma$

As in the definition of chained relations a polynomial from $\mathcal{U}^*$ in condition (Q2s) will be called $(\gamma, \delta, \overline{B})$-*good.*

We now can state and prove Lemma 64 that the property to be strongly chained is preserved under certain transformations of $\overline{\beta}$ and $\overline{B}$. We will use it prove that one of the conditions, (S7), of a $\beta$-strategy (see Section 9.3) remains true when the $\beta$-strategy is being transformed.

**Lemma 64** *Let $R$ be a subdirect product of $\mathbb{A}_1,\dots,\mathbb{A}_n$, $\beta_i\in\mathsf{Con}(\mathbb{A}_i)$ and $B_i$ a $\beta_i$-block, $i\in[n]$, such that $R$ is strongly chained with respect to $\overline{\beta},\overline{B}$. Let $R'=R\cap(B_1\times\dots\times B_n)$ and $B'_i=\mathrm{pr}_iR'$. Fix $i\in[n]$, $\beta'_i\prec\beta_i$, and let $D_i$ be a $\beta'_i$-block that is a member of a nontrivial as-component of $B'_i/_{\beta'_i}$. Let also $\beta'_j=\beta_j$ and $D_j=B_j$ for $j\neq i$. Then $R$ is strongly chained with respect to $\overline{\beta}',\overline{D}$.*

**Proof:** Let $R''=R\cap(D_1\times\dots\times D_n)$ and $D'_i=\mathrm{pr}_iR''$. Take $I$, $j$ from the definition of being strongly chained. Let $I=[\ell]$; if $|I|>1$ we may consider $R$ as a subdirect product of $\mathrm{pr}_IR$ and $\mathbb{A}_{\ell+1},\dots,\mathbb{A}_n$, so we assume $|I|=1$ and $j=n$ in (Q2s). Let $\alpha,\beta\in\mathsf{Con}(\mathbb{A}_1)$, $\gamma,\delta\in\mathsf{Con}(\mathbb{A}_n)$ be such that $\alpha\prec\beta\le\beta_1$, $\gamma\prec\delta\le\beta_n$. Clearly, we may assume $\alpha=\underline{0}_1$, $\gamma=\underline{0}_n$, and $\beta'_i=\underline{0}_i$. Note that replacing $R$ with the $n+1$-ary relation $\{(\mathbf{a},\mathbf{a}[i])\mid\mathbf{a}\in R\}$ we may assume that $i\notin I\cup\{j\}$. Without loss of generality assume $i=2$. By the assumption $\beta'_2=\underline{0}_2$, the classes of $\beta'_2$ are just elements of $\mathbb{A}_2$, so let $B'_2$ be denoted by $c$. Let $C$ be the as-component of $B'_2$ containing $c$.

To prove the lemma for every $a,b\in D'_1$ with $a\overset{\beta}{\equiv}b$ we have to identify a set $T(a,b,\gamma,\delta,\mathcal{U}^*)$ as in conditions (G1),(G2), and for every $\{a',b'\}\in T(a,b,\gamma,\delta,\mathcal{U}^*)$ we need to find a $(\gamma,\delta,\overline{D})$-good polynomial $f$ such that $f(a)=a',f(b)=b'$. In fact, we rather find all the sets $T$ minimal among the sets of the form $T(a,b,\gamma,\delta,\mathcal{U}^*)$ and that satisfy the conditions of Lemma 43(4). Note that such minimal sets exist for $\overline{\beta},\overline{B}$, as well as, $(\gamma,\delta,\overline{B})$-good polynomials by the assumption that $R$ is strongly chained with respect to $\overline{\beta},\overline{B}$. We need to change such a set $T$ and change the polynomials so that they fit the new requirements. We divide the proof into two cases, depending on whether or not $Q=\mathrm{pr}_{12}R'$ is linked. First, we consider the case when $Q$ is not linked, this case is relatively easy.

CLAIM 1. Let $Q'=Q\cap(\mathsf{umax}(\mathrm{pr}_1Q)\times C)$ be not linked and $\mathsf{lk}_1,\mathsf{lk}_2$ link congruences of $Q$. Then $\mathsf{lk}_2=\underline{0}_2$ and either $\beta\le\mathsf{lk}_1$ or $\beta$ is trivial on $D_1$.

Relation $Q$ is a subalgebra of $R\cap(B_1\times B_2)$ and is polynomially closed in $\mathrm{pr}_{12}R$ by Lemma 60. By the Congruence Lemma 63 if $Q'$ is not linked then $Q$ is the graph of a mapping $\varphi:\mathrm{pr}_1Q\to C$. This means $\mathsf{lk}_2=\underline{0}_2$ and $\mathsf{lk}_1$ is the restriction of a congruence $\eta$ of $\mathbb{A}_1$ onto $\mathrm{pr}_1Q$. If $\beta\le\eta$ then obtain the first option of the conclusion of the claim, otherwise $\mathsf{lk}_1\cap\beta=\underline{0}_1$ and we have the second option.

Note that if $\beta\le\mathsf{lk}_1$ then any $\overline{B}$-preserving polynomial that maps a pair of $\beta$-related elements from $D'_1$ on a $(\alpha,\beta)$-subtrace from $D'_1$ is also $\overline{D}$-preserving, because $\mathsf{lk}_2=\underline{0}_2$; the result follows. If $\beta$ is trivial on $D'_1$, there is nothing to prove. Therefore we may assume $Q'$ is linked.

We start with choosing a $\beta$-block required in the chaining conditions, and studying some of its properties. Observe that since $c$ is as-maximal in $B'_2$, the set

$D_1'$ also contains as-maximal elements of $B_1'$. Therefore by Lemma 26 $\mathsf{umax}(D_1') \subseteq \mathsf{umax}(B_1')$. Let $E$ be a $\beta$-block such that $E'' = E \cap D_1' \neq \varnothing$, $E \cap \mathsf{umax}(D_1') \neq \varnothing$ (and so $E''$ satisfies the requirements of the chaining conditions), and let $E' = E \cap B_1'$. Consider $R^* = R' \cap (B_1 \times C \times B_3 \times \cdots \times B_n)$. Note that $R^*$ is not necessarily a subalgebra. Let $B_i^* = \mathrm{pr}_i R^*$, $i \in [n]$, and $E^* = E \cap B_1^*$. By the Maximality Lemma 27(4) $\mathsf{amax}(E^*)$ is a union of as-components of $E'$. Indeed, let $a \in E^*$ and let $\mathbf{a} \in R^*$ be such that $\mathbf{a}[1] = a$ and $\mathbf{a}[2] \in C$; let also $b \in E'$ with $a \sqsubseteq_{as} b$ in $E'$. Then by the Maximality Lemma 27(4) there is $\mathbf{b} \in R'$ such that $\mathbf{b}[1] = b$ and $\mathbf{a} \sqsubseteq_{as} \mathbf{b}$ in $R'$. In particular, $\mathbf{a}[2] \sqsubseteq_{as} \mathbf{b}[2]$ implying $\mathbf{b}[2] \in C$. Also, by Proposition 35, since $Q$ is linked and $\mathsf{umax}(E^*) \subseteq \mathsf{umax}(B_1')$, we have $\mathsf{umax}(E^*) \times C \subseteq Q$, and therefore $\mathsf{umax}(E^*) = \mathsf{umax}(E'') \subseteq \mathsf{umax}(E')$. In particular, $\mathsf{amax}(E'')$ is a union of as-components of $E'$. The last inclusion here is because $E^*$ contains some as-maximal elements of $E'$.

First we prove condition (Q1s) for $\overline{\beta}'$ and $\overline{D}$.

Claim 2. For any $a, b, a', b' \in E''$ such that $a, b$ belong to the same as-component of $E''$ there is a $(\gamma, \delta, \overline{D})$-good polynomial $f$ with $f(\{a', b'\}) = \{a, b\}$.

Consider relation $S$, a subdirect product of $\mathbb{A}_1 \times \mathbb{A}_1 \times \mathbb{A}_2 \times \cdots \times \mathbb{A}_n$, produced from by $(a', b', \mathbf{a})$, where $\mathbf{a} \in \mathrm{pr}_{\{2,\dots,n\}} R''$, as follows:

$$S = \{f(a), f(b), f(\mathbf{a})) \mid f \text{ is a unary polynomial of } R \text{ with } f(\delta) \subseteq \gamma\}.$$

It is not difficult to see that $S$ is a subalgebra, and, in particular it contains all the tuples of the form $(\mathbf{b}[1], \mathbf{b}[1], \mathbf{b}[2], \dots, \mathbf{b}[n])$ for $\mathbf{b} \in R$. Let $S' = S \cap \overline{B}$, and $S'' = S \cap \overline{D}$. Every tuple from $S'$ or from $S''$ corresponds to a $\overline{B}$- or $\overline{D}$-preserving polynomial. Therefore it suffices to prove that $(a, b) \in \mathrm{pr}_{12} S''$. Let $F$ be the as-component of $E''$ containing $a, b$; as observed above $F$ is also an as-component of $E'$. By the assumption of (Q2s) $F^2 \subseteq \mathrm{pr}_{12} S'$ and $(e, e) \in \mathrm{pr}_{12} S''$ for any $e \in F$, since $F \times C \subseteq Q'$. We consider relation $P = \mathrm{pr}_{123} S'$. As $F^2 \subseteq P' = \mathrm{pr}_{12} P$, $(a, b)$ is as-maximal in $P'$. Therefore it suffices to show that $\mathsf{amax}(P)$ is linked when considered as subdirect product of $P'$ and $B_2'$. Since $(e, e) \in \mathrm{pr}_{12} S''$ for any $e \in F$, all pairs of this form are linked in $P$. Then $(e, d, a'') \in P$ for any $e, d \in F$ and some $a'' \in B_2'$, and $(e, e, c'') \in P$ for some $c'' \in C$. Since $F^2 \subseteq P'$, $(e, e) \sqsubseteq_{as} (e, d)$, and by the Maximality Lemma 27(4) $a''$ can be chosen from $C$, and so this implies that $(e, d)$ and $(e, e)$ are also linked. Claim 2 is proved.

Now we extend the result above to pairs from $\mathsf{umax}(E^*)$. We prove the result in two steps. First, we show that for any $a', b' \in E^*$ and any $a, b \in \mathsf{umax}(E^*)$ there is a sequence of $\overline{B}$-preserving polynomials $f_1, \dots, f_k$ such that $f_1(\{a', b'\}), \dots, f_k(\{a', b'\}) \subseteq E^*$ form a chain connecting $a$ and $b$, $f_i(\mathbb{A}_1)$ is an $(\alpha, \beta)$-minimal set, and $f_i(c) \in C$ for $i \in [k]$. Then we prove that $f_1, \dots, f_k$ can be chosen in such a way that

$f_1(\{a', b'\}), \ldots, f_k(\{a', b'\}) \subseteq E''$ and $f_1(c) = \cdots = f_k(c) = c$. Clearly, it suffices to prove in the case when $b$ is as-maximal in $E^*$.

By the assumption there are $a = a_1, a_2, \ldots, a_k = b$, $a_1, \ldots, a_k \in E'$ and $(\gamma, \delta, \overline{B})$-good polynomials $f_1, \ldots, f_{k-1}$ such that $f_i(\mathbb{A}_1)$ is a $(\alpha, \beta)$-minimal set and $f_i(\{a', b'\}) = \{a_i, a_{i+1}\}$, and also $f_i(c) \in B'_2$. We need to show that $a_1, \ldots, a_{k-1}$ and $f_1, \ldots, f_{k-1}$ can be chosen such that $f_i(c) \in C$. Choose $\mathbf{a}, \mathbf{b} \in R''$ such that $\mathbf{a}[1] = a, \mathbf{b}[1] = b$ and $\mathbf{a}[2] = \mathbf{b}[2] = c$. This is possible because $\mathsf{umax}(E^*) = \mathsf{umax}(E'')$. Now let $g_i(x) = \mathsf{maj}(\mathbf{a}, f_i(x), \mathbf{a})$ and $h_i(x) = \mathsf{maj}(\mathbf{a}, \mathbf{b}, f_i(x))$. By Lemma 43 $g_i, h_i$ are $(\gamma, \delta, \overline{B})$-good polynomials, and for each of them either $\{b_i, b_{i+1}\} = g_i(\{a', b'\})$ ($\{c_i, c_{i+1}\} = h_i(\{a', b'\})$) is an $(\alpha, \beta)$-subtrace, or $g_i(\beta) \subseteq \alpha$ ($h_i(\beta) \subseteq \alpha$), that is $g_i(a') = g_i(b')$ (respectively, $h_i(a') = h_i(b')$). The polynomials $g_i, h_i$ satisfying the first option form a sequence of $(\alpha, \beta)$-subtraces connecting $a$ with $\mathsf{maj}(a, b, a)$ — by subtraces of the form $\{b_i, b_{i+1}\}$, — and $\mathsf{maj}(a, b, a)$ with $\mathsf{maj}(a, b, b)$ — by subtraces of the form $\{c_i, c_{i+1}\}$. Also, by Theorem 31 $\mathsf{maj}(a, b, b)$ belongs to the as-component of $E^*$ (and therefore of $E'$ and $E''$) containing $b$. Therefore by Claim 2 this sequence of polynomials and subtraces can be continued to connect $\mathsf{maj}(a, b, b)$ to $b$. Finally, by the same theorem $g_i(c) = \mathsf{maj}(c, f_i(c), c) \in C$ and $h_i(c) = \mathsf{maj}(c, c, f_i(c)) \in C$.

For the second step we assume that $a$ and $b$ are connected with $(\alpha, \beta)$-subtraces $\{a_i, a_{i+1}\}$, $i \in [k-1]$ witnessed by $(\gamma, \delta, \overline{B})$-good polynomials $f_i$ such that $c_i = f_i(c) \in C$. We need to show that $f_i$ can be chosen such that $f_i(c) = c$. Suppose that $c_i \neq c$ for some $i \in [k-1]$. Since $c_i$ and $c$ belong to the same as-component, there is an as-path $c_i = d_1, \ldots, d_\ell = c$ in $C$. We show that if there is a sequence of $(\alpha, \beta)$-subtraces $\{b_j, b_{j+1}\}$ witnessed by polynomials $g_j$ such that $g_j(c) = c$ whenever $f_j(c) = c$, and $f_i(c) = d_t$, there are also $(\alpha, \beta)$-subtraces $\{b'_j, b'_{j+1}\}$ such that $b'_1 = a$ and $b'_k$ is in the as-component containing $b$, witnessed by polynomials $g'_1, \ldots, g'_k$ such that $g'_i(c) = d_{t+1}$ and $g'_j(c) = c$ whenever $g_j(c) = c$.

As is easily seen, it suffices to find a ternary term operation $p$ such that $p(a, a, b)$ belongs to the as-component containing $b$, and $p(d_{t+1}, d_t, d_t) = d_{t+1}$. Indeed, if such a term operation exists, then we set $g'_j(x) = p(\mathbf{a}, \mathbf{a}, g_j(x))$, where $\mathbf{a}$ is as in the first step above, for $j \in [k-1] - \{i\}$, and $\{b'_j, b'_{j+1}\} = g'_j(\{a', b'\})$. We have $g'_1(a') = p(a, a, g_1(a')) = a$ and $g'_j(c) = p(c, c, g_j(c)) = c$ whenever $g_j(c) = c$. Finally, since $g'_\ell(b) = p(a, a, b)$ belongs to the as-component containing $b$, we can use Claim 2 as before to connect $p(a, a, b)$ to $b$. For $g'_i$ we set $g'_i(x) = p(\mathbf{a}', \mathbf{a}'', g_i(x))$ where $\mathbf{a}', \mathbf{a}'' \in R''$ are such that $\mathbf{a}'[1] = \mathbf{a}''[1] = a$ and $\mathbf{a}'[2] = d_{t+1}, \mathbf{a}''[2] = d_t$. Note that such $\mathbf{a}', \mathbf{a}''$ exist, because $\mathsf{umax}(E^*) \times C \subseteq Q$. It follows from the assumption about $p$ that $g'_i$ is as required.

If $d_t \leq d_{t+1}$, then $p(x, y, z) = z \cdot x$ fits the requirements. If $d_t d_{t+1}$ is an affine edge, consider the relation $S \subseteq \mathbb{A}_1 \times \mathbb{A}_2$ generated by $\{(a, d_t), (a, d_{t+1}), (b, d_t)\}$.

Let $\mathbb{B} = \mathsf{Sg}(a,b)$ and $\mathbb{C} = \mathsf{Sg}(d_t, d_{t+1})$; then $\mathbb{B} \times \{d_t\}, \{a\} \times \mathbb{C} \subseteq S$. By Lemma 32, as $d_t d_{t+1}$ is a thin affine edge, $\mathsf{umax}(\mathbb{B}) \times \{d_{t+1}\} \subseteq S$. There is $b'$ with $b \sqsubseteq_{as} b'$ in $\mathbb{B}$ such that $b' \in \mathsf{umax}(\mathbb{B})$. Therefore there is a term operation $p$ with $p(a,a,b) = b'$ and $p(d_{t+1}, d_t, d_t) = d_{t+1}$, as required. □

# 9 Strategies and solutions

## 9.1 The grand scheme

In this section we describe the 'grand scheme' of solving CSPs. We start with introducing two preprocessing steps for our algorithm.

We call a CSP instance $\mathcal{P} = (V, \mathcal{C})$ *subdirectly irreducible* if it is 1-minimal and $\mathbb{A}_v$ is subdirectly irreducible for every $v \in V$.

**Lemma 65 (Folklore)** *Every CSP instance can be reduced in polynomial time to an equivalent subdirectly irreducible one.*

In this section all instances we consider are assumed subdirectly irreducible. The monolith of $\mathbb{A}_v$ is denoted by $\mu_v$.

Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance and for $X \subseteq V$, $|X| = 2$, there is a constraint $C^X = \langle X, R^X \rangle$, where $R^X$ is the set of partial solutions on $X$. We use the notation from the end of Section 6.2. Recall that $\mathcal{W}^{\mathcal{P}}(\overline{\beta})$ denotes the set of triples $(v, \alpha, \beta)$ such that $v \in V$, $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$, and $\alpha \prec \beta \le \beta_v$. If $\beta_v = \underline{1}_v$ for all $v \in V$, we set $\mathcal{W}^{\mathcal{P}}\mathcal{W}^{\mathcal{P}}(\overline{\beta})$. Also, let $W_{v,\alpha\beta,\overline{\beta}}$ to be the set of $w \in V$ such that for some $(w, \gamma, \delta) \in \mathcal{W}^{\mathcal{P}}(\overline{\beta})$ the prime factors $(\alpha, \beta)$ and $(\gamma, \delta)$ cannot be separated in $R^{\{v,w\}}$. Let $\overline{\beta}$, $\beta_v \in \mathsf{Con}(\mathbb{A}_v)$, $v \in V$, be a collection of congruences. Let $\mathcal{W}'(\overline{\beta})$ (and respectively $\mathcal{W}'$) denote the set of triples $(v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})$ (respectively, from $\mathcal{W}$) with $\zeta(\alpha, \beta) = \underline{1}_v$.

We say that algebra $\mathbb{A}_v$ is *semilattice free* if it does not contain semilattice edges. Let $\mathsf{size}(\mathcal{P})$ denote the maximal size of domains of $\mathcal{P}$ that are not semilattice free and $\mathsf{MAX}(\mathcal{P})$ be the set of variables $v \in V$ such that $|\mathbb{A}_v| = \mathsf{size}(\mathcal{P})$ and $\mathbb{A}_v$ is not semilattice free. Finally, for $Y \subseteq V$ let $\mu_v^Y = \mu_v$ if $v \in Y$ and $\mu_v^Y = \underline{0}_v$ otherwise. Recall that by $\mathcal{P}/_{\overline{\mu}^Y}$ we denote the instance $(V, \mathcal{C}^{\overline{\mu}^Y})$ constructed as follows: the domain of $v \in V$ is $\mathbb{A}_v/_{\mu_v^Y}$; for every constraint $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$, the set $\mathcal{C}^{\overline{\mu}^Y}$ includes the constraint $\langle \mathbf{s}, R/_{\overline{\mu}^Y_{\mathbf{s}}} \rangle$.

Instance $\mathcal{P}$ is said to be *block-minimal* if for every $(v, \alpha, \beta) \in \mathcal{W}$ (here $\beta_v = \underline{1}_v$, $v \in V$)

(BM1) for every $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ if $(v, \alpha, \beta) \notin \mathcal{W}'$, and the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}/_{\overline{\mu}^Y}$ otherwise, where $Y = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$, is minimal;

(BM2) if $(v,\alpha,\beta)\in\mathcal{W}'$, then for every $(w,\gamma,\delta)\in\mathcal{W}-\mathcal{W}'$ the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}/_{\overline{\mu}^Y}$, where $Y=\mathsf{MAX}(\mathcal{P})-(W_{v,\alpha\beta,\overline{\beta}}\cap W_{w,\gamma\delta,\overline{\beta}})$ is minimal.

The definition of block-minimality is designed in such a way that block-minimality can be efficiently established. Observe that $W_{v,\alpha\beta,\overline{\beta}}$ can be large, even equal to $V$. However if $(v,\alpha,\beta)\not\in\mathcal{W}'$ by Theorem 58 the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ splits into a union of disjoint problems over smaller domains. On the other hand, if $(v,\alpha,\beta)\in\mathcal{W}'$ then $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ may not be decomposable. Since we need an efficient procedure of establishing block-minimality, this explains the complications introduced in (BM1),(BM2).

For an instance $\mathcal{P}$ we say that an instance $\mathcal{P}'$ is *strictly smaller* than instance $\mathcal{P}$ if $\mathsf{size}(\mathcal{P}')<\mathsf{size}(\mathcal{P})$.

**Lemma 66** *Let $\mathcal{P}=(V,\mathcal{C})$ be a (2,3)-minimal instance. Then $\mathcal{P}$ can be transformed to an equivalent block-minimal instance $\mathcal{P}'$ by solving a quadratic number of strictly smaller CSPs.*

**Proof:** To establish block-minimality of $\mathcal{P}$, for every $(v,\alpha,\beta)\in\mathcal{W}$ (let $W=W_{v,\alpha\beta}$), we need to check if the problems given in conditions (BM1),(BM2) are minimal. If they are then $\mathcal{P}$ is block-minimal, otherwise some tuples can be removed from some constraint relation $R$ (the set of tuples that remain in $R$ is always a subalgebra, as is easily seen), and the instance $\mathcal{P}$ tightened, in which case we need to repeat the procedure with the tightened instance. Therefore we just need to show how to reduce solving those subproblems to solving strictly smaller CSPs.

For $C=\langle\mathbf{s},R\rangle\in\mathcal{C}$ and $\mathbf{a}\in R$ let $\mathcal{P}'$ be the problem obtained as follows: fix the values of variables from $\mathbf{s}\cap W$, or from $\mathbf{s}\cap W\cap W_{w,\gamma\delta}$ in the case of (BM2) to those of $\mathbf{a}$. If the resulting problem is $\mathcal{P}''$ then set $\mathcal{P}'=\mathcal{P}''/_{\overline{\mu}^Y}$, where $Y$ is either empty, if $(v,\alpha,\beta)\not\in\mathcal{W}'$, or $Y=\mathsf{MAX}(\mathcal{P})-\mathbf{s}$, if $(v,\alpha,\beta)\in\mathcal{W}'$ in (BM1), or $Y=\mathsf{MAX}(\mathcal{P})-(W\cap M_{w,\gamma\delta})$ in (BM2). In the first case, by Theorem 58 $\mathcal{P}'$ is a disjoint union of instances $\mathcal{P}_1,\dots,\mathcal{P}_\ell$ and $\mathsf{size}(\mathcal{P}_i)<\mathsf{size}(\mathcal{P})$. In the second case the domains of variables from $\mathbf{s}\cap W$ have cardinality 1, and the domain of each of the remaining variables either is semilattice free, or is smaller than $\mathsf{size}(\mathcal{P})$. Finally, in the last case the domain of each of the variables outside of $W\cap W_{w,\gamma\delta}$ is either semilattice free or smaller than $\mathsf{size}(\mathcal{P})$. Also, by Theorem 58 $\mathcal{P}_{W\cap W_{w,\gamma\delta}}$ is a disjoint union of instances with domains of smaller size. Let $\mathcal{P}_1,\dots,\mathcal{P}_k$ be these disjoint instances. Then $\mathcal{P}'$ can be reduced to solving the instances $\mathcal{P}'_1,\dots,\mathcal{P}'_k$ obtained from $\mathcal{P}'$ by restricting $\mathcal{P}'_{W\cap W_{w,\gamma\delta}}$ to $\mathcal{P}_i$. This completes the proof. □

Let $\mathcal{P}=(V,\mathcal{C})$ be a subdirectly irreducible (2,3)-minimal instance. Let $\mathsf{Center}(\mathcal{P})$ denote the set of variables $v\in V$ such that $\zeta(\underline{0}_v,\mu_v)=\underline{1}_v$. Let $\mu^*_v=\mu_v$ if $v\in\mathsf{MAX}(\mathcal{P})\cap\mathsf{Center}(\mathcal{P})$ and $\mu^*_v=\underline{0}_v$ otherwise.

We consider several cases and indicate what kind of reductions or solution algorithms we intend to use in each case.

**Case 1: Semilattice free domains.** If all domains of $\mathcal{P}$ are semilattice free then $\mathcal{P}$ can be solved in polynomial time, using the few subalgebras algorithm, as shown in [20].

**Case 2: Collapsing trivial centralizers.** If $\mu_v^* = \underline{0}_v$ for all $v \in V$, block-minimality guarantees that a solution exists, and we can use Lemma 66 to solve the instance.

**Theorem 67** *If $\mathcal{P}$ is subdirectly irreducible, (2,3)-minimal, block-minimal, and $\mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P}) = \varnothing$, then $\mathcal{P}$ has a solution.*

**Case 3: Nontrivial centralizers.** If $\mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P}) = \varnothing$, we first solve the problem $\mathcal{P}/_{\mu^*}$, and then use Theorem 68 to reduce $\mathcal{P}$ to a strictly smaller instance. An efficient way to establish 1-minimality of $\mathcal{P}/_{\mu^*}$ is given in Theorem 69.

**Theorem 68** *If $\mathcal{P}/_{\overline{\mu}^*}$ is 1-minimal, then $\mathcal{P}$ can be reduced in polynomial time to a strictly smaller instance.*

With the reductions above a solution algorithm goes as shown in Algorithm 2, we reproduce it here for convenience.

**Theorem 69** *Algorithm SolveCSP (Algorithm 2) correctly solves every instance from $\mathrm{CSP}(\mathcal{A})$ and runs in polynomial time.*

**Proof:** By the results of [20] the algorithm correctly solves the given instance $\mathcal{P}$ in polynomial time if the conditions of Step 1 are true. Lemma 66 implies that Steps 4 and 12 can be completed by recursing to strictly smaller instances.

Next we show that the for-loop in Steps 8-14 checks if $\mathcal{P}^* = \mathcal{P}/_{\overline{\mu}^*}$ is globally 1-minimal. For this we need to verify that a value $a$ is flagged if and only if $\mathcal{P}^*$ has no solution $\varphi$ with $\varphi(v) = a$, and therefore if no values are flagged then $\mathcal{P}^*$ is globally 1-minimal. If $\varphi(v) = a$ for some solution $\varphi$ of $\mathcal{P}^*$, then $\varphi$ is a solution $\mathcal{P}'$ constructed in Step 9. In this case Steps 11,12 cannot result in an empty instance. Suppose $a \in \mathbb{A}_v/_{\mu_v^*}$ is not flagged. If $\mathsf{size}(\mathcal{P}'') < \mathsf{size}(\mathcal{P})$ this means that $\mathcal{P}''$ and therefore $\mathcal{P}'$ has a solution. Otherwise this means that establishing block-minimality of $\mathcal{P}''$ is successful. In this case $\mathcal{P}''$ has a solution by Theorem 67, because $\mathsf{MAX}(\mathcal{P}'') \cap \mathsf{Center}(\mathcal{P}'') = \varnothing$. This in turn implies that $\mathcal{P}'$ has a solution.

**Algorithm 2** Procedure SolveCSP

**Require:** A CSP instance $\mathcal{P} = (V, \mathcal{C})$ over $\mathcal{A}$
**Ensure:** A solution of $\mathcal{P}$ if one exists, 'NO' otherwise
1: **if** all the domains are semilattice free **then**
2:     Solve $\mathcal{P}$ using the few subpowers algorithm and RETURN the answer
3: **end if**
4: Transform $\mathcal{P}$ to a subdirectly irreducible, block-minimal and (2,3)-minimal instance
5: $\mu^*_v = \mu_v$ for $v \in \mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P})$ and $\mu^*_v = \underline{0}_v$ otherwise
6: $\mathcal{P}^* = \mathcal{P}/_{\overline{\mu}^*}$
7: /* the global 1-minimality of $\mathcal{P}^*$
8: **for** every $v \in V$ and $a \in \mathbb{A}_v/_{\mu^*_v}$ **do**
9:     $\mathcal{P}' = \mathcal{P}^*_{(v,a)}$    /* Add constraint $\langle (v), \{a\} \rangle$ fixing the value of $v$ to $a$
10:     Transform $\mathcal{P}'$ to a subdirectly irreducible, (2,3)-minimal instance $\mathcal{P}''$
11:     If $\mathsf{size}(\mathcal{P}'') < \mathsf{size}(\mathcal{P})$ call SolveCSP on $\mathcal{P}''$ and flag $a$ if $\mathcal{P}''$ has no solution
12:     Establish block-minimality of $\mathcal{P}''$; if the problem changes, return to Step 10
13:     If the resulting instance is empty, flag the element $a$
14: **end for**
15: If there are flagged values, tighten the instance by removing the flagged elements and start over
16: Use Theorem 68 to reduce $\mathcal{P}$ to an instance $\mathcal{P}'$ with $\mathsf{size}(\mathcal{P}') < \mathsf{size}(\mathcal{P})$
17: Call SolveCSP on $\mathcal{P}'$ and RETURN the answer

Observe also that the set of unflagged values for each variable $v \in V$ is a subalgebra of $\mathbb{A}/_{\mu^*}$. Indeed, the set of solutions of $\mathcal{P}^*$ is a subalgebra $\mathcal{S}^*$ of $\prod_{v\in V} \mathbb{A}/_{\mu^*}$, and the set of unflagged values is the projection of $\mathcal{S}^*$ of the coordinate position $v$.

Finally, if Steps 8–15 are completed without restarts, Steps 16,17 can be completed by Theorem 68 and recursing on $\mathcal{P}'$ with $\mathsf{size}(\mathcal{P}') < \mathsf{size}(\mathcal{P})$.

To see that the algorithm runs in polynomial time it suffices to observe that

(1) The number of restarts in Steps 4 and 15 is at most linear, as the instance becomes smaller after every restart; therefore the number of times Steps 4–15 are executed together is at most linear.

(2) The number of iterations of the for-loop in Steps 8–14 is linear.

(3) The number of restarts in Steps 10 and 12 is at most linear, as the instance becomes smaller after every iteration.

(4) Every call of SolveCSP when establishing block-minimality in Steps 4, and 12 is made on an instance strictly smaller than $\mathcal{P}$, and therefore depth of recursion is bounded by $\mathsf{size}(\mathcal{P})$ in Step 4,11,12 and 17.

Thus a more thorough estimation gives a bound on the running time of $O(n^{3k})$, where $k$ is the maximal size of an algebra in $\mathcal{A}$. □

## 9.2 Proof of Theorem 68

Following [43] let $\mathcal{P} = (V, \mathcal{C})$ be an instance and $p_v \colon \mathbb{A}_v \to \mathbb{A}_v$, $v \in V$. Mappings $p_v$, $v \in V$, are said to be *consistent* if for any $\langle \mathbf{s}, R\rangle \in \mathcal{C}$, $\mathbf{s} = (v_1, \dots, v_k)$, and any tuple $\mathbf{a} \in R$ the tuple $(p_{v_1}(\mathbf{a}[1]), \dots, p_{v_k}(\mathbf{a}[k]))$ belongs to $R$. It is easy to see that the composition of two families of consistent mappings is also a consistent mapping. For consistent idempotent mappings $p_v$ by $p(\mathcal{P})$ we denote the *retraction* of $\mathcal{P}$, that is, $\mathcal{P}$ restricted to the images of $p_v$. In this case $\mathcal{P}$ has a solution if and only if $p(\mathcal{P})$ has, see [43].

Let $\varphi$ be a solution of $\mathcal{P}/_{\overline{\mu}^*}$. We define $p_v^\varphi : \mathbb{A}_v \to \mathbb{A}_v$ as follows: $p_v^\varphi = q_v^k$, where $q_v(a) = a \cdot b_v$, element $b_v$ is any element of $\varphi(v)$, and $k$ is such that $q_v^k$ is idempotent for all $v \in V$. Note that by Corollary 52 this mapping is properly defined even if $\mu_v^* \neq \underline{0}_v$.

**Lemma 70** *Mappings $p_v^\varphi$, $v \in V$, are consistent.*

**Proof:** Take any $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$. Since $\varphi$ is a solution of $\mathcal{P}/_{\overline{\mu}^*}$, there is $\mathbf{b} \in R$ such that $\mathbf{b}[v] \in \varphi(v)$ for $v \in \mathbf{s}$. Then for any $\mathbf{a} \in R$, $q(\mathbf{a}) = \mathbf{a} \cdot \mathbf{b} \in R$, and this product does not depend on the choice of $\mathbf{b}$, as it follows from Corollary 52. Iterating this operation also produces a tuple from $R$. □

We would like to use the above reduction to reduce $\mathcal{P}$ to a problem $\mathcal{P}'$ such that $\mathsf{size}(\mathcal{P}') < \mathsf{size}(\mathcal{P})$. If $\varphi$ is such that for $v \in \mathsf{MAX}(\mathcal{P})$ there is $a \in \mathbb{A}_v$ with $a^{\mu_v^*} \leq \varphi(v)$ and $a \notin \varphi(v)$, then $|p_v^{\varphi}(\mathbb{A}_v)| < |\mathbb{A}_v|$. Also, observe that if $|p_v^{\varphi}(\mathbb{A}_v)| = |\mathbb{A}_v|$, then $p_v^{\varphi}$ is the identity mapping, that is $p_v^{\varphi}(\mathbb{A}_v) = \mathbb{A}_v$. If $\mathbb{A}_v$ is semilattice free then $p_v^{\varphi}$ is the identity mapping by Proposition 21. Let $V^*$ be the set of variables $v \in V$ such that $\mathbb{A}_v/_{\mu_v^*}$ is not semilattice free.

**Lemma 71** *There are consistent mappings $p_v$, $v \in V$, such that for any $v \in V^*$ we have $|p_v(\mathbb{A}_v)| < |\mathbb{A}_v|$. Moreover, such mappings can be found solving a linear number of instances of the form $(\mathcal{P}_{(v,a^{\mu_v^*})})/_{\overline{\mu}^*}$.*

**Proof:** Since $\mathcal{P}/_{\overline{\mu}^*}$ is globally 1-minimal, for any $a \in \mathbb{A}_v/_{\mu_v^*}$ there is a solution $\varphi$ with $\varphi(v) = a$, and it can be found solving the instance $(\mathcal{P}_{(v,a^{\mu_v^*})})/_{\overline{\mu}^*}$. For every $v \in V^*$ choose $a \in \mathbb{A}_v$ such that there is $b \in \mathbb{A}_v$ and $b \leq a$, $a \overset{\mu_v^*}{\not\equiv} b$, and let $\varphi_v$ be a solution of $\mathcal{P}/_{\overline{\mu}^*}$ with $\varphi_v(v) = a^{\mu_v^*}$. Then $|p_v^{\varphi_v}(\mathbb{A}_v)| < |\mathbb{A}_v|$ and $|p_w^{\varphi_v}(\mathbb{A}_w)| < |\mathbb{A}_w|$ or $p_w^{\varphi_v}$ is the identity mapping for any $w \in V^*$. Therefore the composition of the $p^{\varphi_w}$ for all $w \in V^*$ is as required. $\square$

Theorem 68 now follows by observing that if $\mathbb{A}_v/_{\mu_v^*}$ is semilattice free then $\mathbb{A}_v$ itself is semilattice free.

In order to use Theorem 68 we however need to argue that $p(\mathcal{P})$ is a problem over a class of algebras omitting type **1**. Let $f$ be a weak near-unanimity term of the class $\mathcal{A}$. Then $p \circ f$ is a weak near-unanimity term of $p(\mathcal{A}) = \{p(\mathbb{A}) \mid \mathbb{A} \in \mathcal{A}\}$. Moreover, if $\mathbb{A}$ is semilattice free then $p(\mathbb{A}) = \mathbb{A}$.

## 9.3 Strategies

In this section similar to strategies related to the concepts of consistency and minimality we introduce strategies of some sort that will be used to prove Theorem 67. We start with some necessary definitions.

Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal and block-minimal instance over $\mathcal{A}$. For $(v, \alpha, \beta) \in \mathcal{W}$, if $(v, \alpha, \beta) \notin \mathcal{W}'$, then $\mathcal{S}_{W_{v,\alpha\beta,\overline{\beta}}}$ denotes the set of solutions of $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$, and if $(v, \alpha, \beta) \in \mathcal{W}'$, then $\mathcal{S}_{W_{v,\alpha\beta,\overline{\beta}},Y}$ denotes the set of solutions of $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}/_{\overline{\mu}^Y}$ for an appropriate $Y$.

Let $\beta_v \in \mathsf{Con}(\mathbb{A}_v)$ and let $B_v$ be a $\beta_v$-block, $\overline{\beta} = (\beta_v \mid v \in V)$, $\overline{B} = (B_v \mid v \in V)$. Let $\mathcal{R} = \{R_{C,v,\alpha\beta} \mid C = \langle \mathbf{s}, R\rangle \in \mathcal{C}, (v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})\}$ be a collection of relations such that $R_{C,v,\alpha\beta}$ is a subalgebra of $\mathrm{pr}_{\mathbf{s}\cap W_{v,\alpha\beta,\overline{\beta}}} R$. Let $C = \langle \mathbf{s}, R\rangle$, $(v, \alpha, \beta) \in \mathcal{W}$, and $W = W_{v,\alpha\beta,\overline{\beta}}$. Let $\mathbf{a}$ be a tuple from $\mathrm{pr}_X R$ for $X \subseteq \mathbf{s}$, or from $\mathrm{pr}_X \mathcal{S}_W$, $X \subseteq W$, if $(v, \alpha, \beta) \notin \mathcal{W}'$, or from $\mathrm{pr}_X \mathcal{S}_{W,Y}$ if

$(v, \alpha, \beta) \in \mathcal{W}'$, where $X \subseteq W$ and $Y$ is a set specified in the condition of block-minimality. Tuple $\mathbf{a}$ is said to be $\mathcal{R}$-*compatible* if for any $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$, (let $U = W_{w,\gamma\delta,\overline{\beta}}$) $\mathrm{pr}_{X\cap U}\mathbf{a} \in \mathrm{pr}_{X\cap U} R_{C,w,\gamma\delta}$ or $\mathrm{pr}_{X\cap U}\mathbf{a} \in \mathrm{pr}_{X\cap U} R_{C,w,\gamma\delta}/_{\overline{\mu}^Y}$ for an appropriate set $Y$. By $R^{\mathcal{R}}, \mathcal{S}_W^{\mathcal{R}}, \mathcal{S}_{W,Y}^{\mathcal{R}}$ we denote the set of all $\mathcal{R}$-compatible tuples from the corresponding relation. Also, let $\mathcal{P}^{\mathcal{R}} = (V, \mathcal{C}^{\mathcal{R}})$ denote the problem instance obtained from $\mathcal{P}$ replacing every constraint $\langle \mathbf{s}, R\rangle \in \mathcal{C}$ with $\langle \mathbf{s}, R^{\mathcal{R}}\rangle$.

The collection $\mathcal{R}$ is called a $\overline{\beta}$-*strategy* with respect to $\overline{B}$ if it satisfies the following conditions for every $(v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})$, and every $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ (let $W = W_{w,\alpha\beta,\overline{\beta}}$):

(S1) the relations $\mathsf{umax}(R^{X,\mathcal{R}})$, where $R^{X,\mathcal{R}}$ consists of $\mathcal{R}$-compatible tuples from $R^X$ for $X \subseteq V$, $|X| \le 2$, form a nonempty $(2,3)$-strategy for $\mathcal{P}^{\mathcal{R}}$;

(S2) for every $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$ (let $U = W_{w,\gamma\delta}$) and every $\mathbf{a} \in \mathsf{umax}(\mathrm{pr}_{\mathbf{s}\cap W\cap U} R_{C,v,\alpha\beta})$ it holds: if $(w, \gamma, \delta) \notin \mathcal{W}'$ then $\mathbf{a}$ extends to an $\mathcal{R}$-compatible solution $\varphi$ of $\mathcal{P}_U$; otherwise if $(v, \alpha, \beta) \notin \mathcal{W}'$ then $\mathbf{a}$ extends to an $\mathcal{R}$-compatible solution of $\mathcal{P}_U/_{\overline{\mu}^{Y_1}}$ with $Y_1 = \mathsf{MAX}(\mathcal{P}) - (W \cap U)$; and if $(v, \alpha, \beta) \in \mathcal{W}'$ then $\mathbf{a}$ extends to an $\mathcal{R}$-compatible solution of $\mathcal{P}_U/_{\overline{\mu}^{Y_2}}$, where $Y_2 = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$;

(S3) $R \cap \overline{B}_{\mathbf{s}} \neq \varnothing$ and for any $I \subseteq \mathbf{s}$ any $\mathcal{R}$-compatible tuple $\mathbf{a} \in \mathsf{umax}(\mathrm{pr}_I R)$ extends to an $\mathcal{R}$-compatible tuple $\mathbf{b} \in R$.

(S4) the relation $R_{C,v,\alpha\beta}$ is a subalgebra of $\mathrm{pr}_{\mathbf{s}\cap W} R$, and $\mathsf{umax}(R_{C,v,\alpha\beta}) \subseteq \mathsf{umax}(\mathrm{pr}_{\mathbf{s}\cap W} R)$; if $(v, \alpha, \beta) \notin \mathcal{W}'$ then the relation $\mathcal{S}_W^{\mathcal{R}}$ is a subalgebra of $\mathcal{S}_W$, and $\mathsf{umax}(\mathcal{S}_W^{\mathcal{R}}) \subseteq \mathsf{umax}(\mathcal{S}_W)$; if $(v, \alpha, \beta) \in \mathcal{W}'$ then for any $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta}) - \mathcal{W}'$ the relations $\mathcal{S}_{W,Y_1}^{\mathcal{R}}, \mathcal{S}_{W,Y_2}^{\mathcal{R}}$ are subalgebras of $\mathcal{S}_{W,Y_1}, \mathcal{S}_{W,Y_2}$, respectively, and $\mathsf{umax}(\mathcal{S}_{W,Y_1}^{\mathcal{R}}) \subseteq \mathsf{umax}(\mathcal{S}_{W,Y_1})$, $\mathsf{umax}(\mathcal{S}_{W,Y_2}^{\mathcal{R}}) \subseteq \mathsf{umax}(\mathcal{S}_{W,Y_2})$, where $Y_1 = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$ and $Y_2 = \mathsf{MAX}(\mathcal{P}) - (W \cap W_{w,\gamma\delta})$;

(S5) for every $w \in \mathbf{s}$ and every $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$ (let $U = W_{w,\gamma\delta}$) with $w \in \mathbf{s}\cap U$ it holds $\mathsf{umax}(\mathrm{pr}_w R_{C,w,\gamma\delta}) = \mathsf{umax}(\mathrm{pr}_w R_{C,v,\alpha\beta})$, let $A_{\mathcal{R},w}$ denote the subalgebra generated by this set, $\mathsf{umax}(A_{\mathcal{R},w})$ is as-closed in $\mathsf{umax}(\mathrm{pr}_w(R \cap \overline{B}))$;

(S6) for every $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$ with $\mathbf{s} \cap W_{w,\gamma\delta} \neq \varnothing$ the set of $\mathcal{R}$-compatible tuples from $R_{C,w,\gamma\delta}$ is polynomially closed in $\mathrm{pr}_{\mathbf{s}\cap W_{w,\gamma\delta}} R$;

(S7) relation $R$ is strongly chained with respect to $\overline{\beta}, \overline{B}$; if $(v, \alpha, \beta) \notin \mathcal{W}'$, relation $\mathcal{S}_W$ is strongly chained with respect to $\overline{\beta}, \overline{B}$; if $(v, \alpha, \beta) \in \mathcal{W}'$, for any $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta}) - \mathcal{W}'$ the relations $\mathcal{S}_{W,Y_1}, \mathcal{S}_{W,Y_2}$, $Y_1 = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$, $Y_2 = \mathsf{MAX}(\mathcal{P}) - (W \cap W_{w,\gamma\delta})$, are strongly chained with respect to $\overline{\beta}, \overline{B}$.

Conditions (S1)–(S3) are the conditions we actually want to maintain when transforming a strategy, and these are the ones that provide the desired results. However, to prove that (S1)–(S3) are preserved under transformations of a strategy we also need more technical conditions (S4)–(S7).

We now show how we plan to use $\overline{\beta}$-strategies. Let $\mathcal{P}$ be a subdirectly irreducible, (2,3)-minimal, and block-minimal instance, $\beta_v = \underline{1}_v$ and $B_v = \mathbb{A}_v$ for $v \in V$. Then as is easily seen the collection of relations $\mathcal{R} = \{R_{C,v,\alpha\beta} \mid (v,\alpha,\beta) \in \mathcal{W}(\overline{\beta}), C \in \mathcal{C}\}$ given by $R_{C,v,\alpha\beta} = \mathrm{pr}_{\mathbf{s}\cap W_{v,\alpha\beta,\overline{\beta}}} R$ for $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ is a $\overline{\beta}$-strategy with respect to $\overline{B}$. Also, by (S3) a $\overline{\gamma}$-strategy with $\gamma_v = \underline{0}_v$ for all $v \in V$ gives a solution of $\mathcal{P}$. Our goal is therefore to show that a $\overline{\beta}$-strategy for any $\overline{\beta}$ can be 'reduced', that is, transformed to a $\overline{\beta}'$-strategy for some $\overline{\beta}' < \overline{\beta}$. Note that this reduction of strategies is where the condition $\mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P}) = \varnothing$ is used. Indeed, suppose that $\beta_v = \mu_v^*$. Then by conditions (S1)–(S7) we only have information about solutions to problems of the form $\mathcal{P}_W/_{\overline{\mu}^*}$ or something very close to that. Therefore this barrier cannot be penetrated. We consider two cases.

CASE 1. There are $v \in V$ and $\alpha \prec \beta_v$ nontrivial on $B_v$, $\mathsf{typ}(\alpha,\beta_v) = \mathbf{2}$. This case is considered in Section 10.1.

CASE 2. For all $v \in V$ and $\alpha \prec \beta_v$ nontrivial on $B_v$, $\mathsf{typ}(\alpha,\beta_v) \in \{\mathbf{3},\mathbf{4},\mathbf{5}\}$. This case is considered in Section 10.2.

# 10 Proof of Theorem 67

In the remaining part of the paper we prove Theorem 67.

## 10.1 Tightening affine factors

In this section we consider Case 1 of tightening strategies: there is $\alpha \in \mathsf{Con}(\mathbb{A}_v)$ for some $v \in V$ such that $\alpha \prec \beta_v$.

Let $\mathcal{P} = (V,\mathcal{C})$ be a block-minimal instance with subdirectly irreducible domains, $\overline{\beta} = (\beta_v \in \mathsf{Con}(\mathbb{A}_v) \mid v \in V)$ and $\overline{B} = (B_v \mid B_v \text{ is a } \beta_v\text{-block}, v \in V)$. We use notation from Section 9. Let also $\mathcal{R} = \{R_{C,v,\alpha\beta}\}$ be a $\overline{\beta}$-strategy for $\overline{B}$. We select $v \in V$ and $\alpha,\beta \in \mathsf{Con}(\mathbb{A}_v)$ with $\alpha \prec \beta = \beta_v$, $\mathsf{typ}(\alpha,\beta) = \mathbf{2}$, and an $\alpha$-block $B \in B_v/_\alpha$. In this section we show how $\mathcal{R}$ can be transformed to a $\overline{\beta}'$-strategy $\mathcal{R}'$ for $\overline{B}'$ such that $\beta'_w \le \beta_w$, $B'_w \subseteq B_w$ for $w \in V$, and $\beta'_v = \alpha$, $B'_v = B$.

First of all we identify variables $w \in V$ for which $\beta'_w$ has to be different from $\beta_w$. Since $\mathcal{P}$ is (2,3)-minimal, for every $u,w \in V$ there is $C^{\{u,w\}} = \langle (u,w), R^{\{u,w\}}\rangle \in \mathcal{C}$. For $w \in W_{v,\alpha\beta}$ (we omit $\overline{\beta}$ from $W_{v,\alpha\beta,\overline{\beta}}$ here) consider

$R^{*,\{v,w\}} = R^{\{v,w\}} \cap (B_v \times B_w)$, $R^{\{v,w\},\mathcal{R}}$ the set of all $\mathcal{R}$-compatible pairs from $R^{\{v,w\}}$, $R'^{\{v,w\}} = R^{\{v,w\}}/_\alpha$, and $R'^{\{v,w\},\mathcal{R}} = R^{\{v,w\},\mathcal{R}}/_\alpha$. By (S5) for $\mathcal{R}$ we have that $\mathsf{umax}(\mathrm{pr}_v R'^{\{v,w\},\mathcal{R}})$ is as-closed in $B^*_v/_\alpha$, where $B^*_v = \mathrm{pr}_v R^{*,\{v,w\}}$; since $\mathsf{typ}(\alpha,\beta) = \mathbf{2}$, this implies $\mathrm{pr}_v R'^{\{v,w\},\mathcal{R}} = B^*_v/_\alpha$. Therefore by the Congruence Lemma 63 either $B^*_v/_\alpha \times \mathsf{umax}(\mathrm{pr}_w(R'^{\{v,w\},\mathcal{R}}) \subseteq R'^{\{v,w\},\mathcal{R}}$ or $R'^{\{v,w\},\mathcal{R}}$ is the graph of a mapping $\nu_w : \mathsf{umax}(\mathrm{pr}_w(R'^{\{v,w\},\mathcal{R}}) \to B^*_v/_\alpha$. Let $U \subseteq W_{v,\alpha\beta}$ be the set of variables for which the latter holds, and let $\alpha_w$ be the corresponding congruence of $\mathbb{A}_w$, extension of the kernel of $\nu_w$. Let $\beta'_v = \alpha, B'_v = B$ and $\beta'_w = \alpha_w, B'_w = \nu_w^{-1}(B)$ for $w \in U$, and $\beta'_w = \beta_w, B'_w = B_w$ for $w \in V - U$.

Now we are in a position to define the new strategy. Let $\mathcal{R}'$ be the following collection of relations. We omit subscript $\overline{\beta}$.

(R1) $\mathcal{R}' = \{R'_{C,w,\gamma\delta} \mid C = \langle \mathbf{s}, R\rangle \in \mathcal{C}, (w,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')\}$;

(R2) for every $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, $(u,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')$,

  (a) if $(v,\alpha,\beta) \notin \mathcal{W}'$, $R'_{C,u,\gamma\delta} = \{\mathbf{a} \in R_{C,u,\gamma\delta} \mid$ there is a $\mathcal{R}$-compatible solution $\varphi$ of $\mathcal{P}_{W_{v,\alpha\beta_v}}$, $\varphi(v) \in B'_v$, and $\varphi(w) = \mathbf{a}[w]$ for $w \in \mathbf{s} \cap W_{v,\alpha\beta_v} \cap W_{u,\gamma\delta}\}$;

  (b) if $(v,\alpha,\beta) \in \mathcal{W}', (u,\gamma,\delta) \notin \mathcal{W}'$, $R'_{C,u,\gamma\delta} = \{\mathbf{a} \in R_{C,u,\gamma\delta} \mid$ there is a $\mathcal{R}$-compatible solution $\varphi$ of $\mathcal{P}_{W_{v,\alpha\beta_v}}/_{\overline{\mu}^Y}$ with $Y = \mathsf{MAX}(\mathcal{P}) - W_{u,\gamma\delta}$, such that $\varphi(v) \in B'_v$, and $\varphi(w) = \mathbf{a}[w]$ for $w \in \mathbf{s} \cap W_{v,\alpha\beta_v} \cap W_{u,\gamma\delta}\}$

  (c) if $(v,\alpha,\beta), (u,\gamma,\delta) \in \mathcal{W}'$, $R'_{C,u,\gamma\delta} = \{\mathbf{a} \in R_{C,u,\gamma\delta} \mid$ there is a $\mathcal{R}$-compatible solution $\varphi$ of $\mathcal{P}_{W_{v,\alpha\beta_v}}/_{\overline{\mu}^Y}$ with $Y = \mathsf{MAX}(\mathcal{P}) - (\mathbf{s} \cap W_{u,\gamma\delta})$, such that $\varphi(v) \in B'_v$, and $\varphi(w) = \mathbf{a}[w]$ for $w \in \mathbf{s} \cap W_{v,\alpha\beta_v} \cap W_{u,\gamma\delta}\}$;

Similar to $R^{\mathcal{R}}, \mathcal{S}^{\mathcal{R}}_W, \mathcal{S}^{\mathcal{R}}_{W,Y}$ by $R^{\mathcal{R}'}, \mathcal{S}^{\mathcal{R}'}_W, \mathcal{S}^{\mathcal{R}'}_{W,Y}$ we denote the corresponding sets of $\mathcal{R}'$-compatible tuples. As is easily seen, the sets both types are indeed subalgebras of $R, \mathcal{S}_W, \mathcal{S}_{W,Y}$.

The following three statements show how relations $R'_{C,w,\gamma\delta}$ from $\mathcal{R}'$ are related to $R_{C,w,\gamma\delta}$ from $\mathcal{R}$. They basically amount to saying that either $R'_{C,w,\gamma\delta}$ is the intersection of $R_{C,w,\gamma\delta}$ with a block of a congruence of the projection of $R$, or $\mathsf{umax}(R'_{C,w,\gamma\delta}) = \mathsf{umax}(R_{C,w,\gamma\delta})$

**Lemma 72** *(1) Let $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, and let $\mathcal{S}'_W$, where $W = W_{v,\alpha\beta_v}$, be the set of solutions of $\mathcal{P}_W$ if $(v,\alpha,\beta) \notin \mathcal{W}'$, or the set of solutions of $\mathcal{P}_W/_{\overline{\mu}^{\mathsf{MAX}(\mathcal{P})-\mathbf{s}}}$ if $(v,\alpha,\beta) \in \mathcal{W}'$. For every $U \subseteq \mathbf{s} \cap W$ there is a congruence $\tau_U$ of $\mathrm{pr}_U \mathcal{S}'_W = \mathrm{pr}_U R$ such that $\mathrm{pr}_U \mathcal{S}'^{\mathcal{R}'}_W$ is the intersection of $\mathrm{pr}_U \mathcal{S}'^{\mathcal{R}}_W$ and a $\tau_U$-block.*

*(2) For any $U_1 \subseteq U_2 \subseteq W$ the congruence $\tau_{U_1}$ is the restriction of $\tau_{U_2}$, that is*

$(\mathbf{a},\mathbf{b})\in\tau_{U_1}$ *if and only if for some* $\mathbf{a}',\mathbf{b}'\in\mathcal{S}'_{U_2}$ *with* $\mathrm{pr}_{U_1}\mathbf{a}'=\mathbf{a},\mathrm{pr}_{U_1}\mathbf{b}'=\mathbf{b}$ *it holds* $(\mathbf{a}',\mathbf{b}')\in\tau_{U_2}$.
*(3) For any* $U\subseteq\mathbf{s}\cap W$ *either* $\tau_U|_{\mathrm{pr}_U R\cap\overline{B}}=\overline{\beta}_U|_{\mathrm{pr}_U R\cap\overline{B}}$*, or the algebra* $\mathrm{pr}_U\mathcal{S}'^{\mathcal{R}}_W/_{\tau_U}$ *is isomorphic to* $\mathrm{pr}_v(\mathcal{S}'_W\cap\overline{B})/_\alpha$.
*(4) For any* $(w,\gamma,\delta)\in\mathcal{W}(\overline{\beta})$*,* $X=W_{w,\gamma\delta}$*,* $X'=\mathbf{s}\cap W\cap X$*, let* $\tau=\tau_{X'}$*. Then either* $\mathsf{umax}(\mathrm{pr}_{X'}R'_{C,w,\gamma\delta})=\mathsf{umax}(\mathrm{pr}_{X'}R_{C,w,\gamma\delta})$*, or for a* $\tau$*-block* $S$ *it holds* $\mathrm{pr}_{X'}R'_{C,w,\gamma\delta}\subseteq\mathrm{pr}_{X'}R_{C,w,\gamma\delta}\cap S$ *and* $\mathsf{umax}(\mathrm{pr}_{X'}R'_{C,w,\gamma\delta})$ *is the set of u-maximal elements of* $\mathsf{umax}(\mathrm{pr}_{X'}R_{C,w,\gamma\delta})\cap S$.

If, according to item (3) of the lemma, $\tau_U|_{\mathrm{pr}_U R\cap\overline{B}}=\overline{\beta}_U|_{\mathrm{pr}_U R\cap\overline{B}}$, we say that $\tau_U$ is the *full congruence*; if the latter option of item (3) holds we say that $\tau_U$ is a *maximal congruence*.

**Proof:** (1) If $v\in U$ then $\tau_U$ is $\overline{\beta}'_U$. Otherwise consider $Q=\mathrm{pr}_{U\cup\{v\}}\mathcal{S}'_W$ as a subdirect product of $\mathbb{A}_v$ and $\mathrm{pr}_U\mathcal{S}'_W$. Since this relation is chained by (S7) and $\mathrm{pr}_{U\cap\{v\}}\mathcal{S}'^{\mathcal{R}}_W$ is polynomially closed in $Q$ by Lemma 60(2), the result follows from the Congruence Lemma 63. Indeed, we consider $Q/_\alpha$ as a subdirect product of $\mathrm{pr}_U\mathcal{S}'_W$ and $\mathbb{A}_v/_\alpha$, and choose $\tau_U$ to be the congruence of $\mathrm{pr}_U\mathcal{S}'_W$ identified in the Congruence Lemma 63.

(2) Obvious.

(3) If $v\in U$ then by item (1) $\mathrm{pr}_U\mathcal{S}'^{\mathcal{R}}_W/_{\tau_U}=\mathrm{pr}_U\mathcal{S}'^{\mathcal{R}}_W/_{\overline{\beta}'_U}$, which is isomorphic to $\mathrm{pr}_v(\mathcal{S}'_W\cap\overline{B})/_\alpha$. Otherwise consider relation $Q$ as in item (1). By the Congruence Lemma 63 the restriction of $\tau_U$ is the link congruence of this relation. The result follows.

(4) If $\tau$ is the full congruence then by (S2) for $\mathcal{R}$ we have $\mathsf{umax}(\mathrm{pr}_X R_{C,w,\gamma\delta})=\mathsf{umax}(\mathrm{pr}_X\mathcal{S}'^{\mathcal{R}'}_W)$ and we have the first option. If $\tau$ is a maximal congruence then by (R2) and item (1) there is a $\tau$-block $S$ such that $\mathrm{pr}_{X'}R'_{C,w,\gamma\delta}\subseteq S\cap\mathrm{pr}_{X'}R_{C,w,\gamma\delta}$, and by (S2) $\mathsf{umax}(\mathrm{pr}_{X'}R'_{C,w,\gamma\delta})=\mathsf{umax}(S\cap\mathrm{pr}_{X'}R_{C,w,\gamma\delta})$. □

For $C=\langle\mathbf{s},R\rangle\in\mathcal{C}$ we use $\tau_C$ to denote the congruence $\tau_{\mathbf{s}\cap W}$. Also for $(w,\gamma,\delta)\in\mathcal{W}(\overline{\beta})$ we use $\tau_{C,w,\gamma\delta}$ to denote the congruence $\tau_{\mathbf{s}\cap W\cap W_{w,\gamma\delta}}$.

**Lemma 73** *In the notation above let* $\gamma_u,\delta_u\in\mathsf{Con}(\mathbb{A}_u)$*,* $u\in W'=\mathbf{s}\cap W$ *be such that* $(u,\gamma_u,\delta_u)\in\mathcal{W}(\overline{\beta})$ *and* $(\alpha,\beta_v),(\gamma_u,\delta_u)$ *cannot be separated from each other. Then if* $\tau_C$ *is a maximal congruence, for any polynomial* $f$ *of* $R$*,* $f(\overline{\beta}_{W'})\subseteq\tau_C$ *if and only if* $f(\delta_u)\subseteq\gamma_u$ *for any* $u\in W'$.

**Proof:** Let $\mathcal{S}'_W$ be defined as in Lemma 72. Take a polynomial $f$ of $R$. As $(\gamma_{u_1},\delta_{u_1}),(\gamma_{u_2},\delta_{u_2})$ cannot be separated for any $u_1,u_2\in W'$, it suffices to consider just one variable $u\in W'$. Since $\mathcal{P}$ is a block-minimal instance, the polynomial $f$ can be extended from a polynomial on $\mathrm{pr}_{W'}R$ to a polynomial $f'$ of $\mathcal{S}'_W$, and, in particular, to a polynomial $f''$ of $\mathrm{pr}_{W'\cup\{v\}}\mathcal{S}'_W$. Since $\alpha$ and $\tau_C$ are greater than or equal to the link congruences of $A_{\mathcal{R},v}$ and $\mathrm{pr}_{W'}R^{\mathcal{R}}$ with respect to $\mathrm{pr}_{W'\cup\{v\}}\mathcal{S}'^{\mathcal{R}}_W$, for any $\mathbf{a},\mathbf{b}\in\mathrm{pr}_{W'\cup\{v\}}\mathcal{S}'^{\mathcal{R}}_W$ we have $f''(\mathbf{a}[v])\overset{\alpha}{\equiv}f''(\mathbf{b}[v])$ if and only if $f''(\mathrm{pr}_{W'}\mathbf{a})\overset{\tau_C}{\equiv}f''(\mathrm{pr}_{W'}\mathbf{b})$. In particular, this implies that $f''(\beta_v)\subseteq\alpha$ if and only if $f(\overline{\beta}_{W'})\subseteq\tau_C$. Since the first inclusion holds if and only if $f(\delta_u)\subseteq\gamma_u$, we infer the result. □

**Corollary 74** *For any $(w,\gamma,\delta)\in\mathcal{W}(\overline{\beta})$, $X=W_{w,\gamma\delta}$, $X'=\mathbf{s}\cap W\cap X$, let $\tau=\tau_{X'}$ and $\tau'=\{(\mathbf{a},\mathbf{b})\mid\mathbf{a},\mathbf{b}\in\mathrm{pr}_{\mathbf{s}\cap X}R,(\mathrm{pr}_{X'}\mathbf{a},\mathrm{pr}_{X'}\mathbf{b})\in\tau\}$. Then either $\mathsf{umax}(R'_{C,w,\gamma\delta})=\mathsf{umax}(R_{C,w,\gamma\delta})$, or for a $\tau'$-block $T$ it holds $R'_{C,w,\gamma\delta}\subseteq R_{C,w,\gamma\delta}\cap T$ and $\mathsf{umax}(R'_{C,w,\gamma\delta})$ is the set of u-maximal elements of $\mathsf{umax}(R_{C,w,\gamma\delta})\cap T$.*

**Proof:** By Lemma 72(4) either $\mathsf{umax}(\mathrm{pr}_{X'}R'_{C,w,\gamma\delta})=\mathsf{umax}(\mathrm{pr}_{X'}R_{C,w,\gamma\delta})$, or for a $\tau$-block $S$ it holds $\mathrm{pr}_{X'}R'_{C,w,\gamma\delta}\subseteq\mathrm{pr}_{X'}R_{C,w,\gamma\delta}\cap S$ and $\mathsf{umax}(\mathrm{pr}_{X'}R'_{C,w,\gamma\delta})$ is the set of u-maximal elements of $\mathsf{umax}(\mathrm{pr}_{X'}R_{C,w,\gamma\delta})\cap S$. Then considering $R_{C,w,\gamma\delta}/_{\tau'}$ as a subdirect product of $\mathrm{pr}_{X'}R_{C,w,\gamma\delta}/_{\tau}$ and $\mathrm{pr}_{\mathbf{s}-X'}R_{C,w,\gamma\delta}$, the interval $(\tau,\overline{\beta}_{X'})$ in $\mathrm{pr}_{X'}R/_{\tau}$ can be separated from interval $(\eta,\theta)$ in $\mathsf{Con}(\mathbb{A}_u)$ for any $u\in\mathbf{s}-X'$ by Lemma 73. Then we use the Congruence Lemma 63 to conclude that $S\times\mathsf{umax}(\mathrm{pr}_{\mathbf{s}-X}R_{C,w,\gamma\delta})\subseteq R_{C,w,\gamma\delta}/_{\tau'}$. The result follows. □

Now we are in a position to prove that $\mathcal{R}'$ is a $\overline{\beta}'$-strategy.

**Theorem 75** *In the notation above, $\mathcal{R}'$ is a $\overline{\beta}'$-strategy for $\overline{B}'$.*

**Proof:** Let $W=W_{v,\alpha\beta}$.

For (S1) we consider the collection of constraints $C^X=\langle X,R^X\rangle$, $X\subseteq V$, $|X|=2$, such that $\mathsf{umax}(R^{X,\mathcal{R}})$ constitute a $(2,3)$-strategy for $\mathcal{P}$. Let $R^{X,\mathcal{R}'}$ denote the set of $\mathcal{R}'$-compatible tuples from $R^{X,\mathcal{R}}$. It suffices to show that for any tuple $(a,b)\in\mathsf{umax}(R^{X,\mathcal{R}'})$, $X=\{x,y\}$, and any $w\notin\{x,y\}$ there is $c\in\mathbb{A}_w$ such that $(a,c)\in\mathsf{umax}(R^{\{x,w\},\mathcal{R}'}),(b,c)\in\mathsf{umax}(R^{\{y,w\},\mathcal{R}'})$. By (S1) for $\mathcal{R}$ there is $d\in\mathbb{A}_w$ such that $(a,d)\in\mathsf{umax}(R^{\{x,w\},\mathcal{R}}),(b,d)\in\mathsf{umax}(R^{\{y,w\},\mathcal{R}})$.

Consider the relation $R$ given by

$$R(x,y,w)=R^{\{x,y\}}(x,y)\wedge R^{\{x,w\}}(x,w)\wedge R^{\{y,w\}}(y,w),$$

and let $R^{\mathcal{R}},R^{\mathcal{R}'}$ be the set of $\mathcal{R}$- and $\mathcal{R}'$-compatible tuples from $R$, respectively. By (S4) $R^{\mathcal{R}}$ is a subalgebra of $R$; moreover, by (S1) the binary projections of $R^{\mathcal{R}}$ contain $\mathsf{umax}(R^{\{x,y\},\mathcal{R}})$, $\mathsf{umax}(R^{\{x,w\},\mathcal{R}})$, $\mathsf{umax}(R^{\{y,w\},\mathcal{R}})$, respectively.

First, suppose $w\notin W$. If, say, $x\notin W$ then $R^{\{x,w\},\mathcal{R}}=R^{\{x,w\},\mathcal{R}'}$. If $x\in W$, then by construction either $\mathsf{umax}(R^{\{x,w\},\mathcal{R}})=\mathsf{umax}(R^{\{x,w\},\mathcal{R}'})$ if $\tau_{\{x\}}$ is the full congruence, or $\mathsf{umax}(R^{\{x,w\},\mathcal{R}})\cap(B'_x\times B_w)\subseteq\mathsf{umax}(R^{\{x,w\},\mathcal{R}'}$ otherwise. In either case $(a,d) \in R^{\{x,w\},\mathcal{R}'}$. Similarly, $(b,d) \in R^{\{y,w\},\mathcal{R}'}$. Therefore $(a,b,d) \in R^{\mathcal{R}'}$, and by the Maximality Lemma 27(5) there is always $c$ such that $(a,c) \in \mathsf{umax}(R^{\{x,w\},\mathcal{R}'}), (b,c) \in \mathsf{umax}(R^{\{y,w\},\mathcal{R}'})$. Therefore we may assume that $w \in W$.

If $x \in W$ (or $y \in W$, or $x,y \in W$) then let $\tau_x, \tau_{xw}$ (respectively, $\tau_y, \tau_{yw}$, or $\tau_{xy}$) be as in Lemma 72. If $x \notin W$ (or $y \notin W$) then let $\tau_x = \beta_x$, $\tau_{xw} = \tau_w$, and $\tau_{xy} = \tau_y$ (respectively, $\tau_y = \beta_y, \tau_{yw} = \tau_w$). We view all these congruences interchangeably: as congruences of $R$ in the natural way and their restrictions to $\mathcal{R}$-compatible tuples as congruences of $R^{\mathcal{R}}$, or as congruences of the corresponding projections of $R$ and $R^{\mathcal{R}}$. By Lemma 72 if $\tau'$ is one of these congruences, $\tau'$ is either the full congruence on $R^{\mathcal{R}}$, or $\tau'|_{R^{\mathcal{R}}} \prec (\beta_x \times \beta_y \times \beta_w)|_{R^{\mathcal{R}}}$ in $\mathsf{Con}(R)$; in the latter case we will say that $\tau'$ is maximal. Also let $\tau = \tau_{xy} \wedge \tau_{xw} \wedge \tau_{yw}$. Again by Lemma 72 $R^{\mathcal{R}}/_{\tau}$ is a module. We start with an auxiliary claim.

CLAIM. Let $\mathbb{A}$ be an algebra and $\beta$ in $\mathsf{Con}(\mathbb{A})$ and $\mathbb{A}'$ is a subalgebra of a $\beta$-block. Let also $\alpha < \beta|_{\mathbb{A}'}$ be a congruence of $\mathbb{A}'$ such that $\mathbb{A}'/_{\alpha}$ is a module. Let $\mathbb{B}, \mathbb{C}$ be subalgebras of $\mathbb{A}'$ such that $\mathbb{B} \cap \mathbb{C} \neq \varnothing$ and $\mathbb{B} \cap \mathbb{C}$ contains a u-maximal element of an $\alpha$-block of $\mathbb{A}'$, $\mathbb{B}/_{\alpha} = \mathbb{A}'/_{\alpha}$, $\mathbb{C}/_{\alpha} = \mathbb{A}'/_{\alpha}$, and $\mathbb{B}, \mathbb{C}$ are polynomially closed in $\mathbb{A}$. Then $\mathbb{B} \cap \mathbb{C}/_{\alpha} = \mathbb{A}'/_{\alpha}$.

Clearly it suffices to consider the case $\alpha \prec \beta|_{\mathbb{A}'}$ since then we use induction on an irreducible chain from $\alpha$ to $\beta|_{\mathbb{A}'}$ in $\mathsf{Con}(\mathbb{A}')$. Let $a \in \mathbb{B} \cap \mathbb{C}$ be u-maximal in an $\alpha$-block of $\mathbb{A}'$. Let $a' \in \mathbb{B}$ and $a' \stackrel{\alpha}{\not\equiv} a$. For any $(\alpha,\beta)$-trace $T$ that contains $a$ and any $b \in T$ there is a polynomial $f$ of $\mathbb{A}$ such that $f(a) = a$ and $f(a') = b$. Since $\mathbb{B}$ is polynomially closed, for any $c \in \mathsf{Sg}(a,b)$ such that $a \sqsubseteq_{as} c$, we have $c \in \mathbb{B}$, and $c$ is u-maximal in $\mathbb{A}'$. In a similar way $c \in \mathbb{C}$. Therefore it suffices to show that for any $\alpha$-block $D \subseteq \mathbb{A}'$ there is a $b \in D$ such that $\{a,b\}$ is an $(\alpha,\beta)$-subtrace.

Since $\mathsf{typ}(\alpha,\beta) = \mathbf{2}$, there is a $(\alpha,\beta)$-subtrace $\{d,e\}$ such that $d \stackrel{\alpha}{\equiv} a$ and $e \in D$. Then by Lemma 56 there is also a polynomial $g$ such that $g(\mathbb{A})$ is an $(\alpha,\beta)$-minimal set and $g(a) = a$, $g(e) \in D$. The result follows.

Let $R_a^*, R_b^*$ be the sets of tuples $\mathbf{a} \in R^{\mathcal{R}}$ satisfying $\mathbf{a}[x] = a, \mathbf{a}[y] = b$, respectively; note that $(a,b,d) \in R_x^* \cap R_y^*$. We consider several cases of what the congruences introduced earlier can be.

Suppose first that, say, $\tau_x$ is the full congruence and $\tau_{xw} = \tau_w$. As $\tau_{yw} \leq \tau_w$, if $\tau_{yw}$ is a full congruence, $(a,d) \in R^{\{x,w\},\mathcal{R}'}, (b,d) \in R^{\{y,w\},\mathcal{R}'}$, and we are done. If $\tau_y$ is maximal while $\tau_w$ is not , then again $(a,d) \in R^{\{x,w\},\mathcal{R}'}, (b,d) \in R^{\{y,w\},\mathcal{R}'}$ as any tuple $(b,x) \in R^{\{y,w\},\mathcal{R}}$ also belongs to $R^{\{y,w\},\mathcal{R}'}$ in this case. If both $\tau_y$ and $\tau_w$ are maximal, then, as $\tau_{yw} \leq \tau_y \wedge \tau_w$ and all three congruences are maximal, $\tau_w$ (viewed as a congruence of $\mathrm{pr}_w R^{\mathcal{R}}$) is the link congruence with respect to $\mathrm{pr}_{yw} R^{\mathcal{R}}$, and so $(b,d) \in R^{\{y,w\},\mathcal{R}'}$, implying also, since $(a,b) \in \mathrm{pr}_{xy} R^{\mathcal{R}}$, that

$(a, d) \in R^{\{x,w\},\mathcal{R}'}$.

If $\tau_w$ is maximal while $\tau_y$ is not, let $B^*_z = \mathrm{pr}_z R^{\mathcal{R}}$ for $z \in \{x, y, w\}$. By the Congruence Lemma 63 $\mathsf{umax}(B^*_x) \times B^*_w/_{\tau_w} \subseteq \mathrm{pr}_{xw} R^{\mathcal{R}}$ and $\mathsf{umax}(B^*_y) \times B^*_w/_{\tau_w} \subseteq \mathrm{pr}_{yw} R^{\mathcal{R}}$. Therefore $R^*_a/_{\tau_w} = R^*_b/_{\tau_w} = R^{\mathcal{R}}/_{\tau_w}$ and since $(a, b)$ is u-maximal in a $\tau_{xy}$-block, $d$ can be chosen such that $(a, b, d)$ is u-maximal in a $\tau_w$-block. Then the result follows by the Claim above. Now, let $\tau_{yw} \prec \tau_y = \tau_w = \beta_w$. Again by the Congruence Lemma 63 $\mathsf{umax}(B^*_x) \times \mathrm{pr}_{yw} R^{\mathcal{R}}/_{\tau_{yw}} \subseteq R^{\mathcal{R}}$ and therefore $R^*_a/_{\tau_{yw}} = R^{\mathcal{R}}/_{\tau_{yw}}$. Also, let $Q$ be the union of all $\tau_{yw}$-blocks of $R^{\mathcal{R}}$ whose intersection with $R^*_b$ is nonempty and $R^{**}_a = R^*_a \cap Q$. Note that $R^{\{y,w\},\mathcal{R}'} \cap \mathrm{pr}_{yw} Q \neq \varnothing$. Then $R^{**}_a/_{\tau_{yw}} = Q/_{\tau_{yw}}$ and $R^*_b/_{\tau_{yw}} = Q/_{\tau_{yw}}$. Then as before the result follows by the Claim.

If $\tau_x$ is maximal, while $\tau_y$ is not, the result follows by one of the previous arguments. If $\tau_x, \tau_y$ are both maximal, then $(a, d) \in R^{\{x,w\},\mathcal{R}'}$, $(b, d) \in R^{\{y,w\},\mathcal{R}'}$, and we are done.

Due to symmetries between $x$ and $y$, the only remaining case is when $\tau_{xw}$ and $\tau_{yw}$ are maximal, while $\tau_x, \tau_y, \tau_w$ are not. If $(v, \alpha, \beta) \notin \mathcal{W}'$ or $w \notin \mathsf{MAX}(\mathcal{P})$, that is, $\mu'_w = \underline{0}_w$, then the required $c \in B^*_w$ exists, since $(a, b)$ can be extended to a solution from $\mathcal{S}^{\mathcal{R}'}_W$ or $\mathcal{S}^{\mathcal{R}'}_{W,Y}$, $Y = \mathsf{MAX}(\mathcal{P}) - \{x, y\}$ by construction. Suppose that $(v, \alpha, \beta) \in \mathcal{W}'$ and $\mu'_w = \mu_w$. Let $Q_x$ be a subalgebra of the product $\mathbb{A}_x \times \mathbb{A}_w \times \mathbb{A}_v/_\alpha$ that consists of all triples $(a', b', c')$ such that there is a solution $\varphi \in \mathcal{S}_{W,Y}$ for $Y = \mathsf{MAX}(\mathcal{P}) - \{x, w\}$ with $\varphi(x) = a', \varphi(w) = b'$, and $\varphi(v) \in c'$. By block-minimality $Q_x$ is indeed a subdirect product and by (S2) for $\mathcal{R}$ we have $\mathsf{umax}(R^{\{x,w\},\mathcal{R}}) \subseteq \mathrm{pr}_{xw}(Q_x \cap (B^*_x \times B^*_w \times B^*_v/_\alpha)$ and $\mathrm{pr}_v(Q_x \cap (B^*_x \times B^*_w \times B^*_v/_\alpha) = B^*/_\alpha$. Relation $Q_y$ is defined in a similar way. Let also

$$Q(x, y, w, v) = Q_x(x, w, v) \wedge Q_y(y, w, v),$$

and $Q' = Q \cap \overline{B}$. Let $Q^a = \{\mathbf{a} \in Q' \mid \mathbf{a}[x] = a\}$, $Q^b = \{\mathbf{a} \in Q' \mid \mathbf{a}[y] = b\}$, and $\alpha' = \beta_x \times \beta_y \times \beta_w \times \alpha$. By the assumption that $\tau_x, \tau_y$ are full congruences $Q^a/_\alpha = Q^b/_\alpha = Q'/_\alpha$. Therefore, if we prove that $(a, b) \in \mathrm{pr}_{xy} Q'$, we obtain the result by the Claim.

To this end consider the relations

$$S(x, y, w, v, v') = Q_x(x, w, v) \wedge Q_y(y, w, v'),$$

and $S' = S \cap \overline{B}$. In a similar way we define $S^a = \{\mathbf{a} \in S' \mid \mathbf{a}[x] = a\}$, $S^b = \{\mathbf{a} \in S' \mid \mathbf{a}[y] = b\}$. By (S1) there are $d \in B^*_w$ and $e', e'' \in B^*_v$ such that $(a, b, d, e', e'') \in S'$. Also by construction (R2) there are $a' \in B^*_x, b' \in B^*_y$, $d_1, d_2 \in B^*_w, d_1 \stackrel{\mu_w}{\equiv} d_2$ and $e_1, e_2 \in B^*_v, e_1 \stackrel{\alpha}{\equiv} e_2$, such that $(a, b', d_1, e_1), (a', b, d_2, e_2) \in S'$. Recall that $B^*_v/_\alpha$ is a module. Let $\delta$ be the diagonal congruence of $B^*_v/_\alpha \times$

$B^*_v/_\alpha$, that is, a congruence such that $\Delta = \{(c_1, c_2), \mid c_1 \stackrel{\alpha}{\equiv} c_2\}$ is a $\delta$-block. Let $\delta'$ be the congruence of $S'$ given by $\mathbf{c} \stackrel{\delta'}{\equiv} \mathbf{d}$ if and only if $\mathrm{pr}_{vv'}\mathbf{c} \stackrel{\delta}{\equiv} \mathrm{pr}_{vv'}\mathbf{d}$. Note that $\delta' < \overline{\beta}$ in $\mathsf{Con}(S)$. Let $\Delta'$ be the $\delta'$-block corresponding to $\Delta$. Since $S^a \cap S^b \neq \varnothing$ and $S^a \cap \Delta' \neq \varnothing$, $S^b \cap \Delta' \neq \varnothing$, by the Claim we have $S^a \cap S^b \cap \Delta' \neq \varnothing$. The result follows.

For (S2) take $C = \langle \mathbf{s}, R\rangle$ and $(w_1, \gamma_1, \delta_1), (w_2, \gamma_2, \delta_2) \in \mathcal{W}(\overline{\beta}')$ and let $W_1 = W_{w_1,\gamma_1\delta_1}$, $W_2 = W_{w_2,\gamma_2\delta_2}$, $U = \mathbf{s} \cap W_1 \cap W_2$, and $W = W_{v,\alpha\beta_v}$, as before. Let $\mathbf{a} \in \mathsf{umax}(\mathrm{pr}_U R'_{C,w_1,\gamma_1\delta_1})$. Depending on whether or not $(w_1, \gamma_1, \delta_1), (w_2, \gamma_2, \delta_2) \in \mathcal{W}'$ we need to show that $\mathbf{a}$ can be extended to a solution of $\mathcal{P}_{W_2}/_{\overline{\mu}^Y}$, where $Y$ is either empty, if $(w_2, \gamma_2, \delta_2) \notin \mathcal{W}'$, or $Y = \mathsf{MAX}(\mathcal{P}) - W_1$ if $(w_1, \gamma_1, \delta_1) \notin \mathcal{W}'$ and $(w_2, \gamma_2, \delta_2) \in \mathcal{W}'$, and $Y = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$ if $(w_1, \gamma_1, \delta_1), (w_2, \gamma_2, \delta_2) \in \mathcal{W}'$. The three cases are quite similar so we will consider them simultaneously. Let $\mathcal{P}'_{W'}$ denote the problem $\mathcal{P}_{W'}/_{\overline{\mu}^Y}$ for a set $W' \subseteq V$ and $\mathcal{S}'_{W'}$ denote its set of solutions. Then we need to show that $\mathbf{a} \in \mathrm{pr}_U \mathcal{S}'^{\mathcal{R}'}_{W_2}$.

If $\tau_{C',w_2,\gamma_2\delta_2}$ is the full congruence of $R_{C',w_2,\gamma_2\delta_2}$ for all $C' \in \mathcal{C}$, then $\mathsf{umax}(\mathcal{S}'^{\mathcal{R}}_{W_2}) = \mathsf{umax}(\mathcal{S}'^{\mathcal{R}'}_{W_2})$ and there is nothing to prove. Otherwise let us consider the set $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ of all $\mathcal{R}$-compatible solutions of $\mathcal{P}_{W\cap W_2}/_{\overline{\mu}^Y}$.

CLAIM 1. $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ and $\mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$ are nonempty.

The set $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ is nonempty, as it contains $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}$, which is nonempty by (S2) for $\mathcal{R}$. If $(v, \alpha, \beta) \notin \mathcal{W}'$, then $\mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$ contains $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}'}_{W}$ or its factor modulo $\overline{\mu}^Y$, which is nonempty. Suppose $(w_2, \gamma_2, \delta_2) \notin \mathcal{W}'$ and $C_1 = \langle \mathbf{s}_1, R_1\rangle \in \mathcal{C}$ is a constraint such that $\tau_{C_1,w_2,\gamma_2\delta_2}$ is nontrivial on $R_{C_1,w_2,\gamma_2\delta_2}$ and $\mathbf{b} \in \mathsf{umax}(R'_{C_1,w_2,\gamma_2\delta_2})$. By (S2) for $\mathcal{R}$ tuple $\mathbf{b}$ can be extended to a solution $\varphi$ of $\mathcal{P}_W/_{\overline{\mu}^Y}$, $Y = \mathsf{MAX}(\mathcal{P}) - W_2$. Then $\varphi(v) \in B'_v$ and therefore for any $C_2 = \langle \mathbf{s}_2, R_2\rangle \in \mathcal{C}$ and any $(u, \eta, \theta) \in \mathcal{W}(\overline{\beta})$ we have $\varphi(\mathbf{s}_2 \cap W \cap W_{u,\eta\theta}) \in \mathrm{pr}_{\mathbf{s}\cap W\cap W_{u,\eta\theta}} R'_{C_2,u,\eta\theta}/_{\overline{\mu}^Y}$, that is, $\varphi(W \cap W_2) \in \mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$. Suppose now that $(v, \alpha, \beta), (w_2, \gamma_2, \delta_2) \in \mathcal{W}'$. If $(w_1, \gamma_1, \delta_1) \notin \mathcal{W}'$, then we apply the argument above to the problem $\mathcal{P}_W/_{\overline{\mu}^{\mathsf{MAX}(\mathcal{P})-W_1}}$. If $(w_1, \gamma_1, \delta_1) \in \mathcal{W}'$, then we consider the problem $\mathcal{P}_W/_{\overline{\mu}^{\mathsf{MAX}(\mathcal{P})-\mathbf{s}}}$.

We would like to define a congruence similar to $\tau_C$ on $\mathcal{S}'_{W\cap W_2}$. It cannot be done in the same straightforward way, since $\mathcal{S}'_{W\cap W_2} \neq \mathrm{pr}_{W\cap W_2}\mathcal{S}'_W$, so we define it as follows. For $C' = \langle \mathbf{s}', R'\rangle$ and $(u, \eta, \theta) \in \mathcal{W}(\overline{\beta})$ let $W' = W_{u,\eta\theta,\overline{\beta}} \cap W \cap W_2 \cap \mathbf{s}'$ and $\tau'_{C'}(u, \eta\theta)$ denote the restriction of $\tau_{C'}$ on $R^*_{C',u,\eta\theta} = \mathrm{pr}_{W'} R_{C',u,\eta\theta}/_{\overline{\mu}^Y}$, that is, $(\mathbf{b}, \mathbf{c}) \in \tau'_{C'}(u, \eta\theta)$ if there are $\mathbf{b}', \mathbf{c}' \in R_{C',v,\alpha\beta_v}/_{\overline{\mu}^Y}$ such that $\mathrm{pr}_{W'}\mathbf{b}' = \mathbf{b}$, $\mathrm{pr}_{W'}\mathbf{c}' = \mathbf{c}$, and $(\mathbf{b}', \mathbf{c}') \in \tau_{C'}$. By Lemma 72 $\tau'_{C'}(u, \eta\theta)$ is either the full congruence on $R^*_{C',u,\eta\theta}$, or a maximal one. In the latter case $\mathsf{typ}(\tau'_{C'}(u, \eta\theta), \overline{\beta}_{W'}) = \mathbf{2}$. We extend the congruences $\tau'_{C'}(u, \eta\theta)$ to congruences of $\mathcal{S}'_{W\cap W_2}$ using $\tau'_{C'}(u, \eta\theta) \times \prod_{x\in(W\cap W_2)-W'} \underline{0}_x$. Then the set $\mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$ of $\mathcal{R}'$-compatible tuples from $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ is a block of

$$\tau = \bigwedge_{C' \in \mathcal{C}, (u,\eta,\theta) \in \mathcal{W}(\overline{\beta})} \tau'_{C'}(u, \eta\theta), \tag{1}$$

let it be denoted by $\mathcal{S}^*$. Note that $\tau$ is a congruence of $\mathcal{S}'_{W\cap W_2}$, but the interval $(\tau, \overline{\beta}_{W\cap W_2})$ is not necessarily simple. By the observation above and Lemma 29 $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}/_\tau$ is term equivalent to a module. We need to prove that there is $\varphi \in \mathcal{S}'^{\mathcal{R}}_{W_2}$ such that $\varphi(U) = \mathbf{a}$ and $\varphi(W \cap W_2) \in \mathcal{S}^*$. In fact we prove a stronger statement, namely, that for any u-maximal tuple $\mathbf{b}$ from $\mathrm{pr}_{W_2 - W}\mathcal{S}'^{\mathcal{R}}_{W_2}$ there is a solution $\varphi$ satisfying the required conditions and such that $\varphi(W_2 - W) = \mathbf{b}$. However, to formulate it precisely we need two additional constructions.

Let $W \cap W_2 = \{x_1, \dots, x_k\}$ and $X = W \cap U = \{x_1, \dots, x_\ell\}$, and $X' = \{y_1, \dots, y_\ell\}$. Let

$$Q(x_1, \dots, x_k, y_1, \dots, y_\ell) = \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}(x_1, \dots, x_k) \wedge \mathrm{pr}_X \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}(y_1, \dots, y_\ell) \wedge \bigwedge_{i=1}^{\ell} (x_i = y_i),$$

and its factor $Q' = Q/_{\tau'}$, where $\tau' = \tau \times \underline{0}_{\mathrm{pr}_{X'}\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}}$. Let $\eta_1, \eta_2$ denote the link congruences of $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}/_\tau$ and $\mathrm{pr}_{X'}\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ with respect to $Q'$, and let $\eta'$ denote the congruence of $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$, the full preimage of $\eta_1$, that is, $\eta'/_\tau = \eta_1$. Then, as is easily seen, since $\mathrm{pr}_{W\cap W_2} Q/_\tau$ is a module, $(\mathbf{b}, \mathbf{c}) \in \eta'$ if and only if there are $\mathbf{b}', \mathbf{c}' \in \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ such that $(\mathbf{b}, \mathbf{b}'), (\mathbf{c}, \mathbf{c}') \in \tau$ and $\mathrm{pr}_X \mathbf{b}' = \mathrm{pr}_X \mathbf{c}'$.

Now, let $W_2 = \{x_1, \dots, x_m\}$ (recall that $W \cap W_2 = \{x_1, \dots, x_k\}$) and define a relation $S(x_1, \dots, x_m, y_1, \dots, y_\ell)$ as follows:

$$S(x_1, \dots, x_m, y_1, \dots, y_\ell) = \mathcal{S}'^{\mathcal{R}}_{W_2}(x_1, \dots, x_m) \wedge \mathrm{pr}_X \mathcal{S}'^{\mathcal{R}}_{W_2}(y_1, \dots, y_\ell) \wedge \bigwedge_{i=1}^{\ell} (x_i = y_i),$$

and let $S' = S/_{\tau''}$, where $\tau'' = \tau \times \underline{0}_{\mathrm{pr}_{W_2 - W}\mathcal{S}'^{\mathcal{R}}_{W_2}} \times \underline{0}_{\mathrm{pr}_{X'}\mathcal{S}'^{\mathcal{R}}_{W_2}}$. Similar to $Q$, let $\theta_1, \theta_2$ be the link congruences of $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}/_\tau$ and $\mathrm{pr}_H S$, where $H = X' \cup \{x_{k+1}, \dots, x_m\}$, with respect to $S'$, and let $\theta'$ denote the congruence of $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}$ such that $\theta_1 = \theta'/_\tau$. Then immediately by the definition $(\mathbf{b}, \mathbf{c}) \in \theta'$ if and only if there are $\mathbf{b}', \mathbf{c}' \in \mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}$ and $\mathbf{d} \in \mathrm{pr}_H S$ such that $(\mathbf{b}, \mathbf{b}'), (\mathbf{c}, \mathbf{c}') \in \tau$, $\mathrm{pr}_X \mathbf{b}' = \mathrm{pr}_X \mathbf{c}' = \mathrm{pr}_X \mathbf{d}$, and $(\mathrm{pr}_{(W\cap W_2)-X}\mathbf{b}', \mathbf{d}), (\mathrm{pr}_{(W\cap W_2)-X}\mathbf{c}', \mathbf{d}) \in \mathcal{S}'^{\mathcal{R}}_{W_2}$.

We are interested in congruences $\eta_1$ and $\theta_1$. The first of them indicates which $\tau$-blocks extensions of $\mathrm{pr}_X \mathbf{a}$ can belong to. The second congruence also indicates to which $\tau$-blocks extensions of $\mathbf{a}$ to a solution from $\mathcal{S}'^{\mathcal{R}}_{W_2}$ can belong to. Clearly, $\theta_1 \subseteq \eta_1$. We prove however, that in both cases the set of attainable $\tau$-blocks is the same. This essentially means that if a $\tau$-block can be extended to a solution from $\mathcal{S}'^{\mathcal{R}}_{W_2}$, it can be extended in an almost arbitrary way.

CLAIM 2. (1) $S'' = (\mathrm{pr}_{W\cap W_2} S)/_\tau$ is a union of $\eta_1$-blocks;
(2) $\theta_1 = \eta_1|_{S''}$;
(3) let $D$ be a $\theta_1$-block and $E$ the corresponding $\theta_2$-block, then $D \times \mathsf{umax}(E) \subseteq S'$;
(4) for any $\mathbf{b} \in \mathcal{S}'^{\mathcal{R}}_{W_2}$ such that $\mathrm{pr}_{X\cup(W_2-W)}\mathbf{b}$ is u-maximal in a $\theta_2$-block and any $\mathbf{b}' \in \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ such that $\mathbf{b}' \stackrel{\theta'}{\equiv} \mathrm{pr}_{W\cap W_2}\mathbf{b}$ there is $\mathbf{b}'' \in S$ such that $\mathrm{pr}_{W\cap W_2}\mathbf{b}'' \stackrel{\tau}{\equiv} \mathbf{b}'$ and $\mathrm{pr}_{X\cup(W_2-W)}\mathbf{b}'' = \mathrm{pr}_{X\cup(W_2-W)}\mathbf{b}$.

(1) It follows by Proposition 35 for any $\theta_1$-block $D'$ and a $\theta_2$-block $E'$ with $S' \cap (D' \times E') \neq \varnothing$, that $D' \times \mathsf{umax}(E') \subseteq S'$.

Let $\mathbf{b} \in S$, $D'$ be the $\theta_1$-block containing $(\mathrm{pr}_{W\cap W_2}\mathbf{b})^\tau$ and $E'$ the corresponding $\theta_2$-block. Then by the Maximality Lemma 27(5) there is $\mathbf{b}' \in S$ such that $\mathrm{pr}_{W\cap W_2}\mathbf{b}' \stackrel{\tau}{\equiv} \mathrm{pr}_{W\cap W_2}\mathbf{b}$ and $\mathrm{pr}_H\mathbf{b}'$ is u-maximal in $E'$. We assume that $\mathbf{b}$ satisfies this condition. Let also $D$ be the $\eta_1$-block containing $(\mathrm{pr}_{W\cap W_2}\mathbf{b})^\tau$. Note that $D' \subseteq D$.

Suppose there is $\mathbf{c} \in D$ such that $(\mathbf{c}', \mathrm{pr}_H\mathbf{b}) \in S$ for no $\mathbf{c}' \in \mathbf{c}$. We will derive a contradiction. Take some u-maximal $\mathbf{c}'$ from $\mathbf{c}$. Since $\mathrm{pr}_{W\cap W_2}\mathbf{b} \stackrel{\tau}{\not\equiv} \mathbf{c}'$, there is $C' = \langle \mathbf{s}', R'\rangle \in \mathcal{C}$ and $(u, \chi, \xi) \in \mathcal{W}(\overline{\beta})$ such that $(\mathrm{pr}_Z\mathbf{b}, \mathrm{pr}_Z\mathbf{c}') \not\in \tau^*$, where $\tau^* = \tau'_{C'}(u, \chi\xi)$ and $Z = W_{u,\chi\xi} \cap W \cap W_2 \cap \mathbf{s}'$. Choose a pair $\mathbf{b}, \mathbf{c}$ in such a way that the number of such constraints and triples is minimal. We will find a polynomial $f$ of $\mathcal{S}'_{W_2}$ such that (roughly speaking) $f(\mathrm{pr}_{W\cap W_2}\mathbf{b}), f(\mathbf{c}') \in \mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}$, and $f(\mathbf{c}'), \mathbf{c}'$ differ on few constraints and triples. Let $\mathbf{b}'' = (\mathrm{pr}_Z\mathbf{b})^{\tau^*}$, $\mathbf{c}'' = (\mathrm{pr}_Z\mathbf{c}')^{\tau^*}$. Since the interval $(\tau^*, \overline{\beta}'')$ has type **2**, where $\overline{\beta}'' = \overline{\beta}_Z$, $\{\mathbf{b}'', \mathbf{c}''\}$ is a subtrace of that interval. Let $(x, \gamma_x, \delta_x) \in \mathcal{W}(\overline{\beta})$, $x \in Z$, be such that $(\alpha, \beta_v)$ and $(\gamma_x, \delta_x)$ cannot be separated in $R^{\{v,x\}}$.

By (S7) for $\mathcal{R}$ $\mathcal{S}'_{W_2}$ is strongly chained. Since $\tau^* \prec \overline{\beta}''$ on $\mathrm{pr}_Z\mathcal{S}'^{\mathcal{R}}_{W_2}$ if we consider $\mathcal{S}'_{W_2}$ as a subdirect product of $\mathrm{pr}_Z\mathcal{S}'_{W_2}$ and $\mathbb{A}_y$, $y \in W_2 - Z$, by Lemma 47 there is a $(\tau^*, \overline{\beta}'')$-collapsing polynomial $f$ of $\mathcal{S}'_{W_2}$ for $\overline{\beta}, \overline{B}$. By Lemma 73 $(\tau^*, \overline{\beta}'')$ cannot be separated from $(\gamma_x, \delta_x)$ for $x \in W \cap W_2$. For any $z \in W_2$ and any $\eta, \theta \in \mathsf{Con}(\mathbb{A}_z)$ such that $\eta \prec \theta \leq \beta_z$ consider $\mathrm{pr}_{Z\cup\{z\}}\mathcal{S}'_{W_2}$. If $(\eta, \theta)$ can be separated from $(\alpha, \beta_v)$ (or $(\alpha, \beta_v)$ can be separated from $(\eta, \theta)$) then $(\tau^*, \overline{\beta}'')$ can be separated from $(\eta, \theta)$ (or the other way round). In particular, $f$ can be chosen to satisfy the following conditions

(a) $f(\mathrm{pr}_Z(\mathcal{S}'_{W_2}))$ is a $(\tau^*, \overline{\beta}'')$-minimal set and $f(\mathbb{A}_y)$ is a $(\gamma_y, \delta_y)$-minimal set for $y \in W \cap W_2$;

(b) for every $z \in W_2 - W$, $|f(B_z)| = 1$; and

(c) $f$ is idempotent.

Since $\{\mathbf{b}'', \mathbf{c}''\}$ is a $(\tau^*, \overline{\beta}'')$-subtrace of $(\mathrm{pr}_Z R_{C',u,\chi\xi})/_{\tau^*}$, by condition (Q2s) of being strongly chained (S7) for $\mathcal{R}$ polynomial $f$ can be chosen such that

(d) $\mathbf{b}'' = f(\mathbf{b}'')$, $\mathbf{c}'' = f(\mathbf{c}'')$.

Moreover, let $\mathbf{b}^* \in \mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}$ be such that there is a tuple $\mathbf{d} \in \mathcal{S}'^{\mathcal{R}}_{W_2}$ such that $\mathrm{pr}_{W_2-W}\mathbf{d} = \mathrm{pr}_{W_2-W}\mathbf{b}$, $\mathrm{pr}_{W\cap W_2}\mathbf{d} = \mathbf{b}^*$, $\mathrm{pr}_Z\mathbf{b}^* \in \mathbf{b}''$, and $\mathbf{d}$ is u-maximal in $\mathcal{S}'_{W_2}\cap\overline{B}$. Such a tuple exists, because since for any $\gamma', \delta' \in \mathsf{Con}(\mathbb{A}_z)$, $z \in W_2 - W$, with $\gamma' \prec \delta'$, and any $x \in W \cap W_2$, the interval $(\gamma_x, \delta_x)$ and therefore $(\tau^*, \overline{\beta}'')$ can be separated from $(\gamma', \delta')$, by the Congruence Lemma 63 $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}/_\tau \times \mathsf{umax}(\mathrm{pr}_{W_2-W}\mathcal{S}'^{\mathcal{R}}_{W_2}) \subseteq \mathcal{S}'^{\mathcal{R}}_{W_2}$. By Lemma 56 polynomial $f$ can be chosen such that

(e) $f(\mathbf{d}) = \mathbf{d}$.

Let $\mathbf{c}'$ be a u-maximal tuple from the $\tau$-block $\mathbf{c} \subseteq \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$, and let $\mathbf{c}^* = f(\mathbf{c}')$, and $\mathbf{c}^\dagger$ a tuple from $\mathsf{Sg}(\mathbf{b}^*, \mathbf{c}^*)$ such that $\mathbf{b}^* \sqsubseteq_{as} \mathbf{c}^\dagger$ and $\mathbf{c}^* \stackrel{\tau}{\equiv} \mathbf{c}^\dagger$. Note that it suffices to prove that $\varrho = (\mathbf{c}^\dagger, \mathrm{pr}_{W_2-W}\mathbf{b}) \in \mathcal{S}'^{\mathcal{R}}_{W_2}$. Indeed, since $\mathrm{pr}_Z\mathbf{c}^* \stackrel{\tau^*}{\equiv} \mathrm{pr}_Z\mathbf{c}'$ and $\mathbf{c}'$ agrees with $\mathbf{c}^\dagger$ modulo $\tau'_{C''}(u, \chi'\xi')$ for every $C'' \in \mathcal{C}$ and $(u', \chi'\xi') \in \mathcal{W}(\overline{\beta})$, for which $\mathbf{b}$ and $\mathbf{c}'$ agree, we obtain a contradiction with the choice of $\mathbf{b}, \mathbf{c}$.

Take any $C'' = \langle \mathbf{s}'', R''\rangle \in \mathcal{C}$ and $(x, \chi', \xi') \in \mathcal{W}(\overline{\beta})$; we show that $\varrho(\mathbf{s}'' \cap W_2) \in \mathrm{pr}_{\mathbf{s}''\cap W_2}R_{C'',x,\chi'\xi'}$. Let $U' = \mathbf{s}'' \cap W \cap W_2$ and $U'' = \mathbf{s}'' \cap W_2$. Since $\mathbf{c}' \in \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$, we have $\mathrm{pr}_{U'}\mathbf{c}' \in \mathsf{umax}(\mathrm{pr}_{U'}R_{C'',x,\chi'\xi'})$. By (S2) $\mathrm{pr}_{U'}\mathbf{c}'$ can be extended to an $\mathcal{R}$-compatible solution $\sigma$ from $\mathcal{S}'_{W_2}$. By the choice of $f$, property (b), $f(\mathrm{pr}_{U''-U'}\sigma) = \mathrm{pr}_{U''-U'}\mathbf{d} = \mathrm{pr}_{U''-U'}\mathbf{b}$, and $f(\mathrm{pr}_{U'}\sigma) = \mathrm{pr}_{U'}\mathbf{c}^*$ by definition of $\mathbf{c}^*$. Since $\mathrm{pr}_{U'}\mathbf{b}^* \sqsubseteq_{as} \mathrm{pr}_{U'}\mathbf{c}^\dagger$ in $\mathsf{Sg}(\mathrm{pr}_{U'}\mathbf{b}^*, \mathrm{pr}_{U'}\mathbf{c}^*)$, we have $\mathrm{pr}_{U''}\mathbf{b}^* \sqsubseteq_{as} \mathrm{pr}_{U''}\varrho$ in $\mathsf{Sg}(\mathrm{pr}_{U''}\mathbf{b}^*, \mathrm{pr}_{U''}\mathbf{c}^*)$, implying by (S6) that $\mathrm{pr}_{U''}\varrho \in R_{C'',x,\chi'\xi'}$. As this is true for every constraint $C''$, $\varrho$ is an $\mathcal{R}$-compatible solution.

(2) The proof of item (1) shows in particular that for any $\mathbf{b} \in S$ and $\mathbf{c} \in (\mathrm{pr}_{W\cap W_2}S)/_{\eta_1}$ we also have $(\mathrm{pr}_{W\cap W_2}\mathbf{b})^\tau \stackrel{\theta_1}{\equiv} \mathbf{c}$, as both $\mathbf{b}^\tau$ and $(\mathbf{c}, \mathrm{pr}_H\mathbf{b})$ belong to $S'$.

(3) follows from Proposition 35.

(4) follows from (1) and (3).

If we show that $\mathbf{a}$ can be extended to $\mathbf{b} \in S$ such that $\mathrm{pr}_U\mathbf{b} = \mathbf{a}$ and such that $\mathrm{pr}_H\mathbf{b}$ is u-maximal in a $\theta_2$-block, Claim 2 implies (S2) for $\mathcal{R}'$. Indeed, suppose there is $\mathbf{b} \in S$ satisfying the above conditions. Since $\mathrm{pr}_X\mathbf{a} \in \mathrm{pr}_X\mathcal{S}'^{\mathcal{R}'}_W$, there is $\mathbf{c}' \in \mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$ such that $\mathrm{pr}_X\mathbf{a} = \mathrm{pr}_X\mathbf{c}'$. In particular, this means that $\mathbf{c} = \mathbf{c}'^\tau \in D$, where $D$ is the $\eta_1$-, and therefore $\theta_1$-block containing $(\mathrm{pr}_{W\cap W_2}\mathbf{b})^\tau$, because $\mathrm{pr}_X\mathbf{c}' = \mathrm{pr}_X\mathbf{b} = \mathrm{pr}_X\mathbf{a}$. Then as $(\mathbf{c}, \mathrm{pr}_H\mathbf{b}) \in S'$ by Claim 2(4), this means that there is $\mathbf{d} \in \mathcal{S}'^{\mathcal{R}}_{W_2}$ such that $\mathrm{pr}_U\mathbf{d} = \mathbf{a}$ and $\mathrm{pr}_{W\cap W_2}\mathbf{d} \stackrel{\tau}{\equiv} \mathbf{c}'$; that is, $\mathbf{d} \in \mathcal{S}'^{\mathcal{R}'}_{W_2}$.

Next we show such a $\mathbf{b}$ exists. Recall that $S^*$ denotes the $\tau$-block of $\mathrm{pr}_{W\cap W_2}Q$

that contains $\mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$, and in particular, $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}'}_W$. Observe that

$$S^* \cap \mathrm{pr}_{W\cap W_2} S = S^* \cap \mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2} \neq \varnothing.$$

Indeed, let $\mathbf{d} \in S$ be such that $\mathrm{pr}_U\mathbf{d} = \mathbf{a}$. Then $\mathrm{pr}_{W\cap W_2}\mathbf{d} \in \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ and $\mathbf{d}^\tau$ belongs to the same $\eta_1$-block as $S^*$, because $(S^*/_\tau, \mathrm{pr}_X\mathbf{a}) \in Q'$. By Claim 2(1) $S^*/_\tau \in (\mathrm{pr}_{W\cap W_2} S)/_\tau$, proving the observation.

Let $D$ be the $\theta_1$-block containing $S^*/_\tau$ and $E$ the corresponding $\theta_2$-block. By what is proved above $\mathrm{pr}_H\mathbf{d} \in E$. We now only need to show that $\mathbf{d}$ can be chosen such that $\mathrm{pr}_H\mathbf{d} \in \mathsf{umax}(E)$. Let $\pi$ be the congruence on $\mathrm{pr}_H S$ given by $\mathbf{c} \stackrel{\pi}{\equiv} \mathbf{d}$ if and only if $\mathrm{pr}_{X'}\mathbf{c} \stackrel{\tau_X}{\equiv} \mathrm{pr}_{X'}\mathbf{d}$. Then $\mathrm{pr}_H S/_\pi$ is isomorphic to $\mathrm{pr}_X(R_{C,w_1,\gamma_1\delta_1})/_{\tau_X}$, in particular, it is a module if $\tau_X$ is maximal, and 1-element otherwise. Let $\tau'_X$ denote the congruence of $\mathrm{pr}_U R_{C,w_1,\gamma_1\delta_1}$ given by $\mathbf{c} \stackrel{\tau'_X}{\equiv} \mathbf{d}$ if and only if $\mathrm{pr}_X\mathbf{c} \stackrel{\tau_X}{\equiv} \mathrm{pr}_X\mathbf{d}$. As is easily seen, if $G$ is a $\pi$-block of $\mathrm{pr}_H S$ then $\mathrm{pr}_U G$ is a $\tau'_X$-block of $\mathrm{pr}_U R_{C,w_1,\gamma_1\delta_1}$. Therefore, as $\mathbf{a}$ is u-maximal in $\mathrm{pr}_U R'_{C,w_1,\gamma_1\delta_1}$, by the Maximality Lemma 27(5) it can be extended to a u-maximal tuple $\mathbf{a}'$ in a $\pi$-block $G$. Since $E/_\pi$ is a module, by Lemma 26 $\mathbf{a}'$ is also u-maximal in $E$. The result follows.

The proof of (S3) is similar to that of (S2), except we only need to consider one constraint relation rather than the set of solutions to a subproblem.

For (S4) observe that every $R'_{C,w,\gamma\delta}$ is obtained as the intersection of $R_{C,w,\gamma\delta}$ with a block of $\tau_{C,w,\gamma\delta}$, and therefore is a subalgebra. Also, since $R_{C,w,\gamma\delta}/\tau_{C,w,\gamma\delta}$ is a module, by Lemma 26 $\mathsf{umax}(R'_{C,w,\gamma\delta}) \subseteq \mathsf{umax}(R_{C,w,\gamma\delta})$ proving the first part of (S4).

To prove the rest of (S4) let $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$, $U = W_{w,\gamma\delta}$ and $\mathcal{S}'_U$ be one of the sets $\mathcal{S}_U$ if $(w,\gamma,\delta) \notin \mathcal{W}'$, or $\mathcal{S}_{U,Y}$ for $Y = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$ for some $C = \langle \mathbf{s}, R\rangle$ or $Y = \mathsf{MAX}(\mathcal{P}) - W_{u,\eta\theta}$ for some $(u,\eta,\theta) \notin \mathcal{W}'$, if $(w,\gamma,\delta) \in \mathcal{W}'$. As in the proof of condition (S2) we consider the congruence $\tau$ constructed as in (1) with $U$ in place of $W_2$. Let $Q$ be the $\tau$-block of $\mathrm{pr}_{W\cap U}\mathcal{S}'_U$ containing $\mathcal{R}'$-compatible tuples. By (S4) for $\mathcal{R}$ there is a tuple $\mathbf{a} \in \mathcal{S}'^{\mathcal{R}}_U$ that is in $\mathsf{umax}(\mathcal{S}'_U)$. Since $\mathrm{pr}_{W\cap U}\mathcal{S}'_U/_\tau$ is a module, by the Maximality Lemma 27(4) and Lemma 22 there is an as-path $\mathbf{a} = \mathbf{a}_1, \dots, \mathbf{a}_k$ in $\mathcal{S}'^{\mathcal{R}}_U$ such that $\mathrm{pr}_{W\cap U}\mathbf{a}_k \in Q$. The tuple $\mathbf{a}_k$ belongs to $\mathcal{S}'^{\mathcal{R}'}_U$ and to $\mathsf{umax}(\mathcal{S}'_U)$.

For (S5), the existence of $A_{\mathcal{R}',w}$ follows from (S3). Also, as in the proof of (S4) $\mathsf{umax}(A_{\mathcal{R}',w}) \subseteq \mathsf{umax}(A_{\mathcal{R},w})$. The result now follows from (S5) for $\mathcal{R}$.

For (S6) consider $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, and let $w \in \mathbf{s}$. Let $f$ be a polynomial of $R$, and let $\mathbf{a}, \mathbf{b} \in R$ be tuples satisfying the conditions of polynomial closeness. Let $\mathbf{c} \in \mathsf{Sg}(\mathbf{a}, f(\mathbf{b}))$ be such that $\mathbf{a} \sqsubseteq_{as} \mathbf{c}$ in $\mathsf{Sg}(\mathbf{a}, f(\mathbf{b}))$. By (S6) for $\mathcal{R}$, $\mathbf{c}$ is $\mathcal{R}$-compatible. It suffices to show that $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{c}$ is in the same $\tau_C$ block as $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{a}$. However, this is straightforward, because $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{a} \stackrel{\tau_C}{\equiv} \mathrm{pr}_{\mathbf{s}\cap W}\mathbf{b}$, and as $f(\mathbf{a}) = \mathbf{a}$, we also have $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{a} \stackrel{\tau_C}{\equiv} f(\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{b})$. Since $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{c} \in \mathsf{Sg}(\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{a}, f(\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{b}))$, it follows $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{c} \stackrel{\tau_C}{\equiv} \mathrm{pr}_{\mathbf{s}\cap W}\mathbf{a}$.

Finally, (S7) follows from Lemma 64. □

### 10.2 Tightening for non-affine factors

Let $\mathcal{P} = (V, \mathcal{C})$ be a block-minimal instance, let $\mathcal{R}$ be a $\overline{\beta}$-strategy with respect to $\overline{B}$. Take $v \in V$ and $\alpha \in \mathsf{Con}(\mathbb{A}_v)$ with $\alpha \prec \beta_v$ such that $\mathsf{typ}(\alpha, \beta_v) \neq \mathbf{2}$. We tighten $\mathcal{R}$ in two steps. In the first step we restrict $B_v$ to the subalgebra generated by an as-component of $A_{\mathcal{R},v}$ obtaining a collection of relations that satisfies all the properties of a strategy except (S5) and (S6). In the second step we restrict the same domain to one $\alpha$-block and restore (S5) and (S6). Let $D$ be an as-component of $A_{\mathcal{R},v}/_\alpha$, by (S5) $D$ is also an as-component of $B_v/_\alpha$. Note that if $\mathsf{typ}(\alpha, \beta_v) \in \{\mathbf{4}, \mathbf{5}\}$ then by Lemma 39 any as-component of $B_v/_\alpha$ is a singleton and Step 2 is not needed. Conditions (S5),(S6) in this case are proved as in Step 2.

#### 10.2.1 Step 1.

Let $D' = \{a \in A_{\mathcal{R},v} \mid a^\alpha \in D\}$ and let $\widehat{D} = \mathsf{Sg}(D')$. We consider the problem $\mathcal{P}'$ obtained from $\mathcal{P}$ by restricting the domain of $v$ to $\widehat{D}$ and the domain of $w \in V - \{v\}$ to $A_{\mathcal{R},w}$. We first show that $\mathcal{P}'$ can be converted to a nonempty (2,3)-minimal instance that also satisfies some additional conditions.

In order to do that we introduce a family of binary relations, and then prove that this family is a (2,3)-strategy of $\mathcal{P}'$. For $x, y \in V$, let

$$\begin{aligned} Q^x &= \{a \in \mathsf{amax}(A_{\mathcal{R},x}) \mid \text{ there is } d \in D \text{ such that } (d,a) \in R^{\{v,x\},\mathcal{R}}/_\alpha\}, \\ &\text{and} \\ Q^{xy} &= \{(a,b) \in \mathsf{amax}(R^{\{x,y\},\mathcal{R}}) \mid \text{ there is } d \in D \text{ such that} \\ &\qquad (d,a) \in R^{\{v,x\},\mathcal{R}}/_\alpha, (d,b) \in R^{\{v,y\},\mathcal{R}}/_\alpha\}. \end{aligned}$$

In particular $Q^v = \mathsf{amax}(D')$. We say that a tuple $\mathbf{a}$ on a set $U \subseteq V$ (where $U$ can be, e.g. a subset of $\mathbf{s}$ for a constraint $C = \langle \mathbf{s}, R\rangle$, or a subset of $W_{w,\gamma\delta}$ for some $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$) is *Q-compatible* if $(\mathbf{a}[x], \mathbf{a}[y]) \in Q^{xy}$ for any $x, y \in U$.

**Proposition 76** *(1) For any $x, y, z \in V$ and any $(a, b) \in Q^{xy}$ there is $c \in \mathsf{amax}(A_{\mathcal{R},z})$ such that $(a, c) \in Q^{xz}$ and $(b, c) \in Q^{yz}$.*
*(2) For any $C = \langle \mathbf{s}, R\rangle$ let $R^{\mathcal{R}}$ denote the set of $\mathcal{R}$-compatible tuples from $R$. For any $I \subseteq \mathbf{s}$ and any Q-compatible $\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_I R^{\mathcal{R}})$, there is $\mathbf{a}' \in \mathsf{amax}(R^{\mathcal{R}})$ that is Q-compatible, and $\mathrm{pr}_I \mathbf{a}' = \mathbf{a}$.*
*(3) For any $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$, any $U \subseteq W_{w,\gamma\delta}$, and any $\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_U \mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma\delta}})$,*

*where $\mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma\delta}}$ is the set of solutions of $(\mathcal{P}_{W_{w,\gamma\delta}})/_{\overline{\mu}^Y}$ for some set $Y$ from the definition of block-minimality, there is $\mathbf{a}' \in \mathsf{amax}(\mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma\delta}})$ that is Q-compatible and* $\mathrm{pr}_U\mathbf{a}' = \mathbf{a}$.

**Proof:** For $x, y \in V$ let $Q^{vxy}$ denote the set of tuples $(d, a, b)$ such that $(d, a) \in R^{\{v,x\},\mathcal{R}}/_\alpha$, $(d, b) \in R^{\{v,y\},\mathcal{R}}/_\alpha$, $(a, b) \in R^{\{x,y\},\mathcal{R}}$.

CLAIM 1. The set $Q^x$ is as-closed in $A_{R,x}$, and $Q^{xy}$ is as-closed if $R^{\{x,y\},\mathcal{R}}$.

Let $(a, b) \in Q^{xy}$. By the Maximality Lemma 27(3) either $Q^{vxy}$ contains a subdirect product of $D$ and $\mathsf{as}(a, b)$, or $(D \times \mathsf{as}(a, b)) \cap Q^{vxy} = \varnothing$. Since $(a, b) \in Q^{xy}$ the former option holds. For the first part of the claim observe that $\mathrm{pr}_x\mathsf{as}(a, b) = \mathsf{as}(a)$.

CLAIM 2. For any $x, y \in V$, $Q^{xy}$ is a subdirect product of $Q^x \times Q^y$.

Let $a \in Q^x$, then there is $d \in D$ with $(d, a) \in R^{\{v,x\},\mathcal{R}}$. By (S1) for $\mathcal{R}$, $\{\mathsf{umax}(R^{X,\mathcal{R}})\}_{X\subseteq V,|X|=2}$ is (2,3)-strategy, and there is $b \in \mathbb{A}_y$ with $(d, a, b) \in Q^{vxy}$; then $(a, b) \in Q^{xy}$.

We prove (2), the proof of (3) is basically identical, and we explain how to modify this proof to prove (1).

By induction on $i$ we prove that a Q-compatible tuple $\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_I R^{\mathcal{R}})$ can be found for any $I \subseteq \mathbf{s}$, $|I| = i$. Moreover, for any Q-compatible $\mathbf{a}' \in \mathsf{amax}(\mathrm{pr}_{I-\{u\}}R^{\mathcal{R}})$, a Q-compatible $\mathbf{b} \in \mathsf{amax}(\mathrm{pr}_I R^{\mathcal{R}})$ can be found such that $\mathrm{pr}_{I-\{u\}}\mathbf{b} = \mathbf{a}'$.

First we consider the case $u \neq v$. For $i = 2$ the existence of $\mathbf{a}$ follows from Claim 2. So, suppose such a tuple exists for any $I \subseteq \mathbf{s}$ with $|I| \le i$. Let $I \subseteq \mathbf{s}$, $|I| = i + 1$, $y \in I$, $I' = I - \{y\}$, and $\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_{I'}R^{\mathcal{R}})$ is Q-compatible. Let also $t \in I'$, $I'' = I' - \{t\}$. Without loss of generality assume $I = \{x_1, \dots, x_{i+1}\}$, $y = x_{i+1}$, $t = x_i$. Consider the relation given by

$$Q(x_1, \dots, x_i, z_1, \dots, z_i) = \exists y \ \ \mathrm{pr}_{I''}R^{\mathcal{R}}(x_1, \dots, x_{i-1}) \wedge \bigwedge_{j=1}^{i} \left(R^{\{x_j,v\},\mathcal{R}}/_\alpha(x_j, z_j) \wedge R^{\{v,x_{i+1}\},\mathcal{R}}/_\alpha(z_j, y) \wedge R^{\{x_j,x_{i+1}\},\mathcal{R}}(x_j, y)\right). \quad (2)$$

It suffices to prove that $\mathbf{a}'' = (\mathbf{a}, \mathbf{e}) \in Q$, where $\mathbf{e}[j] \in D$ for each $j \in [i]$, since this would mean that there is a $c \in \mathbb{A}_{\mathcal{R},x_{i+1}}$ with the required properties.

Observe first that $\mathsf{umax}(\mathrm{pr}_{I'}R^{\mathcal{R}}) \subseteq \mathrm{pr}_{I'}Q$. Indeed, any $\mathbf{b} \in \mathrm{pr}_{I'}R^{\mathcal{R}}$ by (S3) can be extended to $\mathbf{b}' \in \mathrm{pr}_I R^{\mathcal{R}}$; then the values of the variables $z_j$ can be chosen by (2,3)-consistency and (S1). This also implies that $\mathsf{umax}(\mathrm{pr}_{I''}Q) = \mathsf{umax}(\mathrm{pr}_{I''}R^{\mathcal{R}})$. Since $\alpha \prec \beta_v$, by (S7) for $\mathcal{R}$ and the Congruence Lemma 63 for any $w \in V$ the relation $R^{\{v,w\},\mathcal{R}}/_\alpha$ either contains $D \times \mathsf{umax}(Q^w)$, or $R^{\{v,w\},\mathcal{R}}/_\alpha \cap$

$(D \times Q^w)$ is the graph of a mapping $\kappa_w : Q^w \to D$. Let the set of variables for which the latter option holds be denoted by $Z$. By construction, for any $w_1, w_2 \in Z$, and any $(c,d) \in Q^{w_1w_2}$, we have $\kappa_{w_1}(c) = \kappa_{w_2}(d)$. For any $w \in V - Z$, the set $Q^w$ is as-closed, and $D \times \mathsf{umax}(Q^w) \subseteq R^{\{v,w\},\mathcal{R}}/_\alpha$.

Let $J = I' \cap Z$ and, if $J \neq \varnothing$, let $d = \kappa_w(\mathbf{a}[w])$ for any $w \in J$. If $J = \varnothing$, but $x_{i+1} \in Z$, then let $(\mathrm{pr}_{I''}\mathbf{a}, c)$ be an extension of $\mathrm{pr}_{I''}\mathbf{a}$ to a $Q$-compatible tuple from $\mathrm{pr}_{I\cup\{x_{i+1}\}}R$ and set $d = \kappa_{x_{i+1}}(c)$. If $x_{i+1} \notin Z$, then let $d$ be any element of $D$. Then $d$ is such that $(\mathbf{a}[w], d) \in R^{\{w,v\},\mathcal{R}}/_\alpha$ for any $w \in I'$.

Consider the tuple $\mathbf{b} = (\mathbf{a}, d, \dots, d)$; we show that it satisfies the conditions of the 2-Decomposition Theorem 30 with $X = I''$. Note that we cannot replace $I''$ with $I'$ here, because in order to apply the 2-Decomposition Theorem 30 $\mathbf{a}$ has to be as-maximal $\mathrm{pr}_{I'}Q$, which may not be true. By what is observed before, $\mathrm{pr}_{I''}\mathbf{a} \in \mathrm{pr}_{I''}Q$, and for any $s_1, s_2 \in I'$ we have $(\mathbf{a}[s_1], \mathbf{a}[s_2]) \in \mathrm{pr}_{s_1s_2}Q$. We now show that for any of the remaining pairs of variables $x, z \in \{x_1, \dots, x_i, z_1, \dots, z_i\}$ $(\mathbf{b}[x], \mathbf{b}[z]) \in \mathrm{pr}_{xz}Q$. Let $x \in \{x_1, \dots, x_i\}$ and $z \in \{z_1, \dots, z_i\}$. If $x \neq x_i$, then by the inductive hypothesis $\mathrm{pr}_{I''}\mathbf{a}$ can be extended to some value $c$ of $y$ such that $(\mathbf{a}[s], c) \in Q^{sy}$ for any $s \in I''$. Then there is $c' \in \mathbb{A}_{x_i}$ such that $(c', c) \in Q^{x_iy}$. Also, $(c,d) \in Q^{yv}$ and $c'$ can be chosen such that $(c', d) \in Q^{x_iv}$. If $x = x_i$, then find a value $c$ for $y$ such that $(\mathbf{a}[x_i], c) \in Q^{x_i,x_{i+1}}$, and then extend $c$ to a tuple on $I'' \cup \{y\}$ by induction hypothesis. The values of $z_j$ can be set using (2,3)-consistency. Finally, if $x, z \in \{z_1, \dots, z_i\}$, we proceed as in one of the previous cases.

By the 2-Decomposition Theorem 30 there is $\mathbf{b}' \in Q$ such that $\mathrm{pr}_{I''}\mathbf{a} \sqsubseteq_{as} \mathrm{pr}_{I''}\mathbf{b}'$ in $Q' = \mathrm{pr}_{I''}Q$, $(\mathbf{a}[x], \mathbf{a}[z]) \sqsubseteq_{as} (\mathbf{b}'[x], \mathbf{b}'[z])$ in $R^{\{x,z\},\mathcal{R}}$ for any $x, z \in I'$, and $d \sqsubseteq_{as} \mathbf{b}'[z_j]$ in $A_{\mathcal{R},v}/_\alpha$ for any $j \in [i]$.

Let $\mathsf{lk}_1, \mathsf{lk}_2$ be the link congruences of $Q', \mathbb{A}_{x_i}$ with respect to $\mathrm{pr}_{I'}Q$. Since $\mathsf{umax}(\mathrm{pr}_{I'}R^{\mathcal{R}}) \subseteq \mathrm{pr}_{I'}Q$, the link congruences of $\mathsf{umax}(\mathrm{pr}_{I''}R^{\mathcal{R}}), \mathbb{A}_{x_i}$ with respect to $\mathrm{pr}_{I'}R^{\mathcal{R}}$ are smaller than $\mathsf{lk}_1, \mathsf{lk}_2$. Therefore the $\mathsf{lk}_1$- and $\mathsf{lk}_2$-blocks $A, B$ containing $\mathrm{pr}_{I''}\mathbf{a}$ and $\mathbf{a}[x_i]$, respectively, are such that $Q'' = \mathrm{pr}_{I'}Q \cap (A \times B) \neq \varnothing$. Choose $\mathbf{a}' \in \mathrm{pr}_{I'}R^{\mathcal{R}} \cap (A \times B)$ such that $\mathbf{a} \sqsubseteq_{as} \mathbf{a}'$ in $\mathrm{pr}_{I'}R^{\mathcal{R}}$ and $\mathbf{a}'$ is as-maximal in $Q''$. As is easily seen, such a tuple exists by the Rectangularity Corollary 34, because, since $Q''$ is linked, any $\mathbf{a}' \in Q''$ such that $\mathrm{pr}_{I''}\mathbf{a}'$ is as-maximal in $A$ and $\mathbf{a}'[x_i]$ is as-maximal in $B$ is as-maximal in $Q''$. Now, consider

$$S(x_1, \dots, x_i, z_1, \dots, z_i) = \exists y A(x_1, \dots, x_{i-1}) \wedge B(x_i)$$
$$\wedge \bigwedge_{j=1}^{i} R^{\mathcal{R},\{x_j,v\}}/_\alpha(x_j, z_j) \wedge R^{\mathcal{R},\{v,x_{i+1}\}}/_\alpha(z_j, y) \wedge R^{\mathcal{R},\{x_j,x_{i+1}\}}(x_j, y).$$

By the same argument as before, there is $\mathbf{c} \in S$ such that $\mathbf{a}' \sqsubseteq_{as} \mathrm{pr}_{I'}\mathbf{c}$ in $\mathrm{pr}_{I'}S$ and $\mathbf{c}[z_j] \in D$ for $j \in [i]$. Since $\mathbf{a}'$ is as-maximal in $\mathrm{pr}_{I'}S$, we also have $\mathrm{pr}_{I'}\mathbf{c} \sqsubseteq_{as} \mathbf{a}'$

in $\mathrm{pr}_{I'}S$. Therefore there is $c \in A_{\mathcal{R},x_{i+1}}$ such that $(\mathbf{a}'[x], c) \in Q^{xx_{i+1}}$ for every $x \in I'$. By the 2-Decomposition Theorem 30 applied to $R^{\mathcal{R}}$, there is $\mathbf{a}'' \in \mathrm{pr}_I R$ such that $\mathrm{pr}_{I'}\mathbf{a}'' = \mathbf{a}'$ and $c \sqsubseteq_{as} \mathbf{a}''[x_{i+1}]$. Finally, since $\mathbf{a}$ is as-maximal in $\mathrm{pr}_{I'}R^{\mathcal{R}}$, there is an as-path from $\mathbf{a}'$ to $\mathbf{a}$ in $\mathrm{pr}_{I'}R^{\mathcal{R}}$ and we complete the proof by the Maximality Lemma 27(4).

Next, we consider the case $u = v$. Let $I = \{x_1, \ldots, x_i, v\}$, $I' = \{x_1, \ldots, x_i\}$ and $I'' = \{x_1, \ldots, x_{i-1}\}$. Note that by reordering the variables we may assume that if $x_i \in Z$ then $Z \cap \{x_1, \ldots, x_{i-1}\} \neq \varnothing$. By the induction hypothesis there is $c \in A_{\mathcal{R},v}$ such that $(\mathrm{pr}_{I''}\mathbf{a}, c)$ belongs to $\mathsf{amax}(\mathrm{pr}_{I''\cup\{v\}}R^{\mathcal{R}})$ and is Q-compatible, in particular, $c^\alpha \in D$. We consider the tuple $(\mathbf{a}, c^\alpha)$ and relation $R' = \mathrm{pr}_I R^{\mathcal{R}}/_\alpha$. We have $\mathbf{a} \in \mathrm{pr}_{I'}R'$. For any $j \in [i-1]$ we have $(\mathbf{a}[j], c) \in Q^{x_j v}$ by the choice of $c$. If $x_i \notin Z$ then there is $c' \stackrel{\alpha}{\equiv} c$ such that $(\mathbf{a}[x_i], c') \in R^{\{x_i,v\},\mathcal{R}}$. By (S3) for $\mathcal{R}$, $(\mathbf{a}[x_i], c')$ extends to a tuple from $R^{\mathcal{R}}$, therefore $(\mathbf{a}[x_i], c) \in \mathrm{pr}_{x_i v}R'$. If $x_i \in Z$, then there is also some $j \in I'' \cap Z$. Then since $(\mathbf{a}[j], \mathbf{a}[i]) \in Q^{x_j x_i}$, we have $\kappa_{x_j}(\mathbf{a}[x_j]) = \kappa_{x_i}(\mathbf{a}[x_i]) = c$. Therefore there is $c' \stackrel{\alpha}{\equiv} c$ such that $(\mathbf{a}[x_i], c') \in R^{\{x_i,v\},\mathcal{R}}$ and we continue as before. By the 2-Decomposition Theorem 30 there is $\mathbf{b} \in R'$ such that $\mathrm{pr}_{I'}\mathbf{b} = \mathbf{a}$ and $c \sqsubseteq_{as} \mathbf{b}[v]$ in $A_{\mathcal{R},v}/_\alpha$, that is, $\mathbf{b}[v] \in D$. Therefore there is $\mathbf{c} \in \mathrm{pr}_I R$ such that $\mathrm{pr}_{I'}\mathbf{c} = \mathbf{a}$ and $\mathbf{c}[v]^\alpha \in D$, as required.

Item (3) can be proved in the same way, as $\mathcal{P}$ is block-minimal. For (1) we need to make two changes. First, we apply the argument above for $i = 2$ and stop before the last application of the 2-Decomposition Theorem 30. Second, we need to consider the case when extending a pair from $Q^{xy}$ by a value of $v$. More precisely, let $(a, b) \in Q^{xy}$, $x, y \in V$, we need to find $c \in A_{\mathcal{R},v}$ such that $c^\alpha \in D$ and $(a, c) \in Q^{xv}$, $(b, c) \in Q^{yv}$. Let

$$Q(x, y, v) = R^{\{x,y\},\mathcal{R}}(x, y) \wedge R^{\{x,v\},\mathcal{R}}(x, v) \wedge R^{\{y,v\},\mathcal{R}}(y, v).$$

By (2,3)-consistency and construction $(a, b) \in \mathrm{pr}_{xy}Q$, $(a, d_1) \in \mathrm{pr}_{xv}Q$, $(b, d_2) \in \mathrm{pr}_{yv}Q$, where $d_1^\alpha = d_2^\alpha \in D$. The relation $Q' = Q/_\alpha$ then satisfies the conditions: $(a, b) \in \mathrm{pr}_{xy}Q'$, $(a, d) \in \mathrm{pr}_{xv}Q'$, $(b, d) \in \mathrm{pr}_{yv}Q'$ where $d = d_1^\alpha = d_2^\alpha$. By the 2-Decomposition Theorem 30 there is a tuple $(a, b, d') \in Q'$ such that $d \sqsubseteq_{as} d'$, that is, $d' \in D$. Therefore $Q$ contains a tuple $(a, b, c)$ for some $c$ with $c^\alpha = d'$. This $c$ is as required. □

Let $\mathcal{P}''$ be the problem obtained from $\mathcal{P}'$ as follows: establish (2,3)-minimality and, for any $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$, establish minimality of $\mathcal{P}'_{W_{w,\gamma\delta}}/_{\overline{\mu}^Y}$, where $Y$ is one of the sets specified in the definition of block-minimality, until the instance does not change any longer. Let $A''_{\mathcal{R},w}$ be the domains of $w \in V$ for $\mathcal{P}''$, for $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ let $R''$ denote the corresponding constraint relation of $\mathcal{P}''$; for $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$ let $\mathcal{S}''_{W_{w,\gamma\delta},Y}$ denote the set of solutions of $\mathcal{P}''_{W_{w,\gamma\delta}}/_{\overline{\mu}^Y}$; finally $R''_{C,w,\gamma\delta}$ denote the set of tuples from $R_{C,w,\gamma\delta}$ extending to a solution of $\mathcal{P}''_{W_{w,\gamma\delta}}/_{\overline{\mu}^Y}$.

The next corollary follows straightforwardly from Proposition 76, because establishing (2,3)-minimality or minimality never eliminates an as-maximal Q-compatible tuples.

**Corollary 77** *The sets $A''_{\mathcal{R},w}$, $R''$, $\mathcal{S}''_{W_{w,\gamma\delta},Y}$, and $R''_{C,w,\gamma\delta}$ contain all the as-maximal Q-compatible tuples from $A_{\mathcal{R},w}$, $R$, $\mathcal{S}_{W_{w,\gamma\delta},Y}$, and $R_{C,w,\gamma\delta}$, respectively.*

Now we can show that the collection of relations

$$\mathcal{R}'' = \{R''_{C,w,\gamma\delta} \mid C \in \mathcal{C}, (w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})\}$$

is almost a $\overline{\beta}$-strategy.

**Theorem 78** *The collection of relations $\mathcal{R}''$ constructed as above satisfies all the conditions of a $\overline{\beta}$-strategy with respect to $\overline{B}$, except (S5) and (S6).*

**Proof:** Condition (S1) follows from Corollary 77. Conditions (S2),(S3) follow from Corollary 77 and construction (establishing block-minimality). For (S4) every relation $R''_{C,w,\gamma\delta}$ is a subalgebra in $R_{C,w,\gamma\delta}$ and therefore in $\mathrm{pr}_{\mathbf{s}\cap W_{w,\gamma\delta}} R$ by construction. Moreover, as by Proposition 76 $R''_{C,w,\gamma\delta}$ contains an element as-maximal in $R_{C,w,\gamma\delta}$, we have $\mathsf{umax}(R''_{C,w,\gamma\delta}) \subseteq \mathsf{umax}(R_{C,w,\gamma\delta})$. Finally, condition (S7) follows from (S7) for $\mathcal{R}$. □

We will need another property of $\mathcal{P}''$. Unfortunately, Q-compatible tuples are not very helpful in establishing properties (S5),(S6), since those are properties of u-maximal fragments of relations. Therefore we need to extend Q-compatibility to u-maximal elements. Similar to Q-compatibility we make the following definition. For $x, y \in V$, let

$$P^x = \{a \in \mathsf{umax}(A_{\mathcal{R},x}) \mid \text{ there is } d \in D \text{ such that } (d,a) \in R^{\{v,x\},\mathcal{R}}/_{\alpha}\},$$

and

$$P^{xy} = \{(a,b) \in \mathsf{umax}(R^{\{x,y\},\mathcal{R}}) \mid \text{ there is } d \in D \text{ such that } (d,a) \in R^{\{v,x\},\mathcal{R}}/_{\alpha}, (d,b) \in R^{\{v,y\},\mathcal{R}}/_{\alpha}\}.$$

In particular $P^v = \mathsf{umax}(\widehat{D})$. Note that these relations are different from $Q^x, Q^{xy}$ in that they consist of u-maximal elements and pairs, rather than as-maximal as $Q^x, Q^{xy}$. We say that a tuple $\mathbf{a}$ on a set $U \subseteq V$ (where $U$ can be, e.g. a subset of $\mathbf{s}$ for a constraint $C = \langle \mathbf{s}, R\rangle$, or a subset of $W_{w,\gamma\delta}$ for some $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$) is *P-compatible* if $(\mathbf{a}[x], \mathbf{a}[y]) \in P^{xy}$ for any $x, y \in U$.

Let $\mathcal{P}'''$ be the instance obtained as follows: First, restrict the domains and relations of $\mathcal{P}$ to the sets of P-compatible tuples they contain; let the new relation for $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ be denoted by $R^{\dagger}$. Second, establish (2,3)-minimality and block-minimality of the resulting instance. Let $R''', R'''_{C,w,\gamma\delta}, \mathcal{S}'''_{W_{w,\gamma\delta},Y}$ denote the relations induced by $\mathcal{P}'''$. Note that the domains and relations of $\mathcal{P}'''$ are not necessarily subalgebras.

**Lemma 79** *Let $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$.*

*(1) Relations $R''', R'''_{C,w,\gamma\delta}, \mathcal{S}'''_{W_{w,\gamma\delta},Y}$ are nonempty, and $R''' \subseteq R''$, $R'''_{C,w,\gamma\delta} \subseteq R''_{C,w,\gamma\delta}$, $\mathcal{S}'''_{W_{w,\gamma\delta},Y} \subseteq \mathcal{S}''_{W_{w,\gamma\delta},Y}$.*

*(2) Relations $R'''_{C,w,\gamma\delta}$ and $\mathcal{S}'''_{W_{w,\gamma,\delta,\overline{\beta}},Y}$ are as-closed in $R_{C,w,\gamma\delta}$, and in $\mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma,\delta}}$, respectively (recall that $\mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma,\delta,\overline{\beta}}}$ is the set of $\mathcal{R}$-compatible tuples from $\mathcal{S}_{W_{w,\gamma,\delta,\overline{\beta}},Y}$).*

**Proof:** (1) The inclusions $R''' \subseteq R''$, $R'''_{C,w,\gamma\delta} \subseteq R''_{C,w,\gamma\delta}$, $\mathcal{S}'''_{W_{w,\gamma\delta,\overline{\beta}},Y} \subseteq \mathcal{S}''_{W_{w,\gamma\delta,\overline{\beta}},Y}$ follow from the construction, as $R^\dagger \subseteq R'$. On the other hand, as every Q-compatible tuple from $R$ belongs to $R^\dagger$, the nonemptyness follows by Corollary 77.

(2) First we observe that for any $x, y \in V$ the relation $P^{xy}$, and therefore $P^x$, is as-closed in $\mathsf{umax}(R^{\{x,y\},\mathcal{R}})$. This can be done in the same way as in the proof of Proposition 76. Let $(a,b) \in P^{xy}$ and $(a',b') \in \mathsf{umax}(R^{\{x,y\},\mathcal{R}})$ be such that $(a,b) \sqsubseteq_{as} (a',b')$. We need to find $d \in D$ such that $(a',d) \in R^{\{x,v\},\mathcal{R}}/_\alpha$, $(b',d) \in R^{\{y,v\},\mathcal{R}}/_\alpha$. Let

$$Q(x,y,v) = R^{\{x,y\},\mathcal{R}}(x,y) \wedge R^{\{x,v\},\mathcal{R}}/_\alpha(x,v) \wedge R^{\{y,v\},\mathcal{R}}/_\alpha(y,v),$$

which is a subalgebra of $\mathbb{A}_x \times \mathbb{A}_y \times \mathbb{A}_v/_\alpha$. Since $(a,b) \in P^{xy}$ there is $c \in D$ with $(a,b,c) \in Q$. By (S1) for $\mathcal{R}$ we have $(a',b') \in \mathrm{pr}_{xy}Q$, moreover, $(a,b) \sqsubseteq_{as} (a',b')$ in $\mathrm{pr}_{xy}Q$. By the Maximality Lemma 27(4) there is $d \in A_{\mathcal{R},v}$ such that $(a',b',d^\alpha) \in Q$ and $(a,b,c) \sqsubseteq_{as} (a',b',d^\alpha)$ in $Q$. Therefore $d^\alpha \in D$.

It suffices to prove the statement for relations of the form $R'''_{C,w,\gamma\delta}$, because relations $\mathcal{S}'''_{W_{w,\gamma,\delta},Y}$ are pp-definable through $R'''_{C,w,\gamma\delta}$, and we can use Lemma 60(2). Since $P^{xy}$ are as-closed, every relation $R^\dagger_{C,w,\gamma\delta}$ is also as-closed in $\mathsf{umax}(R_{C,w,\gamma\delta})$. We prove by induction that this property is preserved as (2,3)-minimality and block-minimality is being established. The observation about the relations $R^\dagger_{C,w,\gamma\delta}$ establishes the base case. For the induction step, let $R^\ddagger_{C,w,\gamma\delta}$ and $\mathcal{S}^\ddagger_{W_{w,\gamma\delta},Y}$ denote the current state of the corresponding relations, and they are as-closed. There are two cases for the induction step.

In the first case we make a step to enforce (2,3)-minimality, that is, for some $x,y,z \in V$ we check whether or not some $(a,b) \in R^{\ddagger\{x,y\}}$ can be extended by $c \in B_z$ such that $(a,c) \in R^{\ddagger\{x,z\}}$ and $(b,c) \in R^{\ddagger\{y,z\}}$. Let $(a,b) \in R^{\ddagger\{x,y\}}$ be such that there are $c \in B_z$ with $(a,c) \in R^{\ddagger\{x,z\}}$, $(b,c) \in R^{\ddagger\{y,z\}}$, and let

$(a', b') \in R^{\ddagger\{x,y\}}$ such that $(a, b) \sqsubseteq_{as} (a', b')$ in $R^{\ddagger\{x,y\}}$. Then there is $c' \in B_z$ such that $(a', c') \in R^{\{x,z\}}$, $(b', c') \in R^{\{y,z\}}$. Similar to part (1) let

$$Q(x, y, z) = \mathsf{Sg}(R^{\ddagger\{x,y\}})(x, y) \wedge R^{\{x,v\},\mathcal{R}}(x, v) \wedge R^{\{y,v\},\mathcal{R}}(y, v),$$

We have $\mathsf{Sg}(R^{\ddagger\{x,y\}}) = \mathrm{pr}_{xy} Q$ by (S1) for $\mathcal{R}$, and $(a, b, c), (a', b', c') \in Q$. By the Maximality Lemma 27(4) we may assume that $(a, b, c) \sqsubseteq_{as} (a', b', c')$ in $Q$. Since $R^{\ddagger\{x,z\}}, R^{\ddagger\{x,z\}}$ are as-closed, we have $(a', c') \in R^{\ddagger\{x,z\}}$, $(b', c') \in R^{\ddagger\{x,z\}}$

In the second case we solve a subproblem of the form $\mathcal{P}^{\ddagger}_{W_{w,\gamma\delta}}/_{\overline{\mu}^Y}$. Let $U = W_{w,\gamma\delta}$, let $\mathcal{S}^{\ddagger}_{W_{w,\gamma\delta},Y}$ be the corresponding set of solutions. Take $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ and $(u, \eta, \theta) \in \mathcal{W}(\overline{\beta})$, let $U' = \mathbf{s} \cap U \cap W_{u,\eta\theta}$. Suppose $\mathbf{a} \in R^{\ddagger}_{C,u,\eta\theta}$ be such that there is a solution $\varphi \in \mathcal{S}^{\ddagger}_{W_{w,\gamma\delta,\overline{\beta}},Y}$ extending $\mathrm{pr}_{U'}\mathbf{a}$, and let $\mathbf{b} \in R^{\ddagger}_{C,u,\eta\theta}$ is such that $\mathbf{a} \sqsubseteq_{as} \mathbf{b}$ in $R^{\ddagger}_{C,u,\eta\theta}$. We need to show that $\mathrm{pr}_{U'}\mathbf{b}$ is extendible to a solution from $\mathcal{S}^{\ddagger}_{W_{w,\gamma\delta},Y}$. Let $\psi \in \mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma\delta},Y}$ be a solution extending $\mathrm{pr}_{U'}\mathbf{b}$. Since $\mathrm{pr}_{U'}\mathbf{a} \sqsubseteq_{as} \mathrm{pr}_{U'}\mathbf{b}$ by the Maximality Lemma 27(4) we may assume that $\varphi \sqsubseteq_{as} \psi$ in $\mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma\delta,\overline{\beta}},Y}$. Since by the induction hypothesis $\mathcal{S}^{\ddagger}_{W_{w,\gamma\delta,\overline{\beta}},Y}$ is as-closed, the result follows. □

### 10.2.2 Step 2.

In this step we tighten the 'near-strategy' $\mathcal{R}''$ in a way similar to that from Section 10.1. We start with showing that the domains of all variables in $W_{v,\alpha\beta}$ have to be tightened.

**Lemma 80** *For every $w \in W = W_{v,\alpha\beta_v}$ there is a congruence $\alpha_w \in \mathsf{Con}(\mathbb{A}_w)$ with $\alpha_w \prec \beta_w$, and such that $\mathcal{S}_{vw} \cap (B_v \times B_w)$ is aligned with respect to $(\alpha, \alpha_w)$, that is, for any $(a_1, a_2), (b_1, b_2) \in \mathcal{S}_{vw} \cap (B_v \times B_w)$, $a_1 \stackrel{\alpha}{\equiv} b_1$ if and only if $a_2 \stackrel{\alpha_w}{\equiv} b_2$.*

**Proof:** It suffices to show that the link congruences $\mathsf{lk}_1, \mathsf{lk}_2$ of $Q = \mathcal{S}_{vw}$ viewed as a subdirect product of $\mathbb{A}_v \times \mathbb{A}_w$ are such that $\beta_v \wedge \mathsf{lk}_1 \le \alpha$ and $\beta_w \wedge \mathsf{lk}_2 < \beta_w$. Since $w \in W$ there are $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_w)$ such that $\gamma \prec \delta \le \beta_w$ and $(\alpha, \beta_v)$ and $(\gamma, \delta)$ cannot be separated. By Lemma 48 it follows that $\beta_v \wedge \mathsf{lk}_1 \le \alpha$ and $\mathsf{lk}_2 \wedge \delta \le \gamma$. We set $\alpha_w = \beta_w \wedge \mathsf{lk}_2 < \beta_w$. Since $B_w/_{\alpha_w}$ is isomorphic to $B_v/_{\alpha}$, $\alpha_w \prec \beta_w$. □

Let $\beta'_v = \alpha$, $\beta'_w = \alpha_w$ for $w \in W = W_{v,\alpha\beta_v}$, and $\beta'_w = \beta_w$ for $w \in V - W$. Lemma 80 implies that there is an isomorphism $\nu_w : B_v/_{\beta'_v} \to B_w/_{\beta'_w}$. Choose an as-maximal $\beta'_v$-block $B$, an element of $D$ from Step 1 and set $B'_v = B$, $B'_w = \nu_w(B)$ for $w \in V - W$, and $B'_w = B_w$ for $w \in V - W$. Let $\mathcal{P}^*$ be the problem instance obtained from $\mathcal{P}''$ as follows: first restrict the domain of

$w \in W$ in $\mathcal{P}''$ to $B'_w$, then establish the (2,3)-minimality of the resulting problem, and finally, establish the minimality of all problems of the form $\mathcal{P}''_{W_{w,\gamma\delta}}/_{\overline{\mu}^Y}$ for $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$, where $Y$ is a set specified in the definition of block-minimality for $\mathcal{P}$.

Let $\mathcal{R}^*$ be the following collection of relations;

(T1) $\mathcal{R}^* = \{R^*_{C,w,\gamma\delta} \mid C = \langle \mathbf{s}, R\rangle \in \mathcal{C}, (w,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')\}$;

(T2) for every $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, $(u,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')$, $R^*_{C,u,\gamma\delta} = \mathrm{pr}_{\mathbf{s}\cap W_{u,\gamma\delta}} R^*$, where $R^*$ is the constraint relation of $\mathcal{P}^*$ obtained from $R$.

**Lemma 81** *(1) For every constraint $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, and every $\mathbf{a} \in \mathsf{umax}(R'')$ such that $\mathbf{a}[u] \in \nu_u(D)$ for $u \in \mathbf{s} \cap W$ there is a tuple $\mathbf{b} \in \mathsf{umax}(R'')$ such that $\mathrm{pr}_{\mathbf{s}-W}\mathbf{b} = \mathrm{pr}_{\mathbf{s}-W}\mathbf{a}$ and $\mathbf{b}[u] \in B'_u$ for $u \in \mathbf{s}\cap W$.*

*(2) Let $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')$, $W' = W_{w,\gamma\delta}$, and $\mathbf{a} \in \mathsf{umax}(R''_{C,w,\gamma\delta})$ such that $\mathbf{a}[u] \in \nu_u(D)$ for $u \in \mathbf{s}\cap W' \cap W$. Then there is $\mathbf{b} \in \mathsf{umax}(R''_{C,w,\gamma\delta}))$ such that $\mathrm{pr}_{(\mathbf{s}\cap W')-W}\mathbf{b} = \mathrm{pr}_{(\mathbf{s}\cap W')-W}\mathbf{a}$ and $\mathbf{b}[u] \in B'_u$ for $u \in \mathbf{s}\cap W'\cap W$.*

*(3) Let $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')$ and $W' = W_{w,\gamma\delta}$. Let $\mathcal{S}''_{W'}$ be the set of solutions of $\mathcal{P}''_{W'}/_{\overline{\mu}^Y}$, where $Y = \varnothing$ if $(w,\gamma,\delta) \notin \mathcal{W}'$, and is one of the sets specified in the definition of block-minimality otherwise. For every solution $\varphi \in \mathsf{umax}(\mathcal{S}''_{W'})$ such that $\varphi[u] \in \nu_u(D)$ for $u \in W'\cap W$ there is a solution $\psi \in \mathsf{umax}(\mathcal{S}''_{W'})$ such that $\psi(u) = \varphi(u)$ for $u \in W' - W$ and $\psi(u) \in B'(u)$ for $u \in W'\cap W$.*

**Proof:** (1) Let $U_1 = \mathbf{s}\cap W$ and $U_2 = \mathbf{s} - W$. If $U_1 = \varnothing$ there is nothing to prove; assume $U_1 \neq \varnothing$. It suffices to consider $Q = R''/_{\overline{\alpha}'}$ where $\alpha'_u = \beta'_u$ if $u \in U_1$ and $\alpha'_u = \underline{0}_u$ otherwise. So, we assume $\beta'_u = \underline{0}_u$ for all $u \in U_1$. Then for any $u_1, u_2 \in U_1$ and any $\mathbf{d} \in R$ such that $\mathbf{d}[u_1] \in B_{u_1}, \mathbf{d}[u_2] \in B_{u_2}$ we have $\mathbf{d}[u_2] = \nu_{u_2}\circ\nu_{u_1}^{-1}(\mathbf{d}[u_1])$. Therefore we may assume that $|U_1| = 1$, say, $U_1 = \{u\}$.

Considering $R''$ as a subalgebra of $\mathbb{A}_u \times \mathrm{pr}_{\mathbf{s}-\{u\}}R$, the result follows by the Congruence Lemma 63. Indeed, since there is a $\alpha_u\beta_u$-collapsing polynomial $f$ of $R$, that is, $f(\overline{\beta}_{\mathbf{s}-\{u\}}) \subseteq \underline{0}_{\mathbf{s}-\{u\}}$, there are no $\eta, \theta \in \mathsf{Con}(\mathrm{pr}_{\mathbf{s}-\{u\}}R)$ with $\eta \prec \theta \leq \overline{\beta}_{\mathbf{s}-\{u\}}$ such that $(\alpha_u, \beta_u)$ cannot be separated from $(\eta,\theta)$.

(2) and (3) are proved in essentially the same way. □

To show that $\mathcal{P}^*$ has the desirable properties, in particular, it is nonempty, we consider a collection of unary and binary relations similar to $Q^x, Q^{xy}$ from Step 1. For $x, y \in V$ let $T^x, T^{xy}$ denote the following sets:

$$T^x = \{a \in \mathsf{amax}(A''_{\mathcal{R},x}) \mid (a,c) \in \mathsf{amax}(R''^{\{x,v\},\mathcal{R}}) \text{ for some } c \in B\};$$
$$T^{xy} = \{(a,b) \in \mathsf{amax}(R''^{\{x,y\},\mathcal{R}}) \mid (a,c) \in \mathsf{amax}(R''^{\{x,v\},\mathcal{R}}), (b,c) \in \mathsf{amax}(R''^{\{y,v\},\mathcal{R}}) \text{ for some } c \in B\}.$$

A tuple $\mathbf{a}$ over a set of variables $U \subseteq V$ is said to be *T-compatible* if for any $x, y \in U$, $(\mathbf{a}[x], \mathbf{a}[y]) \in T^{xy}$. The following lemma provides the main structural result necessary for proving that $\mathcal{R}^*$ is a $\overline{\beta}^*$-strategy.

**Lemma 82** *Let $S$ be one of the relations $R$, $R_{C,w,\gamma\delta}$, $\mathcal{S}_{W_{w,\gamma,\delta},Y}$, and $S''$, $S^*$ the corresponding relations $R''$, $R''_{C,w,\gamma\delta}$, or $\mathcal{S}''_{W_{w,\gamma,\delta},Y}$, and $R^*$, $R^*_{C,w,\gamma\delta}$, or $\mathcal{S}^*_{W_{w,\gamma,\delta},Y}$, respectively, for some $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ and $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$, where $Y$ is a set from the definition of block-minimality; and let $U$ be its set of coordinate positions.*

*(1) Every Q-compatible $\mathbf{a} \in \mathsf{amax}(S)$ such that $\mathbf{a}[u] \in B'_u$ for $u \in U \cap W$ is also T-compatible.*

*(2) $S^*$ contains all the as-maximal T-compatible tuples from $S''$.*

*(3) If $S^* = R^*_{C,w,\gamma\delta}$, then $\mathsf{umax}(S^*)$ is as-closed in $\mathsf{umax}(R_{C,w,\gamma\delta} \cap \overline{B}')$.*

**Proof:** (1) By Corollary 77, if, say $x \in W$ and $(a,b) \in Q^{xy}$ are such that $a \in B'_x$, then $(a,b) \in T^{xy}$, which can be proved as in the proof of Proposition 76(1). Therefore, it suffices to prove that if $x, y \notin W$, then $T^{xy} = Q^{xy}$. This can be done in the same way as in the proof of Proposition 76. Let $(a,b) \in Q^{xy}$, we need to find $c \in B'_v$ such that $(a,c) \in Q^{xv}$, $(b,c) \in Q^{yv}$. Let

$$Q(x,y,v) = R^{\{x,y\},\mathcal{R}}(x,y) \wedge R^{\{x,v\},\mathcal{R}}(x,v) \wedge R^{\{y,v\},\mathcal{R}}(y,v),$$

which is a subalgebra of $\mathbb{A}_x \times \mathbb{A}_y \times \mathbb{A}_v$. By (2,3)-consistency of $\mathcal{P}''$ $Q^{xy} \subseteq \mathrm{pr}_{xy}Q, Q^{xv} \subseteq \mathrm{pr}_{xv}Q, Q^{yv} \subseteq \mathrm{pr}_{xv}Q$. Let $Q' = Q/_{\alpha}$. Since each of $R^{\{x,y\},\mathcal{R}}$, $R^{\{x,v\},\mathcal{R}}$, $R^{\{y,v\},\mathcal{R}}$ is polynomially closed in the corresponding constraint relation $R^{\{x,y\}}$, $R^{\{x,v\}}$, or $R^{\{y,v\}}$ of $\mathcal{P}$, $Q$ is polynomially closed in

$$R^{\{x,y\}}(x,y) \wedge R^{\{x,v\}}(x,v) \wedge R^{\{y,v\}}(y,v),$$

as well, and so is $Q'$ in

$$R^{\{x,y\}}(x,y) \wedge R^{\{x,v\}}/_{\alpha}(x,v) \wedge R^{\{y,v\}}/_{\alpha}(y,v).$$

Let $Q^\dagger = \mathrm{pr}_{xy}(Q' \cap (B_x \times B_y \times D))$, By the Congruence Lemma 63 either $\mathsf{umax}(Q^\dagger) \times D \subseteq Q'$ or there is $\eta \in \mathsf{Con}(R^{\{x,y\}})$ such that $\mathsf{umax}(Q' \cap (B_x \times B_y \times D))$ is the graph of a mapping $\tau : \mathsf{umax}(Q^\dagger) \to D$. In the former case we are done, because then $(a,b,B'_v) \in Q'$, and therefore $(a,b,c) \in Q$ for some $c \in B'_v$. The latter case is not possible, because by the Congruence Lemma 63 there is $\theta \in \mathsf{Con}(R^{\{x,y\}})$ such that $\eta \prec \theta \le \beta_x \times \beta_y$, and $(\alpha, \beta_v)$ and $(\eta,\theta)$ cannot be separated. This however is not the case, since $x, y \notin W$.

(2) Similar to Proposition 76 and Corollary 77 it suffices to prove that for any $X \subseteq U$, any T-compatible $\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_X S'')$ can be extended to a T-compatible $\mathbf{b} \in \mathsf{amax}(S'')$. In fact, since $S''$ contains all the Q-compatible tuples, and therefore all the T-compatible tuples from $S$, it suffices to prove the statement for $S$, rather than for $S''$. We show that for any $w \in U - X$ tuple $\mathbf{a}$ can be extended to a T-compatible tuple $\mathbf{c} \in \mathsf{amax}(\mathrm{pr}_{X\cup\{w\}} S)$. By Proposition 76 there is a Q-compatible $\mathbf{b}' \in S$ with $\mathbf{a} = \mathrm{pr}_X \mathbf{b}'$. If $X \cap W \neq \varnothing$ or $(X \cup \{w\}) \cap W = \varnothing$, we can set $\mathbf{c} = \mathrm{pr}_{X\cup\{w\}} \mathbf{b}'$.

Suppose that $X \cap W = \varnothing$ and $w \in W$. Then we proceed similar to part (1). Let $X = \{x_1, \dots, x_k\}$, let

$$Q(x_1, \dots, x_k, w) = \mathrm{pr}_X S(x_1, \dots, x_k) \wedge \bigwedge_{i=1}^{k} R^{\{x_i,w\},\mathcal{R}}(x_i, w),$$

By (2,3)-consistency of $\mathcal{P}''$ and Proposition 76 $Q^{x_i w} \subseteq \mathrm{pr}_{x_i w} Q$, and by Corollary 77 $S' \subseteq \mathrm{pr}_X Q$, where $S'$ is the set of all T-compatible (equivalently, Q-compatible) tuples from $\mathrm{pr}_X S$. Let $Q' = Q/_{\alpha}$. Since each of $R^{\{x_i,w\},\mathcal{R}}$ is polynomially closed in $R^{\{x_i,w\}}$ and $\mathrm{pr}_X S(x_1, \dots, x_k)$ is polynomially closed in itself, by Lemma 60(2) $Q$ is polynomially closed in

$$\mathrm{pr}_X S(x_1, \dots, x_k) \wedge \bigwedge_{i=1}^{k} R^{\{x_i,w\}}(x_i, w),$$

as well, and so is $Q'$ in

$$\mathrm{pr}_X S(x_1, \dots, x_k) \wedge \bigwedge_{i=1}^{k} R^{\{x_i,w\}}/_{\alpha_w}(x_i, w).$$

Let $Q^\dagger = \mathrm{pr}_X \left( Q' \cap \left( \prod_{i=1}^{k} B_{x_i} \times D \right) \right)$. Now we can finish the proof in the same way as in part (1).

(3) Let $U = \mathbf{s} \cap W_{w,\delta\gamma}$. Observe first, that every tuple from $\mathsf{umax}(R^*_{C,w,\gamma\delta})$ is P-compatible (see Section 10.2.1). If we prove that $R^*_{C,w,\gamma\delta}$ contains every P-compatible tuple $\mathbf{a}$ from $R''_{C,w,\gamma\delta}$ such that $\mathbf{a}[u] \in B'_u$ for every $u \in U \cap W$, by Lemma 79(2) the result follows. As in the proof of Lemma 79 we proceed by induction the restriction of the problem $\mathcal{P}''$ being converted to a (2,3)-minimal and block-minimal instance.

Let $R''$ and $R''_{C,w,\gamma\delta}$ denote the relations associated with the instance $\mathcal{P}''$. Let $R^\dagger$ and $R^\dagger_{C,w,\gamma\delta}$ denote the relations obtained from $R''$ and $R''_{C,w,\gamma\delta}$ in the first step of converting $\mathcal{P}''$ to $\mathcal{P}^*$, that is, restricting the domains. By Lemma 81 relations

$R^\dagger, R^\dagger_{C,w,\gamma\delta}$ contain all the necessary P-compatible relations. This provides the base case. For the induction step we again consider two cases. We denote the current constraint relations by $R^\ddagger_{C,w,\gamma\delta}$ and the ones from the (2,3)-strategy by $R^{\ddagger X}$.

In the first case we enforce (2,3)-minimality for $x, y, z \in V$. Let $(a, b) \in R^{\ddagger\{x,y\}}$ be a P-compatible tuple. Then there is $c_1, c_2 \in B_z$ such that $(a, c_1) \in R''^{\{x,z\}}$, $(b, c_2) \in R''^{\{y,z\}}$ are P-compatible. As in the proof of item (1) of this lemma, we can argue that $c_1 = c_2$ can be assumed. If $z \notin W$, the pairs $(a, c_1), (b, c_2)$ are as required. Otherwise, $c_1, c_2$ can be chosen from $B'_z$ by Lemma 81.

In the second case let $(u, \eta, \theta) \in \mathcal{W}(\overline{\beta})$ and $X = W_{u,\eta\theta}$; we solve a problem of the form $\mathcal{P}''_X/_{\overline{\mu}^Y}$, let $\mathcal{S}^\ddagger_X$ be the set of solutions of this problem. Let also $U' = \mathbf{s} \cap X$. We need to show that for any P-compatible $\mathbf{a} \in \mathsf{umax}(R^\ddagger_{C,w,\gamma\delta})$ with $\mathbf{a}[u] \in B'_u$ for $u \in U$ the tuple $\mathrm{pr}_{U'}\mathbf{a}$ can be extended to a P-compatible solution $\varphi \in \mathcal{S}^\ddagger_X$. Since $\mathbf{a} \in \mathsf{umax}(R''_{C,w,\gamma\delta})$, the tuple $\mathrm{pr}_{U'}\mathbf{a}$ can be extended to a u-maximal solution $\varphi \in \mathcal{S}''_X$. If $U' \cap W \neq \varnothing$ or $X \cap W = \varnothing$, solution $\varphi$ is as required. Otherwise by Lemma 81 $\varphi$ can be chosen P-compatible and such that $\varphi(x) \in B'_x$ for $x \in X \cap W$; that is $\varphi \in \mathcal{S}^\ddagger_X$ by the induction hypothesis. □

Now we are ready to prove that $\mathcal{R}^*$ is a $\overline{\beta}'$-strategy.

**Theorem 83** *$\mathcal{R}^*$ is a $\overline{\beta}'$-strategy with respect to $\overline{B}'$.*

**Proof:** (S1) follows directly from the construction, since the relations $R^{X,\mathcal{R}^*}$ result from establishing (2,3)-minimality of $\mathcal{P}^*$, and they are nonempty by Lemma 82(1). Conditions (S2) and (S3) are also by construction. Condition (S4) also holds by construction, as all the relations of the form $R^*_{C,w,\gamma\delta}$ are subalgebras. Also, each of them contains a Q-compatible element, which is as-maximal in $R_{C,w,\gamma\delta}$, implying that $\mathsf{umax}(R^*_{C,w,\gamma\delta}) \subseteq \mathsf{umax}(R_{C,w,\gamma\delta})$.

For (S5) the existence of $A_{\mathcal{R}^*,w}$ for $w \in V$ follows from the construction, and the as-closeness of $\mathsf{umax}(A_{\mathcal{R}^*,w})$ follows from Lemma 82(3). Condition (S6) follows from Lemma 82(3) and (S6) for $\mathcal{R}$ as well. Finally, condition (S7) holds by Lemma 64. □

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# Index

# Notation