A dichotomy theorem for nonuniform CSPs

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Abstract

In this paper we prove the Dichotomy Conjecture on the complexity of nonuniform constraint satisfaction problems posed by Feder and Vardi.

Contents

1 Introduction

In a Constraint Satisfaction Problem (CSP) the question is, given two similar relational structures, decide whether or not there is a homomorphism from the first structure to the second one. If the target structure $\mathbf{B}$ is fixed, the input of the problem is only the first structure; this restricted problem is referred to as the *nonuniform CSP* and denoted $\mathrm{CSP}(\mathbf{B})$. A systematic study of the complexity of this problem was started by Schaefer in [40] who showed that for every 2-element structure $\mathbf{B}$ the problem $\mathrm{CSP}(\mathbf{B})$ is either solvable in polynomial time or is NP-complete. Schaefer also asked about the complexity of $\mathrm{CSP}(\mathbf{B})$ for larger structures. The next step in the study of $\mathrm{CSP}(\mathbf{B})$ was made in the seminal paper by Feder and Vardi [26, 27], who apart from considering numerous aspects of the problem, posed the *Dichotomy Conjecture* that states that for every finite relational structure $\mathbf{B}$ the problem $\mathrm{CSP}(\mathbf{B})$ is either solvable in polynomial time or is NP- complete. This conjecture has become a focal point of the decision CSP research and most of the effort in this area revolves to some extent around the Dichotomy Conjecture.

The Dichotomy Conjecture was approached with different methods, however, the most effective one turned out to be the *algebraic approach* that associates to every relational structure its (universal) algebra of polymorphisms. The method was first developed in a series of papers by Jeavons and coauthors [31, 32, 33] and then refined by Bulatov, Krokhin, Barto, Kozik, Maroti, Zhuk and others [4, 7, 5, 3, 20, 12, 22, 36, 37, 41, 43, 42]. While the complexity of $\mathrm{CSP}(\mathbf{B})$ has been solved for some interesting classes of structures such as graphs [28], the algebraic approach allowed the researchers to confirm the Dichotomy Conjecture in a number of more general cases: for structures of size up to 7 [10, 13, 35, 43, 42], so called conservative structures [11, 14, 15, 2], and some classes of digraphs [6]. It also allowed to design the main classes of CSP algorithms [5, 19, 17, 8, 30], and refine the exact complexity of the CSP [1, 7, 24, 34].

In this paper we confirm the Dichotomy Conjecture for arbitrary finite structures. More precisely we prove the following

Theorem 1 *For any finite relational structure* $\mathbf{B}$ *the problem* $\mathrm{CSP}(\mathbf{B})$ *is either solvable in polynomial time or is NP-complete.*

The proved criterion matches the algebraic form of the Dichotomy Conjecture suggested in [20]. The hardness part of the conjecture has been known for long time. Therefore the main achievement of this paper is a polynomial time algorithm for problems satisfying the polynomial time conditions. More specifically, we suggest such an algorithm for structures that are cores, and this implies a general dichotomy due to results of [20].

Using the algebraic language we can state the result in a stronger form. Let $\mathbb{A}$ be a finite idempotent algebra and let $\mathrm{CSP}(\mathbb{A})$ denote the union of problems $\mathrm{CSP}(\mathbf{B})$ such that every term operation of $\mathbb{A}$ is a polymorphism of $\mathbf{B}$. Problem $\mathrm{CSP}(\mathbb{A})$ is no longer a nonuniform CSP, and Theorem 1 allows for problems $\mathrm{CSP}(\mathbf{B}) \subseteq \mathrm{CSP}(\mathbb{A})$ to have different solution algorithms even when $\mathbb{A}$ meets the tractability conditions. We show that the solution algorithm only depends on the algebra $\mathbb{A}$.

Theorem 2 *For a finite idempotent algebra that satisfies the conditions of the Dichotomy Conjecture there is a uniform solution algorithm for* $\mathrm{CSP}(\mathbb{A})$.

The paper is structured as follows. We start with preliminaries, where apart from the main definitions and notation, we remind some of the results of [12, 16, 17] related to colored graphs of algebras and relational structures and also some of their properties. In Sections 2.3–2.5 we advance these results a little further. Then in Section 3 we introduce a method of separating factors in congruence lattices using polynomial operations of the algebra. This method constitutes the basis for our algorithm. Some preliminary versions of this approach can be found in [9, 18]. In Section 4 we introduce another helpful operator on congruence lattices that is similar to the well studied centralizer operator, although the precise relationship between the two is not quite clear. In particular, it allows to split certain CSPs into smaller ones.

In Section 5 we give a description of the algorithm, and prove its running time and partially soundness. In very broad strokes the algorithm works as follows. If none of the domains of $\mathcal{P}$ contains a semilattice edge in the sense of colored graphs of algebras, then $\mathcal{P}$ can be solved by the few subalgebras algorithm [8, 30], as shown in [17]. Otherwise in most cases the problem can be solved by establishing some sort of minimality condition, called *block-minimality* similar to that in [18]. The problematic case when block-minimality does not provide a solution, or rather when it cannot be established is roughly speaking when the domains of the instance have nontrivial centers in the sense of the commutator theory. In this case we show in Section 5.2 that a solution of a problem $\mathcal{P}'$ obtained from $\mathcal{P}$ by replacing some of its domains with quotient algebras modulo their centers allows one to reduce the number of semilattice edges in those domains.

The key ingredient of our result is presented in Section 5.3. There for block-minimal instances we introduce strategies that are in certain aspects similar to strategies used to solve problems of bounded width, but allow us to approach general CSPs. Then in Section 6 we show, Theorem 45, that if for a CSP instance $\mathcal{P}$ satisfying the block-minimality conditions such a strategy exists, one can improve (tighten) the strategy to obtain a solution of $\mathcal{P}'$ needed to reduce semilattice edges. This theorem is the most difficult and technically involved part of the proof. Tightening of a strategy works by (effectively) reducing domains of the CSP to a class of a maximal congruence, and then repeating the process as long as possible. The main cases of tightening considered are: when the interval formed by the maximal congruence used and the full congruence is Abelian, and when it is non-Abelian, Sections 6.2 and 6.3, respectively. In the two cases we use quite different transformations of the strategy. In the Abelian case the argument is based on the rectangularity of relations understood in a general sense, while in the non-Abelian case the transformation is similar to that used for bounded width CSPs in [17].

2 Preliminaries

2.1 Universal algebra and CSP: notation and agreements

We assume familiarity with the basics of universal algebra and the algebraic approach to the CSP. For reference please use [23, 38] and [4, 3, 20, 22, 21, 16].

By $[n]$ we denote the set $\{1,\dots,n\}$. For sets $A_1,\dots,A_n$ tuples from $A_1 \times \dots \times A_n$ are denoted in boldface, say, $\mathbf{a}$; the ith component of $\mathbf{a}$ is referred to as $\mathbf{a}[i]$. An n-ary relation R over sets $A_1,\dots,A_n$ is any subset of $A_1 \times \dots \times A_n$. For $I = \{i_1,\dots,i_k\} \subseteq [n]$ by $\mathrm{pr}_I\mathbf{a}, \mathrm{pr}_I R$ we denote the *projections* $\mathrm{pr}_I\mathbf{a} = (\mathbf{a}[i_1],\dots,\mathbf{a}[i_k])$, $\mathrm{pr}_I R = \{\mathrm{pr}_I\mathbf{a} \mid \mathbf{a} \in R\}$ of tuple $\mathbf{a}$ and relation R. If $\mathrm{pr}_i R = A_i$ for each $i \in [n]$, relation R is said to be a *subdirect product* of $A_1 \times \dots \times A_n$. It will be convenient to use $\overline{A}$ for $A_1 \times \dots \times A_n$, or for $\prod_{v\in V} A_v$ if the sets V and A_v are clear from the context. For $I \subseteq [n]$ or $I \subseteq V$ we will use $\overline{A}_I$, for $\prod_{i\in I} A_i$, or if I is clear from the context just $\overline{A}$.

Algebras will be denoted by $\mathbb{A}, \mathbb{B}$ etc.; we often do not distinguish between subuniverses and subalgebras. For $B \subseteq \mathbb{A}$ the subalgebra generated by B is denoted $\mathsf{Sg}(B)$. For $C \subseteq \mathbb{A}^2$ the congruence generated by C is denoted $\mathsf{Cg}(C)$. The equality relation and the full congruence of algebra $\mathbb{A}$ are denoted $\underline{0}_{\mathbb{A}}$ and $\underline{1}_{\mathbb{A}}$, respectively. Often when we need to use one of these trivial congruences of an algebra indexed in some way, say, $\mathbb{A}_i$, we write $\underline{0}_i, \underline{1}_i$ for $\underline{0}_{\mathbb{A}_i}, \underline{1}_{\mathbb{A}_i}$. The set of all polynomials (unary polynomials) of $\mathbb{A}$ is denoted by $\mathsf{Pol}(\mathbb{A})$ and $\mathsf{Pol}_1(\mathbb{A})$, respectively. We frequently use operations on subalgebras of direct products of algebras, say, $R \subseteq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$. If f is such an operation (say, k-ary) then we denote its component-wise action also by f, e.g. $f(a_1, \dots, a_k)$ for $a_1, \dots, a_k \in \mathbb{A}_i$. In the same way we denote the action of f on projections of R, e.g. $f(\mathbf{a}_1, \dots, \mathbf{a}_k)$ for $I \subseteq [n]$ and $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathrm{pr}_I R$. What we mean will always be clear from the context. We use similar agreements for collections of congruences. If $\alpha_i \in \mathsf{Con}(\mathbb{A}_i)$ then $\overline{\alpha}$ denotes the congruence $\alpha_1 \times \dots \times \alpha_n$ of R. If $I \subseteq [n]$ we use $\overline{\alpha}_I$ to denote $\prod_{i \in I} \alpha_i$. If it does not lead to a confusion we write $\overline{\alpha}$ for $\overline{\alpha}_I$. Sometimes α_i are specified for i from a certain set $I \subseteq [n]$, then by $\overline{\alpha}$ we mean the congruence $\prod_{i \in [n]} \alpha'_i$ where $\alpha'_i = \alpha_i$ if $i \in I$ and α'_i is the equality relation otherwise. For example, if $\alpha \in \mathsf{Con}(\mathbb{A}_1)$ then $R/_\alpha$ means the factor of R modulo $\alpha \times \underline{0}_2 \times \dots \times \underline{0}_n$. For $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ we write $\alpha \prec \beta$ if $\alpha < \beta$ and $\alpha \le \gamma \le \beta$ in $\mathsf{Con}(\mathbb{A})$ implies $\gamma = \alpha$ or $\gamma = \beta$. In this paper all algebras are finite, idempotent and omit type **1**.

The (*nonuniform*) *Constraint Satisfaction Problem* (*CSP*) associated with a relational structure $\mathbf{B}$ is the problem $\mathrm{CSP}(\mathbf{B})$, in which, given a structure $\mathbf{A}$ of the same signature as $\mathbf{B}$, the goal is to decide whether or not there is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$. For a class of similar algebras $\mathcal{A} = \{\mathbb{A}_i \mid i \in I\}$ for some set I an *instance* of $\mathrm{CSP}(\mathcal{A})$ is a triple $(V, \delta, \mathcal{C})$, where V is a set of variables; $\delta : V \to \mathcal{A}$ is a *type function* that associates every variable with a *domain* in $\mathcal{A}$. Finally, $\mathcal{C}$ is a set of *constraints*, i.e. pairs $\langle \mathbf{s}, R \rangle$, where $\mathbf{s} = (v_1, \dots, v_k)$ is a tuple of variables from V, the *constraint scope*, and $R \in \mathsf{Inv}(\mathcal{A})$, a subset of $A_{\delta(v_1)} \times \dots \times A_{\delta(v_k)}$, the *constraint relation*. The goal is to find a *solution*, that is a mapping $\varphi : V \to \bigcup \mathcal{A}$ such that $\varphi(v) \in \mathbb{A}_{\delta(v)}$ and for every constraint $\langle \mathbf{s}, R \rangle$, $\varphi(\mathbf{s}) \in R$. It is easy to see that if $\mathcal{A}$ is a class containing just one algebra $\mathbb{A}$, then $\mathrm{CSP}(\mathcal{A})$ can be viewed as the union of $\mathrm{CSP}(\mathbf{A})$ for all relational structures $\mathbf{A}$ invariant under the operations of $\mathbb{A}$. To simplify the notation we always write $\mathbb{A}_v$ rather than $\mathbb{A}_{\delta(v)}$, because the mapping δ is always clear from the context. This also allows us to simplify the notation for instances to $\mathcal{P} = (V, \mathcal{C})$.

The set of solutions of a CSP instance $\mathcal{P} = (V, \mathcal{C})$ will be denoted by $\mathcal{S}_\mathcal{P}$, or just $\mathcal{S}$ if $\mathcal{P}$ is clear from the context. For $W \subseteq V$ by $\mathcal{P}_W$ we denote the *restriction* of $\mathcal{P}$ onto W, that is, the instance $(W, \mathcal{C}_W)$, where for each $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$, the set $\mathcal{C}_W$ includes the constraint $C_W = \langle \mathbf{s} \cap W, \mathrm{pr}_{\mathbf{s} \cap W} R \rangle$. The set of solutions of $\mathcal{P}_W$ will be denoted by $\mathcal{S}_W$. For $v \in V$ and a subalgebra $\mathbb{B}$ of $\mathbb{A}_v$ by $\mathcal{P}_{(v, \mathbb{B})}$ we denote the instance $\mathcal{P}$ with an extra constraint $\langle \{v\}, \mathbb{B} \rangle$; note that this is essentially equivalent to reducing the domain of v, and this is how we usually consider this construction. For $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ let R' be a subalgebra of R and $C' = \langle \mathbf{s}, R' \rangle$. The instance obtained from $\mathcal{P}$ replacing C with C' is denoted by $\mathcal{P}_{C \to C'}$. The transformation of $\mathcal{P}$ by reducing the domain of a variable $v \in V$ or reducing a constraint $C \in \mathcal{C}$, that is, transforming $\mathcal{P}$ into $\mathcal{P}_{(v, \mathbb{B})}$ or $\mathcal{P}_{C \to C'}$ in such a way that the new instance has a solution if and only if $\mathcal{P}$ does, will be called *tightening* of $\mathcal{P}$. Let α_v be a congruence of $\mathbb{A}_v$ for each $v \in V$. By $\mathcal{P}/_{\overline{\alpha}}$ we denote the instance $(V, \mathcal{C}^{\overline{\alpha}})$ constructed as follows: the domain of $v \in V$ is $\mathbb{A}_v/_{\alpha_v}$; for every constraint

$C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, the set $\mathcal{C}^{\overline{\alpha}}$ includes the constraint $\langle \mathbf{s}, R/_{\overline{\alpha}_\mathbf{s}}\rangle$.

Instance $\mathcal{P}$ is said to be *minimal* (or *globally minimal*) if for every $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ and every $\mathbf{a} \in R$ there is a solution $\varphi \in \mathcal{S}$ such that $\varphi(\mathbf{s}) = \mathbf{a}$. Instance $\mathcal{P}$ is said to be *1-minimal* if for every $v \in V$ and every constraint $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ such that $v \in \mathbf{s}$, $\mathrm{pr}_v R = \mathcal{S}_v$. Instance $\mathcal{P}$ is said to be *(2,3)-consistent* if it has a *(2,3)-strategy*, that is, a collection of relations R^X, $X \subseteq V$, $|X| = 2$ satisfying the following conditions:

– for every $X \subseteq V$ with $|X| \le 2$ and any constraint $C = \langle \mathbf{s}, R\rangle$ $\mathrm{pr}_{\mathbf{s}\cap X} R^X \subseteq \mathrm{pr}_{\mathbf{s}\cap X} R$;

– for every $X = \{u, v\} \subseteq V$, any $w \in V - X$ and any $(a, b) \in R^X$, there is $c \in \mathbb{A}_w$ such that $(a, c) \in R^{\{u,w\}}$ and $(b, c) \in R^{\{v,w\}}$.

We will always assume that a (2,3)-consistent instance has a constraint $C^X = \langle X, \mathcal{S}_X\rangle$ for every $X \subseteq V$, $|X| = 2$. Then clearly $R^X \subseteq \mathcal{S}_X$. Let the collection of relations R^X be denoted $\mathcal{R}$. A tuple $\mathbf{a}$ whose entries are indexed with elements of $W \subseteq V$ such that $\mathrm{pr}_X \mathbf{a} \in R^X$ for any $X \subseteq W$, $|X| = 2$, will be called $\mathcal{R}$-*compatible*. If a (2,3)-consistent instance $\mathcal{P}$ with a (2,3)-strategy $\mathcal{R}$ satisfies the additional condition

– for every constraint $C = \langle \mathbf{s}, R\rangle$ of $\mathcal{P}$ every tuple $\mathbf{a} \in R$ is $\mathcal{R}$-compatible,

it is called *(2,3)-minimal*. Any instance can be transformed to a 1-minimal, (2,3)-consistent, or (2,3)-minimal instance in polynomial time using the standard constraint propagation algorithms (see, e.g. [25]). These algorithms tighten the instance.

2.2 Coloured graphs

All the results in this subsection are stated for a single algebra $\mathbb{A}$. However, they can be readily generalized for any finite class of algebras $\mathcal{A}$. To do that one just needs to consider the product of all algebras in $\mathcal{A}$. Throughout the paper we apply results of this subsection to the class $\mathsf{HS}(\mathbb{A})$ for a finite algebra $\mathbb{A}$.

In [12, 22] we introduced a local approach to the structure of finite algebras. As we use this approach throughout the paper, we present it here in some details, see also [16]. For the sake of the definitions below we slightly abuse terminilogy and by a module mean the full idempotent reduct of a module.

For an algebra $\mathbb{A}$ graph $\mathcal{G}(\mathbb{A})$ is defined as follows. The vertex set is the universe A of $\mathbb{A}$. A pair ab of vertices is an *edge* iff there exists a congruence θ of $\mathsf{Sg}(a, b)$, other than the full congruence and a term operation f of $\mathbb{A}$ such that either $\mathsf{Sg}(a, b)/_\theta$ is a module and f is an affine operation on it, or f is a semilattice operation on $\{a^\theta, b^\theta\}$, or f is a majority operation on $\{a^\theta, b^\theta\}$. (Note that we use the same operation symbol in this case.) Usually, θ is chosen to be a maximal congruence of $\mathsf{Sg}(a, b)$.

If there are a congruence θ and a term operation f of $\mathbb{A}$ such that f is a semilattice operation on $\{a^\theta, b^\theta\}$ then ab is said to have the *semilattice type*. An edge ab is of *majority type* if there are a congruence θ and a term operation f such that f is a majority operation on $\{a^\theta, b^\theta\}$ and there is no semilattice term operation on $\{a^\theta, b^\theta\}$. Finally, ab has the *affine type* if there are θ and f such that f is an affine operation on $\mathsf{Sg}(a, b)/_\theta$ and $\mathsf{Sg}(a, b)/_\theta$ is a module; in particular it implies that there is no semilattice or majority operation on $\{a^\theta, b^\theta\}$. In all cases we say that congruence θ *witnesses* the type of edge ab. Observe that a pair ab can still be an edge of more than one type as witnessed by different congruences.

Omitting type **1** can be characterized as follows.

Theorem 3 ([12, 16]) *An idempotent algebra $\mathbb{A}$ omits type* **1** *iff $\mathcal{G}(\mathbb{B})$ is connected for every subalgebra $\mathbb{B}$ of $\mathbb{A}$.*

For the sake of the dichotomy conjecture, it suffices to consider *reducts* of an algebra $\mathbb{A}$ omitting type **1**, that is, algebras with the same universe but reduced set of term operations, as long as reducts also omit type **1**. In particular, we are interested in reducts of $\mathbb{A}$, in which semilattice and majority edges are subalgebras.

Theorem 4 ([12, 16]) *Let $\mathbb{A}$ be an algebra such that $\mathcal{G}(\mathbb{B})$ is connected for all subalgebras of $\mathbb{B}$ of $\mathbb{A}$, and let ab be an edge of $\mathcal{G}(\mathbb{A})$ of the semilattice or majority type witnessed by congruence θ, and $R_{ab} = a^\theta \cup b^\theta$. Let also F_{ab} denote set of term operations of $\mathbb{A}$ preserving R_{ab}, and $\mathbb{A}' = (A, F_{ab})$. Then $\mathcal{G}(\mathcal{B}')$ is cannected for all subalgebras $\mathbb{B}'$ of $\mathbb{A}'$.*

An algebra $\mathbb{A}$ such that $a^\theta \cup b^\theta$ is a subuniverse of $\mathbb{A}$ for every semilattice or majority edge ab of $\mathbb{A}$ is called *sm-smooth*. In the rest of the paper all algebras are assumed to be sm-smooth.

The next statement uniformizes the operations witnessing the type of edges.

Theorem 5 ([12, 16]) *Let $\mathbb{A}$ be an idempotent algebra. There are term operations f, g, h of $\mathbb{A}$ such that f is a semilattice operation on $\{a^\theta, b^\theta\}$ if ab is a semilattice edge; g is a majority operation on $\{a^\theta, b^\theta\}$ if ab is a majority edge; h is an affine operation operation on $\mathsf{Sg}(a, b)/_\theta$ if ab is an affine edge, where θ witnesses the type of the edge. Moreover, f, g, h can be chosen such that*

(1) $f(x, f(x, y)) = f(x, y)$ *for all* $x, y \in \mathbb{A}$*;*

(2) $g(x, g(x, y, y), g(x, y, y)) = g(x, y, y)$ *for all* $x, y \in \mathbb{A}$*;*

(3) $h(h(x, y, y), y, y) = h(x, y, y)$ *for all* $x, y \in \mathbb{A}$.

There is a term operation t such that for any affine edge ab and a majority, edge cd witnessed by congruences η and θ, respectively, $t(a,b) \stackrel{\eta}{\equiv} a$ and $t(c,d) \stackrel{\theta}{\equiv} d$.

Unlike majority and affine operations, for a semilattice edge ab and a congruence θ of $\mathsf{Sg}(a,b)$ witnessing that, there can be semilattice operations acting differently on $\{a^\theta, b^\theta\}$, which corresponds to the two possible orientations of ab. In every such case by fixing operation f from Theorem 5 we effectively choose one of the two orientations. In this paper we do not really care about what orientation is preferable.

In [16] we introduced a stronger notion of edge. A pair ab of elements of algebra $\mathbb{A}$ is called a *thin semilattice edge* if ab is a semilattice edge, and the congruence witnessing that is the equality relation. In other words, $f(a,a) = a$ and $f(a,b) = f(b,a) = f(b,b) = b$. We denote the fact that ab is a thin semilattice edge by $a \leq b$. Thin semilattice edges allow us to introduce a directed graph $\mathcal{G}_s(\mathbb{A})$, whose vertices are the elements of $\mathbb{A}$, and the arcs are the thin semilattice edges. We then can define *semilattice-connected* and *strongly semilattice-connected* components of $\mathcal{G}_s(\mathbb{A})$. We will also use the natural order on the set of strongly semilattice-connected components of $\mathcal{G}_s(\mathbb{A})$: for components A, B, we write $A \leq B$ if there is a directed path in $\mathcal{G}_s(\mathbb{A})$ connecting a vertex from A with a vertex from B. Elements from the maximal strongly connected components (or simply *maximal components*) of $\mathcal{G}_s(\mathbb{A})$ are called *maximal* elements of $\mathbb{A}$ and the set of all such elements is denoted by $\max(\mathbb{A})$. A directed path in $\mathcal{G}_s(\mathbb{A})$ is called a *semilattice path* or *s-path*. If there is an s-path from a to b we write $a \sqsubseteq b$.

Proposition 6 ([12, 16]) *Let $\mathbb{A}$ be an algebra. There is a binary term operation f of $\mathbb{A}$ such that f is a semilattice operation on $\{a^\theta, b^\theta\}$ for every semilattice edge ab of $\mathbb{A}$, where congruence θ witnesses that, and, for any $a, b \in \mathbb{A}$, either $a = f(a,b)$ or the pair $(a, f(a,b))$ is a thin semilattice edge of $\mathbb{A}$. Operation f with this property will be denoted by a dot (think multiplication).*

Let operatons g, h be as in Theorem 5. A pair ab from $\mathbb{A}$ is called a *thin majority edge* if (a) it is a majority edge, let congruence θ witness this, (b) for any $c \in b^\theta$, $b \in \mathsf{Sg}(a,c)$, (c) $g(a,b,b) = b$, and (d) there exists a ternary term operation g' such that $g'(a,b,b) = g'(b,a,b) = g'(b,b,a) = b$. Finally, a pair ab is called a *thin affine edge* if (a) it is an affine edge, let congruence θ witness this, (b) for any $c \in b^\theta$, $b \in \mathsf{Sg}(a,c)$, (c) $h(b,a,a) = b$, (d) there exists a ternary term operation h' such that $h'(b,a,a) = h'(a,a,b) = b$, and (e) a is maximal in $\mathsf{Sg}(a,b)$. Note that the operations h, g from Theorem 5 do not have to be majority or affine operations on thin edges; thin edges even do not have to be closed under g, h. Thin edges of all types are oriented. We therefore can define yet another directed graph, $\mathcal{G}'(\mathbb{A})$, in which the arcs are the thin edges of all types.

Lemma 7 ([16]) *Let $\mathbb{A}$ be an algebra.*

(1) Let ab be a semilattice or majority edge in $\mathbb{A}$, and θ the congruence of $\mathsf{Sg}(a,b)$ witnessing that. Then there is $b' \in b^\theta$ such that ab' is a thin semilattice or majority edge, respectively.

(2) Let ab be an affine edge, and θ the congruence of $\mathsf{Sg}(a,b)$ witnessing that. Then there are $a' \in a^\theta$ and $b' \in b^\theta$ such that $a \sqsubseteq a'$ in a^θ and $a'b'$ is a thin affine edge.

The following simple properties of thin edges will be useful. Note that a subdirect product of algebras (a relation) is also an algebra, and so edges and thin edges can be defined for relations as well.

Lemma 8 ([16]) *(1) Let $\mathbb{A}$ be an algebra and ab a thin edge. Then ab is a thin edge in any subalgebra of $\mathbb{A}$ containing a, b, and $a^\theta b^\theta$ is a thin edge in $\mathbb{A}/_\theta$ for any congruence θ.*

(2) Let R be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$, $I \subseteq [n]$, and $\mathbf{ab}$ a thin edge in R. Then $\mathrm{pr}_I\mathbf{a}\mathrm{pr}_I\mathbf{b}$ is a thin edge in $\mathrm{pr}_I R$ of the same type as $\mathbf{ab}$.

We will need stronger versions of Lemmas 18 and 20 of [16]. Let $\mathcal{A}$ be a finite class of algebras closed under taking subalgebras and factor-algebras.

Lemma 9 *(1) Let ab be a thin majority edge of algebra $\mathbb{A} \in \mathcal{A}$. There is a term operation t_{ab} such that $t_{ab}(a,b) = b$ and $t_{ab}(c,d) \stackrel{\theta_{cd}}{\equiv} c$ for all affine edges cd of $\mathbb{A}' \in \mathcal{A}$, where the type of cd is witnessed by congruence θ_{cd}.*

(2) Let ab be a thin affine edge of algebra $\mathbb{A} \in \mathcal{A}$. There is a term operation h_{ab} such that $h_{ab}(a,a,b) = b$ and $h_{ab}(d,c,c) \stackrel{\theta_{cd}}{\equiv} d$ for all affine edges cd of $\mathbb{A}' \in \mathcal{A}$, where the type of cd is witnessed by congruence θ_{cd}. Moreover, $h_{ab}(x,c',d')$ is a permutation of $\mathsf{Sg}(c,d)/_{\theta_{cd}}$ for any $c', d' \in \mathsf{Sg}(c,d)$.

(3) Let ab and cd be thin edges in algebras $\mathbb{A}, \mathbb{A}' \in \mathcal{A}$, respectively. If they have different types there is a binary term operation t such that $t(a,b) = a$, $t(c,d) = d$. If both edges are affine then there is a term operation h' such that $h'(a,a,b) = b$ and $h'(d,c,c) = d$.

Proof: (1) Let $c_1d_1, \dots, c_\ell d_\ell$ be a list of all affine edges of algebras in $\mathcal{A}$, $c_i, d_i \in \mathbb{A}_i$ and $\theta_{c_id_i}$ the corresponding congruences. Set $\mathbf{c} = (c_1, \dots, c_\ell), \mathbf{d} = (d_1, \dots, d_\ell)$. Let R be the subalgebra of $\mathbb{A} \times \prod_{i=1}^{\ell} \mathbb{A}_i$ generated by $(a, \mathbf{c}), (b, \mathbf{d})$. Pair ab is also a majority edge, let it be witnessed by a congruence θ. By Theorem 5

$$\begin{pmatrix} b' \\ \mathbf{c}' \end{pmatrix} = t\left(\begin{pmatrix} a \\ \mathbf{c} \end{pmatrix}, \begin{pmatrix} b \\ \mathbf{d} \end{pmatrix}\right) \in R,$$

where $b' \in b^\theta$ and $\mathbf{c}'[i] \stackrel{\theta_{c_i d_i}}{\equiv} \mathbf{c}[i]$, as t is the first projection on $\mathsf{Sg}(c_i, d_i)/_{\theta_{c_i d_i}}$ and a second projection on $\mathsf{Sg}(a, b)/_\theta$. Then as $b \in \mathsf{Sg}(a, b')$, we get $(b, \mathbf{c}'') \in R$ for some $\mathbf{c}''$ such that $\mathbf{c}''[i] \stackrel{\theta_{c_i d_i}}{\equiv} \mathbf{c}[i]$. The result follows.

(2) We use the notation from item (1) except ab now is a thin affine edge and R is generated by $(a, \mathbf{d}), (a, \mathbf{c}), (b, \mathbf{c})$. By condition (a) of the definition of thin affine edges,

$$\begin{pmatrix} b' \\ \mathbf{d}' \end{pmatrix} = h\left(\begin{pmatrix} a \\ \mathbf{d} \end{pmatrix}, \begin{pmatrix} a \\ \mathbf{c} \end{pmatrix}, \begin{pmatrix} b \\ \mathbf{c} \end{pmatrix}\right) \in R,$$

where $b' \in b^\theta$ and $\mathbf{d}'[i] \stackrel{\theta_{c_i d_i}}{\equiv} \mathbf{d}[i]$, as h is a Mal'tsev operation on $\mathsf{Sg}(a, b)/_\theta$ and on $\mathsf{Sg}(c_i, d_i)/_{\theta_{c_i d_i}}$. Then as $b \in \mathsf{Sg}(a, b')$, by condition (b) we get $(b, \mathbf{d}'') \in R$ for some $\mathbf{d}''$ such that $\mathbf{d}''[i] \stackrel{\theta_{c_i d_i}}{\equiv} \mathbf{d}[i]$. The first result follows.

Let now h_{ab} be the term operation we constructed and $c', d' \in \mathsf{Sg}(c_i, d_i)$, $i \in [\ell]$. Since $\mathbb{B} = \mathsf{Sg}(c_i, d_i)/_{\theta_{c_i d_i}}$ is a module, in particular, it is an Abelian algebra and $h_{ab}(x, c^*, c^*) = x$ for all $c^* \in \mathbb{B}$, the second result follows.

(3) Follows from [16], Lemmas 15,18,19,20. □

2.3 Maximality

A directed path in $\mathcal{G}'(\mathbb{A})$ is called an *asm-path*, if there is an asm-path from a to b we write $a \sqsubseteq_{asm} b$. If all edges of this path are semilattice or affine, it is called an *affine-semilattice path* or an *as-path*, if there is an as-path from a to b we write $a \sqsubseteq_{as} b$. Similar to maximal components, we consider strongly connected components of $\mathcal{G}'(\mathbb{A})$ with majority edges removed, and the natural partial order on such components. The maximal components will be called *as-components*, and the elements from as-components are called *as-maximal*; the set of all as-maximal elements of $\mathbb{A}$ is denoted by $\mathsf{amax}(\mathbb{A})$. If a is an as-maximal element, the as-component containing a is denoted $\mathsf{as}(a)$. An alternative way to define as-maximal elements is as follows: a is as-maximal if for every $b \in \mathbb{A}$ such that $a \sqsubseteq_{as} b$ it also holds that $b \sqsubseteq_{as} a$. Finally, element $a \in \mathbb{A}$ is said to be *universally maximal* (or *u-maxmal* for short) if for every $b \in \mathbb{A}$ such that $a \sqsubseteq_{asm} b$ it also holds that $b \sqsubseteq_{asm} a$. The set of all u-maximal elements of $\mathbb{A}$ is denoted $\mathsf{umax}(\mathbb{A})$.

Proposition 10 ([16]) *Let $\mathbb{A}$ be an algebra. Then*

(1) any $a, b \in \mathbb{A}$ are connected in $\mathcal{G}'(\mathbb{A})$ with an undirected path;

(2) any $a, b \in \max(\mathbb{A})$ (or $a, b \in \mathsf{amax}(\mathbb{A})$, or $a, b \in \mathsf{umax}(\mathbb{A})$) are connected in $\mathcal{G}'(\mathbb{A})$ with a directed path.

Proof: Item (2) is only proved in [16] for maximal and as-maximal elements; so we prove it here for u-maximal elements as well. Let $a', b' \in \mathbb{A}$ be maximal elements of $\mathbb{A}$ such that $a \sqsubseteq a'$ and $b \sqsubseteq b'$. Then by Proposition 10 for maximal elements $a' \sqsubseteq_{asm} b'$, and, as b is u-maximal, $b' \sqsubseteq_{asm} b$. $\square$

Since for every $a \in \mathbb{A}$ there is a maximal $a' \in \mathbb{A}$ such that $a \sqsubseteq a'$, Proposition 10 implies that there is only one u-maximal component. U-maximality has an additional useful property, it is somewhat hereditary, as it made precise in the following

Lemma 11 *Let B be a subalgebra containing a u-maximal element. Then every element u-maximal in B is also u-maximal in $\mathbb{A}$. In particular, if α is a congruence of $\mathbb{A}$ and B is a u-maximal α-block, then $\mathsf{umax}(B) \subseteq \mathsf{umax}(\mathbb{A})$.*

Proof: Let $a \in B$ be an element u-maximal in $\mathbb{A}$, let $b \in \mathsf{umax}(B)$. For any $c \in \mathbb{A}$ with $b \sqsubseteq_{asm} c$ we also have $c \sqsubseteq_{asm} a$. Finally, since $b \in \mathsf{umax}(B)$ and $a \in B$, we have $a \sqsubseteq_{asm} b$. For the second part of the lemma we need to find a u-maximal element in B. Let $b \in \mathsf{umax}(\mathbb{A})$. Then as B is u-maximal in $\mathbb{A}/_\alpha$ applying Lemma 7 we get that there is $a' \in B$ such that $b \sqsubseteq_{asm} a'$. Clearly, $a' \in \mathsf{umax}(\mathbb{A})$. $\square$

Let $\mathbb{A}$ be an algebra and $a \in \mathbb{A}$. By $\mathrm{Ft}_\mathbb{A}(a)$ we denote the set of elements a is connected to (in terms of semilattice paths); similarly, by $\mathrm{Ft}^{as}_\mathbb{A}(a)$ and $\mathrm{Ft}^{asm}_\mathbb{A}(a)$ we denote the set of elements a is as-connected and asm-connected to. Also, $\mathrm{Ft}_\mathbb{A}(C) = \bigcup_{a\in C} \mathrm{Ft}_\mathbb{A}(a)$ ($\mathrm{Ft}^{as}_\mathbb{A}(C) = \bigcup_{a\in C} \mathrm{Ft}^{as}_\mathbb{A}(a), \mathrm{Ft}^{asm}_\mathbb{A}(C) = \bigcup_{a\in C} \mathrm{Ft}^{asm}_\mathbb{A}(a)$, respectively) for $C \subseteq \mathbb{A}$. Note that if a is an as-maximal element then $\mathsf{as}(a) = \mathrm{Ft}^{as}_\mathbb{A}(a)$, and $a \in \mathrm{Ft}^{asm}_\mathbb{A}(b)$ for any $b \in \mathbb{A}$. We will need the following statements.

Lemma 12 ([17]) *Let R be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $I \subseteq [n]$.*

(1) For any $\mathbf{a} \in R$, $\mathbf{b} \in \mathrm{pr}_I R$ with $\mathrm{pr}_I \mathbf{a} \le \mathbf{b}$, there is $\mathbf{b}' \in R$ such that $\mathbf{a} \le \mathbf{b}'$ and $\mathrm{pr}_I \mathbf{b}' = \mathbf{b}$.

(2) For any $\mathbf{a} \in R$, $\mathbf{b} \in \mathrm{pr}_I R$ such that $\mathrm{pr}_I \mathbf{ab}$ is a thin majority edge there is $\mathbf{b}' \in R$ such that $\mathbf{ab}'$ is a thin majority edge, and $\mathrm{pr}_I \mathbf{b}' = \mathbf{b}$.

(3) For any $\mathbf{a} \in R$, $\mathbf{b} \in \mathrm{pr}_I R$ such that $\mathrm{pr}_I \mathbf{ab}$ is a thin affine edge there are $\mathbf{a}', \mathbf{b}' \in R$ such that $\mathbf{a} \sqsubseteq \mathbf{a}'$, $\mathbf{a}'\mathbf{b}'$ is a thin affine edge, and $\mathrm{pr}_I \mathbf{a}' = \mathrm{pr}_I \mathbf{a}$, $\mathrm{pr}_I \mathbf{b}' = \mathbf{b}$.

(4) For any $\mathbf{a} \in R$, and an s-path (as-path, asm-path) $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \mathrm{pr}_I R$ with $\mathrm{pr}_I \mathbf{a} = \mathbf{b}_1$, there is an s-path (as-path,asm-path, respectively) $\mathbf{b}'_1, \ldots, \mathbf{b}'_\ell \in R$ such that $\mathrm{pr}_I \mathbf{b}'_\ell = \mathbf{b}_\ell$.

(5) For any $\mathbf{b} \in \max(\mathrm{pr}_I R)$ ($\mathbf{b} \in \mathsf{amax}(\mathrm{pr}_I R)$, $\mathbf{b} \in \mathsf{umax}(\mathrm{pr}_I R)$) there is $\mathbf{b}' \in \max(R)$ ($\mathbf{b}' \in \mathsf{amax}(R)$, $\mathbf{b}' \in \mathsf{umax}(R)$, respectively), such that $\mathrm{pr}_I \mathbf{b}' =$

$\mathbf{b}$. *In particular,* $\mathrm{pr}_{[n]-I}\mathbf{b}' \in \max(\mathrm{pr}_{[n]-I}R)$ *(*$\mathrm{pr}_{[n]-I}\mathbf{b}' \in \mathsf{amax}(\mathrm{pr}_{[n]-I}R)$*,* $\mathrm{pr}_{[n]-I}\mathbf{b}' \in \mathsf{umax}(\mathrm{pr}_{[n]-I}R)$*, respectively).*

Proof: Items (1) and (3) are proved in [17], and the parts of (4) and (5) are only proved for s- and as-paths, and, respectively, for maximal and as-maximal elements. Items (4) and (5) for asm-paths and u-maximal elements follow from (2).

(2) Observe that it suffices to consider binary relations R. Indeed, R can be viewed as a subdirect product of $\mathrm{pr}_I R \times \mathrm{pr}_{[n]-I} R$. So, suppose $n = 2$ and $I = \{1\}$. We have $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = b_1$. Let θ be a maximal congruence of $\mathsf{Sg}(a_1, b_1)$ witnessing that $a_1 b_1$ is an majority edge. Choose $\mathbf{b}'' = (b_1', b_2) \in R$ such that $b_1' \in b_1^\theta$ and $R' = \mathsf{Sg}(\mathbf{a}, \mathbf{b}'')$ is minimal possible with this condition. It suffices to prove the lemma for R', since $b_1 \in \mathsf{Sg}(a_1, b_1')$ and so $b_1 \in \mathrm{pr}_1 R'$, and $a_1 b_1$ is still a thin majority edge. This means that (b_1', b_2) can be chosen such that $b_1' = b_1$. Also, by taking $\begin{pmatrix} b_1 \\ b_2' \end{pmatrix} = g\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right)$ we may assume by Theorem 5 that $g(a_2, b_2, b_2) = b_2$. As is easily seen, the pair $(a_1, a_2)(b_1, b_2)$ is a majority edge as witnessed by congruence $\theta' = \theta \times \underline{1}_{\mathbb{A}_2'}$ where $\mathbb{A}_2' = \mathsf{Sg}(a_2, b_2)$. By the choice of $\mathbf{b}''$ he pair (b_1, b_2) belongs to $\mathsf{Sg}((a_1, a_2), (c_1, c_2))$ for any $(c_1, c_2) \in (b_1, b_2)^{\theta'}$, and it only remains to prove condition (d) of the definition of thin majority edges.

Let g' be the operation from condition (d) for $a_1 b_1$. Then

$$g'\left(\begin{pmatrix} (a_1, a_2) \\ (b_1, b_2) \\ (b_1, b_2) \end{pmatrix}, \begin{pmatrix} (b_1, b_2) \\ (a_1, a_2) \\ (b_1, b_2) \end{pmatrix}, \begin{pmatrix} (b_1, b_2) \\ (b_1, b_2) \\ (a_1, a_2) \end{pmatrix}\right) = \begin{pmatrix} (b_1, b_2') \\ (b_1, b_2'') \\ (b_1, b_2''') \end{pmatrix}.$$

Since $b_1 \in \mathsf{Sg}(a_1, b_1')$, there is a term operation r_1 such that

$$r_1\left(\begin{pmatrix} (a_1, a_2) \\ (b_1, b_2) \\ (b_1, b_2) \end{pmatrix}, \begin{pmatrix} (b_1, b_2') \\ (b_1, b_2'') \\ (b_1, b_2''') \end{pmatrix}\right) = \begin{pmatrix} (b_1, b_2) \\ (b_1, b_2^*) \\ (b_1, b_2^{**}) \end{pmatrix}.$$

Repeating this for the second and third coordinate positions we obtain a ternary operation g'' such that

$$\begin{aligned} g''\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) &= g''\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \\ &= g''\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \end{aligned}$$

confirming property (d). □

We complete this section with an auxiliary statement that will be needed later.

Lemma 13 *Let $\alpha \prec \beta$, $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, let B a β-block and* $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$. *Then $B/_\alpha$ is term equivalent to a module. In particular, every pair of elements of $B/_\alpha$ is a thin affine edge in $\mathbb{A}/_\alpha$.*

Proof: As $\mathbb{A}$ is an idempotent algebra that generates a variety omitting type $\mathbf{1}$, and (α, β) is a simple interval in $\mathsf{Con}(\mathbb{A})$ of type $\mathbf{2}$, by Theorem 7.11 of [29] there is a term operation of $\mathbb{A}$ that is Mal'tsev on $B/_\alpha$. Since β is Abelian on $B/_\alpha$, we get the result. □

2.4 Quasi-decomposition and quasi-majority

We make use of the property of quasi-2-decomposability proved in [17].

Theorem 14 ([17]) *If R is an n-ary relation, $X \subseteq [n]$, tuple $\mathbf{a}$ is such that $\mathrm{pr}_J \mathbf{a} \in \mathrm{pr}_J R$ for any $J \subseteq [n]$, $|J| = 2$, and $\mathrm{pr}_X \mathbf{a} \in \mathsf{amax}(\mathrm{pr}_X R)$, there is a tuple $\mathbf{b} \in R$ with $\mathrm{pr}_J \mathbf{b} \in \mathrm{Ft}^{as}_{\mathrm{pr}_J R}(\mathrm{pr}_J \mathbf{a})$ for any $J \subseteq [n]$, $|J| = 2$, and $\mathrm{pr}_X \mathbf{b} = \mathrm{pr}_X \mathbf{a}$.*

One useful implication of Theorem 14 is the existence of term operation resembling a majority function. We state this theorem for finite classes of algebras rather than a single algebra, because it concerns as-components that in subalgebras of products may have complicated structure.

Theorem 15 *Let $\mathcal{A}$ be a finite class of finite similar sm-smooth algebras omitting type $\mathbf{1}$. There is a term operation maj of $\mathcal{A}$ such that for any $\mathbb{A} \in \mathcal{A}$ and any $a, b \in \mathbb{A}$, $\mathsf{maj}(a, a, b), \mathsf{maj}(a, b, a), \mathsf{maj}(b, a, a) \in \mathrm{Ft}^{as}_{\mathbb{A}}(a)$.*

In particular, if a is as-maximal, then $\mathsf{maj}(a, a, b), \mathsf{maj}(a, b, a), \mathsf{maj}(b, a, a)$ belong to the as-component of $\mathbb{A}$ containing a.

Proof: Let $\{a_1, b_1\}, \dots, \{a_n, b_n\}$ be a list of all pairs of elements from algebras of $\mathcal{A}$, let $a_i, b_i \in \mathbb{A}_i$. Define relation R to be a subdirect product of $\mathbb{A}_1^3 \times \dots \times \mathbb{A}_n^3$ generated by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, where for every $i \in [n]$, $\mathrm{pr}_{3i-2,3i-1,3i} \mathbf{a}_1 = (a_i, a_i, b_i)$, $\mathrm{pr}_{3i-2,3i-1,3i} \mathbf{a}_2 = (a_i, b_i, a_i)$, $\mathrm{pr}_{3i-2,3i-1,3i} \mathbf{a}_3 = (b_i, a_i, a_i)$. In other words the triples $(\mathbf{a}_1[3i-2], \mathbf{a}_2[3i-2], \mathbf{a}_3[3i-2])$, $(\mathbf{a}_1[3i-1], \mathbf{a}_2[3i-1], \mathbf{a}_3[3i-1])$, $(\mathbf{a}_1[3i], \mathbf{a}_2[3i], \mathbf{a}_3[3i])$ have the form $(a_i, a_i, b_i), (a_i, b_i, a_i), (b_i, a_i, a_i)$, respectively. Therefore it suffices to show that R contains a tuple $\mathbf{b}$ such that $a_i \sqsubseteq_{as} \mathbf{b}[j]$, where $j \in \{3i, 3i-1, 3i-2\}$. However, since $(a_{i_1}, a_{i_2}) \in \mathrm{pr}_{j_1 j_2} R$ for any $i_1, i_2 \in [n]$ and $j_1 \in \{3i_1, 3i_1 - 1, 3i_1 - 2\}$, $j_2 \in \{3i_2, 3i_2 - 1, 3i_2 - 2\}$, this follows from Theorem 14. □

A function maj satisfying the properties from Theorem 15 will be called a *quasi-majority function*.

2.5 Rectangularity

Let R be a subdirect product of $\mathbb{A}_1, \mathbb{A}_2$. By $R[c], R^{-1}[c']$ for $c \in \mathbb{A}_1, c' \in \mathbb{A}_2$ we denote the sets $\{b \mid (c,b) \in R\}, \{a \mid (a,c) \in R\}$, respectively, and for $C \subseteq \mathbb{A}_1, C' \subseteq \mathbb{A}'_1$ we use $R[C] = \bigcup_{c \in C} R[c]$, $R^{-1}[C'] = \bigcup_{c' \in C'} R^{-1}[c']$, respectively. Binary relations $\mathsf{tol}_1, \mathsf{tol}_2$ on $\mathbb{A}_1, \mathbb{A}_2$ given by $\mathsf{tol}_1(R) = \{(a,b) \mid R[a] \cap R[b] \neq \varnothing\}$ and $\mathsf{tol}_2(R) = \{(a,b) \mid R^{-1}[a] \cap R^{-1}[b] \neq \varnothing\}$, respectively, are called *link tolerances* of R. They are tolerances of $\mathbb{A}_1$, $\mathbb{A}_2$, respectively, that is invariant reflexive and symmetric relations. The transitive closures $\mathsf{lk}_1, \mathsf{lk}_2$ of $\mathsf{tol}_1(R), \mathsf{tol}_2(R)$ are called *link congruences*, and they are, indeed, congruences. Relation R is said to be *linked* if the link congruences are full congruences.

Lemma 16 ([17]) *Let R be a subalgebra of $\mathbb{A}_1 \times \mathbb{A}_2$ and let $a \in \mathbb{A}_1$ and $B = R[a]$. For any $b \in \mathbb{A}_1$ such that ab is thin edge, and any $c \in R[b] \cap B$, $\mathrm{Ft}^{as}_B(c) \subseteq R[b]$.*

Proof: The case when $a \leq b$ or ab is affine is considered in [17], so suppose that ab is majority. Let $D \subseteq \mathrm{Ft}^{as}_B(c)$ such that $D \subseteq R[b]$. Set D is nonempty, as $c \in D$. If $D \neq \mathrm{Ft}^{as}_B(c)$, there are $b_1 \in D$ and $b_2 \in \mathrm{Ft}^{as}_B(c) - D$ such that $b_1 b_2$ is a thin edge. By Lemma 9 there is a term operation t such that $t(a,b) = b$ and $t(b_2, b_1) = b_2$. Then $\begin{pmatrix} b \\ b_2 \end{pmatrix} = t\left(\begin{pmatrix} a \\ b_2 \end{pmatrix}, \begin{pmatrix} b \\ b_1 \end{pmatrix}\right) \in R$. The result follows. □

Proposition 17 ([17]) *Let $R \leq \mathbb{A}_1 \times \mathbb{A}_2$ be a linked subdirect product and let B_1, B_2 be as-components of $\mathbb{A}_1, \mathbb{A}_2$, respectively, such that $R \cap (B_1 \times B_2) \neq \varnothing$. Then $B_1 \times B_2 \subseteq R$.*

Corollary 18 *Let R be a subdirect product of $\mathbb{A}_1$ and $\mathbb{A}_2$, $\mathsf{lk}_1, \mathsf{lk}_2$ the link congruences, and let B_1, B_2 be as-components of a lk_1-block and a lk_2-block, respectively, such that $R \cap (B_1 \times B_2) \neq \varnothing$. Then $B_1 \times B_2 \subseteq R$.*

Proposition 19 *Let R be a subdirect product of $\mathbb{A}_1$ and $\mathbb{A}_2$, $\mathsf{lk}_1, \mathsf{lk}_2$ the link congruences, and let B_1 be an as-component of a lk_1-block and $B'_2 = R[B_1]$; let $B_2 = \mathsf{umax}(B'_2)$. Then $B_1 \times B_2 \subseteq R$.*

Proof: Let B'_2 be a subset of a lk_2-block C. By Lemma 12(5) B'_2 contains an as-maximal element a of C. By Corollary 18 $B_1 \times \{a\} \subseteq R$. It then suffices to show that $B_1 \times \mathrm{Ft}^{asm}_{B'_2}(a) \subseteq R$.

Suppose for $D \subseteq \mathrm{Ft}^{asm}_{B'_2}(a)$ it holds $B_1 \times D \subseteq R$. If $D \neq \mathrm{Ft}^{asm}_{B'_2}(a)$, there are $b_1 \in D$ and $b_2 \in \mathrm{Ft}^{asm}_{B'_2}(a) - D$ such that $b_1 b_2$ is a thin edge. By Lemma 16 $B_1 \times \{b_2\} \subseteq R$; the result follows. □

3 Separating congruences

In this section we introduce and study the relationship between prime intervals in the congruence lattice of an algebra, or in congruence lattices of factors in a subdirect products. It was first introduced in [9] and used in the CSP research in [18].

3.1 Special polynomials, mapping pairs

We start with several technical results. They demonstrate the connection between minimal sets of an algebra $\mathbb{A}$ and the structure its graph $\mathcal{G}'(\mathbb{A})$. Let $\mathbb{A}$ be an algebra and let $Q^{\mathbb{A}}_{ab}$, $a, b \in \mathbb{A}$, denote the subdirect product of $\mathbb{A}^2$ generated by $\{(x,x) \mid x \in \mathbb{A}\} \cup \{(a,b)\}$.

Lemma 20 *(1)* $Q^{\mathbb{A}}_{ab} = \{(f(a), f(b)) \mid f \in \mathsf{Pol}_1(\mathbb{A})\}$.
(2) For any $f \in \mathsf{Pol}_1(\mathbb{A})$, $(f(a), f(b)) \in \mathsf{tol}_1(Q^{\mathbb{A}}_{ab})$. *In particular,* $\mathsf{lk}_1(Q^{\mathbb{A}}_{ab}) = \mathsf{Cg}(a,b)$*; denote this congruence by* α.
(3) $Q^{\mathbb{A}}_{ab} \subseteq \mathsf{Cg}(a,b)$.
(4) Let B_1, B_2 *be* α*-blocks, and* C_1, C_2 *as-components of* B_1, B_2 *respectively such that* $f(a) \in C_1$ *and* $f(b) \in C_2$ *for a polynomial* f*. Then* $C_1 \times C_2 \subseteq Q^{\mathbb{A}}_{ab}$.

Proof: (1) follows directly from the definitions.

(2) Take $f \in \mathsf{Pol}_1(\mathbb{A})$ and let $f(x) = g(x, a_1, \ldots, a_k)$ for a term operation g of $\mathbb{A}$. Then $\begin{pmatrix} f(a) \\ f(b) \end{pmatrix} = g\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_k \\ a_k \end{pmatrix}\right) \in R$ and $\begin{pmatrix} f(b) \\ f(b) \end{pmatrix} = g\left(\begin{pmatrix} b \\ b \end{pmatrix}, \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_k \\ a_k \end{pmatrix}\right) \in R$. Thus $(f(a), f(b)) \in \mathsf{tol}_1(Q^{\mathbb{A}}_{ab})$.

(3) follows from (1), and (4) follows from (2),(3), and Corollary 18. □

We say that a is *α-maximal* for a congruence $\alpha \in \mathsf{Con}(\mathbb{A})$ if a is as-maximal in the subalgebra a^α.

Corollary 21 *Let* $\alpha \in \mathsf{Con}(\mathbb{A})$ *and* $\underline{0} \prec \alpha$*. Then for any* $a, b \in \mathbb{A}$ *with* $a \stackrel{\alpha}{\equiv} b$ *and any* α*-maximal* $c, d \in \mathbb{A}$, $c \neq d$*, with* $c \stackrel{\alpha}{\equiv} d$*, belonging to the same as-component of* c^α*, there is* $f \in \mathsf{Pol}_1(\mathbb{A})$ *such that* $c = f(a)$, $d = f(b)$.

Proof: The result follows from Lemma 20(4). □

Recall that for $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ with $\alpha \prec \beta$ a pair $\{a, b\}$ is called an (α,β)-*subtrace* if $(a,b) \in \beta - \alpha$ and $a, b \in U$ for some (α,β)-minimal set U.

Corollary 22 *Let* $\alpha \in \mathsf{Con}(\mathbb{A})$ *and* $\underline{0} \prec \alpha$*, and let* $c, d \in \mathbb{A}$, $c \stackrel{\alpha}{\equiv} d$*, be* α*-maximal.*

(1) If c, d belong to the same as-component of c^α, then $\{c, d\}$ is a $(\underline{0}, \alpha)$-subtrace.

(2) If there is a $(\underline{0}, \alpha)$-subtrace $\{c', d'\}$ such that $c' \in \mathsf{as}(c)$ and $d' \in \mathsf{as}(d)$ then $\{c, d\}$ is a $(\underline{0}, \alpha)$-subtrace as well.

Proof: (1) Take any $(\underline{0}, \alpha)$-minimal set U, $U = g(\mathbb{A})$, g idempotent, and $a, b \in U$ with $a \stackrel{\alpha}{\equiv} b$. By Corollary 21 there is $f \in \mathsf{Pol}_1(\mathbb{A})$ with $c = f(a)$, $d = f(b)$. Since $U' = f \circ g(\mathbb{A}) = f \circ g(U)$ and $f \circ g(\alpha) \not\subseteq \underline{0}$, the set U' is a $(\underline{0}, \alpha)$-minimal set.

(2) As in (1) one can argue that $(c', d') \in Q^{\mathbb{A}}_{ab}$, that is, $Q^{\mathbb{A}}_{ab} \cap (\mathsf{as}(c) \times \mathsf{as}(d)) \neq \varnothing$. We then complete by Lemma 20(4). □

Lemma 23 *For any $\alpha \in \mathsf{Con}(\mathbb{A})$ with $\underline{0} \prec \alpha$ such that $|D| > 1$ for some as-component D of an α-block, the prime factor $\underline{0} \prec \alpha$ has type* **2** *or* **3***.*

Proof: Let $a, b \in D$ for an as-component D of an α-block. Then by Corollary 21 there is a polynomial f such that $f(a) = b$ and $f(b) = a$. Also, a, b belong to some $(\underline{0}, \alpha)$-minimal set. This rules out types **4** and **5**. Since $\mathbb{A}$ omits type **1**, this only leaves types **2** and **3**. □

Lemma 24 *Let $\alpha \in \mathsf{Con}(\mathbb{A})$ with $\underline{0} \prec \alpha$ be such that some α-block contains a semilattice or majority edge. Then the prime factor $(\underline{0}, \alpha)$ has type* **3***,* **4** *or* **5***.*

Proof: We need to show that $(\underline{0}, \alpha)$ does not have type **2**. Let B the α-block containing a semilattice or majority edge. Then B contains a non-Abelian subalgebra, which implies $(\underline{0}, \alpha)$ is also non-Abelian. □

3.2 Separation

Let $\mathbb{A}$ be an algebra, and let $\alpha \prec \beta$ and $\gamma \prec \delta$ be prime intervals in $\mathsf{Con}(\mathbb{A})$. We say that $\alpha \prec \beta$ can be *separated* from $\gamma \prec \delta$ if there is a unary polynomial $f \in \mathsf{Pol}_1(\mathbb{A})$ such that $f(\beta) \not\subseteq \alpha$, but $f(\delta) \subseteq \gamma$. The polynomial f in this case is said to *separate* $\alpha \prec \beta$ from $\gamma \prec \delta$.

Since we often consider relations rather than single algebras, we also introduce separability in a slightly different way. Let R be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$. Let $i, j \in [n]$ and let $\alpha_i \prec \beta_i$, $\alpha_j \prec \beta_j$ be prime intervals in $\mathsf{Con}(\mathbb{A}_i)$ and $\mathsf{Con}(\mathbb{A}_j)$, respectively. Interval $\alpha_i \prec \beta_i$ can be separated from $\alpha_j \prec \beta_j$ if there is a unary polynomial f of R such that $f(\beta_i) \not\subseteq \alpha_i$ but $f(\beta_j) \subseteq \alpha_j$. Similarly, the polynomial f in this case is said to *separate* $\alpha_i \prec \beta_i$ from $\alpha_j \prec \beta_j$

First, we observe a connection between separation in a single algebra and in relations.

Lemma 25 *Let R be the binary equality relation on $\mathbb{A}$. Prime interval $\alpha \prec \beta$ can be separated from $\gamma \prec \delta$ as intervals in $\mathsf{Con}(\mathbb{A})$ if and only if $\alpha \prec \beta$ can be separated from $\gamma \prec \delta$ in R.*

Proof: Note that for any polynomial f its action on the first and second projections of R are the same polynomial of $\mathbb{A}$. Therefore $\alpha \prec \beta$ can be separated from $\gamma \prec \delta$ in $\mathsf{Con}(\mathbb{A})$ if and only if, there is $f \in \mathsf{Pol}_1(\mathbb{A})$, $f(\beta) \not\subseteq \alpha$ while $f(\delta) \subseteq \gamma$. This condition can be expressed as follows: there is $f \in \mathsf{Pol}_1(R)$, $f(\beta) \not\subseteq \alpha$ while $f(\delta) \subseteq \gamma$, which precisely means $\alpha \prec \beta$ cannot be separated from $\gamma \prec \delta$ in R. $\square$

In what follows when proving results about separation we will always assume that we deal with a relation — a subdirect product — and that the prime intervals in question are from congruence lattices of different factors of the subdirect product. If this is not the case, one can duplicate the factor containing the prime intervals and apply Lemma 25.

Let R be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $I \subseteq [n]$, and let f be a polynomial of $\mathrm{pr}_I R$, that is, there are a term operation g of R and $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathrm{pr}_I R$ such that $f(x_1, \ldots, x_\ell) = g(x_1, \ldots, x_\ell, \mathbf{a}_1, \ldots, \mathbf{a}_k)$. The tuples $\mathbf{a}_i$ can be extended to tuples $\mathbf{a}'_i \in R$. Then the polynomial of R given by $f(x_1, \ldots, x_\ell) = g(x_1, \ldots, x_\ell, \mathbf{a}'_1, \ldots, \mathbf{a}'_k)$ is said to be an *extension* of f to a polynomial of R.

Lemma 26 *Let R be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $i, j \in [n]$, and $\alpha_i \prec \beta_i$, $\alpha_j \prec \beta_j$ for $\alpha_i, \beta_i \in \mathsf{Con}(\mathbb{A}_i)$, $\alpha_j, \beta_j \in \mathsf{Con}(\mathbb{A}_j)$. Let also a unary polynomial f of R separates $\alpha_i \prec \beta_i$ from $\alpha_j \prec \beta_j$. Then f can be chosen idempotent and such that $f(\mathbb{A}_i)$ is a (α_i, β_i)-minimal set.*

Proof: Let g be a polynomial separating i from j. Since $g(\beta_i) \not\subseteq \alpha_i$, there is a (α_i, β_i)-minimal set U such that $g(\beta_i|_U) \not\subseteq \alpha_i$. Let $V = g(U)$, clearly, V is a (α_i, β_i)-minimal set. Let h be a unary polynomial such that h maps V onto U and $h \circ g|_U$ is the identity mapping. Let also h' be an extension of h to a polynomial of R. Then $h' \circ g$ separates i from j. Now f can be chosen to be an appropriate power of $h' \circ g$. $\square$

For a subdirect product $R \subseteq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ the relation 'cannot be separated' on prime quotients of the $\mathbb{A}_i$s is clearly reflexive and transitive. If the algebras $\mathbb{A}_i$ are Mal'tsev, it is also symmetric (for partial results see [9, 18]). Moreover, it can be shown that it remains symmetric when the $\mathbb{A}_i$s contain no majority edges. In the general case however the situation is more complicated. Next we introduce conditions that make the 'cannot be separated' relation to some extent symmetric, as it will be demonstrated in Lemma 29.

Let $\beta_i \in \mathsf{Con}(\mathbb{A}_i)$ and let B_i be a β_i-block for $i \in [n]$. Let also $\mathcal{U}$ be a set of unary polynomials of R, $i \in [n]$, and $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_i)$ with $\alpha \prec \beta \le \beta_i$. Let

$T_{\mathbb{A}_i}(a',b';\alpha,\beta,\overline{\beta},\overline{B},\mathcal{U}) \subseteq \beta/_\alpha$ for $a',b' \in B'_i/_\alpha$, $(a',b') \in \beta-\alpha$, denote the set of pairs $(a,b) \in \beta/_\alpha$ such that there is a polynomial $g \in \mathcal{U}$ satisfying the following conditions: $g(\{a',b'\}) = \{a,b\}$ and $g(\mathbb{A}_i)$ is a (α,β)-minimal set. Note that these conditions imply that $\{a,b\}$ is a (α,β)-subtrace. We say that α and β are *$\mathcal{U}$-chained* in R if for any $a',b' \in B'_i/_\alpha$ with $(a',b') \in \beta-\alpha$,

(G1) For a $\beta/_\alpha$-block E let $E' = E\cap \mathrm{pr}_i(R\cap\overline{B})/_\alpha$. Then $(a,b) \in T_{\mathbb{A}_i}(a',b';\alpha,\beta,\overline{\beta},\overline{B},\mathcal{U})$ for any a,b from the same as-component of E'.

(G2) For any $\beta/_\alpha$-block E, and any $a,b \in \mathsf{umax}(E')$, where $E' = E \cap \mathrm{pr}_i(R \cap \overline{B})/_\alpha$, there is a sequence $a = a_1,\dots,a_k = b$ in E' such that $\{a_i,a_{i+1}\} \in T_{\mathbb{A}_i}(a',b';\alpha,\beta,\overline{\beta},\overline{B},\mathcal{U})$ for any $i \in [k-1]$.

Also, if for elements a,b and any a',b' there is a sequence of elements satisfying (G1),(G2), we say that a and b are *subtrace connected*; congruences $\alpha,\beta,\overline{\beta}$, congruence classes $\overline{B}$, and set of polynomials $\mathcal{U}$ will always be clear from the context in this case. Observe that $\mathcal{U}$-chaining amounts to saying that polynomials from $\mathcal{U}$ do not allow any congruences of β-blocks viewed as subalgebras between α and β, at least where u-maximal elements are concerned.

A unary polynomial f is said to be *$\overline{B}$-preserving* if $f(\overline{B}) \subseteq \overline{B}$. We call relation R *chained* with respect to $\overline{\beta},\overline{B}$ if

(Q1) for any $\alpha,\beta \in \mathsf{Con}(\mathbb{A}_i)$, $i \in [n]$, such that $\alpha \prec \beta \le \beta_i$, congruences α and β are $\mathcal{U}_B$-chained in R, where $\mathcal{U}_B$ is the set of all $\overline{B}$-preserving polynomials of R

(Q2) for any $\alpha,\beta \in \mathsf{Con}(\mathbb{A}_i)$, $\gamma,\delta \in \mathsf{Con}(\mathbb{A}_j)$, $i,j \in [n]$, such that $\alpha \prec \beta \le \beta_i$, $\gamma \prec \delta \le \beta_j$, and (α,β) can be separated from (γ,δ), congruences α and β are $\mathcal{U}^*$-chained in R, where $\mathcal{U}^*$ is the set of all $\overline{B}$-preserving polynomials of R such that $g(\delta) \subseteq \gamma$.

Polynomials from $\mathcal{U}^*$ in condition (Q2) will be called *$(\gamma,\delta,\overline{B})$-good*.

Lemma 27 *(1) Any constant polynomial from $\overline{B} \cap R$ is $(\gamma,\delta,\overline{B})$-good.*

(2) If f is a k-ary term function of R and $g_1,\dots,g_k$ are $(\gamma,\delta,\overline{B})$-good polynomials, then $f(g_1(x),\dots,g_k(x))$ is $(\gamma,\delta,\overline{B})$-good.

(3) Let $T(a',b')$ denote $T_{\mathbb{A}_i}(a',b';\alpha,\beta,\overline{\beta},\overline{B},\mathcal{U})$ for $\mathcal{U} \in \{\mathcal{U}_{\overline{B}},\mathcal{U}^*\}$. If $\{a,b\} \in T(a',b')$ then $T(a,b) \subseteq T(a',b')$.

(4) Let $E' = E \cap \mathrm{pr}_i(R \cap \overline{B})/_\alpha$, where E is a $\beta/_\alpha$-block, and $B'_i = \mathrm{pr}_i(R \cap \overline{B})$. If E' contains a nontrivial as-component, then there is a set $T \subseteq \beta/_\alpha$ such that $T \subseteq T(a',b')$ for any $a',b' \in B'_i/_\alpha$, $a' \overset{\beta/\alpha}{\equiv} b'$ and T satisfies conditions (G1),(G2) for $T(a',b')$.

(5) Let $a',b' \in B'_i/_\alpha$, $a' \overset{\beta/\alpha}{\equiv} b'$ be such that $T(a',b')$ is minimal among sets of this

form. Then for any $(a,b)\in T(a',b')$ *there is* $h\in\mathcal{U}$ *such that* h *is idempotent and* $h(a)\stackrel{\alpha}{\equiv}a, h(b)\stackrel{\alpha}{\equiv}b$.

Proof: Items (1),(2) are straightforward.

(3) Let $\{a'',b''\}\in T(a,b)$. Then there are polynomials $f,g\in\mathcal{U}$ with $\{a,b\}=f(\{a',b'\})$ and $\{a'',b''\}=g(\{a,b\})$. Then $g\circ f\in\mathcal{U}$ by item (2) or definition and $g\circ f(\{a',b'\})=\{a'',b''\}$.

(4) Take $a,b\in C$ where C is a nontrivial as-component in E'. By (G2) $\{a,b\}\in T(a',b')$ for any appropriate a',b'. Therefore by (3) $T=T(a,b)\subseteq T(a',b')$.

(5) Let $\{a,b\}\in T(a',b')$. Then by (3) $T(a,b)\subseteq T(a',b')$, and therefore by the minimality of $T(a',b')$ we get $T(a,b)=T(a',b')$. The result follows by definition of $T(a',b')$. □

The next lemma shows how we will use the property of being chained.

Lemma 28 *Let* R *be a subdirect product of* $\mathbb{A}_1,\dots,\mathbb{A}_n$ *chained with respect to* $\overline{\beta},\overline{B}$*, where* $\beta_i\in\mathsf{Con}(\mathbb{A}_i)$ *and* B_i *is a* β_i*-block, and* $R'=R\cap(B_1\times\dots\times B_n)$*,* $B'_i=\mathrm{pr}_iR'$*. Let also* lk *be the link congruence of* B'_i *with respect to* $\mathrm{pr}_{ij}R'$ *for some* $i,j\in[n]$*, and* $\delta=\mathsf{Cg}(\mathsf{lk})$ *the congruence of* $\mathbb{A}_i$ *generated by* lk*. Then for any* $\gamma\in\mathsf{Con}(\mathbb{A}_i)$ *with* $\gamma\prec\delta$ *it holds* $(\delta/_\gamma)|_{\mathsf{umax}(E)}=(\mathsf{lk}/_\gamma)|_{\mathsf{umax}(E)}$ *for every* $\delta|_{B'_i}$*-block* E*.*

Proof: If $\gamma\prec\delta$ then by the choice of δ there are $a,b\in B'_i$ with $(a,b)\in\delta-\gamma$. Let E be the intersection of B'_i with a δ-block. By condition (Q1) for any $a',b'\in\mathsf{umax}(E)$ there is a sequence $a'=a_1,\dots,a_k=b'$ such that $\{a_\ell,a_{\ell+1}\}=f_\ell(\{a,b\})$ for some $\overline{B}$-preserving polynomial f_ℓ for each $\ell\in[k-1]$. This means that $(a_\ell,a_{\ell+1})\in\mathsf{lk}$, and so $(a',b')\in\mathsf{lk}$. □

The following lemma establishes the weak symmetricity of separability relation mentioned before.

Lemma 29 *Let* R *be a subdirect product of* $\mathbb{A}_1\times\dots\times\mathbb{A}_n$*,* $\beta_i\in\mathsf{Con}(\mathbb{A}_i)$*,* B_i *a* β_i*-block such that* R *is chained with respect to* $\overline{\beta},\overline{B}$*;* $R'=R\cap\overline{B}$*,* $B'_i=\mathrm{pr}_iR'$*. Let also* $\alpha\prec\beta\le\beta_1$*,* $\gamma\prec\delta=\beta_2$*, where* $\alpha,\beta\in\mathsf{Con}(\mathbb{A}_1)$*,* $\gamma,\delta\in\mathsf{Con}(\mathbb{A}_2)$*. If* $B'_2/_\gamma$ *has a nontrivial as-component and* (α,β) *can be separated from* (γ,δ)*, then there is a* $\overline{B}$*-preserving polynomial* g *such that* $g(\beta)\subseteq\alpha$ *and* $g(\delta)\not\subseteq\gamma$*.*

Proof: As is easily seen, we can assume that α,γ are equality relations. By Theorem 2.9(3) of [29] it suffices to show that for some $c,d\in B'_2$ there is a $\overline{B}$-preserving polynomial h of R and an (α,β)-subtrace $\{a,b\}\subseteq B'_1$ such that

(1) h is idempotent;

(2) $h(a) = h(b)$;

(3) $h(c) = c, h(d) = d$.

We consider two cases.

CASE 1. There is an element c from a nontrivial as-component of B'_2 such that $(a, c, \mathbf{b}) \in R'$ for some $a \in B'$, a β-block having a nontrivial intersection with B'_1 (that is, $|B' \cap B'_1| > 1$).

Let T_1 be the minimal set of (α, β)-subtraces as in Lemma 27(4) for $\mathcal{U}^*$, the set of $(\gamma, \delta, \overline{B})$-good polynomials. Take an (α, β)-subtrace $\{a, b\} \in T_1$. Let C' be a nontrivial as-component of B'_2 containing c and $d \in C'$, $c \neq d$.

By $Q^* \subseteq \mathbb{A}_1^2 \times \mathbb{A}_2^2 \times R$ we denote the relation generated by $\{(a, b, c, d, \mathbf{a})\} \cup \{(x, x, y, y, \mathbf{z}) \mid \mathbf{z} \in R, \mathbf{z}[1] = x, \mathbf{z}[2] = y\}$, where $\mathbf{a}$ is an arbitrary element from R'. Let $Q = \mathrm{pr}_{1234} Q^*$ and $Q' = \mathrm{pr}_{1234}(Q^* \cap (B'_1 \times B'_1 \times B'_2 \times B'_2 \times \overline{B}))$. Observe that Q is exactly the set of quadruples $(f(a), f(b), f(c), f(d))$ for unary polynomials f of R and Q' is exactly the set of quadruples $(f(a), f(b), f(c), f(d))$ for $\overline{B}$-preserving unary polynomials f of R. We prove that Q' contains a quadruple of the form (a', a', c, d); the result then follows.

Let also $Q_1 = \mathrm{pr}_{1,2} Q = Q_{ab}^{\mathbb{A}_1}$, $Q_2 = \mathrm{pr}_{3,4} Q = Q_{cd}^{\mathbb{A}_2}$; set $Q'_1 = \mathrm{pr}_{1,2} Q'$, $Q'_2 = \mathrm{pr}_{3,4} Q'$, Let $\mathsf{lk}_1, \mathsf{lk}_2$ denote the link congruences of Q' viewed as a subdirect product of Q'_1 and Q'_2. Note that these congruences may be different from the link congruences of Q restricted to $Q_1 \cap (B'_1 \times B'_1)$, $Q_2 \cap (B'_2 \times B'_2)$, respectively. We show that (a', a') for some $a' \in B'_1$ is as-maximal in a lk_1-block, (c, d) is as-maximal in a lk_2-block, and $Q' \cap (\mathsf{as}(a', a') \times \mathsf{as}(c, d)) \neq \varnothing$. By Corollary 18 this implies the result.

CLAIM 1. (c, d) is as-maximal in a lk_2-block.

Relation Q' contains tuples $(a, b, c, d), (a, b, c', c'), (a, a, c', c'), (a, a, c, c)$ for some $c' \in B'_2$. Indeed, $(a, b, c, d) \in Q'$ by definition, $(a, a, c, c) \in Q$ because $(a, c, \mathbf{b}) \in R$, and $(a, b, c', c'), (a, a, c', c')$ can be chosen to be the images of (a, b, c, d) and (a, a, c, c), respectively, under a $\overline{B}$-preserving polynomial g^{ab} such that $g^{ab}(a) = a$, $g^{ab}(b) = b$ and $g^{ab}(\delta) \subseteq \gamma$. Such a polynomial exists because R is chained and because (α, β) can be separated from (γ, δ). This implies that $(c, d) \stackrel{\mathsf{lk}_2}{\equiv} (c, c)$. Note that the congruences $\alpha \times \beta$ and $\gamma \times \delta$ restricted to the appropriate sets are congruences of Q'_1, Q'_2 generated by $((a, b), (a, a))$ and $((c, d), (c, c))$, respectively. Indeed, in the case of, say, $\alpha \times \beta$, relation Q'_1 consists of pairs $(g(a), g(b))$ for a $\overline{B}$-preserving unary polynomial g of $\mathbb{A}_1$. Since $(a, b) \stackrel{\alpha \times \beta}{\equiv} (a, a)$, for any $(a', b') \in Q'_1$ it holds that

$$(a', b') = (g(a), g(b)) \stackrel{\alpha \times \beta}{\equiv} (g(a), g(a)) = (a', a').$$

For Q'_2 and $\gamma \times \delta$ the argument is similar. Since $(a,b),(a,a)$ are in the same lk_1-block, $(\alpha\times\beta)|_{Q'_1} \subseteq \mathsf{lk}_1$; similarly, $(\gamma\times\delta)|_{Q'_2} \subseteq \mathsf{lk}_2$.

If for some $e,e' \in B'_2$ we have $(e,e) \stackrel{\mathsf{lk}_2}{\equiv} (e',e')$, then, as (e,e') generates δ, for any (γ,δ)-subtrace $\{e'',e'''\} \in T_{\mathbb{A}_2}(e,e') = T_{\mathbb{A}_2}(e,e';\gamma,\delta,\overline{\beta},\overline{B},\mathcal{U}_{\overline{B}})$ there is a $\overline{B}$-preserving polynomial f' with $f'(\{e,e'\}) = \{e'',e'''\}$. Applying this polynomial to the tuples witnessing that $(e,e) \stackrel{\mathsf{lk}_2}{\equiv} (e',e')$ we get $(e'',e'') \stackrel{\mathsf{lk}_2}{\equiv} (e''',e''')$. Therefore all tuples of the form (x,x), $x \in \mathsf{umax}(B'_2)$, are lk_2-related. Since $\{c,d\}$ is a (γ,δ)-subtrace from $T_{\mathbb{A}_2}(c,d) \subseteq T_{\mathbb{A}_2}(e,e')$, this implies that $\mathsf{lk}_2|_{Q''} = \delta^2|_{Q''}$, where $Q'' = Q'_2 \cap (\mathsf{umax}(B'_2)\times\mathsf{umax}(B'_2))$. In particular, $C'\times C'$, where C' is the as-component of B'_2 containing c,d, is contained in Q'_2, and is contained in a lk_2-block. All elements of $C'\times C'$ are as-maximal in Q''.

Otherwise, since the inclusion $(\gamma\times\delta)|_{Q'_2} \subseteq \mathsf{lk}_2$ implies that if $(c_1,d_1) \stackrel{\mathsf{lk}_2}{\equiv} (c_2,d_2)$ then $(c_1,c_1) \stackrel{\mathsf{lk}_2}{\equiv} (c_2,c_2)$, we have $\mathsf{lk}_2|_{Q''} = (\gamma\times\delta)|_{Q''}$. In particular, $\{c\}\times C'$ is contained in a lk_2-block. Since c,d are as-maximal, (c,d) is as-maximal in this lk_2-block. Claim 1 is proved.

By Lemma 12(5) there is an element (a',b') as-maximal in Q'_1 such that $(a',b',c,d) \in Q'$. If $a'=b'$ then we are done. Otherwise by Lemma 27(3) $\{a',b'\}$ is an (α,β)-subtrace from T_1, $(a',c) \in R$ because $\mathrm{pr}_{1,3}Q \subseteq R$, and we can replace a,b with a',b'. Note also that a',b' are as-maximal in $E = a'^{\beta} \cap B'_1$. We use a,b for a',b' from now on.

Claim 2. (a,a) is as-maximal in a lk_1-block.

Since Q'_1 consists of constant pairs and (α,β)-subtraces, and since $\mathsf{umax}(E)$ belongs to a block of the transitive closure of T_1, $Q''_1 = Q'_1 \cap (\mathsf{umax}(E)\times\mathsf{umax}(E))$ belongs to a linked block of $Q'_1 \cap (E\times E)$. Let E' be the as-components of E containing a. Then $E'^2 \subseteq Q'_1$.

By the assumption for any (α,β)-subtrace $(a',b') \in T_1 \cap E^2$ there is a $\overline{B}$-preserving polynomial $g^{a'b'}$ satisfying $g^{a'b'}(a') = a'$, $g^{a'b'}(b') = b'$, and $g^{a'b'}(B'_2) = \{c'\} \subseteq B'_2$. Therefore $(a',b',c',c'),(a',a',c',c'),(b',b',c',c') \in Q'$. The second two tuples imply that $(a',a') \stackrel{\mathsf{lk}_1}{\equiv} (b',b')$, and therefore $(a'',a'') \stackrel{\mathsf{lk}_1}{\equiv} (b'',b'')$ for any $a'',b'' \in E$. Also, the first two tuples indicate that every pair $(a'',b'') \in Q''_1$ is lk_1-related to the pair (a'',a''), which shows that $\mathsf{lk}_1|_{Q''_1}$ is the full congruence. Then since $E'^2 \subseteq Q''_1$ and E'^2 is an as-maximal component of Q''_1, Claim 2 is proved.

Now, as $(a,a,c,c) \in Q'$, $Q' \cap (E'\times E'\times\{c\}\times C') \neq \varnothing$. By Corollary 18 $(Q'_1 \cap (E'\times E')) \times (Q'_2 \cap (\{c\}\times C')) \subseteq Q'$, in particular $(a,a,c,d) \in Q$. Thus, there is a polynomial h such that $h(a) = h(b) = a$ and $h(c) = c, h(d) = d$.

CASE 2. For every element c from a nontrivial as-component of B'_2 and any $a \in B'_1$ such that $(a, c) \in R$ element a belongs to a β-block having a trivial intersection with B'_1.

We use the same elements $c, d \in C'$, an as-component of B'_2. For any (α, β)-subtrace $(a, b) \in T_1$ choose $c', d' \in B'_2$ such that $(a, c'), (b, d') \in \mathrm{pr}_{12}R'$. (Recall that we are assuming α and γ to be equality relatons.) If $c' = d'$, that is, $(b, c') \in R$, choose d' to be an arbitrary element from B'_2. By Lemmas 20, 27(5) and because R is chained there is an idempotent $\overline{B}$-preserving polynomial g such that $g(c') = c, g(d') = d$. Let $g(a) = a'$, $g(b) = b'$. Then $(a', c) \in R$ and $b' \stackrel{\beta}{\equiv} a'$. Since g is $\overline{B}$-preserving, $b' \in B'_1$. Since $|a'^{\beta} \cap B'_1| = 1$, $a' = b'$, and g can be chosen as h. □

3.3 Collapsing polynomials

We say that prime factors (α, β) and (γ, δ) *cannot be separated* if (α, β) cannot be separated from (γ, δ) and (γ, δ) cannot be separated from (α, β). In this section we introduce and prove the existence of polynomials that collapse all prime intervals in congruence lattices of factors of a subproduct, except for a set of factors that cannot be separated from each other.

Lemma 30 *Let $\mathbb{A}$ be an algebra.*
(1) If prime intervals $\alpha \prec \beta$ and $\gamma \prec \delta$ are projective, then they cannot be separated.
(2) If $\alpha \prec \beta$ and $\gamma \prec \delta$ cannot be separated, then a set U is a (α, β)-minimal set if and only if it is an (γ, δ)-minimal set.

Proof: (1) Follows from [29], Lemma 6.2.

(2) Let Q be the binary equality relation on $\mathbb{A}$ and f its polynomial such that $f(\mathbb{A}) = U$ and $f(\beta) \not\subseteq \alpha$. Since (α, β) cannot be separated from (γ, δ), we have $f(\delta) \not\subseteq \gamma$ and therefore U contains a (γ, δ)-minimal set U'. If $U' \neq U$, there is a polynomial g with $g \circ f(\delta) \not\subseteq \gamma$ and $g \circ f(\mathbb{A}) = U'$. In particular, $|f(U)| < |U|$, and so $g \circ f(\beta) \subseteq \alpha$; a contradiction with the assumption that (γ, δ) cannot be separated from (α, β). □

Let R be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $i \in [n]$, and $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_i)$ such that $\alpha \prec \beta$. Let also $\beta_j \in \mathsf{Con}(\mathbb{A}_j)$, $j \in [n]$, be such that $\alpha \prec \beta \leq \beta_i$; let also B_j be a β_j-block. We call an idempotent unary polynomial f of R *$\alpha\beta$-collapsing for $\overline{\beta}, \overline{B}$* if f is $\overline{B}$-preserving, $f(\beta) \not\subseteq \alpha$, $f(\delta|_{B_j}) \subseteq \gamma|_{B_j}$ for every $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_j)$, $j \in [n]$, with $\gamma \prec \delta \leq \beta_j$, and such that (α, β) can be separated from (γ, δ) or (γ, δ) can be separated from (α, β), and $|f(R)|$ is minimal possible.

Lemma 31 *Let R, α, β, and β_i, $i \in [n]$, be as above, and let R be chained with respect to $\overline{\beta}, \overline{B}$; and let $R' = R \cap \overline{B}$. Then if $\beta = \beta_i$, $\mathrm{pr}_i R'/_\alpha$ contains a nontrivial as-component, then an $\alpha\beta$-collapsing polynomial for $\overline{\beta}, \overline{B}$ exists.*

Proof: Suppose $i = 1$, let $B'_1 = \mathrm{pr}_1 R'$ and C be a nontrivial as-component of $B'_1/_\alpha$. Take a (α, β)-subtrace $\{a, b\} \subseteq B'_1$ such that $a^\alpha, b^\alpha \in C$. Since R is chained with respect to $\overline{\beta}, \overline{B}$, by Lemma 27(5) there is a $\overline{B}$-preserving idempotent polynomial f of R such that $f(\mathbb{A}_1)$ is an (α, β)-minimal set and $a^\alpha, b^\alpha \in f(\mathbb{A}_1)/_\alpha$. Let polynomial f be such that $f(R)$ is minimal possible. We show that f if $\alpha\beta$-collapsible.

Let for $j \in [n]$ and $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_j)$ such that $\gamma \prec \delta \leq \beta_j$, and $(\alpha, \beta), (\gamma, \delta)$ can be separated. Since R is chained, by Lemma 29 there is an idempotent unary polynomial $f_{j\gamma\delta}$ of R such that $f_{j\gamma\delta}(\mathbb{A}_1)$ is an (α, β)-minimal set with $a^\alpha, b^\alpha \in f_{j\gamma\delta}(\mathbb{A}_1)/_\alpha$ and $f_{j\gamma\delta}(\delta|_{B_j}) \subseteq \gamma|_{B_j}$. Then if $f(\delta|_{B_j}) \not\subseteq \gamma$, then let $g = f \circ f_{j\gamma\delta}$. We have $g(\beta) \not\subseteq \alpha$, but $g(\delta|_{B_j}) \subseteq \gamma$ implying $|g(R)| < |f(R)|$, a contradiction with minimality of $f(R)$. □

3.4 Separation and minimal sets

In this section we show a connection between the fact that two prime intervals cannot be separated, their types, and link congruences.

Lemma 32 *Let R be a subdirect product of $\mathbb{A}$ and $\mathbb{B}$ and let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\gamma, \delta \in \mathsf{Con}(\mathbb{B})$ be such that $\alpha \prec \beta$, $\gamma \prec \delta$, and $(\alpha, \beta), (\gamma, \delta)$ cannot be separated. Let also $\mathsf{lk}_1, \mathsf{lk}_2$ be the link congruences of $\mathbb{A}, \mathbb{B}$, respectively. If $\mathsf{typ}(\alpha, \beta) \neq \mathbf{2}$ then $\mathsf{lk}_1 \wedge \beta \leq \alpha$, $\mathsf{lk}_2 \wedge \delta \leq \gamma$.*

Proof: Assume $\alpha = \underline{0}_\mathbb{A}$, $\gamma = \underline{0}_\mathbb{B}$. Let f be a unary polynomial such that $f(\mathbb{A}_1) = U_1$, $f(\mathbb{A}_2) = U_2$ are $(\underline{0}_1, \alpha_1)$- and $(\underline{0}_2, \alpha_2)$-minimal sets, respectively. By $N_1 = \{0, 1\}$ we denote the only trace of U_1; by T_1 we denote the tail of U_1. By Lemma 4.15 of [29] there is a polynomial $p(x, y)$ with $p(\mathbb{A}_1, \mathbb{A}_1) = U$ and such that p is a semilattice operation on N, say, $p(0, 1) = 0$, and p is a semilattice operation on $\{0, a\}, \{1, a\}$ with $p(a, 0) = p(a, 1) = a$ for any $a \in T_1$. Such an operation is called a *pseudo-meet* operation on U_1. There are two cases.

CASE 1. $\mathsf{typ}(\gamma, \delta) \neq \mathbf{2}$.

Let $N_2 = \{0', 1'\}$ be the trace of U_2 and T_2 the tail of U_2. We may assume $p(x, p(x, y)) = p(x, y)$. Observe first that p preserves N_2. Indeed, otherwise $p(x, x)$ is not a permutation, as $p(0', 0'), p(0', 1'), p(1', 0'), p(1', 1')$ belong to the same β_2-block, and if they do not belong to N_2 then they are all equal, a contradiction with the assumption that (α_1, β_1) and (α_2, β_2) cannot be separated.

Suppose first p is a projection, say, the first projection on N_2. If $(\{0\} \times N_2) \cap R \neq \varnothing$, say, $(0,a) \in R$, then $f'(x) = p\left(x, \begin{pmatrix} 0 \\ a \end{pmatrix}\right)$ satisfies the conditions: $f'(N_1) = \{0\}$, that is, $f'(\alpha_1) \subseteq \underline{0}_1$, and $f'(x) = x$ on N_2; a contradiction that (γ,δ) cannot be separated from (α,β). If $(\{1\} \times N_2) \cap R \neq \varnothing$, say, $(1,a) \in R$, then $f'(x) = p\left(\begin{pmatrix} 1 \\ a \end{pmatrix}, x\right)$ satisfies the conditions: $f'(x) = x$ on N_1, that is, $f'(\beta_1) \not\subseteq \underline{0}_1$, and $f'(N_2) = \{a\}$ on N_2; a contradiction that (α,β) cannot be separated from (γ,δ). Therefore, for some $a \in T_1$, $(a,1') \in R$. The operation $f' = p\left(x, \begin{pmatrix} a \\ 1' \end{pmatrix}\right)$ is the projection on N_2 and $f'(N_1) = \{a\}$; a contradiction again.

Suppose now that p is a semilattice operation on N_2. Let $1'$ be the neutral element of p. If $(a,1') \in R$ for some $a \in T_1$, then $f'(x) = p\left(x, \begin{pmatrix} a \\ 1' \end{pmatrix}\right)$ is the projection on N_2, and $f'(N_1) = \{a\}$. If $(a,1') \in R$ for no $a \in T_1$, then we continue as follows. If $(0,1')$ or $(1,0')$ belong to R, then one of the operations $p\left(x, \begin{pmatrix} 0 \\ 1' \end{pmatrix}\right)$ and $p\left(x, \begin{pmatrix} 1 \\ 0' \end{pmatrix}\right)$ contradicts the assumption that i,j cannot be separated. Therefore $R \cap (U_1 \times U_2) \subseteq \{(1,1')\} \cup ((\{0\} \cup T_1) \times (\{0'\} \cup T_2))$.

Suppose that either $\mathsf{lk}_1 \cap \beta \neq \underline{0}_{\mathbb{A}}$ or $\mathsf{lk}_2 \cap \delta \neq \underline{0}_{\mathbb{B}}$, where $\mathsf{lk}_1, \mathsf{lk}_2$ are the link congruences of $\mathbb{A}, \mathbb{B}$ with respect to R. Assume the latter. Let $N = \{a,b\}$ be a $(\underline{0}_{\mathbb{B}},\delta)$-trace, and $a_1,\dots,a_k \in \mathbb{A}$ and $b_1,\dots,b_{k+1} \in \mathbb{B}$ with $a = b_1, b = b_{k+1}$, and $(a_i,b_i),(a_i,b_{i+1}) \in R$. Take a polynomial f of R such that $U_1 = f(\mathbb{A})$, $U_2 = f(\mathbb{B})$ are $(\underline{0}_{\mathbb{A}},\beta)$-, and $(\underline{0}_{\mathbb{B}},\delta)$-minimal sets, respectively, and such that U_2 is a $(\underline{0}_{\mathbb{B}},\delta)$-minimal set containing N as a trace and $f(a) = a, f(b) = b$. Then, as is easily seen, $R \cap (U_1 \times U_2)$ does not have the form described above. Thus, $\mathsf{tol}_1(R) \cap \beta = \underline{0}_{\mathbb{A}}$ and $\mathsf{tol}_2(R) \cap \delta = \underline{0}_{\mathbb{B}}$.

CASE 2. $\mathsf{typ}(\underline{0}_{\mathbb{B}},\delta) = \mathbf{2}$.

As in Case 1, since $p(x,x) = x$ on U_2, operation p preserves every trace of U_2. Let N_2 be a trace in U_2. Then N_2 is polynomially equivalent to a one-dimensional vector space over $\mathrm{GF}(q)$ where q is a prime power. Since p is idempotent, it can be represented in the form $\gamma x + (1-\gamma)y$, $\gamma \in \mathrm{GF}(q)$. We may assume that $\gamma = 1$. Indeed, if $\gamma = 0$ then consider $p(y,x)$ instead of $p(x,y)$. Otherwise, the operation

$$\underbrace{p(p \dots p}_{q-1 \text{ times}}(x,y), y \dots, y)$$

satisfies the required conditions. Now we can complete the proof as in Case 1. $\square$

The proof of Lemma 32 also implies

Corollary 33 *Let R be a subdirect product of $\mathbb{A}$ and $\mathbb{B}$ and let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\gamma, \delta \in \mathsf{Con}(\mathbb{B})$ be such that $\alpha \prec \beta$, $\gamma \prec \delta$, and $(\alpha, \beta), (\gamma, \delta)$ cannot be separated. Then* $\mathsf{typ}(\alpha, \beta) = \mathsf{typ}(\gamma, \delta)$.

4 Centralizers and decomposition of CSPs

In this section we introduce an operator on congruence lattices, study its properties andits connection to decompositions of CSPs.

4.1 Quasi-Centralizer

For an algebra $\mathbb{A}$, a term operation $f(x, y_1, \ldots, y_k)$, and $\mathbf{a} \in \mathbb{A}^k$, let $f^{\mathbf{a}}(x) = f(x, \mathbf{a})$. Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\alpha \leq \beta$, and let $\zeta(\alpha, \beta) \subseteq \mathbb{A}^2$ denote the following binary relation: $(a, b) \in \zeta(\alpha, \beta)$ if an only if, for any term operation $f(x, y_1, \ldots, y_k)$, any $i \in [k]$, and any $\mathbf{a}, \mathbf{b} \in \mathbb{A}^k$ such that $\mathbf{a}[i] = a$, $\mathbf{b}[i] = b$, and $\mathbf{a}[j] = \mathbf{b}[j]$ for $j \neq i$, it holds $f^{\mathbf{a}}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{b}}(\beta) \subseteq \alpha$.

Lemma 34 *For any $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\alpha \leq \beta$.*

(1) $\zeta(\alpha, \beta)$ is an equivalence relation.

(2) $\zeta(\alpha, \beta)$ is the greatest binary relation δ satisfying the condition: for any term operation $f(x, y_1, \ldots, y_k)$, any $i \in [k]$, and any $\mathbf{a}, \mathbf{b} \in \mathbb{A}^k$ such that $(\mathbf{a}[i], \mathbf{b}[i]) \in \delta$ for $j \in [k]$, it holds $f^{\mathbf{a}}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{b}}(\beta) \subseteq \alpha$.

(3) $\zeta(\alpha, \beta)$ is a congruence of $\mathbb{A}$.[1]

Proof: (1) $\zeta(\alpha, \beta)$ is clearly reflexive and symmetric. Suppose $(a, b), (b, c) \in \zeta(\alpha, \beta)$. Let $f(x, y_1, \ldots, y_k)$ be a term operation, $i \in [k]$, and $\mathbf{a}, \mathbf{c} \in \mathbb{A}^k$ such that $\mathbf{a}[i] = a$, $\mathbf{c}[i] = c$ and $\mathbf{a}[j] = \mathbf{c}[j]$ for $j \neq i$. Let $\mathbf{b} \in \mathbb{A}^k$ be such that $\mathbf{b}[i] = b$ and $\mathbf{b}[j] = \mathbf{a}[j]$ for $j \neq i$. Then $f^{\mathbf{a}}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{b}}(\beta) \subseteq \alpha$, which is if and only if $f^{\mathbf{c}}(\beta) \subseteq \alpha$.

(2) As is easily seen, δ is reflexive. Choosing $\mathbf{a}, \mathbf{b}$ in item (2) that differ in only one position, we show that $\delta \subseteq \zeta(\alpha, \beta)$.

Let us show the reverse inclusion. Let $f, \mathbf{a}, \mathbf{b}$ be as in the lemma, except $(\mathbf{a}[i], \mathbf{b}[i]) \in \zeta(\alpha, \beta)$, rather than δ. Set $\mathbf{a}_i \in \mathbb{A}^k$, $i \in \{0, \ldots, k\}$, as follows: $\mathbf{a}_i[j] = \mathbf{a}[j]$ for $j \leq i$ and $\mathbf{a}_i[j] = \mathbf{b}[j]$ for $j > i$. Then $f^{\mathbf{a}_i}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{a}_{i+1}}(\beta) \subseteq \alpha$. Thus, $f^{\mathbf{a}}(\beta) \subseteq \alpha$ if and only if $f^{\mathbf{b}}(\beta) \subseteq \alpha$.

[1]Congruence $\zeta(\alpha, \beta)$ appeared in [29], but completely inconsequentially, they did not study it at all. It is easy to see, thanks to K.Kearnes, that $\zeta(\alpha, \beta)$ is greater than the commutator of α and β, but the reverse inclusion is unclear.

(3) By (1) $\zeta(\alpha,\beta)$ is an equivalence relation, so, we only need to show it is preserved by term operations. Let $g(z_1,\dots,z_m)$ be a term operation and $\mathbf{a},\mathbf{b}\in\mathbb{A}^m$ such that $(\mathbf{a}[i],\mathbf{b}[i])\in\zeta(\alpha,\beta)$ for $i\in[m]$. Let also $a=g(\mathbf{a})$ and $b=g(\mathbf{b})$. We show that $(a,b)\in\zeta(\alpha,\beta)$. Take a term operation $f(x,y_1,\dots,y_k)$, $i\in[k]$, and $\mathbf{a}',\mathbf{b}'\in\mathbb{A}^k$ such that $\mathbf{a}'[i]=a,\mathbf{b}'[i]=b$, and $(\mathbf{a}'[j],\mathbf{b}'[j])\in\zeta(\alpha,\beta)$ for $j\neq i$. Without loss of generality, $i=k$. Let also

$$h(x,y_1,\dots,y_{k-1},z_1,\dots,z_m)=f(x,y_1,\dots,y_{k-1},g(z_1,\dots,z_m)),$$

and $\mathbf{a}''=(\mathbf{a}'[1],\dots,\mathbf{a}'[k-1],\mathbf{a}[1],\dots,\mathbf{a}[m]),\mathbf{b}''=(\mathbf{b}'[1],\dots,\mathbf{b}'[k-1],\mathbf{b}[1],\dots,\mathbf{b}[m])$. Then $(\mathbf{a}''[j],\mathbf{b}''[j])\in\zeta(\alpha,\beta)$ for all $j\in[k+m-1]$. Therefore $f^{\mathbf{a}'}(\beta)=h^{\mathbf{a}''}(\beta)\subseteq\alpha$ if and only if $f^{\mathbf{b}'}(\beta)=h^{\mathbf{b}''}(\beta)\subseteq\alpha$. □

Next we prove several properties of quasi-centralizer similar to some extent to the properties of the regular centralizer.

Proposition 35 *If $\zeta(\alpha,\beta)\geq\beta$, then (α,β) has type* **2***, and for any β-blocks B,C such that $B\leq C$ in $\mathbb{A}/_\beta$ and they belong to the same $\zeta(\alpha,\beta)$-block, there is an injective mapping $\sigma\colon B/_\alpha\to C/_\alpha$ such that for any $a\in B/_\alpha$, $a\leq\sigma(a)$ and $a\not\leq b$ for any other $b\in C$.*

Proof: Clearly, we may assume $\alpha=\underline{0}$. Suppose first that $\mathsf{typ}(\underline{0},\beta)\neq\mathbf{2}$. Take any $(\underline{0},\beta)$-minimal set U, its only trace N, and a pseudo-meet operation p on U. Then the polynomial $p(x,0)$ does not collapse β, as $f(0,0)=0$, $f(1,0)=1$, while the polynomial $p(x,1)$ does, a contradiction with the assumption $\zeta(\alpha,\beta)\geq\beta$.

Suppose now that $\mathsf{typ}(\underline{0},\beta)=\mathbf{2}$. Then by Corollary 22(1) for any $a,b\in\mathbb{A}$ with $a\stackrel{\beta}{\equiv}b$, there is a $(\underline{0},\beta)$-minimal set U such that $a,b\in U$.

Let B,C be β-blocks and $B\leq C$ in $\mathbb{A}/_\beta$. By Lemma 7(1) for any $a\in\mathbb{A}$ there is $b\in C$ with $a\leq b$. Suppose the statement of the proposition is not true. Then there are two possibilities.

1. For some $a\in B$ and $b,c\in C$, $a\leq b,a\leq c$. Let f be a polynomial such that $U=f(\mathbb{A})$ is a $(\underline{0},\beta)$-minimal set and $b,c\in U$. Consider $g_1(x)=a\cdot f(x)$ and $g_2(x)=b\cdot f(x)$. Clearly, $g_1(b)=b,g_1(c)=c$, so $g_1(\beta)\not\subseteq\underline{0}$. On the other hand, $g_2(b)=b=g_2(c)$, that is, $g_2(\beta)\subseteq\underline{0}$, as $|g_2(\mathbb{A})|\leq|U|$, a contradiction with the assumption $(a,b)\in(\underline{0},\beta)$.

2. For some $a\in C$ and $b,c\in B$, $b\leq a,c\leq a$. Let f be a polynomial such that $U=f(\mathbb{A})$ is a $(\underline{0},\beta)$-minimal set and $b,c\in U$. Consider $g_1(x)=f(x)\cdot a$ and $g_2(x)=f(x)\cdot b$. Clearly, $g_1(b)=g_1(c)=a$, so $g_1(\beta)\subseteq\underline{0}$, as $|g_1(\mathbb{A})|\leq|U|$. On the other hand, $g_2(b)=b,g_2(c)=c$, that is, $g_2(\beta)\not\subseteq\underline{0}$ a contradiction again. □

Corollary 36 *Let $\zeta(\alpha,\beta)=\underline{1}$, $a,b,c\in\mathbb{A}$ and $b\stackrel{\beta}{\equiv}c$. Then $ab\stackrel{\alpha}{\equiv}ac$.*

Proof: We have $ab \stackrel{\beta}{\equiv} ac$ and $a \leq ab, ac$. By Proposition 35 $ab \stackrel{\alpha}{\equiv} ac$. $\square$

Remark 37 *Recently, Payne [39] developed a polynomial time algorithm for the following class of algebras: Every algebra $\mathbb{A}$ from this class has a congruence α such that $\mathbb{A}/_\alpha$ is a semilattice, and the interactions between α-blocks satisfy a certain condition. It seems that Lemma 35 is similar to what this condition can provide.*

Lemma 38 *Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ such that $\alpha \prec \beta \leq \zeta = \zeta(\alpha, \beta)$ and let B, C be β-blocks from the same ζ-block such that BC is a thin edge in $\mathbb{A}/_\beta$. For any $b \in B, c \in C$ such that bc is a thin edge the polynomial $f(x) = x \cdot c$ if $b \leq c$, $f(x) = t_{bc}(x, c)$ if bc is majority, and $f(x) = h_{bc}(x, b, c)$ if bc is affine, where t_{ab}, h_{ab} are the operations from Lemma 9, is an injective mapping from $B/_\alpha$ to $C/_\alpha$.*

Proof: We can assume that α is the equality relation. Suppose $f(a_1) = f(a_2)$ for some $a_1, a_2 \in B$. Since $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$, by Corollary 22(1) every pair of elements of B is an (α, β)-subtrace. Let f' be an idempotent unary polynomial such that $f'(a_1) = a_1$, $f'(a_2) = a_2$, and $f'(\mathbb{A})$ is an (α, β)-minimal set.

If $b \leq c$, let $g(x, y) = f'(x) \cdot y$. Then $g_c = g(x, c) = f(x)$ on $\{a_1, a_2\}$, that is, $g_c(a_1) = g_c(a_2)$ implying $g_c(\beta) \subseteq \alpha$. On the other hand, $g_b(x) = f'(x)$ on $\{a_1, a_2\}$ implying $g_b(\beta) \not\subseteq \alpha$, a contradiction with the assumption $b \stackrel{\zeta}{\equiv} c$.

If bc is a thin majority edge, set $g(x, y) = t_{bc}(f'(x), y)$. Then polynomials $g_c(x)$ and g_b on $\{a_1, a_2\}$ act as in the previous case, and we have a contradiction again. Finally, if bc is a thin affine edge, we consider the polynomials $g(x, y, z) = h_{bc}(f'(x), y, z)$ and $g_{bc}(x) = g(x, b, c)$, $g_{a_1a_1}(x) = g(x, a_1, a_1)$. Again, $g_{bc}(a_1) = f(a_1) = f(a_2) = g_{bc}(a_2)$, while

$$g_{a_1a_1}(a_1) = h_{bc}(f'(a_1), a_1, a_1) = a_1 \neq h_{bc}(f'(a_2), a_1, a_1) = g_{a_1a_1}(a_2),$$

since by Lemma 9 $h_{bc}(x, a_1, a_1)$ is a permutation. This implies that $g_{bc}(\beta) \subseteq \alpha$ and $g_{a_1a_1}(\beta) \not\subseteq \alpha$, a contradiction. $\square$

Lemma 39 *Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ be such that $\alpha \prec \beta$ and $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$, and $\zeta = \zeta(\alpha, \beta)$. Then for any β-blocks B_1, B_2 that belong to the same ζ-block C and such that $B_1 \sqsubseteq_{asm} B_2$ and $B_2 \sqsubseteq_{asm} B_1$ in $C/_\beta$, $|B_1/_\alpha| = |B_2/_\alpha|$.*

Proof: Since there is an asm-path from B_1 to B_2 and back, the result follows from Lemma 38. $\square$

Let $\mathbb{A}$ be an idempotent algebra and $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\alpha \prec \beta$. Element $a \in \mathbb{A}$ is said to be $\alpha\beta$-*minimal* if it belongs to an (α, β)-trace. Let $\mathsf{Z}_{\mathbb{A}}(\alpha, \beta)$ denote the set of all $\alpha\beta$-minimal elements of $\mathbb{A}$.

Lemma 40 *Let $\alpha, \beta, \gamma, \delta \in \mathsf{Con}(\mathbb{A})$ be such that $\gamma \prec \delta \leq \beta$, $\alpha \prec \beta$, intervals $(\alpha, \beta), (\gamma, \delta)$ cannot be separated, and and $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$; let B be a β-block. If $a \in \mathsf{Z}_{\mathbb{A}}(\gamma, \delta) \cap B$ then for any $b \in B$ such that $a \sqsubseteq_{asm} b$ in B, $b \in \mathsf{Z}_{\mathbb{A}}(\gamma, \delta)$.*

In particular, $\mathsf{umax}(B) \subseteq \mathsf{Z}_{\mathbb{A}}(\gamma, \delta)$.

Moreover, if $a \stackrel{\alpha}{\equiv} b$, $a \sqsubseteq_{asm} b$ in a^α, and f is a polynomial such that $f(a) = a$, $f(\mathbb{A})$ is an (α, β)-minimal set, and N its trace with $a \in N$, then there is a polynomial g such that $g(b) = b$, $g(\mathbb{A})$ is an (α, β)-minimal set, N' is its trace containing b and $N'/_\alpha = N/_\alpha$.

Proof: Let f be an idempotent unary polynomial of $\mathbb{A}$ such that $a \in N$, a trace in $U = f(\mathbb{A})$, a (γ, δ)-minimal set. Note that $f(B) \subseteq B$ and $f(\beta) \not\subseteq \alpha$. It suffices to consider the case when ab is a thin edge.

Depending on the type of the edge ab we set $f'(x) = f(x) \cdot b$, $f'(x) = t_{ab}(f(x), b)$, or $f'(x) = h_{ab}(f(x), a, b)$, if ab is semilattice, majority or affine, respectively. Note also that by Lemma 9 $f'(a) = b$, and therefore if $f'(\delta) \not\subseteq \gamma$ we have $f'(\mathbb{A})$ is a (γ, δ)-minimal set, and b belongs to it.

Since (α, β) and (γ, δ) cannot be separated, there are $a_1, a_2 \in B/_\alpha$ such that $a_1 \neq a_2$ and $f(a_1) = a_1, f(a_2) = a_2$. Since $a_1 a_2$ is an affine edge in $\mathbb{B}/_\alpha$, depending on the type of ab we have:

– if $a \leq b$, then $f'(a_i) = a_i \cdot b^\alpha = a_i$ for $i = 1, 2$;

– if ab is majority, then $f'(a_i) = t_{ab}(a_i, b^\alpha) = a_i$, as $a_i \stackrel{\beta/\alpha}{\equiv} b^\alpha$ for $i = 1, 2$;

– if ab is affine, then $h_{ab}(x, a^\alpha, b^\alpha)$ is a permutation on $B/_\alpha$, in particular, $f'(a_1) \neq f'(a_2)$.

In either case we obtain $f'(\beta) \not\subseteq \alpha$, implying $f'(\delta) \not\subseteq \gamma$.

For the last claim of the lemma it suffices to notice that if $a \stackrel{\alpha}{\equiv} b$ we have $f'(x) \stackrel{\alpha}{\equiv} f(x)$ for $x \in B$. □

4.2 Decompositon of CSPs

In this section we show that if intervals in congruence lattices of domains in a CSP instance cannot be separated, they induce certain decomposition of the instance or its subinstances. The components of this decomposition are instances over smaller domains, which are, actually, blocks of the corresponding quasi-centralizers.

Let $R \leq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $i, j \in [n]$, and $\alpha_i \in \mathsf{Con}(\mathbb{A}_i)$, $\alpha_j \in \mathsf{Con}(\mathbb{A}_j)$. The coordinate positions i, j are said to be *$\alpha_i\alpha_j$-aligned in R* if, for any $(a, c), (b, d) \in \mathrm{pr}_{ij} R$, $(a, b) \in \alpha_i$ if and only if $(c, d) \in \alpha_j$. Or in other words, the link congruences of $\mathbb{A}_i, \mathbb{A}_j$ with respect to $\mathrm{pr}_{ij} R$ are no greater than α_i, α_j, respectively.

Lemma 41 *Let $R \leq \mathbb{A}_1 \times \mathbb{A}_2$, $\alpha_i, \beta_i \in \mathsf{Con}(\mathbb{A}_i)$, $\alpha_i \prec \beta_i$, for $i = 1, 2$. If (α_1, β_1) and (α_2, β_2) cannot be separated from each other, then the coordinate positions 1,2 are $\zeta(\alpha_1, \beta_1)\zeta(\alpha_2, \beta_2)$-aligned in R.*

Proof: Let us assume the contrary, that is, without loss of generality there are $a, b \in \mathbb{A}_1$ and $c, d \in \mathbb{A}_2$ with $(a, c), (b, d) \in R$, $(a, b) \in \zeta(\alpha_1, \beta_1)$, but $(c, d) \notin \zeta(\alpha_2, \beta_2)$. Therefore there is $g(x, y_1, \ldots, y_k)$, a term operation of $\mathbb{A}_2$, $i \in [k]$, and $\mathbf{c}, \mathbf{d} \in \mathbb{A}^k$ with $\mathbf{c}[i] = c$, $\mathbf{d}[i] = d$ and $\mathbf{c}[j] = \mathbf{d}[j]$ for $j \neq i$, such that $g^{\mathbf{c}}(\beta_2) \subseteq \alpha_2$ but $g^{\mathbf{d}}(\beta_2) \not\subseteq \alpha_2$, or the other way round. Extend g to a term operation g of R, and choose $\mathbf{a}, \mathbf{b} \in \mathbb{A}_1^k$ such that $\mathbf{a}[i] = a, \mathbf{b}[i] = b$, $\mathbf{a}[j] = \mathbf{b}[j]$ for $j \neq i$, and $(\mathbf{a}[j], \mathbf{c}[j]), (\mathbf{b}[j], \mathbf{d}[j]) \in R$ for $j \in [k]$. Then $g^{\mathbf{a}}(\beta_1) \subseteq \alpha_1$ if and only if $g^{\mathbf{b}}(\beta_1) \subseteq \alpha_1$. Therefore, there is a polynomial of R that separates (α_1, β_1) from (α_2, β_2) or the other way round, a contradiction. $\Box$

Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance, in particular, for every $X \subseteq V$, $|X| = 2$, it contains a constraint $C^X = \langle X, R^X \rangle$. Let $w_1, w_2, \in V$. We say that w_1, w_2 are $\alpha_1\alpha_2$-aligned in $\mathcal{P}$ if they are $\alpha_1\alpha_2$-*aligned* in $R^{w_1w_2}$. For $\alpha_v \in \mathsf{Con}(\mathbb{A}_v)$, $v \in V$, instance $\mathcal{P}$ is said to be $\overline{\alpha}$-aligned if every w_1, w_2 are $\alpha_{w_1}\alpha_{w_2}$-aligned. This means that there are one-to-one mappings $\varphi_{w_1w_2} : \mathbb{A}_{w_1}/_{\alpha_{w_1}} \to \mathbb{A}_{w_1}/_{\alpha_{w_1}}$ such that whenever $(a, b) \in R^{w_1w_2}$, $b^{\alpha_{w_2}} = \varphi_{w_1w_2}(a^{w_1})$. Observe that since $\mathcal{P}$ is (2,3)-minimal, these mappings are consistent, that is, for any $u, v, w \in V$, $\varphi_{uv} \circ \varphi_{uw} = \varphi_{uw}$. Therefore $\mathcal{P}$ can be represented as a disjoint union of instances $\mathcal{P}_1, \ldots, \mathcal{P}_k$, where k is the number of α_v-blocks for any $v \in V$ and the domain of $v \in V$ of $\mathcal{P}_i$ is the i-th α_v-block.

Let again $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance and let $\overline{\beta}$, $\beta_v \in \mathsf{Con}(\mathbb{A}_v)$, $v \in V$, be a collection of congruences. Let $\mathcal{W}^{\mathcal{P}}(\overline{\beta})$ denote the set of triples (v, α, β) such that $v \in V$, $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$, and $\alpha \prec \beta \leq \beta_v$. Also, $\mathcal{W}^{\mathcal{P}}$ denotes $\mathcal{W}^{\mathcal{P}}(\overline{\beta})$ when $\beta_v = \underline{1}_v$ for all $v \in V$. We will omit the superscript $\mathcal{P}$ whenever it is clear from the context. For every $(v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})$, let Z denote the set of triples $(w, \gamma, \delta) \in \mathcal{W}^{\mathcal{P}}(\overline{\beta})$ such that (α, β) and (γ, δ) cannot be separated in R^{vw}. Slightly abusing the terminology we will also say that (α, β) and (γ, δ) cannot be separated in $\mathcal{P}$. Then let $W_{v,\alpha\beta,\overline{\beta}} = \{w \in V \mid (w, \gamma, \delta) \in Z \text{ for some } \gamma, \delta \in \mathsf{Con}(\mathbb{A}_w)\}$. We will omit the subscript $\overline{\beta}$ whenever possible. The following statement is an easy corollary of Lemma 41.

Theorem 42 *Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance and $(v, \alpha, \beta) \in \mathcal{W}$. For $w \in W_{v,\alpha\beta,\overline{\beta}}$, where $\beta_v = \underline{1}_v$ for $v \in V$, let $(w, \gamma, \delta) \in \mathcal{W}$ be such that (α, β) and (γ, δ) cannot be separated and $\zeta_w = \zeta(\gamma, \delta)$. Then $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ is $\overline{\zeta}$-aligned.*

5 Strategies and solutions

5.1 The grand scheme

In this section we describe the 'grand scheme' of solving CSPs.

We call a CSP instance $\mathcal{P} = (V, \mathcal{C})$ *subdirectly irreducible* if it is 1-minimal and $\mathbb{A}_v$ is sudirectly irreducible for every $v \in V$.

Lemma 43 (Folklore) *Every CSP instance can be reduced in polynomial time to an equivalent subdirectly irreducible one.*

In this section all instances we consider are assumed subdirectly irreducible. The monolith of $\mathbb{A}_v$ is denoted by μ_v.

Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance and for $X \subseteq V$, $|X| = 2$, the constraint $C^X = \langle X, R^X \rangle$ is as in the end of the previous section. Let $\overline{\beta}$, $\beta_v \in \mathsf{Con}(\mathbb{A}_v)$, $v \in V$, be a collection of congruences and $\mathcal{W}(\overline{\beta}), \mathcal{W}$ as in the previous section. Let $\mathcal{W}'(\overline{\beta})$ (and respectively $\mathcal{W}'$) denote the set of triples $(v, \alpha, \beta) \in \mathcal{W}(\overline{\beta})$ (respectively, from $\mathcal{W}$) with $\zeta(\alpha, \beta) = \underline{1}_v$. We say that algebra $\mathbb{A}_v$ is *semilattice free* if it does not contain semilattice edges. Let $\mathsf{size}(\mathcal{P})$ denote the maximal size of domains of $\mathcal{P}$ that are not semilattice free and $\mathsf{MAX}(\mathcal{P})$ be the set of variables $v \in V$ such that $|\mathbb{A}_v| = \mathsf{size}(\mathcal{P})$ and $\mathbb{A}_v$ is not semilattice free. Finally, for $Y \subseteq V$ let $\mu_v^Y = \mu_v$ if $v \in Y$ and $\mu_v^Y = \underline{0}_v$ otherwise. Recall that by $\mathcal{P}/_{\overline{\mu}^Y}$ we denote the instance $(V, \mathcal{C}^{\overline{\mu}^Y})$ constructed as follows: the domain of $v \in V$ is $\mathbb{A}_v/_{\mu_v^Y}$; for every constraint $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$, the set $\mathcal{C}^{\overline{\mu}^Y}$ includes the constraint $\langle \mathbf{s}, R/_{\overline{\mu}_{\mathbf{s}}^Y} \rangle$.

Instance $\mathcal{P}$ is said to be *block-minimal* if for every $(v, \alpha, \beta) \in \mathcal{W}$ (here $\beta_v = \underline{1}_v, v \in V$)

(BM1) for every $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ if $(v, \alpha, \beta) \notin \mathcal{W}'$, and the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}/_{\overline{\mu}^Y}$ otherwise, where $Y = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$, is minimal;

(BM2) if $(v, \alpha, \beta) \in \mathcal{W}'$, then for every $(w, \gamma, \delta) \in \mathcal{W} - \mathcal{W}'$ the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}/_{\overline{\mu}^Y}$, where $Y = \mathsf{MAX}(\mathcal{P}) - (W_{v,\alpha\beta,\overline{\beta}} \cap W_{w,\gamma\delta,\overline{\beta}})$ is minimal.

Observe that $W_{v,\alpha\beta,\overline{\beta}}$ can be equal to V. However if $(v, \alpha, \beta) \notin \mathcal{W}'$ the problem $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ sprlits into a union of disjoint problems over smaller domains. On the other hand, if $(v, \alpha, \beta) \in \mathcal{W}'$ then $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$ may not be decomposable. Since we need an efficient procedure of establishing block-minimality, this explains the complications introduced in (BM2).

For a block-minimal instance $\mathcal{P}$ and $(v, \alpha, \beta) \in \mathcal{W}$, if $(v, \alpha, \beta) \notin \mathcal{W}'$, then $\mathcal{S}_{W_{v,\alpha\beta,\overline{\beta}}}$ denotes the set of solutions of $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}$, and if $(v, \alpha, \beta) \in \mathcal{W}'$, then $\mathcal{S}_{W_{v,\alpha\beta,\overline{\beta}},Y}$ denotes the set of solutions of $\mathcal{P}_{W_{v,\alpha\beta,\overline{\beta}}}/_{\overline{\mu}^Y}$ for an appropriate Y.

For an instance $\mathcal{P}$ we say that an instance $\mathcal{P}'$ is *strictly smaller* than instance $\mathcal{P}$ if $\mathsf{size}(\mathcal{P}') < \mathsf{size}(\mathcal{P})$.

Lemma 44 *Let $\mathcal{P} = (V, \mathcal{C})$ be a (2,3)-minimal instance. Then $\mathcal{P}$ can be transformed to an equivalent block-minimal instance $\mathcal{P}'$ by solving a quadratic number of strictly smaller CSPs.*

Proof: To establish block-minimality for every $(v, \alpha, \beta) \in \mathcal{W}$ (let $W = W_{v,\alpha\beta}$), and we need to check if the problems given in the definition are minimal. If they do then $\mathcal{P}$ is block-minimal, otherwise some tuples can be removed from some constraint relation R (the set of tuples that remain in R is always a subalgebra, as is easily seen), and the instance $\mathcal{P}$ tightened, in which case we need to repeat the procedure with the tightened instance. For $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ and $\mathbf{a} \in R$ let $\mathcal{P}'$ be the problem obtained as follows: fix the values of variables from $\mathbf{s} \cap W$, or from $\mathbf{s} \cap W \cap W_{w,\gamma\delta}$ in the case of (BM2) to those of $\mathbf{a}$. If the resulting problem is $\mathcal{P}''$ then set $\mathcal{P}' = \mathcal{P}''/_{\overline{\mu}^Y}$, where Y is either empty, if $(v, \alpha, \beta) \notin \mathcal{W}'$, or $Y = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$, if $(v, \alpha, \beta) \in \mathcal{W}'$ in (BM1), or $Y = \mathsf{MAX}(\mathcal{P}) - (W \cap M_{w,\gamma\delta})$ in (BM2). In the first case, by Theorem 42 $\mathcal{P}'$ is a disjoint union of instances $\mathcal{P}_1, \dots, \mathcal{P}_\ell$ and $\mathsf{size}(\mathcal{P}_i) < \mathsf{size}(\mathcal{P})$. In the second case the domains of variables from $\mathbf{s} \cap W$ have cardinality 1, and the domains of the remaining variables are less than $\mathsf{size}(\mathcal{P})$. Finally, in the last case the domains of variables outside of $W \cap W_{w,\gamma\delta}$ are smaller than $\mathsf{size}(\mathcal{P})$. Also, by Theorem 42 $\mathcal{P}_{W \cap W_{w,\gamma\delta}}$ is a disjoint union of instances with domains of smaller size. Restricting $\mathcal{P}'$ on each of these domains we complete the proof. □

Let $\mathcal{P} = (V, \mathcal{C})$ be a subdirectly irreducible (2,3)-minimal instance. Let $\mathsf{Center}(\mathcal{P})$ denote the set of variables $v \in V$ such that $\zeta(\underline{0}_v, \mu_v) = \underline{1}_v$. Let $\mu^*_v = \mu_v$ if $v \in \mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P})$ and $\mu^*_v = \underline{0}_v$ otherwise.

We consider several cases and indicate what kind of reductions or solution algorithms we intend to use in each case.

Reduction 1: Semilattice free domains. If no domain of $\mathcal{P}$ contains a semilattice edge then $\mathcal{P}$ can be solved in polynomial time, using the few subalgebras algorithm, as shown in [17].

Reduction 2: Collapsing trivial centralizers. In this case we can use Lemma 44 to solve the instance obtained by factoring modulo $\mu{*}_v$.

Theorem 45 *If $\mathcal{P}$ is subdirectly irreducible, block-minimal, and $\mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P}) = \varnothing$, then $\mathcal{P}/_{\overline{\mu}^*}$ has a solution.*

Reduction 3: Nontrivial centralizers. In this case it is possible to reduce the domains of variables from $\mathsf{Center}(\mathcal{P})$.

Theorem 46 *If $\mathcal{P}/_{\overline{\mu}^*}$ is 1-minimal, then $\mathcal{P}$ can be reduced in polynomial time to a strictly smaller instance.*

With the reductions above a solution algorithm goes as shown in Algorithm 1.

Algorithm 1 Procedure $\mathsf{SolveCSP}$

Require: A CSP instance $\mathcal{P} = (V, \mathcal{C})$ over $\mathcal{A}$
Ensure: A solution of $\mathcal{P}$ if one exists, 'NO' otherwise

```
1:  if all the domains are semilattice free then
2:      Solve P using the few subalgebras algorithm
3:      RETURN the answer
4:  end if
5:  Transform P to a subdirectly irreducible, block-minimal and (2,3)-minimal
    instance
6:  μ*_v = μ_v for v ∈ MAX(P) ∩ Center(P) and μ*_v = 0_v otherwise
7:  P* = P/μ̄*
8:  for every v ∈ V and a ∈ A_v/μ*_v do
9:      P' = P*_(v,a)
10:     Transform P' to a subdirectly irreducible, (2,3)-minimal instance P''
11:     If size(P'') < size(P) call SolveCSP on P'' and flag a if P'' has no solution
12:     Establish block-minimality of P''; if the problem changes, return to Step 10
13:     If the resulting instance is empty, flag the element a of A_v/μ*_v
14: end for
15: If there are flagged values, tighten the instance and start over
16: Use Theorem 46 to reduce P to an instance P' with size(P') < size(P)
17: Call SolveCSP on P' and RETURN the answer
```

Theorem 47 *Algorithm SolveCSP (Algorithm 1) correctly solves every instance from* $\mathrm{CSP}(\mathcal{A})$ *and runs in polynomial time.*

Proof: By the results of [17] the algorithm correctly solves the given instance $\mathcal{P}$ in polynomial time if the conditions of Step 1 are true. Lemma 44 implies that Steps 5 and 12 can be completed by recursing to strictly smaller instances.

Next we show that the for-loop in Steps 8-14 checks if $\mathcal{P}^* = \mathcal{P}/_{\overline{\mu}^*}$ is globally 1-minimal. For that we need to verify that whenever value a is flagged $\mathcal{P}^*$ has no solution φ with $\varphi(v) = a$, and if no values are flagged then $\mathcal{P}^*$ is globally 1-minimal. If $\varphi(v) = a$ for some solution φ of $\mathcal{P}^*$, then φ is a solution $\mathcal{P}'$ constructed in Step 9. In this case Steps 11,12 cannot result in an empty instance. Let $\mathcal{P}'' = (V'', \mathcal{C})$ and $\mathcal{S}'_v, \mathcal{S}''_w$ be domains of $v \in V$ of $\mathcal{P}'$ and of $w \in V''$ of $\mathcal{P}''$. Suppose $a \in \mathbb{A}_v/_{\mu_v^*}$ is not flagged. If $\mathsf{size}(\mathcal{P}'') < \mathsf{size}(\mathcal{P})$ this means that $\mathcal{P}''$ and therefore $\mathcal{P}'$ has a solution. Otherwise this means that establishing block-minimality of $\mathcal{P}''$ is successful. In this case $\mathcal{P}''$ has a solution by Theorem 45, because $\mathsf{MAX}(\mathcal{P}'') \cap \mathsf{Center}(\mathcal{P}'') = \varnothing$. This in turn implies that $\mathcal{P}'$ has a solution.

Finally, if Steps 8–15 are completed without restarts, Steps 16,17 can be completed by Theorem 46 and recursing on $\mathcal{P}'$ with $\mathsf{size}(\mathcal{P}') < \mathsf{size}(\mathcal{P})$.

To see that the algorithm runs in polynomial time it suffices to observe that (1) the number of restarts in Steps 5,12 and 15 is at most linear, as the instance becomes smaller after every restart; (2) the number of instances we recurse on in Steps 5,11,12,16 is linear, as well as the number of restarts in Step 12, and; finally (3) the depth of recursion is bounded by $\mathsf{size}(\mathcal{P})$ in Step 5,11,12 and 17. $\square$

5.2 Proof of Theorem 46

Following [37] let $\mathcal{P} = (V, \mathcal{C})$ be an instance and $p_v \colon \mathbb{A}_v \to \mathbb{A}_v$, $v \in V$. Mappings p_v, $v \in V$, are said to be *consistent* if for any $\langle \mathbf{s}, R\rangle \in \mathcal{C}$, $\mathbf{s} = (v_1, \dots, v_k)$, and any tuple $\mathbf{a} \in R$ the tuple $(p_{v_1}(\mathbf{a}[1]), \dots, p_{v_k}(\mathbf{a}[k]))$ belongs to R. It is easy to see that the composition of two families of consistent mappings is also a consistent mapping. For consistent idempotent mappings p_v by $p(\mathcal{P})$ we denote the *retraction* of $\mathcal{P}$, that is, $\mathcal{P}$ restricted to the images of p_v. In this case $\mathcal{P}$ has a solution if and only if $p(\mathcal{P})$ has, see [37].

Let φ be a solution of $\mathcal{P}/_{\overline{\mu}^*}$. We define $p_v^\varphi : \mathbb{A}_v \to \mathbb{A}_v$ as follows: $p_v^\varphi = q_v^k$, where $q_v(a) = a \cdot b_v$, element b_v is any element of $\varphi(v)$, and k is such that q_v^k is idempotent for all $v \in V$. Note that by Corollary 36 this mapping is properly defined even if $\mu_v^* \neq \underline{0}_v$.

Lemma 48 *Mappings p_v^φ, $v \in V$, are consistent.*

Proof: Take any $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$. Since φ is a solution of $\mathcal{P}/_{\overline{\mu}^*}$, there is $\mathbf{b} \in R$ such that $\mathbf{b}[v] \in \varphi(v)$ for $v \in \mathbf{s}$. Then for any $\mathbf{a} \in R$, $q(\mathbf{a}) = \mathbf{a} \cdot \mathbf{b} \in R$, and this product does not depend on the choice of $\mathbf{b}$, as it follows from Corollary 36. Iterating this operation also produces a tuple from R. $\square$

We would like to use the above reduction to reduce $\mathcal{P}$ to a problem $\mathcal{P}'$ such that $\mathsf{size}(\mathcal{P}') < \mathsf{size}(\mathcal{P})$. If φ is such that for $v \in \mathsf{MAX}(\mathcal{P})$ there is $a \in \mathbb{A}_v$ with $a^{\mu_v^*} \le \varphi(v)$ and $a \notin \varphi(v)$, then $|p_v^\varphi(\mathbb{A}_v)| < |\mathbb{A}_v|$. Also, observe that if $|p_v^\varphi(\mathbb{A}_v)| = |\mathbb{A}_v|$, then p_v^φ is the identity mapping, that is $p_v^\varphi(\mathbb{A}_v) = \mathbb{A}_v$. If $\mathbb{A}^v$ is semilattice free then p_v^φ is the identity mapping by Proposition 6. Let V^* be the set of variables $v \in V$ such that $\mathbb{A}_v/_{\mu_v^*}$ is not semilattice free.

Lemma 49 *There are consistent mappings p_v, $v \in V$, such that for any $v \in V^*$ we have $|p_v(\mathbb{A}_v)| < |\mathbb{A}_v|$. Moreover, such mappings can be found solving a linear number of instances of the form $(\mathcal{P}_{(v,a^{\mu_v^*})})/_{\overline{\mu}^*}$.*

Proof: Since $\mathcal{P}/_{\overline{\mu}^*}$ is globally 1-minimal, for any $a \in \mathbb{A}_v/_{\mu_v^*}$ there is a solution φ with $\varphi(v) = a$, and it can be found solving the instance $(\mathcal{P}_{(v,a^{\mu_v^*})})/_{\overline{\mu}^*}$. For every $v \in V^*$ choose $a \in \mathbb{A}_v$ such that there is $b \in \mathbb{A}_v$ and $b \le a$, $a \stackrel{\mu_v^*}{\not\equiv} b$, and let φ_v be a solution of $\mathcal{P}/_{\overline{\mu}^*}$ with $\varphi_v(v) = a^{\mu_v^*}$. Then $|p_v^{\varphi_v}(\mathbb{A}_v)| < |\mathbb{A}_v|$ and $|p_w^{\varphi_v}(\mathbb{A}_w)| < |\mathbb{A}_w|$ or $p_w^{\varphi_v}$ is the identity mapping for any $w \in V^*$. Therefore the composition of the p^{φ_w} for all $w \in V^*$ is as required. □

Theorem 46 now follows.

In order to use Theorem 46 we however need to argue that $p(\mathcal{P})$ is a problem over a class of algebras omitting type **1**. Let f be a weak near-unanimity term of the class $\mathcal{A}$. Then $p \circ f$ is a weak near-unanimity term of $p(\mathcal{A}) = \{p(\mathbb{A}) \mid \mathbb{A} \in \mathcal{A}\}$. Moreover, if $\mathbb{A}$ is semilattice free then $p(\mathbb{A}) = \mathbb{A}$.

5.3 Strategies

In this section similar to strategies related to the concepts of consistency and minimality we introduce strategies of some sort that will be used to prove Theorem 45. We start with several necessary definitions.

We say that a set A is *as-closed* in algebra $\mathbb{B}$, $A \subseteq \mathbb{B}$, if $A \cap \mathsf{umax}(\mathbb{B}) \neq \varnothing$ and, for every $a, b \in \mathbb{B}$ such that $a \sqsubseteq_{as} b$ in $\mathbb{B}$ and $a \in A \cap \mathsf{umax}(\mathbb{B})$, element b also belongs to A.

Let R be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$ and Q a subalgebra of R. We say that Q is *polynomially closed* in R if for any $i \in [n]$ and any poilynomial f of R the following condition holds: for any $\mathbf{a}, \mathbf{b} \in \mathsf{umax}(Q)$ such that $f(\mathbf{a}) = \mathbf{a}$ and for any $\mathbf{c} \in \mathsf{Sg}(\mathbf{a}, f(\mathbf{b}))$ such that $\mathbf{c}[i] \stackrel{\alpha}{\equiv} f(\mathbf{b}[i])$, the tuple $\mathbf{c}$ belongs to Q.

Remark 50 *Polynomially closed subalgebras of Mal'tsev algebras are congruence blocks. In the general case their structure is more intricate. The intuition (although not entirely correct) is that if for some block B of a congruence β and a congruence α with $\alpha \prec \beta$ the set $B/_{\alpha}$ contains several as-components, a polynomially closed subalgebra contains some of them and has empty intersection with the rest. However, since this is true only for factor sets, and we do not even consider non-as-maximal elements, the actual structure is more 'fractal'.*

The following lemma follows from the definitions and the fact that congruences are invariant under polynomials.

Lemma 51 *(1) Let R be a subdirect product of $\mathbb{A}_1,\dots,\mathbb{A}_n$ and Q_1,Q_2 relations polynomially closed in R, and $Q_1\cap Q_2\cap\mathsf{umax}(Q)\ne\varnothing$. Then $Q_1\cap Q_2$ is polynomially closed in R.*

In particular, let $\beta_i\in\mathsf{Con}(\mathbb{A}_i)$ and B_i a u-maximal β_i-block. Then $Q_1\cap\overline{B}$ is polynomially closed in R.

If Q_1,Q_2 are as-closed in R, then $Q_1\cap Q_2$ is as-closed in R.

(2) Let Q_i be polynomially closed in R_i, $i\in[k]$, and let R,Q be pp-defined through $R_1,\dots,R_k$ and $Q_1,\dots,Q_k$, respectively, by the same pp-formula Φ; that is, $R=\Phi(R_1,\dots,R_k)$ and $Q=\Phi(Q_1,\dots,Q_k)$. Let also $\mathsf{umax}(Q)\cap\mathsf{umax}(R)\ne\varnothing$. Then Q is polynomially closed in R.

If Q_i is as-closed in R_i then Q is as-closed in R.

(3) Let R be a subdirect product of $\mathbb{A}_1,\dots,\mathbb{A}_n$, $\beta_i\in\mathsf{Con}(\mathbb{A}_i)$, $i\in[n]$, and let Q be polynomially closed in R. Then $Q/_{\overline{\beta}}$ is polynomially closed in R.

If Q is as-closed in R then $Q/_{\overline{\beta}}$ is as-closed in $R/_{\overline{\beta}}$.

The following condition is slightly stronger than chaining. We call relation R *strongly chained* with respect to $\overline{\beta},\overline{B}$, where $\beta_i\in\mathsf{Con}(\mathbb{A}_i)$ and B_i is a β_i-block for $i\in[n]$, if

(Q1s) for any $I\subseteq[n]$ and $\alpha,\beta\in\mathsf{Con}(\mathrm{pr}_I R)$ such that $\alpha\prec\beta\le\overline{\beta}_I$, α and β are $\mathcal{U}_B$-chained in R, where $\mathcal{U}_B$ is the set of all $\overline{B}$-preserving polynomials of R

(Q2s) for any $\alpha,\beta\in\mathsf{Con}(\mathrm{pr}_I R)$, $\gamma,\delta\in\mathsf{Con}(\mathbb{A}_j)$, $j\in[n]$, such that $\alpha\prec\beta\le\overline{\beta}_I$, $\gamma\prec\delta\le\beta_j$, and (α,β) can be separated from (γ,δ), and α and β are $\mathcal{U}^*$-chained in R, where $\mathcal{U}^*$ is the set of all $\overline{B}$-preserving polynomials g of R such that $g(\delta)\subseteq\gamma$

As in the definition of chained relations a polynomial from $\mathcal{U}^*$ in condition (Q2s) will be called $(\gamma,\delta,\overline{B})$-*good*.

Let $\mathcal{P}=(V,\mathcal{C})$ be a (2,3)-minimal and block-minimal instance over $\mathcal{A}$. Let $\beta_v\in\mathsf{Con}(\mathbb{A}_v)$ and let B_v be a β_v-block, $\overline{\beta}=(\beta_v\mid v\in V)$, $\overline{B}=(B_v\mid v\in V)$. Let $\mathcal{R}=\{R_{C,v,\alpha\beta}\mid C=\langle\mathbf{s},R\rangle\in\mathcal{C},(v,\alpha,\beta)\in\mathcal{W}(\overline{\beta})\}$ be a collection of relations such that $R_{C,v,\alpha\beta}$ is a subalgebra of $\mathrm{pr}_{\mathbf{s}\cap W_{v,\alpha\beta,\overline{\beta}}}R$. Let $C=\langle\mathbf{s},R\rangle$, $(v,\alpha,\beta)\in\mathcal{W}$, and $W=W_{v,\alpha\beta,\overline{\beta}}$. Let $\mathbf{a}$ be a tuple from $\mathrm{pr}_X R$ for $X\subseteq\mathbf{s}$, or from $\mathrm{pr}_X\mathcal{S}_W$, $X\subseteq W$, if $(v,\alpha,\beta)\notin\mathcal{W}'$, or from $\mathrm{pr}_X\mathcal{S}_{W,Y}$ if $(v,\alpha,\beta)\in\mathcal{W}'$, where $X\subseteq W$ and Y is a set specified in the condition of block-minimality. Tuple $\mathbf{a}$ is said to be $\mathcal{R}$-*compatible* if for any $(w,\gamma,\delta)\in\mathcal{W}(\overline{\beta})$, (let $U=W_{w,\gamma\delta,\overline{\beta}}$) $\mathrm{pr}_{X\cap U}\mathbf{a}\in\mathrm{pr}_{X\cap U}R_{C,w,\gamma\delta}$ or $\mathrm{pr}_{X\cap U}\mathbf{a}\in\mathrm{pr}_{X\cap U}R_{C,w,\gamma\delta}/_{\overline{\mu}^Y}$ for an appropriate set Y. By $R^{\mathcal{R}},\mathcal{S}^{\mathcal{R}}_W,\mathcal{S}^{\mathcal{R}}_{W,Y}$ we denote the set of all $\mathcal{R}$-compatible tuples from the corresponding relation.

The collection $\mathcal{R}$ is called a $\overline{\beta}$-*strategy* with respect to $\overline{B}$ if it satisfies the following conditions for every $(v,\alpha,\beta)\in\mathcal{W}(\overline{\beta})$, and every $C=\langle\mathbf{s},R\rangle\in\mathcal{C}$ (let

$W = W_{w,\alpha\beta,\overline{\beta}}$):

(S1) the relations $\mathsf{umax}(R^{X,\mathcal{R}})$, where $R^{X,\mathcal{R}}$ consists of $\mathcal{R}$-compatible tuples from R^X for $X \subseteq V$, $|X| \le 2$, form a nonempty $(2,3)$-strategy for $\mathcal{P}^{\mathcal{R}}$;

(S2) for every $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$ (let $U = W_{w,\gamma\delta}$) and every $\mathbf{a} \in \mathsf{umax}(\mathrm{pr}_{\mathbf{s}\cap W\cap U} R_{C,v,\alpha\beta})$ it holds: if $(w,\gamma,\delta) \notin \mathcal{W}'$ then $\mathbf{a}$ extends to an $\mathcal{R}$-compatible solution φ of $\mathcal{P}_U$; otherwise if $(v,\alpha,\beta) \notin \mathcal{W}'$ then $\mathbf{a}$ extends to an $\mathcal{R}$-compatible solution of $\mathcal{P}_U/_{\overline{\mu}^{Y_1}}$ with $Y_1 = \mathsf{MAX}(\mathcal{P}) - (W \cap U)$; and if $(v,\alpha,\beta) \in \mathcal{W}'$ then $\mathbf{a}$ extends to an $\mathcal{R}$-compatible solution of $\mathcal{P}_U/_{\overline{\mu}^{Y_2}}$, where $Y_2 = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$;

(S3) $R \cap \overline{B}_{\mathbf{s}} \neq \varnothing$ and for any $I \subseteq \mathbf{s}$ any $\mathcal{R}$-compatible tuple $\mathbf{a} \in \mathsf{umax}(\mathrm{pr}_I R)$ extends to an $\mathcal{R}$-compatible tuple $\mathbf{b} \in R$.

(S4) the relation $R_{C,v,\alpha\beta}$ is a subalgebra of $\mathrm{pr}_{\mathbf{s}\cap W} R$, and $\mathsf{umax}(R_{C,v,\alpha\beta}) \subseteq \mathsf{umax}(\mathrm{pr}_{\mathbf{s}\cap W} R)$; if $(v,\alpha,\beta) \notin \mathcal{W}'$ then the relation $\mathcal{S}_W^{\mathcal{R}}$ is a subalgebra of $\mathcal{S}_W$, and $\mathsf{umax}(\mathcal{S}_W^{\mathcal{R}}) \subseteq \mathsf{umax}(\mathcal{S}_W)$; if $(v,\alpha,\beta) \in \mathcal{W}'$ then for any $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta}) - \mathcal{W}'$ the relations $\mathcal{S}^{\mathcal{R}}_{W,Y_1}, \mathcal{S}^{\mathcal{R}}_{W,Y_2}$ are subalgebras of $\mathcal{S}_{W,Y_1}, \mathcal{S}_{W,Y_2}$, respectively, and $\mathsf{umax}(\mathcal{S}^{\mathcal{R}}_{W,Y_1}) \subseteq \mathsf{umax}(\mathcal{S}_{W,Y_1})$, $\mathsf{umax}(\mathcal{S}^{\mathcal{R}}_{W,Y_2}) \subseteq \mathsf{umax}(\mathcal{S}_{W,Y_2})$, where $Y_1 = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$ and $Y_2 = \mathsf{MAX}(\mathcal{P}) - (W \cap W_{w,\gamma\delta})$;

(S5) for every $w \in \mathbf{s}$ and every $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$ (let $U = W_{w,\gamma\delta}$) with $w \in \mathbf{s} \cap U$ it holds $\mathsf{umax}(\mathrm{pr}_w R_{C,w,\gamma\delta}) = \mathsf{umax}(\mathrm{pr}_w R_{C,v,\alpha\beta})$, let $A_{\mathcal{R},w}$ denote the subalgebra generated by this set, $\mathsf{umax}(A_{\mathcal{R},w})$ is as-closed in $\mathsf{umax}(\mathrm{pr}_w(R \cap \overline{B}))$;

(S6) for every $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$ with $\mathbf{s} \cap W_{w,\gamma\delta} \neq \varnothing$ the set of $\mathcal{R}$-compatible tuples from $R_{C,w,\gamma\delta}$ is polynomially closed in $\mathrm{pr}_{\mathbf{s}\cap W_{w,\gamma\delta}} R$;

(S7) relation R is strongly chained with respect to $\overline{\beta}, \overline{B}$; if $(v,\alpha,\beta) \notin \mathcal{W}'$, relation $\mathcal{S}_W$ is strongly chained with respect to $\overline{\beta}, \overline{B}$; if $(v,\alpha,\beta) \in \mathcal{W}'$, for any $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta}) - \mathcal{W}'$ the relations $\mathcal{S}_{W,Y_1}, \mathcal{S}_{W,Y_2}$, $Y_1 = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$, $Y_2 = \mathsf{MAX}(\mathcal{P}) - (W \cap W_{w,\gamma\delta})$, are strongly chained with respect to $\overline{\beta}, \overline{B}$.

Conditions (S1)–(S3) are the conditions we actually want to maintain when transforming a strategy, and these are the ones that provide the desired results. However, to prove that (S1)–(S3) are preserved under transformations of a strategy we also need more technical conditions (S4)–(S7).

Let $\mathcal{P}$ be a block-minimal instance, $\beta_v = \underline{1}_v$ and $B_v = \mathbb{A}_v$ for $v \in V$. Then as is easily seen the collection of relations $\mathcal{R} = \{R_{C,v,\alpha\beta} \mid (v,\alpha,\beta) \in \mathcal{W}(\overline{\beta})\}$ given by $R_{C,v,\alpha\beta} = \mathrm{pr}_{\mathbf{s}\cap W_{v,\alpha\beta,\overline{\beta}}} R$ for $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ is a $\overline{\beta}$-strategy with respect to $\overline{B}$. Also, by (S3) a $\overline{\gamma}$-strategy with $\gamma_v = \underline{0}_v$ gives a solution of $\mathcal{P}$. Our goal is therefore to show that a $\overline{\beta}$-strategy for any $\overline{\beta}$ can be 'reduced', that is, transformed to a $\overline{\beta}'$-strategy for some $\overline{\beta}' < \overline{\beta}$. Note that this reduction of strategies is where the condition $\mathsf{MAX}(\mathcal{P}) \cap \mathsf{Center}(\mathcal{P}) = \varnothing$. Indeed, suppse that $\beta_v = \mu_v^*$. Then by conditions (S1)–(S7) we only have information about solutions to problems of the form $\mathcal{P}_W/_{\overline{\mu}^*}$ or something very close to that. Therefore this barrier cannot be penetrated. We consider two cases.

CASE 1. There are $v \in V$ and $\alpha \prec \beta_v$ nontrivial on B_v, $\mathsf{typ}(\alpha, \beta_v) = \mathbf{2}$. This case is considered in Section 6.2.

CASE 2. For all $v \in V$ and $\alpha \prec \beta_v$ nontrivial on B_v $\mathsf{typ}(\alpha, \beta_v) \in \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$. This case is considered in Section 6.3.

6 Proof of Theorem 45

In the remaining part of the paper we prove Theorem 45.

6.1 Auxiliary lemmas

We start with several auxiliary lemmas. The last one, Lemma 55 basically proves that condition (S7) remains true when a β-strategy gets tightened, we will use it in the subsequent proofs in a very straightforward way. The first two lemmas study the structure of binary relations that have in their domains a pair of prime intervals of type **2** that cannot be separated. They show that if we restrict ourselves to blocks of the link congruences then this structure is very uniform. The third lemma, Lemma 54, is an important technical result. To explain what it amounts to saying consider this: let $Q \subseteq \mathbb{A}' \times \mathbb{B}'$ be a subdirect product and the link congruence of $\mathbb{A}'$ is the equality relation. Then, clearly, Q is the graph of a mapping $\sigma : \mathbb{B}' \to \mathbb{A}'$, and the kernel of this mapping is the link congruence η of $\mathbb{B}'$ with respect to Q. Suppose now that Q is a subalgebra of R, a subdirect product of $\mathbb{A} \times \mathbb{B}$ such that $\mathbb{A}'$ is a subalgebra of $\mathbb{A}$ and $\mathbb{B}'$ is a subalgebra of $\mathbb{B}$. Then the restriction of the link congruence of $\mathbb{A}$ with respect to R to $\mathbb{A}'$ does not have to be the equality relation, and similarly the restriction of the link congruence of $\mathbb{B}$ to $\mathbb{B}'$ does not have to be η. Most importantly, the restriction of $\mathsf{Cg}(\eta)$, the congruence of $\mathbb{B}$ generated by η, to $\mathbb{B}'$ does not have to be η. Lemma 54 shows, however, that this is exactly what happens when Q and $\mathbb{A}', \mathbb{B}'$ satisfy some additional conditions, such as being chained and polynomially closed.

In the next two lemmas let R be a subdirect product of $\mathbb{A}_1 \times \mathbb{A}_2$, β_1, β_2 congruences of $\mathbb{A}_1, \mathbb{A}_2$ and B_1, B_2 β_1- and β_2-blocks, respectively; R is chained with respect to $(\beta_1, \beta_2), (B_1, B_2)$ and $R^* = R \cap (B_1 \times B_2)$, $B_1^* = \mathrm{pr}_1 R^*, B_2^* = \mathrm{pr}_2 R^*$. Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_1)$, $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_2)$ be such that $\alpha \prec \beta \leq \beta_1$, $\gamma \prec \delta \leq \beta_2$,

$\mathsf{typ}(\alpha,\beta) = \mathsf{typ}(\gamma,\delta) = \mathbf{2}$, and $(\alpha,\beta),(\gamma,\delta)$ cannot be separated. Let also $\zeta_1 = \zeta(\alpha,\beta)|_{B_1^*}$, $\zeta_2 = \zeta(\gamma,\delta)|_{B_2^*}$ and $\mathsf{lk}_1^*, \mathsf{lk}_2^*$ the link congruences of B_1^*, B_2^*, respectively, with respect to R^*. Let F, G be ζ_1-, ζ_2-blocks such that $R^* \cap (F \times G) \neq \varnothing$ and F, G contain nontrivial β- and δ-blocks A, B, respectively. By Lemma 39 all the β-blocks $A' \in F/_\beta$, $A \sqsubseteq_{asm} A'$ in F (respectively, all δ-blocks $B' \in G/_\delta$, $B \sqsubseteq_{asm} B'$ in G) are also nontrivial. Note that by Lemma 41 $\mathsf{lk}_1^* \leq \zeta_1$ and $\mathsf{lk}_2^* \leq \zeta_2$. Let also $D \subseteq F, E \subseteq G$ be blocks of $\mathsf{lk}_1^*, \mathsf{lk}_2^*$ such that $R^* \cap (D \times E) \neq \varnothing$.

Lemma 52 *Let lk_2^* be nontrivial on the δ-block B, such that $B \cap E \neq \varnothing$, that is, $\mathsf{lk}_2^* \wedge \delta$ is not the equality relation on B. Then (1) $\delta|_{\mathsf{umax}(G)} \leq \mathsf{lk}_2^* \vee \gamma|_{B_2^*}$; and (2) any $B' \in G/_\delta$ with $B \sqsubseteq_{asm} B'$ in $G/_\delta$ is nontrivial. In particular, $\mathsf{umax}(D), \mathsf{umax}(E)$ and $\mathsf{umax}(F), \mathsf{umax}(G)$ do not intersect any trivial β- and δ-blocks, respectively.*

Proof: Since $\mathsf{lk}_1^* \leq \zeta_1$ and $\mathsf{lk}_2^* \leq \zeta_2$, (2) follows by Lemma 39. Also, as lk_2^* is nontrivial on a δ-block, we obtain (1) by Lemma 28. □

Lemma 53 *Let $\delta|_{\mathsf{umax}(E)} \leq \mathsf{lk}_2^* \vee \gamma|_{B_2^*}$, and let A be a β-block, $A \cap D \neq \varnothing$ and B a δ-block such that $A \cap D$ belongs to $\mathsf{umax}(D/_{\mathsf{lk}_1^* \wedge \beta})$ and $B \in \mathsf{umax}(E/_\delta)$. (Or, $A \subseteq F, B \subseteq E$ are such that $A \in \mathsf{umax}(F/_\beta)$ and $B \in \mathsf{umax}(G/_\delta)$.) Then either $R \cap (A \times B) = \varnothing$, or for any $c \in A$ with $B \cap R[c] \neq \varnothing$ we have $B/_\gamma \subseteq R[c]/_\gamma$.*

Proof: We prove the lemma for $A \in \mathsf{umax}(D/_{\mathsf{lk}_1^* \wedge \beta})$ and $B \in \mathsf{umax}(E/_\delta)$. The case $A \in \mathsf{umax}(F/_\beta)$, $B \in \mathsf{umax}(G/_\delta)$ follows.

We assume $\gamma = \underline{0}_2$. If B is contained in an as-component of E then the result follows by Corollary 18, since $B/_\gamma$ is a module. Now we show that if the result is true for B then it is also true for $B' \in E/_\delta$ and BB' is a thin edge in $E/_\delta$. Note also that if $B \cap \mathsf{umax}(E) \neq \varnothing$ or $B \cap \mathsf{amax}(E) \neq \varnothing$, then $B \subseteq \mathsf{umax}(E)$ and $B \subseteq \mathsf{amax}(E)$, respectively. Suppose that $\{c\} \times B \subseteq R$ for some $c \in \mathsf{umax}(D)$ and $B \subseteq \mathsf{umax}(E)$; as we observed this is true for some B contained in an as-component of E. We consider 3 cases; together they imply the result.

CASE 1. There is $d \in D$ such that cd is a thin edge and $(d, e) \in R$ for some $e \in B$. Then $\{d\} \times B \subseteq R$.

This case follows from Lemma 16.

CASE 2. There is $B' \subseteq \mathsf{umax}(E)$ such that BB' is a thin edge in $\mathbb{A}_2/_\delta$ and $(c, e') \in R$ for some $e' \in B'$.

Let $b \in B$, $b' \in B'$ be elements such that bb' is a thin edge. Let $f(x)$ be the unary polynomial of R constructed as in Lemma 38, that is,

$f(x) = x \cdot \begin{pmatrix} c \\ b' \end{pmatrix}$, $f(x) = t_{bb'}\left(x, \begin{pmatrix} c \\ b' \end{pmatrix}\right)$, or $f(x) = h_{bb'}\left(x, \begin{pmatrix} c \\ b \end{pmatrix}, \begin{pmatrix} c \\ b' \end{pmatrix}\right)$, depending on the type of bb'. Then $f : B \to B'$ is a bijection, and therefore maps $\{c\} \times B \subseteq R$ onto $\{c\} \times B'$, implying $\{c\} \times B' \subseteq R$.

Before tackling the last case let $R^\dagger = R \cap (\mathsf{umax}(D) \times \mathsf{umax}(E))$; note that $R^\dagger$ is not necessarily linked. Let $\mathsf{lk}^\dagger_1, \mathsf{lk}^\dagger_2$ be equivalence relations on $\mathsf{umax}(D)$ and $\mathsf{umax}(E)$, respectively, defined as the link congruences, except that $\mathsf{umax}(D)$, $\mathsf{umax}(E)$ are not necessarily subalgebras.

CASE 3. Relations $\mathsf{lk}^\dagger_1, \mathsf{lk}^\dagger_2$ are nontrivial and there is $d \in D$ such that cd is a thin edge and $c \stackrel{\mathsf{lk}^\dagger_1}{\not\equiv} d$.

We prove that there is a δ-block B' such that BB' is a thin edge in $\mathbb{A}_2/_\delta$ of the same type as cd and $(d, e) \in R$ for some $e \in B'$. Indeed, consider a tuple $(c, b) \in R$ for some $b \in B$ and the thin edge cd. By Lemma 12(4) there is $e \in \mathbb{A}_2$ such that $(d, e) \in R$ and either be is a thin edge of the same type as cd, or, if cd is affine, b can be chosen this way. Since $d \in D$, we have $e \in E$, and therefore can set $B' = e^\delta$. Then similar to Case 2 depending on the type of cd we consider polynomial $f(x) = x \cdot \begin{pmatrix} d \\ e \end{pmatrix}$, $f(x) = t_{cd}\left(x, \begin{pmatrix} d \\ e \end{pmatrix}\right)$, or $f(x) = h_{cd}\left(x, \begin{pmatrix} c \\ b \end{pmatrix}, \begin{pmatrix} d \\ e \end{pmatrix}\right)$ for some $b \in B$. We have $f(c) = d$ and $f : B \to B'$ is a bijection by Lemma 38, thus proving that $\{d\} \times B' \subseteq R$. □

Now, let again R be a subdirect product of $\mathbb{A}_1 \times \mathbb{A}_2$, β_1, β_2 congruences of $\mathbb{A}_1, \mathbb{A}_2$ and B_1, B_2 β_1- and β_2-blocks, respectively. Also, let R be chained with respect to $(\beta_1, \beta_2), (B_1, B_2)$ and $R^* = R \cap (B_1 \times B_2)$, $B^*_1 = \mathrm{pr}_1 R^*$, $B^*_2 = \mathrm{pr}_2 R^*$. Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_1)$ be such that $\alpha \prec \beta \le \beta_1$. This time we do not assume that $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$.

Lemma 54 *Suppose $\alpha = \underline{0}_1$ and let R' be a polynomially closed subalgebra of R^* and such that $B'_1 = \mathrm{pr}_1 R'$ contains an as-component C of B^*_1 and $R' \cap \mathsf{umax}(R^*) \neq \varnothing$. Then either*

(1) $C \times \mathsf{umax}(B''_2) \subseteq R'$, where $B''_2 = R'[C]$, or

(2) there are $\eta, \theta \in \mathsf{Con}(\mathbb{A}_2)$ with $\eta \prec \theta \le \beta_2$ such that these intervals cannot be separated.

Moreover, in case (2) $R' \cap (C \times \mathsf{umax}(B''_2))$ is the graph of a mapping $\varphi : B''_2 \to C$ such that the kernel of φ is the restriction of η on B''_2.

Proof: Note that if $|C| = 1$, the lemma is trivially true. Let $B'_2 = \mathrm{pr}_2 R'$. We assume $\beta_2|_{B'_2} \neq \lambda|_{B'_2}$ for any congruence $\lambda \le \beta_2$; otherwise replace β_2 with λ. Let $\mathsf{lk}'_1, \mathsf{lk}'_2$ be the link congruences of B'_1, B'_2 with respect to R'. We need to show that either lk'_1 is the equality relation on C or C is in a lk'_1-block. Suppose lk'_1 is nontrivial on C. Let $\eta \leq \beta_2$ be such that $\eta|_{\mathsf{umax}(B'_2)} \subseteq \mathsf{lk}'_2$ and η is maximal among congruences of $\mathbb{A}_2$ with this property. We show that η is the required congruence. If η is the total relation on $\mathsf{umax}(B'_2)$, we are done; otherwise there are two cases.

CASE 1. For some $\theta \in \mathsf{Con}(\mathbb{A}_2)$ with $\eta \prec \theta$ the intervals $(\underline{0}_1, \beta_1), (\eta, \theta)$ can be separated.

In this case we prove that η has to be β_2 and we have option (1) of the lemma. Since R is chained, by Lemma 27(4) there is a set $T \subseteq B^*_1 \times B^*_1$ of $(\underline{0}_1, \beta_1)$-subtraces such that any pair of elements from $\mathsf{umax}(B^*_1)$ belongs to the transitive closure of T, and for any $(a, b) \in T$ there is a $\overline{B}$-preserving polynomial f such that $f(a) = a, f(b) = b$, and $f(\theta|_{B^*_2}) \subseteq \eta$. This means that C belongs to the lk^*_1-block of B^*_1, where lk^*_1 is the link congruence with respect to $R^*/_\eta$. Therefore $C \times \mathsf{umax}(R^*[C])/_\eta \subseteq R^*/_\eta$. Observe that as $R' \subseteq R^*$, the link congruence of B^*_1 with respect to R^* retricted to C contains $\mathsf{lk}'_1|_C$. Therefore, we also have $C \times \mathsf{umax}(R^*[C]) \subseteq R^*$. Note that both $R^*[C]$ and B'_2 contain a u-maximal element from B^*_2. Since $B''_2 \subseteq R^*[C]$, we have $\mathsf{umax}(B''_2) \subseteq \mathsf{umax}(R^*[C])$. Therefore $C \times \mathsf{umax}(B''_2) \subseteq R^*$.

We are going to argue that the same inclusion holds for R'. But first we show that for any thin semilattice or affine edge ab of C and any $c \in \mathsf{umax}(R^*[C])$ there is a polynomial g such that $g(a) = a, g(b) = b$, $f(\theta|_{B^*_2}) \subseteq \eta$, and $g(c) = c$. Note that since R is chained, all such pairs $\{a, b\}$ belong to T. Since every pair of elements of C is a $(\underline{0}_1, \beta_1)$-subtrace, again, as R is chained, and by Lemma 27(5) this is true for some $c \in R^*[C]$. Suppose cc' is a thin edge in $R^*[C]$; by Lemma 16 this implies that $(a, c), (b, c), (a, c'), (b, c') \in R$. Then as in Lemma 40 we find a polynomial satisfying the required properties for c'. Specifically, $g'(x) = g(x) \cdot \begin{pmatrix} a \\ c' \end{pmatrix}$, $g'(x) = t\left(g(x), \begin{pmatrix} a \\ c' \end{pmatrix}\right)$, and $g'(x) = h'\left(g(x), \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} a \\ c' \end{pmatrix}\right)$, where t and h' are operations from Lemma 9(3), depending on the type of cc' and ab.

Now we are back to proving that $C \times \mathsf{umax}(B''_2) \subseteq R'$. Observe that $R'/_{\mathsf{lk}'_1 \times \mathsf{lk}'_2}$ is the graph of a bijective mapping $\varphi : B'_1/_{\mathsf{lk}'_1} \to B'_2/_{\mathsf{lk}'_2}$. Take $a, b \in C$ and $c \in \mathsf{umax}(B''_2)$ such that $(a, c) \in R'$, ab is a thin semilattice or affine edge and $(a, b) \notin \mathsf{lk}'_1$. Let also $(b, d) \in R'$. By what is proved there is a polynomial f of R such that $f(a) = a, f(b) = b$, $f(c) = c \stackrel{\eta}{\equiv} d' = f(d)$, and $f(\theta|_{B^*_2}) \subseteq \eta$. In particular, $(a, d'), (b, d') \in R$. Since $(a, c) \in R'$ and $(a, c) \leq (b, c')$ or $(a, c)(b, c')$ is an affine edge for some $c' \in \mathsf{Sg}(c, d')$, we obtain $(b, c') \in R'$, as R' is polynomially closed. Since $c' \stackrel{\eta}{\equiv} c$ and $\eta \leq \mathsf{lk}'_2$, we get a contradiction with $(a, b) \notin \mathsf{lk}'_1$.

CASE 2. For all $\theta \in \mathsf{Con}(\mathbb{A}_2)$ with $\eta \prec \theta$ the intervals $(\underline{0}_1, \beta_1), (\eta, \theta)$ cannot be separated.

Suppose $\mathsf{lk}'_2|_{B''_2} \neq \eta|_{B''_2}$. Without loss of generality let $\eta = \underline{0}_2$. Then there are $a, b \in B''_2$ and $c \in C$ such that $(c, a), (c, b) \in R'$, or there are $a, b \in C$ and $c \in B''_2$ such that $(a, c), (b, c) \in R'$. Since $\eta|_{B''_2} < \mathsf{lk}'_2|_{B''_2}$, it is easy to see that the former configuration can always be found. If $\mathsf{typ}(\underline{0}_1, \beta) = \mathbf{3}$ then by Lemma 32 such a, b do not exists, as long as $(\underline{0}_1, \beta_1), (\eta, \theta)$ cannot be separated. Finally, if $\mathsf{typ}(\underline{0}_1, \beta) \in \{\mathbf{4}, \mathbf{5}\}$, C is a singleton by Lemma 23, and the result is trivial.

Suppose now that $\mathsf{typ}(\underline{0}_1, \beta_1) = \mathsf{typ}(\underline{0}_2, \theta) = \mathbf{2}$. Since R is chained a, b can be assumed to be from $\mathsf{umax}(B''_2)$, and so $\eta|_{\mathsf{umax}(B''_2)} < \mathsf{lk}'_2|_{(B''_2)}$. Also, $B'_2 = B''_2$ in this case. We prove that $\theta|_{\mathsf{umax}(B''_2)} \subseteq \mathsf{lk}'_2$ producing a contradiction with the choice of η. In this case B^*_1 is a module $C = B^*_1$, and by Lemma 41 $R[B^*_1]$ is a subset of a $\zeta(\underline{0}_2, \theta)$-block. Then $\theta \le \mathsf{lk}''_2 \vee \eta$, where lk''_2 is the link congruence of $\mathbb{A}_2$ with respect to R, and as R is chained, $\theta|_{\mathsf{umax}(E)} \le \mathsf{lk}^*_2 \vee \eta|_{B^*_2}$ for any lk^*_2-block $E \subseteq B^*_2$. Thus we are in the conditions of Lemma 53. Therefore if $(c, d) \in R'$ then $(c, e) \in R$ for any $e \stackrel{\theta}{\equiv} d$ for any $c \in C$ and $d, e \in \mathsf{umax}(R[C])$.

Again, we now extend this property to R'. Since any pair $\{a, b\} \subseteq B^*_2$ with $a \stackrel{\theta}{\equiv} b$ is a (η, θ)-subtrace, as R is chained, for any $a', b' \in B^*_2$ with $a' \stackrel{\theta}{\equiv} b'$ there is a (B_1, B_2)-preserving polynomial f such that $f(a) = a'$ and $f(b) = b'$. Now, use the pairs $(c, a), (c, b) \in R'$. For any $b' \in \mathsf{umax}(B''_2)$ with $b \stackrel{\theta}{\equiv} b'$, let $b'' \in b^\theta$ be such that $h(b, b'', a) = b'$, where h is the function from Theorem 5; such b'' exists because $h(b, x, a)$ is a permutation on every θ-block (recall that a θ-block is a module in this case). Since R is chained, there is a polynomial f such that $f(a) = a, f(b) = b''$ and $f(c) = d$ for some $d \in B^*_1 = C$. The mapping $g(x) = h\left(x, f(x), \begin{pmatrix} d \\ a \end{pmatrix}\right)$ is such that $g\begin{pmatrix} c \\ a \end{pmatrix} = \begin{pmatrix} c \\ a \end{pmatrix}$ and $g\begin{pmatrix} c \\ b \end{pmatrix} = \begin{pmatrix} c \\ b' \end{pmatrix}$, because, again, B^*_1 is a module. Since R' is polynomially closed and $(c, b) \sqsubseteq_{as} (c, b')$ we have $(c, b') \in R'$; and as b' is arbitrary from a^θ, we have $\{c\} \times a^\theta \subseteq R'$. Thus, we have proved the property for a specific θ-block; next we extend it to other θ-blocks.

Suppose $\{c\} \times E \subseteq R'$ for some θ-block E and a θ-block E' is such that for some $a \in E, b \in E' \cap B''_2$, ab is a thin edge and $(d, b) \in R'$ for some $d \in C$. Then by Lemma 38 mapping $g(x)$ that is defined as $x \cdot \begin{pmatrix} d \\ b \end{pmatrix}$, $t_{ab}\left(x, \begin{pmatrix} d \\ b \end{pmatrix}\right)$, $h_{ab}\left(x, \begin{pmatrix} c \\ a \end{pmatrix}, \begin{pmatrix} d \\ b \end{pmatrix}\right)$ depending on the type of ab is injective on E. In particular, if ab is semilattice or majority then g maps $\{c\} \times E$ to $\{c\} \times E'$, $g(c, a) = (c, b), g(c, a') = (c, b')$ and $b \neq b'$ whenever $a \neq a'$; and since all the tuples involved belong to R', $(c, b), (c, b') \in R'$. If ab is affine then g maps $\{c\} \times E$ to $\{d\} \times E'$, and $g(c, a) = (d, b), g(c, a') = (d, b')$ and $b \neq b'$ whenever $a \neq a'$, and $(d, b), (d, b') \in R'$. In either case, lk_2 is nontrivial on E', and applying the argument from the previous paragraph we obtain $\{c\} \times E' \subseteq R'$ or $\{d\} \times E' \subseteq R'$. Therefore $\theta|_{\mathsf{umax}(B_2'')} \subseteq \lambda_2|_{\mathsf{umax}(B_2'')}$, a contradiction with the choice of η. □

Finally we prove that the property to be strongly chained is preserved under certain transformations of $\overline{\beta}$ and $\overline{B}$.

Lemma 55 *Let R be a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$, $\beta_i \in \mathsf{Con}(\mathbb{A}_i)$ and B_i a β_i-block, $i \in [n]$, such that R is strongly chained with respect to $\overline{\beta}, \overline{B}$. Let $R' = R \cap (B_1 \times \cdots \times B_n)$ and $B_i' = \mathrm{pr}_i R'$. Let $i \in [n]$, $\beta_i' \prec \beta_i$, and let D_i be a β_i'-block that is a member of a nontrivial as-component of $B_i'/_{\beta_i'}$. Let also $\beta_j' = \beta_j$ and $D_j = B_j$ for $j \neq i$. Then R is strongly chained with respect to $\overline{\beta}', \overline{D}$.*

Proof: Let $R'' = R \cap (D_1 \times \cdots \times D_n)$ and $D_i' = \mathrm{pr}_i R''$. Take I, j from the definition of being chained. Let $I = [\ell]$; if $|I| > 1$ we may consider R as a subdirect product of $\mathrm{pr}_I R$ and $\mathbb{A}_{\ell+1}, \dots, \mathbb{A}_n$, so we assume $|I| = 1$ and $j = n$ in (Q2s). Let $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_1)$, $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_n)$ be such that $\alpha \prec \beta \leq \beta_1$, $\gamma \prec \delta \leq \beta_n$. Clearly, we may assume $\alpha = \underline{0}_1$, $\gamma = \underline{0}_n$, and $\beta_i' = \underline{0}_i$. Note that replacing R with the $n+1$-ary relation $\{(\mathbf{a}, \mathbf{a}[i]) \mid \mathbf{a} \in R\}$ we may assume that $i \notin I \cup \{j\}$. Without loss of generality assume $i = 2$. By the assumption $\beta_2' = \underline{0}_2$ the classes of β_2' are just elements of $\mathbb{A}_2$, so let B_2' be denoted by c. Let C be the as-component of B_2' containing c. We divide the proof into two cases, depending on whether or not $Q = \mathrm{pr}_{12} R'$ is linked. First, we consider the case when Q is not linked, this case is relatively easy.

CLAIM 1. Let $Q' = Q \cap (\mathsf{umax}(\mathrm{pr}_1 Q) \times C)$ be not linked and $\mathsf{lk}_1, \mathsf{lk}_2$ link congruences of Q. Then $\mathsf{lk}_2 = \underline{0}_2$ and either $\beta \leq \mathsf{lk}_1$ or β is trivial on D_1.

Relation Q is a subalgebra of $R \cap (B_1 \times B_2)$ and is polynomially closed in $\mathrm{pr}_{12} R$ by Lemma 51. By Lemma 54 if Q' is not linked then it is the graph of a mapping $\varphi : \mathrm{pr}_1 Q' \to C$. Since by (Q1s) for $\overline{\beta}, \overline{B}$ and Lemma 28, if $\mathsf{lk}_2 \neq \underline{0}_2$ then lk_2 restricted to C is also nontrivial, this means $\mathsf{lk}_2 = \underline{0}_2$ and lk_1 is the restriction of a congruence η of $\mathbb{A}_1$ onto $\mathrm{pr}_1 Q$. If $\beta \leq \eta$ then obtain the first option, otherwise $\mathsf{lk}_1 \cap \beta = \underline{0}_1$ and we have the second option.

Note that if $\beta \leq \mathsf{lk}_1$ then any $\overline{B}$-preserving polynomial that maps an (α, β)-subtrace from D_1' on a subtrace from D_1' is also $\overline{D}$-preserving, because $\mathsf{lk}_2 = \underline{0}_2$; the result follows. If β is trivial on D_1', there is nothing to prove. Therefore we may assume Q' is linked.

Let E be a β-block such that $E'' = E \cap D_1' \neq \varnothing$ and $E' = E \cap B_1'$. Consider $R^* = R' \cap (B_1 \times C \times B_3 \times \cdots \times B_n)$. Note that R^* is not necessarily a subalgebra. Let $B_i^* = \mathrm{pr}_i R^*$, $i \in [n]$, and $E^* = E \cap B_1^*$. By Lemma 12(4) $\mathsf{amax}(E^*)$ is a union of as-components of E'. Also, by Proposition 19 $\mathsf{umax}(E^*) \times C \subseteq Q$, and therefore $\mathsf{umax}(E^*) = \mathsf{umax}(E'') \subseteq \mathsf{umax}(E')$. In particular, $\mathsf{amax}(E'')$ is a union of as-components of E'. The last inclusion here is because E^* contains some as-maximal elements of E'.

First we prove condition (Q1s) for $\overline{\beta}'$ and $\overline{D}$.

CLAIM 2. For any $a, b, a', b' \in E''$ such that a, b belong to the same as-component of E'' there is a $(\gamma, \delta, \overline{D})$-good polynomial f with $f(\{a', b'\}) = \{a, b\}$.

Consider relation S, a subdirect product of $\mathbb{A}_1 \times \mathbb{A}_1 \times \mathbb{A}_2 \times \dots \times \mathbb{A}_n$, produced from by $(a', b', \mathbf{a})$, where $\mathbf{a} \in \mathrm{pr}_{\{2,\dots,n\}} R''$, as follows:

$$S = \{f(a), f(b), f(\mathbf{a})) \mid f \text{ is a unary polynomial of } R \text{ with } f(\delta) \subseteq \gamma\}.$$

It is not difficult to see that S is a subalgebra, and, in particular it contains all the tuples of the form $(\mathbf{b}[1], \mathbf{b}[1], \mathbf{b}[2], \dots, \mathbf{b}[n])$ for $\mathbf{b} \in R$. Let $S' = S \cap \overline{B}$, and $S'' = S \cap \overline{D}$. Every tuple from S' or from S'' corresponds to a $\overline{B}$- or $\overline{D}$-preserving polynomial. Therefore it suffices to prove that $(a, b) \in \mathrm{pr}_{12} S''$. Let F be the as-component of E'' containing a, b; as observed above F is also an as-component of E'. By the assumption of (Q2s) $F^2 \subseteq \mathrm{pr}_{12} S'$ and $(e, e) \in \mathrm{pr}_{12} S''$ for any $e \in F$, since $F \times C \subseteq Q'$. We consider relation $T = \mathrm{pr}_{123} S'$. As $F^2 \subseteq T' = \mathrm{pr}_{12} T$, (a, b) is as-maximal in this relation. Therefore it suffices to show that $\mathsf{amax}(T)$ is linked when considered as subdirect product of T' and B'_2. Since $(e, e) \in \mathrm{pr}_{12} S''$ for any $e \in F$, all pairs of this form are linked in T. Then $(e, d, a'') \in T$ for any $e, d \in F$ and some $a'' \in B'_2$, and $(e, e, c'') \in T$ for some $c'' \in C$. Since $F^2 \subseteq T'$, $(e, e) \sqsubseteq_{as} (e, d)$, and by Lemma 12(4) a'' can be chosen from C, and so this implies that (e, d) and (e, e) are also linked. Claim 2 is proved.

Now we extend the result above to pairs from $\mathsf{umax}(E^*)$. We prove the result in two steps. First, we show that for any $a', b' \in E^*$ and any $a, b \in \mathsf{umax}(E^*)$ there is a sequence of $\overline{B}$-preserving polynomials $f_1, \dots, f_k$ such that $f_1(\{a', b'\}), \dots, f_k(\{a', b'\}) \subseteq E^*$ form a chain connecting a and b, $f_i(\mathbb{A}_1)$ is an (α, β)-minimal set, and $f_i(c) \in C$ for $i \in [k]$. Then we prove that $f_1, \dots, f_k$ can be chosen in such a way that $f_1(\{a', b'\}), \dots, f_k(\{a', b'\}) \subseteq E''$ and $f_1(c) = \dots = f_k(c) = c$. Clearly, it suffices to prove in the case when b is as-maximal in E^*.

By assumption there are $a = a_1, a_2, \dots, a_k = b$, $a_1, \dots, a_k \in E'$ and $(\gamma, \delta, \overline{B})$-good polynomials $f_1, \dots, f_{k-1}$ such that $f_i(\mathbb{A}_1)$ is a (α, β)-minimal set and $f_i(\{a', b'\}) = \{a_i, a_{i+1}\}$, and also $f_i(c) \in B'_2$. We need to show that $a_1, \dots, a_{k-1}$ and $f_1, \dots, f_{k-1}$ can be chosen such that $f_i(c) \in C$. Choose $\mathbf{a}, \mathbf{b} \in R''$ such that $\mathbf{a}[1] = a, \mathbf{b}[1] = b$ and $\mathbf{a}[2] = \mathbf{b}[2] = c$. Now let $g_i(x) = \mathsf{maj}(\mathbf{a}, f_i(x), \mathbf{a})$ and $h_i(x) = \mathsf{maj}(\mathbf{a}, \mathbf{b}, f_i(x))$. By Lemma 27 g_i, h_i are $(\gamma, \delta, \overline{B})$-good polynomials, and for each of them either $\{b_i, b_{i+1}\} = g_i(\{a', b'\})$ ($\{c_i, c_{i+1}\} = h_i(\{a', b'\})$) is an (α, β)-subtrace, or $g_i(\beta) \subseteq \alpha$ ($h_i(\beta) \subseteq \alpha$), that is $g_i(a') = g_i(b')$ (respectively, $h_i(a') = h_i(b')$). The polynomials g_i, h_i satisfying the first option form a sequence of (α, β)-subtraces connecting a with $\mathsf{maj}(a, b, a)$ — by subtraces of the form $\{b_i, b_{i+1}\}$, — and $\mathsf{maj}(a, b, a)$ with $\mathsf{maj}(a, b, b)$ — by subtraces of the form $\{c_i, c_{i+1}\}$. Also, by Theorem 15 $\mathsf{maj}(a, b, b) \in C$, and so by Claim 2 this sequence of polynomials and subtraces can be continued to connect $\mathsf{maj}(a, b, b)$ to b. Finally, by the same theorem $g_i(c) = \mathsf{maj}(c, f_i(c), c) \in C$ and $h_i(c) = \mathsf{maj}(c, c, f_i(c)) \in C$.

For the second step we assume that a and b are connected with (α, β)-subtraces $\{a_i, a_{i+1}\}$, $i \in [k-1]$ witnessed by polynomials f_i such that $c_i = f_i(c) \in C$. We need to show that f_i can be chosen such that $f_i(c) = c$. Suppose for some $i \in [k-1]$, $c_i \neq c$. Since c_i and c belong to the same as-component, there is an as-path $c_i = d_1, \ldots, d_\ell = c$ in C. We show that if there is a sequence of (α, β)-subtraces $\{b_j, b_{j+1}\}$ witnessed by polynomials g_j such that $g_j(c) = c$ whenever $f_j(c) = c$, and $f_i(c) = d_t$, there are also (α, β)-subtraces $\{b'_j, b'_{j+1}\}$ such that $b'_1 = a$ and b'_k is in the as-component containing b, witnessed by polynomials $g'_1, \ldots, g'_k$ such that $g'_i(c) = d_{t+1}$ and $g'_j(c) = c$ whenever $g_j(c) = c$.

As is easily seen, it suffices to find a ternary term operation p such that $p(a, a, b)$ belongs to the as-component containing b, and $p(d_{t+1}, d_t, d_t) = d_{t+1}$. Indeed, if such a term operation exists, then we set $g'_j(x) = p(\mathbf{a}, \mathbf{a}, g_j(x))$, where $\mathbf{a}$ is as in the first step above, for $j \in [k-1] - \{i\}$, and $\{b'_j, b'_{j+1}\} = g'_j(\{a', b'\})$. We have $g'_1(a') = p(a, a, g_1(a')) = a$ and $g'_j(c) = p(c, c, g_j(c)) = c$ whenever $g_j(c) = c$. Finally, since $g'_\ell(b) = p(a, a, b)$ belongs to the as-component containing b, we can use Claim 2 as before to connect $p(a, a, b)$ to b. For g'_i we set $g'_i(x) = p(\mathbf{a}', \mathbf{a}'', g_i(x))$ where $\mathbf{a}', \mathbf{a}'' \in R''$ are such that $\mathbf{a}'[1] = \mathbf{a}''[1] = a$ and $\mathbf{a}'[2] = d_{t+1}, \mathbf{a}''[2] = d_t$. Note that such $\mathbf{a}', \mathbf{a}''$ exist, because $\mathsf{umax}(E^*) \times C \subseteq Q$. It follows from the assumption about p that g'_i is as required.

If $d_t \leq d_{t+1}$, then $p(x, y, z) = z \cdot x$ fits the requirements. If $d_t d_{t+1}$ is an affine edge, consider the relation $S \subseteq \mathbb{A}_1 \times \mathbb{A}_2$ generated by $\{(a, d_t), (a, d_{t+1}), (b, d_t)\}$. Let $\mathbb{B} = \mathsf{Sg}(a, b)$ and $\mathbb{C} = \mathsf{Sg}(d_t, d_{t+1})$; then $\mathbb{B} \times \{d_t\}, \{a\} \times \mathbb{C} \subseteq S$. By Lemma 16, as $d_t d_{t+1}$ is a thin affine edge, $\mathsf{umax}(\mathbb{B}) \times \{d_{t+1}\} \subseteq S$. There is b' with $b \sqsubseteq_{as} b'$ in $\mathbb{B}$ such that $b' \in \mathsf{umax}(\mathbb{B})$. Therefore there is a term operation p with $p(a, a, b) = b'$ and and $p(d_{t+1}, d_t, d_t) = d_{t+1}$, as required. $\square$

6.2 Tightening affine factors

In this section we consider Case 1 of tightening strategies: there is $\alpha \in \mathsf{Con}(\mathbb{A}_v)$ for some $v \in V$ such that $\alpha \prec \beta_v$.

Let $\mathcal{P} = (V, \mathcal{C})$ be a block-minimal instance with subdirectly irreducible domains, $\overline{\beta} = (\beta_v \in \mathsf{Con}(\mathbb{A}_v) \mid v \in V)$ and $\overline{B} = (B_v \mid B_v \text{ is a } \beta_v\text{-block}, v \in V)$. We use notation from Section 5. Let also $\mathcal{R} = \{R_{C,v,\alpha\beta}\}$ be a $\overline{\beta}$-strategy for $\overline{B}$. We select $v \in V$ and $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$ with $\alpha \prec \beta = \beta_v$, $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$, and an α-block $B \in B_v/_\alpha$. In this section we show how $\mathcal{R}$ can be transformed to a $\overline{\beta}'$-strategy $\mathcal{R}'$ for $\overline{B}'$ such that $\beta'_w \le \beta_w$, $B'_w \subseteq B_w$ for $w \in V$, and $\beta'_v = \alpha$, $B'_v = B$.

First of all we identify variables $w \in V$ for which β'_w has to be different from β_w. Since $\mathcal{P}$ is (2,3)-minimal, for every $u, w \in V$ there is $C^{\{u,w\}} = \langle (u,w), R^{\{u,w\}}\rangle \in \mathcal{C}$. For $w \in W_{v,\alpha\beta}$ (we omit $\overline{\beta}$ from $W_{v,\alpha\beta,\overline{\beta}}$ here) consider $R^{*,\{v,w\}} = R^{\{v,w\}} \cap (B_v \times B_w)$, $R^{\{v,w\},\mathcal{R}}$ the set of all $\mathcal{R}$-compatible pairs from $R^{\{v,w\}}$, $R'^{\{v,w\}} = R^{\{v,w\}}/_\alpha$, and $R'^{\{v,w\},\mathcal{R}} = R^{\{v,w\},\mathcal{R}}/_\alpha$. By (S5) for $\mathcal{R}$ we have that $\mathsf{umax}(\mathrm{pr}_v R'^{\{v,w\},\mathcal{R}})$ is as-closed in $B^*_v/_\alpha$, where $B^*_v = \mathrm{pr}_v R^{*,\{v,w\}}$; since $\mathsf{typ}(\alpha,\beta) = \mathbf{2}$, this implies $\mathrm{pr}_v R'^{\{v,w\},\mathcal{R}} = B^*_v/_\alpha$. Therefore by Lemma 54 either $B^*_v/_\alpha \times \mathsf{umax}(\mathrm{pr}_w(R'^{\{v,w\},\mathcal{R}}) \subseteq R'^{\{v,w\},\mathcal{R}}$ or $R'^{\{v,w\},\mathcal{R}}$ is the graph of a mapping $\nu_w : \mathsf{umax}(\mathrm{pr}_w(R'^{\{v,w\},\mathcal{R}}) \to B^*_v/_\alpha$. Let $U \subseteq W_{v,\alpha\beta}$ be the set of variables for which the latter holds, and let α_w be the corresponding congruence of $\mathbb{A}_w$, extension of the kernel of ν_w. Let $\beta'_v = \alpha, B'_v = B$ and $\beta'_w = \alpha_w, B'_w = \nu_w^{-1}(B)$ for $w \in U$, and $\beta'_w = \beta_w, B'_w = B_w$ for $w \in V - U$.

Now we are in a position to define the new strategy. Let $\mathcal{R}'$ be the following collection of relations. We omit subscript $\overline{\beta}$.

(R1) $\mathcal{R}' = \{R'_{C,w,\gamma\delta} \mid C = \langle \mathbf{s}, R\rangle \in \mathcal{C}, (w,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')\}$;

(R2) for every $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, $(u,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')$,

- (a) if $(v,\alpha,\beta) \notin \mathcal{W}'$, $R'_{C,u,\gamma\delta} = \{\mathbf{a} \in R_{C,u,\gamma\delta} \mid$ there is a $\mathcal{R}$-compatible solution φ of $\mathcal{P}_{W_{v,\alpha\beta_v}}$, $\varphi(v) \in B'_v$, and $\varphi(w) = \mathbf{a}[w]$ for $w \in \mathbf{s} \cap W_{v,\alpha\beta_v} \cap W_{u,\gamma\delta}\}$;
- (b) if $(v,\alpha,\beta) \in \mathcal{W}', (u,\gamma,\delta) \notin \mathcal{W}'$, $R'_{C,u,\gamma\delta} = \{\mathbf{a} \in R_{C,u,\gamma\delta} \mid$ there is a $\mathcal{R}$-compatible solution φ of $\mathcal{P}_{W_{v,\alpha\beta_v}}/_{\overline{\mu}^Y}$ with $Y = \mathsf{MAX}(\mathcal{P}) - W_{u,\gamma\delta}$, such that $\varphi(v) \in B'_v$, and $\varphi(w) = \mathbf{a}[w]$ for $w \in \mathbf{s} \cap W_{v,\alpha\beta_v} \cap W_{u,\gamma\delta}\}$
- (c) if $(v,\alpha,\beta), (u,\gamma,\delta) \in \mathcal{W}'$, $R'_{C,u,\gamma\delta} = \{\mathbf{a} \in R_{C,u,\gamma\delta} \mid$ there is a $\mathcal{R}$-compatible solution φ of $\mathcal{P}_{W_{v,\alpha\beta_v}}/_{\overline{\mu}^Y}$ with $Y = \mathsf{MAX}(\mathcal{P}) - (\mathbf{s} \cap W_{u,\gamma\delta})$, such that $\varphi(v) \in B'_v$, and $\varphi(w) = \mathbf{a}[w]$ for $w \in \mathbf{s} \cap W_{v,\alpha\beta_v} \cap W_{u,\gamma\delta}\}$;

Similar to $R^{\mathcal{R}}, \mathcal{S}^{\mathcal{R}}_W, \mathcal{S}^{\mathcal{R}}_{W,Y}$ by $R^{\mathcal{R}'}, \mathcal{S}^{\mathcal{R}'}_W, \mathcal{S}^{\mathcal{R}'}_{W,Y}$ we denote the corresponding sets of $\mathcal{R}'$-compatible tuples. As is easily seen, the sets both types are indeed subalgebras of $R, \mathcal{S}_W, \mathcal{S}_{W,Y}$.

The following three statements show how relations $R'_{C,w,\gamma\delta}$ from $\mathcal{R}'$ are related to $R_{C,w,\gamma\delta}$ from $\mathcal{R}$. They basically amount to saying that either $R'_{C,w,\gamma\delta}$ is the intersection of $R_{C,w,\gamma\delta}$ with a block of a congruence of the projection of R, or $\mathsf{umax}(R'_{C,w,\gamma\delta}) = \mathsf{umax}(R_{C,w,\gamma\delta})$

Lemma 56 *(1) Let $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, and let $\mathcal{S}'_W$, where $W = W_{v,\alpha\beta_v}$, be the set of solutions of $\mathcal{P}_W$ if $(v,\alpha,\beta) \notin \mathcal{W}'$, or the set of solutions of $\mathcal{P}_W/_{\overline{\mu}^{\mathsf{MAX}(\mathcal{P})-\mathbf{s}}}$ if $(v,\alpha,\beta) \in \mathcal{W}'$. For every $U \subseteq \mathbf{s}\cap W$ there is a congruence τ_U of $\mathrm{pr}_U\mathcal{S}'_W = \mathrm{pr}_U R$ such that $\mathrm{pr}_U\mathcal{S}'^{\mathcal{R}'}_W$ is the intersection of $\mathrm{pr}_U\mathcal{S}'^{\mathcal{R}}_W$ and a τ_U-block.*

(2) For any $U_1 \subseteq U_2 \subseteq W$ the congruence τ_{U_1} is the restriction of τ_{U_2}, that is $(\mathbf{a},\mathbf{b}) \in \tau_{U_1}$ if and only if for some $\mathbf{a}',\mathbf{b}' \in \mathcal{S}'_{U_2}$ with $\mathrm{pr}_{U_1}\mathbf{a}' = \mathbf{a}, \mathrm{pr}_{U_1}\mathbf{b}' = \mathbf{b}$ it holds $(\mathbf{a}',\mathbf{b}') \in \tau_{U_2}$.

(3) For any $U \subseteq \mathbf{s}\cap W$ either $\tau_U|_{\mathrm{pr}_U R\cap\overline{B}} = \overline{\beta}_U|_{\mathrm{pr}_U R\cap\overline{B}}$, or the algebra $\mathrm{pr}_U\mathcal{S}'^{\mathcal{R}}_W/_{\tau_U}$ is isomorphic to $\mathrm{pr}_v(\mathcal{S}'_W \cap \overline{B})/_\alpha$.

(4) For any $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$, $X = W_{w,\gamma\delta}$, $X' = \mathbf{s}\cap W\cap X$, let $\tau = \tau_{X'}$. Then either $\mathsf{umax}(\mathrm{pr}_{X'}R'_{C,w,\gamma\delta}) = \mathsf{umax}(\mathrm{pr}_{X'}R_{C,w,\gamma\delta})$, or for a τ-block S it holds $\mathrm{pr}_{X'}R'_{C,w,\gamma\delta} \subseteq \mathrm{pr}_{X'}R_{C,w,\gamma\delta}\cap S$ and $\mathsf{umax}(\mathrm{pr}_{X'}R'_{C,w,\gamma\delta})$ is the set of u-maximal elements of $\mathsf{umax}(\mathrm{pr}_{X'}R_{C,w,\gamma\delta})\cap S$.

If, according to item (3) of the lemma, $\tau_U|_{\mathrm{pr}_U R\cap\overline{B}} = \overline{\beta}_U|_{\mathrm{pr}_U R\cap\overline{B}}$, we say that τ_U is the *full congruence*; if the latter option of item (3) holds we say that τ_U is a *maximal congruence*.

Proof: (1) If $v \in U$ then τ_U is $\overline{\beta}'_U$. Otherwise consider $Q = \mathrm{pr}_{U\cup\{v\}}\mathcal{S}'_W$ as a subdirect product of $\mathbb{A}_v$ and $\mathrm{pr}_U\mathcal{S}'_W$. Since this relation is chained by (S7) and $\mathrm{pr}_{U\cap\{v\}}\mathcal{S}'^{\mathcal{R}}_W$ is polynomially closed in Q by Lemma 51(2), the result follows from Lemma 54. Indeed, we consider $Q/_\alpha$ as a subdirect product of $\mathrm{pr}_U\mathcal{S}'_W$ and $\mathbb{A}_v/_\alpha$, and choose τ_U to be the conguence of $\mathrm{pr}_U\mathcal{S}'_W$ identified in Lemma 54.

(2) Obvious.

(3) If $v \in U$ then by item (1) $\mathrm{pr}_U\mathcal{S}'^{\mathcal{R}}_W/_{\tau_U} = \mathrm{pr}_U\mathcal{S}'^{\mathcal{R}}_W/_{\overline{\beta}'_U}$, which is isomorphic to $\mathrm{pr}_v(\mathcal{S}'_W \cap \overline{B})/_\alpha$. Otherwise consider relation Q as in item (1). By Lemma 54 the restriction of τ_U is the link congruence of this relation. The result follows.

(4) If τ is the full congruence then by (S2) for $\mathcal{R}$ we have $\mathsf{umax}(\mathrm{pr}_X R_{C,w,\gamma\delta}) = \mathsf{umax}(\mathrm{pr}_X\mathcal{S}'^{\mathcal{R}'}_W)$ and we have the first option. If τ is a maximal congruence then by (R2) and item (1) there is a τ-block S such that $\mathrm{pr}_{X'}R'_{C,w,\gamma\delta} \subseteq S\cap\mathrm{pr}_{X'}R_{C,w,\gamma\delta}$, and by (S2) $\mathsf{umax}(\mathrm{pr}_{X'}R'_{C,w,\gamma\delta}) = \mathsf{umax}(S\cap\mathrm{pr}_{X'}R_{C,w,\gamma\delta})$. □

For $C = \langle\mathbf{s},R\rangle \in \mathcal{C}$ we use τ_C to denote the congruence $\tau_{\mathbf{s}\cap W}$. Also for $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$ we use $\tau_{C,w,\gamma\delta}$ to denote the congruence $\tau_{\mathbf{s}\cap W\cap W_{w,\gamma\delta}}$.

Lemma 57 *In the notation above let $\gamma_u, \delta_u \in \mathsf{Con}(\mathbb{A}_u)$, $u \in W' = \mathbf{s}\cap W$ be such that $(u,\gamma_u,\delta_u) \in \mathcal{W}(\overline{\beta})$ and $(\alpha,\beta_v), (\gamma_u,\delta_u)$ cannot be separated from each other. Then if τ_C is a maximal congruence, for any polynomial f of R, $f(\overline{\beta}_{W'}) \subseteq \tau_C$ if and only if $f(\delta_u) \subseteq \gamma_u$ for any $u \in W'$.*

Proof: Let $\mathcal{S}'_W$ be defined as in Lemma 56. Take a polynomial f of R. As $(\gamma_{u_1},\delta_{u_1}), (\gamma_{u_2},\delta_{u_2})$ cannot be separated for any $u_1,u_2 \in W'$, it suffices to consider just one variable $u \in W'$. Since $\mathcal{P}$ is a block-minimal instance, the polynomial f can be extended from a polynomial on $\mathrm{pr}_{W'}R$ to a polynomial f' of $\mathcal{S}'_W$, and, in particular, to a polynomial f'' of $\mathrm{pr}_{W'\cup\{v\}}\mathcal{S}'_W$. Since α and τ_C are greater than or equal to the link congruences of $A_{\mathcal{R},v}$ and $\mathrm{pr}_{W'}R^{\mathcal{R}}$ with respect to $\mathrm{pr}_{W'\cup\{v\}}\mathcal{S}'^{\mathcal{R}}_W$, for any $\mathbf{a},\mathbf{b} \in \mathrm{pr}_{W'\cup\{v\}}\mathcal{S}'^{\mathcal{R}}_W$ we have $f''(\mathbf{a}[v]) \stackrel{\alpha}{\equiv} f''(\mathbf{b}[v])$ if and only if $f''(\mathrm{pr}_{W'}\mathbf{a}) \stackrel{\tau_C}{\equiv} f''(\mathrm{pr}_{W'}\mathbf{b})$. In particular, this implies that $f''(\beta_v) \subseteq \alpha$ if and only if $f(\overline{\beta}_{W'}) \subseteq \tau_C$. Since the first inclusion holds if and only if $f(\delta_u) \subseteq \gamma_u$, we infer the result. □

Corollary 58 *For any $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$, $X = W_{w,\gamma\delta}$, $X' = \mathbf{s} \cap W \cap X$, let $\tau = \tau_{X'}$ and $\tau' = \{(\mathbf{a},\mathbf{b}) \mid \mathbf{a},\mathbf{b} \in \mathrm{pr}_{\mathbf{s}\cap X}R, (\mathrm{pr}_{X'}\mathbf{a}, \mathrm{pr}_{X'}\mathbf{b}) \in \tau\}$. Then either $\mathsf{umax}(R'_{C,w,\gamma\delta}) = \mathsf{umax}(R_{C,w,\gamma\delta})$, or for a τ'-block T it holds $R'_{C,w,\gamma\delta} \subseteq R_{C,w,\gamma\delta} \cap T$ and $\mathsf{umax}(R'_{C,w,\gamma\delta})$ is the set of u-maximal elements of $\mathsf{umax}(R_{C,w,\gamma\delta}) \cap T$.*

Proof: By Lemma 56(4) either $\mathsf{umax}(\mathrm{pr}_{X'}R'_{C,w,\gamma\delta}) = \mathsf{umax}(\mathrm{pr}_{X'}R_{C,w,\gamma\delta})$, or for a τ-block S it holds $\mathrm{pr}_{X'}R'_{C,w,\gamma\delta} \subseteq \mathrm{pr}_{X'}R_{C,w,\gamma\delta} \cap S$ and $\mathsf{umax}(\mathrm{pr}_{X'}R'_{C,w,\gamma\delta})$ is the set of u-maximal elements of $\mathsf{umax}(\mathrm{pr}_{X'}R_{C,w,\gamma\delta}) \cap S$. Then considering $R_{C,w,\gamma\delta}/_{\tau'}$ as a subdirect product of $\mathrm{pr}_{X'}R_{C,w,\gamma\delta}/_{\tau}$ and $\mathrm{pr}_{\mathbf{s}-X'}R_{C,w,\gamma\delta}$, the interval $(\tau, \overline{\beta}_{X'})$ in $\mathrm{pr}_{X'}R/_{\tau}$ can be separated from interval (η,θ) in $\mathsf{Con}(\mathbb{A}_u)$ for any $u \in \mathbf{s} - X'$ by Lemma 57. Then we use Lemma 54 to conclude that $S \times \mathsf{umax}(\mathrm{pr}_{\mathbf{s}-X}R_{C,w,\gamma\delta}) \subseteq R_{C,w,\gamma\delta}/_{\tau'}$. The result follows. □

Now we are in a position to prove that $\mathcal{R}'$ is a $\overline{\beta}'$-strategy.

Theorem 59 *In the notation above, $\mathcal{R}'$ is a $\overline{\beta}'$-strategy for $\overline{B}'$.*

Proof: Let $W = W_{v,\alpha\beta}$.

For (S1) we consider the collection of constraints $C^X = \langle X, R^X \rangle$, $X \subseteq V$, $|X| = 2$, such that $\mathsf{umax}(R^{X,\mathcal{R}})$ constitute a $(2,3)$-strategy for $\mathcal{P}$. Let $R^{X,\mathcal{R}'}$ denote the set of $\mathcal{R}'$-compatible tuples from $R^{X,\mathcal{R}}$. It suffices to show that for any tuple $(a,b) \in \mathsf{umax}(R^{X,\mathcal{R}'})$, $X = \{x,y\}$, and any $w \notin \{x,y\}$ there is $c \in \mathbb{A}_w$ such that $(a,c) \in \mathsf{umax}(R^{\{x,w\},\mathcal{R}'}), (b,c) \in \mathsf{umax}(R^{\{y,w\},\mathcal{R}'})$. By (S1) for $\mathcal{R}$ there is $d \in \mathbb{A}_w$ such that $(a,d) \in \mathsf{umax}(R^{\{x,w\},\mathcal{R}}), (b,d) \in \mathsf{umax}(R^{\{y,w\},\mathcal{R}})$.

Consider the relation R given by

$$R(x,y,w) = R^{\{x,y\}}(x,y) \wedge R^{\{x,w\}}(x,w) \wedge R^{\{y,w\}}(y,w),$$

and let $R^{\mathcal{R}}, R^{\mathcal{R}'}$ be the set of $\mathcal{R}$- and $\mathcal{R}'$-compatible tuples from R, respectively. By (S4) $R^{\mathcal{R}}$ is a subalgebra of R; moreover, by (S1) the binary projections of $R^{\mathcal{R}}$ contain $\mathsf{umax}(R^{\{x,y\},\mathcal{R}})$, $\mathsf{umax}(R^{\{x,w\},\mathcal{R}})$, $\mathsf{umax}(R^{\{y,w\},\mathcal{R}})$, respectively.

First, suppose $w \notin W$. If, say, $x \notin W$ then $R^{\{x,w\},\mathcal{R}} = R^{\{x,w\},\mathcal{R}'}$. If $x \in W$, then by construction either $\mathsf{umax}(R^{\{x,w\},\mathcal{R}}) = \mathsf{umax}(R^{\{x,w\},\mathcal{R}'})$ if $\tau_{\{x\}}$ is the full congruence, or $\mathsf{umax}(R^{\{x,w\},\mathcal{R}}) \cap (B'_x \times B_w) \subseteq \mathsf{umax}(R^{\{x,w\},\mathcal{R}'}$ otherwise. In either case $(a,d) \in R^{\{x,w\},\mathcal{R}'}$. Similarly, $(b,d) \in R^{\{y,w\},\mathcal{R}'}$. Therefore $(a,b,d) \in R^{\mathcal{R}'}$, and by Lemma 12(5) there is always c such that $(a,c) \in \mathsf{umax}(R^{\{x,w\},\mathcal{R}'}), (b,c) \in \mathsf{umax}(R^{\{y,w\},\mathcal{R}'})$. Therefore we may assume that $w \in W$.

If $x \in W$ (or $y \in W$, or $x, y \in W$) then let τ_x, τ_{xw} (respectively, τ_y, τ_{yw}, or τ_{xy}) be as in Lemma 56. If $x \notin W$ (or $y \notin W$) then let $\tau_x = \beta_x$, $\tau_{xw} = \tau_w$, and $\tau_{xy} = \tau_y$ (respectively, $\tau_y = \beta_y, \tau_{yw} = \tau_w$). We view all these congruences interchangeably: as congruences of R in the natural way and their restrictions to $\mathcal{R}$-compatible tuples as congruences of $R^{\mathcal{R}}$, or as congruences of the corresponding projections of R and $R^{\mathcal{R}}$. By Lemma 56 if τ' is one of these congruences, τ' is either the full congruence on $R^{\mathcal{R}}$, or $\tau'|_{R^{\mathcal{R}}} \prec (\beta_x \times \beta_y \times \beta_w)|_{R^{\mathcal{R}}}$ in $\mathsf{Con}(R)$; in the latter case we will say that τ' is maximal. Also let $\tau = \tau_{xy} \wedge \tau_{xw} \wedge \tau_{yw}$. Again by Lemma 56 $R^{\mathcal{R}}/_{\tau}$ is a module. We start with an auxiliary claim.

CLAIM. Let $\mathbb{A}$ be an algebra and β in $\mathsf{Con}(\mathbb{A})$ and $\mathbb{A}'$ is a subalgebra of a β-block. Let also $\alpha < \beta|_{\mathbb{A}'}$ be a congruence of $\mathbb{A}'$ such that $\mathbb{A}'/_{\alpha}$ is a module. Let $\mathbb{B}, \mathbb{C}$ be subalgebras of $\mathbb{A}'$ such that $\mathbb{B} \cap \mathbb{C} \neq \varnothing$ and $\mathbb{B} \cap \mathbb{C}$ contains a u-maximal element of an α-block of $\mathbb{A}'$, $\mathbb{B}/_{\alpha} = \mathbb{A}'/_{\alpha}, \mathbb{C}/_{\alpha} = \mathbb{A}'/_{\alpha}$, and $\mathbb{B}, \mathbb{C}$ are polynomially closed in $\mathbb{A}$. Then $\mathbb{B} \cap \mathbb{C}/_{\alpha} = \mathbb{A}'/_{\alpha}$.

Clearly it suffices to consider the case $\alpha \prec \beta|_{\mathbb{A}'}$ since then we use induction on an irreducible chain from α to $\beta|_{\mathbb{A}'}$ in $\mathsf{Con}(\mathbb{A}')$. Let $a \in \mathbb{B} \cap \mathbb{C}$ be u-maximal in an α-block of $\mathbb{A}'$. Let $a' \in \mathbb{B}$ and $a' \stackrel{\alpha}{\not\equiv} a$. For any (α, β)-trace T that contains a and any $b \in T$ there is a polynomial f of $\mathbb{A}$ such that $f(a) = a$ and $f(a') = b$. Since $\mathbb{B}$ is polynomially closed, for any $c \in \mathsf{Sg}(a, b)$ such that $a \sqsubseteq_{as} c$, we have $c \in \mathbb{B}$, and c is u-maximal in $\mathbb{A}'$. In a similar way $c \in \mathbb{C}$. Therefore it suffices to show that for any α-block $D \subseteq \mathbb{A}'$ there is a $b \in D$ such that $\{a, b\}$ is an (α, β)-subtrace.

Since $\mathsf{typ}(\alpha, \beta) = \mathbf{2}$, there is a (α, β)-subtrace $\{d, e\}$ such that $d \stackrel{\alpha}{\equiv} a$ and $e \in D$. Then by Lemma 40 there is also a polynomial g such that $g(\mathbb{A})$ is an (α, β)-minimal set and $g(a) = a$, $g(e) \in D$. The result follows.

Let R^*_a, R^*_b be the sets of tuples $\mathbf{a} \in R^{\mathcal{R}}$ satisfying $\mathbf{a}[x] = a, \mathbf{a}[y] = b$, respectively; note that $(a,b,d) \in R^*_x \cap R^*_y$. We consider several cases of what the congruences introduced earlier can be.

Suppose first that, say, τ_x is the full congruence and $\tau_{xw} = \tau_w$. As $\tau_{yw} \le \tau_w$, if τ_{yw} is a full congruence, $(a,d) \in R^{\{x,w\},\mathcal{R}'}, (b,d) \in R^{\{y,w\},\mathcal{R}'}$, and we are done. If τ_y is maximal while τ_w is not , then again $(a,d) \in R^{\{x,w\},\mathcal{R}'}, (b,d) \in R^{\{y,w\},\mathcal{R}'}$ as any tuple $(b,x) \in R^{\{y,w\},\mathcal{R}}$ also belongs to $R^{\{y,w\},\mathcal{R}'}$ in this case. If both τ_y and

τ_w are maximal, then, as $\tau_{yw} \le \tau_y \wedge \tau_w$ and all three congruences are maximal, τ_w (viewed as a congruence of $\mathrm{pr}_w R^{\mathcal{R}}$) is the link congruence with respect to $\mathrm{pr}_{yw} R^{\mathcal{R}}$, and so $(b,d) \in R^{\{y,w\},\mathcal{R}'}$, implying also, since $(a,b) \in \mathrm{pr}_{xy} R^{\mathcal{R}}$, that $(a,d) \in R^{\{x,w\},\mathcal{R}'}$.

If τ_w is maximal while τ_y is not, let $B^*_z = \mathrm{pr}_z R^{\mathcal{R}}$ for $z \in \{x,y,w\}$. By Lemma 54 $\mathsf{umax}(B^*_x) \times B^*_w/_{\tau_w} \subseteq \mathrm{pr}_{xw} R^{\mathcal{R}}$ and $\mathsf{umax}(B^*_y) \times B^*_w/_{\tau_w} \subseteq \mathrm{pr}_{yw} R^{\mathcal{R}}$. Therefore $R^*_a/_{\tau_w} = R^*_b/_{\tau_w} = R^{\mathcal{R}}/_{\tau_w}$ and since (a,b) is u-maximal in a τ_{xy}-block, d can be chosen such that (a,b,d) is u-naximal in a τ_w-block. Then the result follows by the Claim above. Now, let $\tau_{yw} \prec \tau_y = \tau_w = \beta_w$. Again by Lemma 54 $\mathsf{umax}(B^*_x) \times \mathrm{pr}_{yw} R^{\mathcal{R}}/_{\tau_{yw}} \subseteq R^{\mathcal{R}}$ and therefore $R^*_a/_{\tau_{yw}} = R^{\mathcal{R}}/_{\tau_{yw}}$. Also, let Q be the union of all τ_{yw}-blocks of $R^{\mathcal{R}}$ whose intersection with R^*_b is nonempty and $R^{**}_a = R^*_a \cap Q$. Note that $R^{\{y,w\},\mathcal{R}'} \cap \mathrm{pr}_{yw} Q \neq \varnothing$. Then $R^{**}_a/_{\tau_{yw}} = Q/_{\tau_{yw}}$ and $R^*_b/_{\tau_{yw}} = Q/_{\tau_{yw}}$. Then as before the result follows by the Claim.

If τ_x is maximal, while τ_y is not, the result follows by one of the previous arguments. If τ_x, τ_y are both maximal, then $(a,d) \in R^{\{x,w\},\mathcal{R}'}$, $(b,d) \in R^{\{y,w\},\mathcal{R}'}$, and we are done.

Due to symmetries between x and y, the only remaining case is when τ_{xw} and τ_{yw} are maximal, while τ_x, τ_y, τ_w are not. If $(v,\alpha,\beta) \notin \mathcal{W}'$ or $w \notin \mathsf{MAX}(\mathcal{P})$, that is, $\mu'_w = \underline{0}_w$, then the required $c \in B^*_w$ exists, since (a,b) can be extended to a solution from $\mathcal{S}^{\mathcal{R}'}_W$ or $\mathcal{S}^{\mathcal{R}'}_{W,Y}$, $Y = \mathsf{MAX}(\mathcal{P}) - \{x,y\}$ by construction. Suppose that $(v,\alpha,\beta) \in \mathcal{W}'$ and $\mu'_w = \mu_w$. Let Q_x be a subalgebra of the product $\mathbb{A}_x \times \mathbb{A}_w \times \mathbb{A}_v/_\alpha$ that consists of all triples (a',b',c') such that there is a solution $\varphi \in \mathcal{S}_{W,Y}$ for $Y = \mathsf{MAX}(\mathcal{P}) - \{x,w\}$ with $\varphi(x) = a', \varphi(w) = b'$, and $\varphi(v) \in c'$. By block-minimality Q_x is indeed a subdirect product and by (S2) for $\mathcal{R}$ we have $\mathsf{umax}(R^{\{x,w\},\mathcal{R}}) \subseteq \mathrm{pr}_{xw}(Q_x \cap (B^*_x \times B^*_w \times B^*_v/_\alpha)$ and $\mathrm{pr}_v(Q_x \cap (B^*_x \times B^*_w \times B^*_v/_\alpha) = B^*/_\alpha$. Relation Q_y is defined in a similar way. Let also

$$Q(x,y,w,v) = Q_x(x,w,v) \wedge Q_y(y,w,v),$$

and $Q' = Q \cap \overline{B}$. Let $Q^a = \{\mathbf{a} \in Q' \mid \mathbf{a}[x] = a\}$, $Q^b = \{\mathbf{a} \in Q' \mid \mathbf{a}[y] = b\}$, and $\alpha' = \beta_x \times \beta_y \times \beta_w \times \alpha$. By the assumption that τ_x, τ_y are full congruences $Q^a/_\alpha = Q^b/_\alpha = Q'/_\alpha$. Therefore, if we prove that $(a,b) \in \mathrm{pr}_{xy} Q'$, we obtain the result by the Claim.

To this end consider the relations

$$S(x,y,w,v,v') = Q_x(x,w,v) \wedge Q_y(y,w,v'),$$

and $S' = S \cap \overline{B}$. In a similar way we define $S^a = \{\mathbf{a} \in S' \mid \mathbf{a}[x] = a\}$, $S^b = \{\mathbf{a} \in S' \mid \mathbf{a}[y] = b\}$. By (S1) there are $d \in B^*_w$ and $e', e'' \in B^*_v$ such that $(a,b,d,e',e'') \in S'$. Also by construction (R2) there are $a' \in B^*_x, b' \in B^*_y$,

$d_1, d_2 \in B^*_w$, $d_1 \stackrel{\mu_w}{\equiv} d_2$ and $e_1, e_2 \in B^*_v$, $e_1 \stackrel{\alpha}{\equiv} e_2$, such that $(a, b', d_1, e_1), (a', b, d_2, e_2) \in S'$. Recall that $B^*_v/_\alpha$ is a module. Let δ be the diagonal congruence of $B^*_v/_\alpha \times B^*_v/_\alpha$, that is, a congruence such that $\Delta = \{(c_1, c_2), \mid c_1 \stackrel{\alpha}{\equiv} c_2\}$ is a δ-block. Let δ' be the congruence of S' given by $\mathbf{c} \stackrel{\delta'}{\equiv} \mathbf{d}$ if and only if $\mathrm{pr}_{vv'}\mathbf{c} \stackrel{\delta}{\equiv} \mathrm{pr}_{vv'}\mathbf{d}$. Note that $\delta' < \overline{\beta}$ in $\mathsf{Con}(S)$. Let Δ' be the δ'-block corresponding to Δ. Since $S^a \cap S^b \neq \varnothing$ and $S^a \cap \Delta' \neq \varnothing$, $S^b \cap \Delta' \neq \varnothing$, by the Claim we have $S^a \cap S^b \cap \Delta' \neq \varnothing$. The result follows.

For (S2) take $C = \langle \mathbf{s}, R\rangle$ and $(w_1, \gamma_1, \delta_1), (w_2, \gamma_2, \delta_2) \in \mathcal{W}(\overline{\beta}')$ and let $W_1 = W_{w_1,\gamma_1\delta_1}$, $W_2 = W_{w_2,\gamma_2\delta_2}$, $U = \mathbf{s} \cap W_1 \cap W_2$, and $W = W_{v,\alpha\beta_v}$, as before. Let $\mathbf{a} \in \mathsf{umax}(\mathrm{pr}_U R'_{C,w_1,\gamma_1\delta_1})$. Depending on whether or not $(w_1, \gamma_1, \delta_1), (w_2, \gamma_2, \delta_2) \in \mathcal{W}'$ we need to show that $\mathbf{a}$ can be extended to a solution of $\mathcal{P}_{W_2}/_{\overline{\mu}^Y}$, where Y is either empty, if $(w_2, \gamma_2, \delta_2) \notin \mathcal{W}'$, or $Y = \mathsf{MAX}(\mathcal{P}) - W_1$ if $(w_1, \gamma_1, \delta_1) \notin \mathcal{W}'$ and $(w_2, \gamma_2, \delta_2) \in \mathcal{W}'$, and $Y = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$ if $(w_1, \gamma_1, \delta_1), (w_2, \gamma_2, \delta_2) \in \mathcal{W}'$. The three cases are quite similar so we will consider them simultaneously. Let $\mathcal{P}'_{W'}$ denote the problem $\mathcal{P}_{W'}/_{\overline{\mu}^Y}$ for a set $W' \subseteq V$ and $\mathcal{S}'_{W'}$ denote its set of solutions. Then we need to show that $\mathbf{a} \in \mathrm{pr}_U \mathcal{S}'^{\mathcal{R}'}_{W_2}$.

If $\tau_{C',w_2,\gamma_2\delta_2}$ is the full congruence of $R_{C',w_2,\gamma_2\delta_2}$ for all $C' \in \mathcal{C}$, then $\mathsf{umax}(\mathcal{S}'^{\mathcal{R}}_{W_2}) = \mathsf{umax}(\mathcal{S}'^{\mathcal{R}'}_{W_2})$ and there is nothing to prove. Otherwise let us consider the set $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ of all $\mathcal{R}$-compatible solutions of $\mathcal{P}_{W\cap W_2}/_{\overline{\mu}^Y}$.

CLAIM 1. $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ and $\mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$ are nonempty.

The set $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ is nonempty, as it contains $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}$, which is nonempty by (S2) for $\mathcal{R}$. If $(v, \alpha, \beta) \notin \mathcal{W}'$, then $\mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$ contains $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}'}_{W}$ or its factor modulo $\overline{\mu}^Y$, which is nonempty. Suppose $(w_2, \gamma_2, \delta_2) \notin \mathcal{W}'$ and $C_1 = \langle \mathbf{s}_1, R_1\rangle \in \mathcal{C}$ is a constraint such that $\tau_{C_1,w_2,\gamma_2\delta_2}$ is nontrivial on $R_{C_1,w_2,\gamma_2\delta_2}$ and $\mathbf{b} \in \mathsf{umax}(R'_{C_1,w_2,\gamma_2\delta_2})$. By (S2) for $\mathcal{R}$ tuple $\mathbf{b}$ can be extended to a solution φ of $\mathcal{P}_W/_{\overline{\mu}^Y}$, $Y = \mathsf{MAX}(\mathcal{P}) - W_2$. Then $\varphi(v) \in B'_v$ and therefore for any $C_2 = \langle \mathbf{s}_2, R_2\rangle \in \mathcal{C}$ and any $(u, \eta, \theta) \in \mathcal{W}(\overline{\beta})$ we have $\varphi(\mathbf{s}_2 \cap W \cap W_{u,\eta\theta}) \in \mathrm{pr}_{\mathbf{s}\cap W\cap W_{u,\eta\theta}} R'_{C_2,u,\eta\theta}/_{\overline{\mu}^Y}$, that is, $\varphi(W \cap W_2) \in \mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$. Suppose now that $(v, \alpha, \beta), (w_2, \gamma_2, \delta_2) \in \mathcal{W}'$. If $(w_1, \gamma_1, \delta_1) \notin \mathcal{W}'$, then we apply the argument above to the problem $\mathcal{P}_W/_{\overline{\mu}^{\mathsf{MAX}(\mathcal{P})-W_1}}$. If $(w_1, \gamma_1, \delta_1) \in \mathcal{W}'$, then we consider the problem $\mathcal{P}_W/_{\overline{\mu}^{\mathsf{MAX}(\mathcal{P})-\mathbf{s}}}$.

We would like to define a congruence similar to τ_C on $\mathcal{S}'_{W\cap W_2}$. It cannot be done in the same straightforward way, since $\mathcal{S}'_{W\cap W_2} \neq \mathrm{pr}_{W\cap W_2}\mathcal{S}'_W$, so we define it as follows. For $C' = \langle \mathbf{s}', R'\rangle$ and $(u, \eta, \theta) \in \mathcal{W}(\overline{\beta})$ let $W' = W_{u,\eta\theta,\overline{\beta}} \cap W \cap W_2 \cap \mathbf{s}'$ and $\tau'_{C'}(u, \eta\theta)$ denote the restriction of $\tau_{C'}$ on $R^*_{C',u,\eta\theta} = \mathrm{pr}_{W'} R_{C',u,\eta\theta}/_{\overline{\mu}^Y}$, that is, $(\mathbf{b}, \mathbf{c}) \in \tau'_{C'}(u, \eta\theta)$ if there are $\mathbf{b}', \mathbf{c}' \in R_{C',v,\alpha\beta_v}/_{\overline{\mu}^Y}$ such that $\mathrm{pr}_{W'}\mathbf{b}' = \mathbf{b}$, $\mathrm{pr}_{W'}\mathbf{c}' = \mathbf{c}$, and $(\mathbf{b}', \mathbf{c}') \in \tau_{C'}$. By Lemma 56 $\tau'_{C'}(u, \eta\theta)$ is either the full congruence on $R^*_{C',u,\eta\theta}$, or a maximal one. In the latter case $\mathsf{typ}(\tau'_{C'}(u, \eta\theta), \overline{\beta}_{W'}) = \mathbf{2}$.

We extend the congruences $\tau'_{C'}(u,\eta\theta)$ to congruences of $\mathcal{S}'_{W\cap W_2}$ using $\tau'_{C'}(u,\eta\theta)\times\prod_{x\in(W\cap W_2)-W'}\underline{0}_x$. Then the set $\mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$ of $\mathcal{R}'$-compatible tuples from $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ is a block of

$$\tau=\bigwedge_{C'\in\mathcal{C},(u,\eta,\theta)\in\mathcal{W}(\overline{\beta})}\tau'_{C'}(u,\eta\theta), \tag{1}$$

let it be denoted by $\mathcal{S}^*$. Note that τ is a congruence of $\mathcal{S}'_{W\cap W_2}$, but the interval $(\tau,\overline{\beta}_{W\cap W_2})$ is not necessarily simple. By the observation above and Lemma 13 $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}/_\tau$ is term equivalent to a module. We need to prove that there is $\varphi\in\mathcal{S}'^{\mathcal{R}}_{W_2}$ such that $\varphi(U)=\mathbf{a}$ and $\varphi(W\cap W_2)\in\mathcal{S}^*$. In fact we prove a stronger statement, namely, that for any u-maximal tuple $\mathbf{b}$ from $\mathrm{pr}_{W_2-W}\mathcal{S}'^{\mathcal{R}}_{W_2}$ there is a solution φ satisfying the required conditions and such that $\varphi(W_2-W)=\mathbf{b}$. However, to formulate it precisely we need two additional constructions.

Let $W\cap W_2=\{x_1,\dots,x_k\}$ and $X=W\cap U=\{x_1,\dots,x_\ell\}$, and $X'=\{y_1,\dots,y_\ell\}$. Let

$$Q(x_1,\dots,x_k,y_1,\dots,y_\ell)=\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}(x_1,\dots,x_k)\wedge\mathrm{pr}_X\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}(y_1,\dots,y_\ell)\wedge\bigwedge_{i=1}^{\ell}(x_i=y_i),$$

and its factor $Q'=Q/_{\tau'}$, where $\tau'=\tau\times\underline{0}_{\mathrm{pr}_{X'}\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}}$. Let η_1,η_2 denote the link congruences of $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}/_\tau$ and $\mathrm{pr}_{X'}\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ with respect to Q', and let η' denote the congruence of $\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$, the full preimage of η_1, that is, $\eta'/_\tau=\eta_1$. Then, as is easily seen, since $\mathrm{pr}_{W\cap W_2}Q/_\tau$ is a module, $(\mathbf{b},\mathbf{c})\in\eta'$ if and only if there are $\mathbf{b}',\mathbf{c}'\in\mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ such that $(\mathbf{b},\mathbf{b}'),(\mathbf{c},\mathbf{c}')\in\tau$ and $\mathrm{pr}_X\mathbf{b}'=\mathrm{pr}_X\mathbf{c}'$.

Now, let $W_2=\{x_1,\dots,x_m\}$ (recall that $W\cap W_2=\{x_1,\dots,x_k\}$) and define a relation $S(x_1,\dots,x_m,y_1,\dots,y_\ell)$ as follows:

$$S(x_1,\dots,x_m,y_1,\dots,y_\ell)=\mathcal{S}'^{\mathcal{R}}_{W_2}(x_1,\dots,x_m)\wedge\mathrm{pr}_X\mathcal{S}'^{\mathcal{R}}_{W_2}(y_1,\dots,y_\ell)\wedge\bigwedge_{i=1}^{\ell}(x_i=y_i),$$

and let $S'=S/_{\tau''}$, where $\tau''=\tau\times\underline{0}_{\mathrm{pr}_{W_2-W}\mathcal{S}'^{\mathcal{R}}_{W_2}}\times\underline{0}_{\mathrm{pr}_{X'}\mathcal{S}'^{\mathcal{R}}_{W_2}}$. Similar to Q, let θ_1,θ_2 be the link congruences of $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}/_\tau$ and $\mathrm{pr}_H S$, where $H=X'\cup\{x_{k+1},\dots,x_m\}$, with respect to S', and let θ' denote the congruence of $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}$ such that $\theta_1=\theta'/_\tau$. Then immediately by the definition $(\mathbf{b},\mathbf{c})\in\theta'$ if and only if there are $\mathbf{b}',\mathbf{c}'\in\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}$ and $\mathbf{d}\in\mathrm{pr}_H S$ such that $(\mathbf{b},\mathbf{b}'),(\mathbf{c},\mathbf{c}')\in\tau$, $\mathrm{pr}_X\mathbf{b}'=\mathrm{pr}_X\mathbf{c}'=\mathrm{pr}_X\mathbf{d}$, and $(\mathrm{pr}_{(W\cap W_2)-X}\mathbf{b}',\mathbf{d}),(\mathrm{pr}_{(W\cap W_2)-X}\mathbf{c}',\mathbf{d})\in\mathcal{S}'^{\mathcal{R}}_{W_2}$.

We are interested in congruences η_1 and θ_1. The first of them indicates which τ-blocks extensions of $\mathrm{pr}_X\mathbf{a}$ can belong to. The second congruence also indicates to which τ-blocks extensions of $\mathbf{a}$ to a solution from $\mathcal{S}'^{\mathcal{R}}_{W_2}$ can belong to. Clearly,

$\theta_1 \subseteq \eta_1$. We prove however, that in both cases the set of attainable τ-blocks is the same. This essentially means that if a τ-block can be extended to a solution from $\mathcal{S}'^{\mathcal{R}}_{W_2}$, it can be extended in an almost arbitrary way.

CLAIM 2. (1) $S'' = (\mathrm{pr}_{W\cap W_2} S)/_\tau$ is a union of η_1-blocks;
(2) $\theta_1 = \eta_1|_{S''}$;
(3) let D be a θ_1-block and E the corresponding θ_2-block, then $D\times \mathsf{umax}(E) \subseteq S'$;
(4) for any $\mathbf{b} \in \mathcal{S}'^{\mathcal{R}}_{W_2}$ such that $\mathrm{pr}_{X\cup(W_2-W)}\mathbf{b}$ is u-maximal in a θ_2-block and any $\mathbf{b}' \in \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ such that $\mathbf{b}' \stackrel{\theta'}{\equiv} \mathrm{pr}_{W\cap W_2}\mathbf{b}$ there is $\mathbf{b}'' \in S$ such that $\mathrm{pr}_{W\cap W_2}\mathbf{b}'' \stackrel{\tau}{\equiv} \mathbf{b}'$ and $\mathrm{pr}_{X\cup(W_2-W)}\mathbf{b}'' = \mathrm{pr}_{X\cup(W_2-W)}\mathbf{b}$.

(1) It follows by Proposition 19 for any θ_1-block D' and a θ_2-block E' with $S'\cap(D'\times E') \neq \varnothing$, that $D'\times \mathsf{umax}(E') \subseteq S'$.

Let $\mathbf{b} \in S$, D' be the θ_1-block containing $(\mathrm{pr}_{W\cap W_2}\mathbf{b})^\tau$ and E' the corresponding θ_2-block. Then by Lemma 12(5) there is $\mathbf{b}' \in S$ such that $\mathrm{pr}_{W\cap W_2}\mathbf{b}' \stackrel{\tau}{\equiv} \mathrm{pr}_{W\cap W_2}\mathbf{b}$ and $\mathrm{pr}_H\mathbf{b}'$ is u-maximal in E'. We assume that $\mathbf{b}$ satisfies this condition. Let also D be the η_1-block containing $(\mathrm{pr}_{W\cap W_2}\mathbf{b})^\tau$. Note that $D' \subseteq D$.

Suppose there is $\mathbf{c} \in D$ such that $(\mathbf{c}', \mathrm{pr}_H\mathbf{b}) \in S$ for no $\mathbf{c}' \in \mathbf{c}$. We will derive a contradiction. Take some u-maximal $\mathbf{c}'$ from $\mathbf{c}$. Since $\mathrm{pr}_{W\cap W_2}\mathbf{b} \stackrel{\tau}{\not\equiv} \mathbf{c}'$, there is $C' = \langle \mathbf{s}', R'\rangle \in \mathcal{C}$ and $(u,\chi,\xi) \in \mathcal{W}(\overline{\beta})$ such that $(\mathrm{pr}_Z\mathbf{b}, \mathrm{pr}_Z\mathbf{c}') \notin \tau^*$, where $\tau^* = \tau'_{C'}(u,\chi\xi)$ and $Z = W_{u,\chi\xi}\cap W\cap W_2\cap \mathbf{s}'$. Choose a pair $\mathbf{b},\mathbf{c}$ in such a way that the number of such constraints and triples is minimal. We will find a polynomial f of $\mathcal{S}'_{W_2}$ such that (roughly speaking) $f(\mathrm{pr}_{W\cap W_2}\mathbf{b}), f(\mathbf{c}') \in \mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}$, and $f(\mathbf{c}'), \mathbf{c}'$ differ on few constraints and triples. Let $\mathbf{b}'' = (\mathrm{pr}_Z\mathbf{b})^{\tau^*}$, $\mathbf{c}'' = (\mathrm{pr}_Z\mathbf{c}')^{\tau^*}$. Since the interval $(\tau^*, \overline{\beta}'')$ has type **2**, where $\overline{\beta}'' = \overline{\beta}_Z$, $\{\mathbf{b}'', \mathbf{c}''\}$ is a subtrace of that interval. Let $(x,\gamma_x,\delta_x) \in \mathcal{W}(\overline{\beta})$, $x \in Z$, be such that (α,β_v) and (γ_x,δ_x) cannot be separated in $R^{\{v,x\}}$.

By (S7) for $\mathcal{R}$ $\mathcal{S}'_{W_2}$ is strongly chained. Since $\tau^* \prec \overline{\beta}''$ on $\mathrm{pr}_Z\mathcal{S}'^{\mathcal{R}}_{W_2}$ if we consider $\mathcal{S}'_{W_2}$ as a subdirect product of $\mathrm{pr}_Z\mathcal{S}'_{W_2}$ and $\mathbb{A}_y$, $y \in W_2 - Z$, by Lemma 31 there is a $(\tau^*, \overline{\beta}'')$-collapsing polynomial f of $\mathcal{S}'_{W_2}$ for $\overline{\beta}, \overline{B}$. By Lemma 57 $(\tau^*, \overline{\beta}'')$ cannot be separated from (γ_x,δ_x) for $x \in W\cap W_2$. For any $z \in W_2$ and any $\eta,\theta \in \mathsf{Con}(\mathbb{A}_z)$ such that $\eta \prec \theta \leq \beta_z$ consider $\mathrm{pr}_{Z\cup\{z\}}\mathcal{S}'_{W_2}$. If (η,θ) can be separated from (α,β_v) (or (α,β_v) can be separated from (η,θ)) then $(\tau^*, \overline{\beta}'')$ can be separated from (η,θ) (or the other way round). In particular, f can be chosen to satisfy the following conditions

(a) $f(\mathrm{pr}_Z(\mathcal{S}'_{W_2}))$ is a $(\tau^*, \overline{\beta}'')$-minimal set and $f(\mathbb{A}_y)$ is a (γ_y,δ_y)-minimal set for $y \in W\cap W_2$;

(b) for every $z \in W_2 - W$, $|f(B_z)| = 1$; and

(c) f is idempotent.

Since $\{\mathbf{b}'', \mathbf{c}''\}$ is a $(\tau^*, \overline{\beta}'')$-subtrace of $(\mathrm{pr}_Z R_{C',u,\chi\xi})/_{\tau^*}$, by condition (Q2s) of being strongly chained (S7) for $\mathcal{R}$ polynomial f can be chosen such that

(d) $\mathbf{b}'' = f(\mathbf{b}'')$, $\mathbf{c}'' = f(\mathbf{c}'')$.

Moreover, let $\mathbf{b}^* \in \mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}$ be such that there is a tuple $\mathbf{d} \in \mathcal{S}'^{\mathcal{R}}_{W_2}$ such that $\mathrm{pr}_{W_2-W}\mathbf{d} = \mathrm{pr}_{W_2-W}\mathbf{b}$, $\mathrm{pr}_{W\cap W_2}\mathbf{d} = \mathbf{b}^*$, $\mathrm{pr}_Z\mathbf{b}^* \in \mathbf{b}''$, and $\mathbf{d}$ is u-maximal in $\mathcal{S}'_{W_2}\cap\overline{B}$. Such a tuple exists, because since for any $\gamma', \delta' \in \mathsf{Con}(\mathbb{A}_z)$, $z \in W_2 - W$, with $\gamma' \prec \delta'$, and any $x \in W \cap W_2$, the interval (γ_x, δ_x) and therefore $(\tau^*, \overline{\beta}'')$ can be separated from (γ', δ'), by Lemma 54 $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2}/_\tau \times \mathsf{umax}(\mathrm{pr}_{W_2-W}\mathcal{S}'^{\mathcal{R}}_{W_2}) \subseteq \mathcal{S}'^{\mathcal{R}}_{W_2}$. By Lemma 40 polynomial f can be chosen such that

(e) $f(\mathbf{d}) = \mathbf{d}$.

Let $\mathbf{c}'$ be a u-maximal tuple from the τ-block $\mathbf{c} \subseteq \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$, and let $\mathbf{c}^* = f(\mathbf{c}')$, and $\mathbf{c}^\dagger$ a tuple from $\mathsf{Sg}(\mathbf{b}^*, \mathbf{c}^*)$ such that $\mathbf{b}^* \sqsubseteq_{as} \mathbf{c}^\dagger$ and $\mathbf{c}^* \stackrel{\tau}{\equiv} \mathbf{c}^\dagger$. Note that it suffices to prove that $\varrho = (\mathbf{c}^\dagger, \mathrm{pr}_{W_2-W}\mathbf{b}) \in \mathcal{S}'^{\mathcal{R}}_{W_2}$. Indeed, since $\mathrm{pr}_Z\mathbf{c}^* \stackrel{\tau^*}{\equiv} \mathrm{pr}_Z\mathbf{c}'$ and $\mathbf{c}'$ agrees with $\mathbf{c}^\dagger$ modulo $\tau'_{C''}(u, \chi'\xi')$ for every $C'' \in \mathcal{C}$ and $(u', \chi'\xi') \in \mathcal{W}(\overline{\beta})$, for which $\mathbf{b}$ and $\mathbf{c}'$ agree, we obtain a contradiction with the choice of $\mathbf{b}, \mathbf{c}$.

Take any $C'' = \langle \mathbf{s}'', R''\rangle \in \mathcal{C}$ and $(x, \chi', \xi') \in \mathcal{W}(\overline{\beta})$; we show that $\varrho(\mathbf{s}'' \cap W_2) \in \mathrm{pr}_{\mathbf{s}''\cap W_2} R_{C'',x,\chi'\xi'}$. Let $U' = \mathbf{s}'' \cap W \cap W_2$ and $U'' = \mathbf{s}'' \cap W_2$. Since $\mathbf{c}' \in \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$, we have $\mathrm{pr}_{U'}\mathbf{c}' \in \mathsf{umax}(\mathrm{pr}_{U'} R_{C'',x,\chi'\xi'})$. By (S2) $\mathrm{pr}_{U'}\mathbf{c}'$ can be extended to an $\mathcal{R}$-compatible solution σ from $\mathcal{S}'_{W_2}$. By the choice of f, property (b), $f(\mathrm{pr}_{U''-U'}\sigma) = \mathrm{pr}_{U''-U'}\mathbf{d} = \mathrm{pr}_{U''-U'}\mathbf{b}$, and $f(\mathrm{pr}_{U'}\sigma) = \mathrm{pr}_{U'}\mathbf{c}^*$ by definition of $\mathbf{c}^*$. Since $\mathrm{pr}_{U'}\mathbf{b}^* \sqsubseteq_{as} \mathrm{pr}_{U'}\mathbf{c}^\dagger$ in $\mathsf{Sg}(\mathrm{pr}_{U'}\mathbf{b}^*, \mathrm{pr}_{U'}\mathbf{c}^*)$, we have $\mathrm{pr}_{U''}\mathbf{b}^* \sqsubseteq_{as} \mathrm{pr}_{U''}\varrho$ in $\mathsf{Sg}(\mathrm{pr}_{U''}\mathbf{b}^*, \mathrm{pr}_{U''}\mathbf{c}^*)$, implying by (S6) that $\mathrm{pr}_{U''}\varrho \in R_{C'',x,\chi'\xi'}$. As this is true for every constraint C'', ϱ is an $\mathcal{R}$-compatible solution.

(2) The proof of item (1) shows in particular that for any $\mathbf{b} \in S$ and $\mathbf{c} \in (\mathrm{pr}_{W\cap W_2}S)/_{\eta_1}$ we also have $(\mathrm{pr}_{W\cap W_2}\mathbf{b})^\tau \stackrel{\theta_1}{\equiv} \mathbf{c}$, as both $\mathbf{b}^\tau$ and $(\mathbf{c}, \mathrm{pr}_H\mathbf{b})$ belong to S'.

(3) follows from Proposition 19.

(4) follows from (1) and (3).

If we show that $\mathbf{a}$ can be extended to $\mathbf{b} \in S$ such that $\mathrm{pr}_U\mathbf{b} = \mathbf{a}$ and such that $\mathrm{pr}_H\mathbf{b}$ is u-maximal in a θ_2-block, Claim 2 implies (S2) for $\mathcal{R}'$. Indeed, suppose there is $\mathbf{b} \in S$ satisfying the above conditions. Since $\mathrm{pr}_X\mathbf{a} \in \mathrm{pr}_X\mathcal{S}'^{\mathcal{R}'}_W$, there is $\mathbf{c}' \in \mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$ such that $\mathrm{pr}_X\mathbf{a} = \mathrm{pr}_X\mathbf{c}'$. In particular, this means that $\mathbf{c} = \mathbf{c}'^\tau \in D$, where D is the η_1-, and therefore θ_1-block containing $(\mathrm{pr}_{W\cap W_2}\mathbf{b})^\tau$, because $\mathrm{pr}_X\mathbf{c}' = \mathrm{pr}_X\mathbf{b} = \mathrm{pr}_X\mathbf{a}$. Then as $(\mathbf{c}, \mathrm{pr}_H\mathbf{b}) \in S'$ by Claim 2(4), this means that there is $\mathbf{d} \in \mathcal{S}'^{\mathcal{R}}_{W_2}$ such that $\mathrm{pr}_U\mathbf{d} = \mathbf{a}$ and $\mathrm{pr}_{W\cap W_2}\mathbf{d} \stackrel{\tau}{\equiv} \mathbf{c}'$; that is, $\mathbf{d} \in \mathcal{S}'^{\mathcal{R}'}_{W_2}$.

Next we show such a $\mathbf{b}$ exists. Recall that S^* denotes the τ-block of $\mathrm{pr}_{W\cap W_2}Q$ that contains $\mathcal{S}'^{\mathcal{R}'}_{W\cap W_2}$, and in particular, $\mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}'}_{W}$. Observe that

$$S^* \cap \mathrm{pr}_{W\cap W_2}S = S^* \cap \mathrm{pr}_{W\cap W_2}\mathcal{S}'^{\mathcal{R}}_{W_2} \neq \varnothing.$$

Indeed, let $\mathbf{d} \in S$ be such that $\mathrm{pr}_U\mathbf{d} = \mathbf{a}$. Then $\mathrm{pr}_{W\cap W_2}\mathbf{d} \in \mathcal{S}'^{\mathcal{R}}_{W\cap W_2}$ and $\mathbf{d}^\tau$ belongs to the same η_1-block as S^*, because $(S^*/_\tau, \mathrm{pr}_X\mathbf{a}) \in Q'$. By Claim 2(1) $S^*/_\tau \in (\mathrm{pr}_{W\cap W_2}S)/_\tau$, proving the the observation.

Let D be the θ_1-block containing $S^*/_\tau$ and E the corresponding θ_2-block. By what is proved above $\mathrm{pr}_H\mathbf{d} \in E$. We now only need to show that $\mathbf{d}$ can be chosen such that $\mathrm{pr}_H\mathbf{d} \in \mathsf{umax}(E)$. Let π be the congruence on $\mathrm{pr}_H S$ given by $\mathbf{c} \stackrel{\pi}{\equiv} \mathbf{d}$ if and only if $\mathrm{pr}_{X'}\mathbf{c} \stackrel{\tau_X}{\equiv} \mathrm{pr}_{X'}\mathbf{d}$. Then $\mathrm{pr}_H S/_\pi$ is isomorphic to $\mathrm{pr}_X(R_{C,w_1,\gamma_1\delta_1})/_{\tau_X}$, in particular, it is a module if τ_X is maximal, and 1-element otherwise. Let τ'_X denote the congrunce of $\mathrm{pr}_U R_{C,w_1,\gamma_1\delta_1}$ given by $\mathbf{c} \stackrel{\tau'_X}{\equiv} \mathbf{d}$ if and only if $\mathrm{pr}_X\mathbf{c} \stackrel{\tau_X}{\equiv} \mathrm{pr}_X\mathbf{d}$. As is easily seen, if G is a π-block of $\mathrm{pr}_H S$ then $\mathrm{pr}_U G$ is a τ'_X-block of $\mathrm{pr}_U R_{C,w_1,\gamma_1\delta_1}$. Therefore, as $\mathbf{a}$ is u-maximal in $\mathrm{pr}_U R'_{C,w_1,\gamma_1\delta_1}$, by Lemma 12(5) it can be extended to a u-maximal tuple $\mathbf{a}'$ in a π-block G. Since $E/_\pi$ is a module, by Lemma 11 $\mathbf{a}'$ is also u-maximal in E. The result follows.

The proof of (S3) is similar to that of (S2), except we only need to consider one constraint relation rather than the set of solutions to a subproblem.

For (S4) observe that every $R'_{C,w,\gamma\delta}$ is obtained as the intersection of $R_{C,w,\gamma\delta}$ with a block of $\tau_{C,w,\gamma\delta}$, and therefore is a subalgebra. Also, since $R_{C,w,\gamma\delta}/_{\tau_{C,w,\gamma\delta}}$ is a module, by Lemma 11 $\mathsf{umax}(R'_{C,w,\gamma\delta}) \subseteq \mathsf{umax}(R_{C,w,\gamma\delta})$ proving the first part of (S4).

To prove the rest of (S4) let $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$, $U = W_{w,\gamma\delta}$ and $\mathcal{S}'_U$ be one of the sets $\mathcal{S}_U$ if $(w,\gamma,\delta) \notin \mathcal{W}'$, or $\mathcal{S}_{U,Y}$ for $Y = \mathsf{MAX}(\mathcal{P}) - \mathbf{s}$ for some $C = \langle \mathbf{s}, R\rangle$ or $Y = \mathsf{MAX}(\mathcal{P}) - W_{u,\eta\theta}$ for some $(u,\eta,\theta) \notin \mathcal{W}'$, if $(w,\gamma,\delta) \in \mathcal{W}'$. As in the proof of condition (S2) we consider the congruence τ constructed as in (1) with U in place of W_2. Let Q be the τ-block of $\mathrm{pr}_{W\cap U}\mathcal{S}'_U$ containing $\mathcal{R}'$-compatible tuples. By (S4) for $\mathcal{R}$ there is a tuple $\mathbf{a} \in \mathcal{S}'^{\mathcal{R}}_U$ that is in $\mathsf{umax}(\mathcal{S}'_U)$. Since $\mathrm{pr}_{W\cap U}\mathcal{S}'_U/_\tau$ is a module, by Lemmas 12(4) and 7 there is an as-path $\mathbf{a} = \mathbf{a}_1, \dots, \mathbf{a}_k$ in $\mathcal{S}'^{\mathcal{R}}_U$ such that $\mathrm{pr}_{W\cap U}\mathbf{a}_k \in Q$. The tuple $\mathbf{a}_k$ belongs to $\mathcal{S}'^{\mathcal{R}'}_U$ and to $\mathsf{umax}(\mathcal{S}'_U)$.

For (S5), the existence of $A_{\mathcal{R}',w}$ follows from (S3). Also, as in the proof of (S4) $\mathsf{umax}(A_{\mathcal{R}',w}) \subseteq \mathsf{umax}(A_{\mathcal{R},w})$. The result now follows from (S5) for $\mathcal{R}$.

For (S6) consider $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, and let $w \in \mathbf{s}$. Let f be a polynomial of R, and let $\mathbf{a}, \mathbf{b} \in R$ be tuples satisfying the conditions of polynomial closeness. Let $\mathbf{c} \in \mathsf{Sg}(\mathbf{a}, f(\mathbf{b}))$ be such that $\mathbf{a} \sqsubseteq_{as} \mathbf{c}$ in $\mathsf{Sg}(\mathbf{a}, f(\mathbf{b}))$. By (S6) for $\mathcal{R}$, $\mathbf{c}$ is $\mathcal{R}$-compatible. It suffices to show that $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{c}$ is in the same τ_C block as $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{a}$.

However, this is straightforward, because $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{a} \stackrel{\tau_C}{\equiv} \mathrm{pr}_{\mathbf{s}\cap W}\mathbf{b}$, and as $f(\mathbf{a}) = \mathbf{a}$, we also have $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{a} \stackrel{\tau_C}{\equiv} f(\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{b})$. Since $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{c} \in \mathsf{Sg}(\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{a}, f(\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{b}))$, it follows $\mathrm{pr}_{\mathbf{s}\cap W}\mathbf{c} \stackrel{\tau_C}{\equiv} \mathrm{pr}_{\mathbf{s}\cap W}\mathbf{a}$.

Finally, (S7) follows from Lemma 55. □

6.3 Tightening for non-affine factors

Let $\mathcal{P} = (V, \mathcal{C})$ be a block-minimal instance, let $\mathcal{R}$ be a $\overline{\beta}$-strategy with respect to $\overline{B}$. Take $v \in V$ and $\alpha \in \mathsf{Con}(\mathbb{A}_v)$ with $\alpha \prec \beta_v$ such that $\mathsf{typ}(\alpha, \beta_v) \neq \mathbf{2}$. We tighten $\mathcal{R}$ in two steps. In the first step we restrict B_v to the subalgebra generated by an as-component of $A_{\mathcal{R},v}$ obtaining a collection of relations that satisfies all the properties of a strategy except (S5) and (S6). In the second step we restrict the same domain to one α-block and restore (S5) and (S6). Let D be an as-component of $A_{\mathcal{R},v}/_\alpha$, by (S5) D is also an as-component of $B_v/_\alpha$. Note that if $\mathsf{typ}(\alpha, \beta_v) \in \{\mathbf{4}, \mathbf{5}\}$ then by Lemma 23 any as-component of $B_v/_\alpha$ is a singleton and Step 2 is not needed. Conditions (S5),(S6) in this case are proved as in Step 2.

6.3.1 Step 1.

Let $D' = \{a \in A_{\mathcal{R},v} \mid a^\alpha \in D\}$ and let $\widehat{D} = \mathsf{Sg}(D')$. We consider the problem $\mathcal{P}'$ obtained from $\mathcal{P}$ by restricting the domain of v to $\widehat{D}$ and the domain of $w \in V - \{v\}$ to $A_{\mathcal{R},w}$. We first show that $\mathcal{P}'$ can be converted to a nonempty (2,3)-minimal instance that also satisfies some additional conditions.

In order to do that we introduce a family of binary relations, and then prove that this family is a (2,3)-strategy of $\mathcal{P}'$. For $x, y \in V$, let

$$Q^x = \{a \in \mathsf{amax}(A_{\mathcal{R},x}) \mid \text{ there is } d \in D \text{ such that } (d,a) \in R^{\{v,x\},\mathcal{R}}/_\alpha\},$$

and

$$Q^{xy} = \{(a,b) \in \mathsf{amax}(R^{\{x,y\},\mathcal{R}}) \mid \text{ there is } d \in D \text{ such that } (d,a) \in R^{\{v,x\},\mathcal{R}}/_\alpha, (d,b) \in R^{\{v,y\},\mathcal{R}}/_\alpha\}.$$

In particular $Q^v = \mathsf{amax}(D')$. We say that a tuple $\mathbf{a}$ on a set $U \subseteq V$ (where U can be, e.g. a subset of $\mathbf{s}$ for a constraint $C = \langle \mathbf{s}, R\rangle$, or a subset of $W_{w,\gamma\delta}$ for some $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$) is *Q-compatible* if $(\mathbf{a}[x], \mathbf{a}[y]) \in Q^{xy}$ for any $x, y \in U$.

Proposition 60 *(1) For any $x, y, z \in V$ and any $(a,b) \in Q^{xy}$ there is $c \in \mathsf{amax}(A_{\mathcal{R},z})$ such that $(a,c) \in Q^{xz}$ and $(b,c) \in Q^{yz}$.*
(2) For any $C = \langle \mathbf{s}, R\rangle$ let $R^{\mathcal{R}}$ denote the set of $\mathcal{R}$-compatible tuples from R. For any $I \subseteq \mathbf{s}$ and any Q-compatible $\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_I R^{\mathcal{R}})$, there is $\mathbf{a}' \in \mathsf{amax}(R^{\mathcal{R}})$ that is Q-compatible, and $\mathrm{pr}_I \mathbf{a}' = \mathbf{a}$.

(3) For any $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$, any $U \subseteq W_{w,\gamma\delta}$, and any $\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_U \mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma\delta}})$, where $\mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma\delta}}$ is the set of solutions of $(\mathcal{P}_{W_{w,\gamma\delta}})/_{\overline{\mu}^Y}$ for some set Y from the definition of block-minimality, there is $\mathbf{a}' \in \mathsf{amax}(\mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma\delta}})$ that is Q-compatible and $\mathrm{pr}_U \mathbf{a}' = \mathbf{a}$.

Proof: For $x, y \in V$ let Q^{vxy} denote the set of tuples (d,a,b) such that $(d,a) \in R^{\{v,x\},\mathcal{R}}/_\alpha$, $(d,b) \in R^{\{v,y\},\mathcal{R}}/_\alpha$, $(a,b) \in R^{\{x,y\},\mathcal{R}}$.

CLAIM 1. The set Q^x is as-closed in $A_{R,x}$, and Q^{xy} is as-closed if $R^{\{x,y\},\mathcal{R}}$.

Let $(a,b) \in Q^{xy}$. By Lemma 12(3) either Q^{vxy} contains a subdirect product of D and $\mathsf{as}(a,b)$, or $(D \times \mathsf{as}(a,b)) \cap Q^{vxy} = \varnothing$. Since $(a,b) \in Q^{xy}$ the former option holds. For the first part of the claim observe that $\mathrm{pr}_x \mathsf{as}(a,b) = \mathsf{as}(a)$.

CLAIM 2. For any $x, y \in V$, Q^{xy} is a subdirect product of $Q^x \times Q^y$.

Let $a \in Q^x$, then there is $d \in D$ with $(d,a) \in R^{\{v,x\},\mathcal{R}}$. By (S1) for $\mathcal{R}$, $\{\mathsf{umax}(R^{X,\mathcal{R}})\}_{X \subseteq V, |X|=2}$ is (2,3)-strategy, and there is $b \in \mathbb{A}_y$ with $(d,a,b) \in Q^{vxy}$; then $(a,b) \in Q^{xy}$.

We prove (2), the proof of (3) is basically identical, and we explain how to modify this proof to prove (1).

By induction on i we prove that a Q-compatible tuple $\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_I R^{\mathcal{R}})$ can be found for any $I \subseteq \mathbf{s}$, $|I| = i$. Moreover, for any Q-compatible $\mathbf{a}' \in \mathsf{amax}(\mathrm{pr}_{I-\{u\}} R^{\mathcal{R}})$, a Q-compatible $\mathbf{b} \in \mathsf{amax}(\mathrm{pr}_I R^{\mathcal{R}})$ can be found such that $\mathrm{pr}_{I-\{u\}} \mathbf{b} = \mathbf{a}'$.

First we consider the case $u \neq v$. For $i = 2$ the existence of $\mathbf{a}$ follows from Claim 2. So, suppose such a tuple exists for any $I \subseteq \mathbf{s}$ with $|I| \le i$. Let $I \subseteq \mathbf{s}$, $|I| = i+1$, $y \in I$, $I' = I - \{y\}$, and $\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_{I'} R^{\mathcal{R}})$ is Q-compatible. Let also $t \in I'$, $I'' = I' - \{t\}$. Without loss of generality assume $I = \{x_1, \dots, x_{i+1}\}$, $y = x_{i+1}$, $t = x_i$. Consider the relation given by

$$
\begin{aligned}
Q(x_1,\dots,x_i,z_1,\dots,z_i) = \exists y\ \ & \mathrm{pr}_{I''} R^{\mathcal{R}}(x_1,\dots,x_{i-1}) \\
\wedge \bigwedge_{j=1}^{i} & \left(R^{\{x_j,v\},\mathcal{R}}/_\alpha(x_j,z_j) \wedge R^{\{v,x_{i+1}\},\mathcal{R}}/_\alpha(z_j,y) \wedge R^{\{x_j,x_{i+1}\},\mathcal{R}}(x_j,y) \right).
\end{aligned} \tag{2}
$$

It suffices to prove that $\mathbf{a}'' = (\mathbf{a},\mathbf{e}) \in Q$, where $\mathbf{e}[j] \in D$ for each $j \in [i]$, since this would mean that there is a $c \in \mathbb{A}_{\mathcal{R},x_{i+1}}$ with the required properties.

Observe first that $\mathsf{umax}(\mathrm{pr}_{I'} R^{\mathcal{R}}) \subseteq \mathrm{pr}_{I'} Q$. Indeed, any $\mathbf{b} \in \mathrm{pr}_{I'} R^{\mathcal{R}}$ by (S3) can be extended to $\mathbf{b}' \in \mathrm{pr}_I R^{\mathcal{R}}$; then the values of the variables z_j can be chosen by (2,3)-consistency and (S1). This also implies that $\mathsf{umax}(\mathrm{pr}_{I''} Q) = \mathsf{umax}(\mathrm{pr}_{I''} R^{\mathcal{R}})$. Since $\alpha \prec \beta_v$, by (S7) for $\mathcal{R}$ and Lemma 54 for any $w \in V$ the relation $R^{\{v,w\},\mathcal{R}}/_\alpha$ either contains $D \times \mathsf{umax}(Q^w)$, or $R^{\{v,w\},\mathcal{R}}/_\alpha \cap (D \times Q^w)$

is the graph of a mapping $\kappa_w : Q^w \to D$. Let the set of variables for which the latter option holds be denoted by Z. By construction, for any $w_1, w_2 \in Z$, and any $(c,d) \in Q^{w_1 w_2}$, we have $\kappa_{w_1}(c) = \kappa_{w_2}(d)$. For any $w \in V - Z$, the set Q^w is as-closed, and $D \times \mathsf{umax}(Q^w) \subseteq R^{\{v,w\},\mathcal{R}}/_\alpha$.

Let $J = I' \cap Z$ and, if $J \neq \varnothing$, let $d = \kappa_w(\mathbf{a}[w])$ for any $w \in J$. If $J = \varnothing$, but $x_{i+1} \in Z$, then let $(\mathrm{pr}_{I''}\mathbf{a}, c)$ be an extention of $\mathrm{pr}_{I''}\mathbf{a}$ to a Q-compatible tuple from $\mathrm{pr}_{I\cup\{x_{i+1}\}}R$ and set $d = \kappa_{x_{i+1}}(c)$. If $x_{i+1} \notin Z$, then let d be any element of D. Then d is such that $(\mathbf{a}[w], d) \in R^{\{w,v\},\mathcal{R}}/_\alpha$ for any $w \in I'$.

Consider the tuple $\mathbf{b} = (\mathbf{a}, d, \ldots, d)$; we show that it satisfies the conditions of Theorem 14 with $X = I''$. Note that we cannot replace I'' with I' here, because in order to apply Theorem 14 $\mathbf{a}$ has to be as-maximal $\mathrm{pr}_{I'}Q$, which may not be true. By what is observed before, $\mathrm{pr}_{I''}\mathbf{a} \in \mathrm{pr}_{I''}Q$, and for any $s_1, s_2 \in I'$ we have $(\mathbf{a}[s_1], \mathbf{a}[s_2]) \in \mathrm{pr}_{s_1 s_2}Q$. We now show that for any of the remaining pairs of variables $x, z \in \{x_1, \ldots, x_i, z_1, \ldots, z_i\}$ $(\mathbf{b}[x], \mathbf{b}[z]) \in \mathrm{pr}_{xz}Q$. Let $x \in \{x_1, \ldots, x_i\}$ and $z \in \{z_1, \ldots, z_i\}$. If $x \neq x_i$, then by the inductive hypothesis $\mathrm{pr}_{I''}\mathbf{a}$ can be extended to some value c of y such that $(\mathbf{a}[s], c) \in Q^{sy}$ for any $s \in I''$. Then there is $c' \in \mathbb{A}_{x_i}$ such that $(c', c) \in Q^{x_i y}$. Also, $(c, d) \in Q^{yv}$ and c' can be chosen such that $(c', d) \in Q^{x_i v}$. If $x = x_i$, then find a value c for y such that $(\mathbf{a}[x_i], c) \in Q^{x_i, x_{i+1}}$, and then extend c to a tuple on $I'' \cup \{y\}$ by induction hypothesis. The values of z_j can be set using (2,3)-consistency. Finally, if $x, z \in \{z_1, \ldots, z_i\}$, we proceed as in one of the previous cases.

By Theorem 14 there is $\mathbf{b}' \in Q$ such that $\mathrm{pr}_{I''}\mathbf{a} \sqsubseteq_{as} \mathrm{pr}_{I''}\mathbf{b}'$ in $Q' = \mathrm{pr}_{I''}Q$, $(\mathbf{a}[x], \mathbf{a}[z]) \sqsubseteq_{as} (\mathbf{b}'[x], \mathbf{b}'[z])$ in $R^{\{x,z\},\mathcal{R}}$ for any $x, z \in I'$, and $d \sqsubseteq_{as} \mathbf{b}'[z_j]$ in $A_{\mathcal{R},v}/_\alpha$ for any $j \in [i]$.

Let $\mathsf{lk}_1, \mathsf{lk}_2$ be the link congruences of $Q', \mathbb{A}_{x_i}$ with respect to $\mathrm{pr}_{I'}Q$. Since $\mathsf{umax}(\mathrm{pr}_{I'}R^{\mathcal{R}}) \subseteq \mathrm{pr}_{I'}Q$, the link congruences of $\mathsf{umax}(\mathrm{pr}_{I''}R^{\mathcal{R}}), \mathbb{A}_{x_i}$ with respect to $\mathrm{pr}_{I'}R^{\mathcal{R}}$ are smaller than $\mathsf{lk}_1, \mathsf{lk}_2$. Therefore the lk_1- and lk_2-blocks A, B containing $\mathrm{pr}_{I''}\mathbf{a}$ and $\mathbf{a}[x_i]$, respectively, are such that $Q'' = \mathrm{pr}_{I'}Q \cap (A \times B) \neq \varnothing$. Choose $\mathbf{a}' \in \mathrm{pr}_{I'}R^{\mathcal{R}} \cap (A \times B)$ such that $\mathbf{a} \sqsubseteq_{as} \mathbf{a}'$ in $\mathrm{pr}_{I'}R^{\mathcal{R}}$ and $\mathbf{a}'$ is as-maximal in Q''. As is easily seen, such a tuple exists by Corollary 18, because, since Q'' is linked, any $\mathbf{a}' \in Q''$ such that $\mathrm{pr}_{I''}\mathbf{a}'$ is as-maximal in A and $\mathbf{a}'[x_i]$ is as-maximal in B is as-maximal in Q''. Now, consider

$$\begin{aligned} S(x_1, \ldots, x_i, z_1, \ldots, z_i) &= \exists y A(x_1, \ldots, x_{i-1}) \wedge B(x_i) \\ &\wedge \bigwedge_{j=1}^{i} R^{\mathcal{R},\{x_j,v\}}/_\alpha(x_j, z_j) \wedge R^{\mathcal{R},\{v,x_{i+1}\}}/_\alpha(z_j, y) \wedge R^{\mathcal{R},\{x_j,x_{i+1}\}}(x_j, y). \end{aligned}$$

By the same argument as before, there is $\mathbf{c} \in S$ such that $\mathbf{a}' \sqsubseteq_{as} \mathrm{pr}_{I'}\mathbf{c}$ in $\mathrm{pr}_{I'}S$ and $\mathbf{c}[z_j] \in D$ for $j \in [i]$. Since $\mathbf{a}'$ is as-maximal in $\mathrm{pr}_{I'}S$, we also have $\mathrm{pr}_{I'}\mathbf{c} \sqsubseteq_{as} \mathbf{a}'$

in $\mathrm{pr}_{I'}S$. Therefore there is $c \in A_{\mathcal{R},x_{i+1}}$ such that $(\mathbf{a}'[x], c) \in Q^{xx_{i+1}}$ for every $x \in I'$. By Theorem 14 applied to $R^{\mathcal{R}}$, there is $\mathbf{a}'' \in \mathrm{pr}_I R$ such that $\mathrm{pr}_{I'}\mathbf{a}'' = \mathbf{a}'$ and $c \sqsubseteq_{as} \mathbf{a}''[x_{i+1}]$. Finally, since $\mathbf{a}$ is as-maximal in $\mathrm{pr}_{I'}R^{\mathcal{R}}$, there is an as-path from $\mathbf{a}'$ to $\mathbf{a}$ in $\mathrm{pr}_{I'}R^{\mathcal{R}}$ and we complete the proof by Lemma 12(4).

Next, we consider the case $u = v$. Let $I = \{x_1, \dots, x_i, v\}$, $I' = \{x_1, \dots, x_i\}$ and $I'' = \{x_1, \dots, x_{i-1}\}$. Note that by reordering the variables we may assume that if $x_i \in Z$ then $Z \cap \{x_1, \dots, x_{i-1}\} \neq \varnothing$. By the induction hypothesis there is $c \in A_{\mathcal{R},v}$ such that $(\mathrm{pr}_{I''}\mathbf{a}, c)$ belongs to $\mathsf{amax}(\mathrm{pr}_{I''\cup\{v\}}R^{\mathcal{R}})$ and is Q-compatible, in particular, $c^{\alpha} \in D$. We consider the tuple $(\mathbf{a}, c^{\alpha})$ and relation $R' = \mathrm{pr}_I R^{\mathcal{R}}/_{\alpha}$. We have $\mathbf{a} \in \mathrm{pr}_{I'}R'$. For any $j \in [i-1]$ we have $(\mathbf{a}[j], c) \in Q^{x_j v}$ by the choice of c. If $x_i \notin Z$ then there is $c' \stackrel{\alpha}{\equiv} c$ such that $(\mathbf{a}[x_i], c') \in R^{\{x_i,v\},\mathcal{R}}$. By (S3) for $\mathcal{R}$, $(\mathbf{a}[x_i], c')$ extends to a tuple from $R^{\mathcal{R}}$, therefore $(\mathbf{a}[x_i], c) \in \mathrm{pr}_{x_i v}R'$. If $x_i \in Z$, then there is also some $j \in I'' \cap Z$. Then since $(\mathbf{a}[j], \mathbf{a}[i]) \in Q^{x_j x_i}$, we have $\kappa_{x_j}(\mathbf{a}[x_j]) = \kappa_{x_i}(\mathbf{a}[x_i]) = c$. Therefore there is $c' \stackrel{\alpha}{\equiv} c$ such that $(\mathbf{a}[x_i], c') \in R^{\{x_i,v\},\mathcal{R}}$ and we continue as before. By Theorem 14 there is $\mathbf{b} \in R'$ such that $\mathrm{pr}_{I'}\mathbf{b} = \mathbf{a}$ and $c \sqsubseteq_{as} \mathbf{b}[v]$ in $A_{\mathcal{R},v}/_{\alpha}$, that is, $\mathbf{b}[v] \in D$. Therefore there is $\mathbf{c} \in \mathrm{pr}_I R$ such that $\mathrm{pr}_{I'}\mathbf{c} = \mathbf{a}$ and $\mathbf{c}[v]^{\alpha} \in D$, as required.

Item (3) can be proved in the same way, as $\mathcal{P}$ is block-minimal. For (1) we need to make two changes. First, we apply the argument above for $i = 2$ and stop before the last application of Theorem 14. Second, we need to consider the case when extending a pair from Q^{xy} by a value of v. More precisely, let $(a, b) \in Q^{xy}$, $x, y \in V$, we need to find $c \in A_{\mathcal{R},v}$ such that $c^{\alpha} \in D$ and $(a, c) \in Q^{xv}$, $(b, c) \in Q^{yv}$. Let

$$Q(x, y, v) = R^{\{x,y\},\mathcal{R}}(x, y) \wedge R^{\{x,v\},\mathcal{R}}(x, v) \wedge R^{\{y,v\},\mathcal{R}}(y, v).$$

By (2,3)-consistency and construction $(a, b) \in \mathrm{pr}_{xy}Q, (a, d_1) \in \mathrm{pr}_{xv}Q, (b, d_2) \in \mathrm{pr}_{yv}Q$, where $d_1^{\alpha} = d_2^{\alpha} \in D$. The relation $Q' = Q/_{\alpha}$ then satisfies the conditions: $(a, b) \in \mathrm{pr}_{xy}Q', (a, d) \in \mathrm{pr}_{xv}Q', (b, d) \in \mathrm{pr}_{yv}Q'$ where $d = d_1^{\alpha} = d_2^{\alpha}$. By Theorem 14 there is a tuple $(a, b, d') \in Q'$ such that $d \sqsubseteq_{as} d'$, that is, $d' \in D$. Therefore Q contains a tuple (a, b, c) for some c with $c^{\alpha} = d'$. This c is as required. □

Let $\mathcal{P}''$ be the problem obtained from $\mathcal{P}'$ as follows: establish (2,3)-minimality and, for any $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$, establish minimality of $\mathcal{P}'_{W_{w,\gamma\delta}}/_{\overline{\mu}^Y}$, where Y is one of the sets specified in the definition of block-minimality, until the instance does not change any longer. Let $A''_{\mathcal{R},w}$ be the domains of $w \in V$ for $\mathcal{P}''$, for $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ let R'' denote the corresponding constarint relation of $\mathcal{P}''$; for $(w, \gamma, \delta) \in \mathcal{W}(\overline{\beta})$ let $\mathcal{S}''_{W_{w,\gamma\delta},Y}$ denote the set of solutions of $\mathcal{P}''_{W_{w,\gamma\delta}}/_{\overline{\mu}^Y}$; finally $R''_{C,w,\gamma\delta}$ denote the set of tuples from $R_{C,w,\gamma\delta}$ extending to a solution of $\mathcal{P}''_{W_{w,\gamma\delta}}/_{\overline{\mu}^Y}$.

The next corollary follows straightforwardly from Proposition 60, because establishing (2,3)-minimality or minimality never eliminates an as-maximal Q-compatible tuples.

Corollary 61 *The sets $A''_{\mathcal{R},w}$, R'', $\mathcal{S}''_{W_{w,\gamma\delta},Y}$, and $R''_{C,w,\gamma\delta}$ contain all the as-maximal Q-compatible tuples from $A_{\mathcal{R},w}$, R, $\mathcal{S}_{W_{w,\gamma\delta},Y}$, and $R_{C,w,\gamma\delta}$, respectively.*

Now we can show that the collection of relations

$$\mathcal{R}'' = \{R''_{C,w,\gamma\delta} \mid C \in \mathcal{C}, (w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})\}$$

is almost a $\overline{\beta}$-strategy.

Theorem 62 *The collection of relations $\mathcal{R}''$ constructed as above satisfies all the conditions of a $\overline{\beta}$-strategy with respect to $\overline{B}$, except (S5) and (S6).*

Proof: Condition (S1) follows from Corollary 61. Conditions (S2),(S3) follow from Corollary 61 and construction (establishing block-minimality). For (S4) every relation $R''_{C,w,\gamma\delta}$ is a subalgebra in $R_{C,w,\gamma\delta}$ and therefore in $\mathrm{pr}_{\mathbf{s}\cap W_{w,\gamma\delta}} R$ by construction. Moreover, as by Proposition 60 $R''_{C,w,\gamma\delta}$ contains an element as-maximal in $R_{C,w,\gamma\delta}$, we have $\mathsf{umax}(R''_{C,w,\gamma\delta}) \subseteq \mathsf{umax}(R_{C,w,\gamma\delta})$. Finally, condition (S7) follows from (S7) for $\mathcal{R}$. $\Box$

We will need another property of $\mathcal{P}''$. Unfortunately, Q-compatible tuples are not very helpful in establishing properties (S5),(S6), since those are properties of u-maximal fragments of relations. Therefore we need to extend Q-compatibility to u-maximal elements. Similar to Q-compatibility we make the following definition. For $x, y \in V$, let

$$P^x = \{a \in \mathsf{umax}(A_{\mathcal{R},x}) \mid \text{ there is } d \in D \text{ such that } (d,a) \in R^{\{v,x\},\mathcal{R}}/_\alpha\},$$

and

$$P^{xy} = \{(a,b) \in \mathsf{umax}(R^{\{x,y\},\mathcal{R}}) \mid \text{ there is } d \in D \text{ such that } (d,a) \in R^{\{v,x\},\mathcal{R}}/_\alpha, (d,b) \in R^{\{v,y\},\mathcal{R}}/_\alpha\}.$$

In particular $P^v = \mathsf{umax}(\widehat{D})$. Note that these relations are different from Q^x, Q^{xy} in that they consist of u-maximal elements and pairs, rather than as-maximal as Q^x, Q^{xy}. We say that a tuple $\mathbf{a}$ on a set $U \subseteq V$ (where U can be, e.g. a subset of $\mathbf{s}$ for a constraint $C = \langle \mathbf{s}, R\rangle$, or a subset of $W_{w,\gamma\delta}$ for some $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$) is *P-compatible* if $(\mathbf{a}[x], \mathbf{a}[y]) \in P^{xy}$ for any $x, y \in U$.

Let $\mathcal{P}'''$ be the instance obtained as follows: First, restrict the domains and relations of $\mathcal{P}$ to the sets of P-compatible tuples they contain; let the new relation for $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ be denoted by $R^\dagger$. Second, establish (2,3)-minimality and block-minimality of the resulting instance. Let $R''', R'''_{C,w,\gamma\delta}, \mathcal{S}'''_{W_{w,\gamma\delta},Y}$ denote the relations induced by $\mathcal{P}'''$. Note that the domains and relations of $\mathcal{P}'''$ are not necessarily subalgebras.

Lemma 63 *Let $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$.*

(1) Relations $R''', R'''_{C,w,\gamma\delta}, \mathcal{S}'''_{W_{w,\gamma\delta},Y}$ are nonempty, and $R''' \subseteq R''$, $R'''_{C,w,\gamma\delta} \subseteq R''_{C,w,\gamma\delta}$, $\mathcal{S}'''_{W_{w,\gamma\delta},Y} \subseteq \mathcal{S}''_{W_{w,\gamma\delta},Y}$.

(2) Relations $R'''_{C,w,\gamma\delta}$ and $\mathcal{S}'''_{W_{w,\gamma,\delta,\overline{\beta}},Y}$ are as-closed in $R_{C,w,\gamma\delta}$, and in $\mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma,\delta}}$, respectively (recall that $\mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma,\delta,\overline{\beta}}}$ is the set of $\mathcal{R}$-compatible tuples from $\mathcal{S}_{W_{w,\gamma,\delta,\overline{\beta}},Y}$).

Proof: (1) The inclusions $R''' \subseteq R''$, $R'''_{C,w,\gamma\delta} \subseteq R''_{C,w,\gamma\delta}$, $\mathcal{S}'''_{W_{w,\gamma\delta,\overline{\beta}},Y} \subseteq \mathcal{S}''_{W_{w,\gamma\delta,\overline{\beta}},Y}$ follow from the construction, as $R^\dagger \subseteq R'$. On the other hand, as every Q-compatible tuple from R belongs to $R^\dagger$, the nonemptyness follows by Corollary 61.

(2) First we observe that for any $x, y \in V$ the relation P^{xy}, and therefore P^x, is as-closed in $\mathsf{umax}(R^{\{x,y\},\mathcal{R}})$. This can be done in the same way as in the proof of Proposition 60. Let $(a,b) \in P^{xy}$ and $(a',b') \in \mathsf{umax}(R^{\{x,y\},\mathcal{R}})$ be such that $(a,b) \sqsubseteq_{as} (a',b')$. We need to find $d \in D$ such that $(a',d) \in R^{\{x,v\},\mathcal{R}}/_\alpha$, $(b',d) \in R^{\{y,v\},\mathcal{R}}/_\alpha$. Let

$$Q(x,y,v) = R^{\{x,y\},\mathcal{R}}(x,y) \wedge R^{\{x,v\},\mathcal{R}}/_\alpha(x,v) \wedge R^{\{y,v\},\mathcal{R}}/_\alpha(y,v),$$

which is a subalgebra of $\mathbb{A}_x \times \mathbb{A}_y \times \mathbb{A}_v/_\alpha$. Since $(a,b) \in P^{xy}$ there is $c \in D$ with $(a,b,c) \in Q$. By (S1) for $\mathcal{R}$ we have $(a',b') \in \mathrm{pr}_{xy}Q$, moreover, $(a,b) \sqsubseteq_{as} (a',b')$ in $\mathrm{pr}_{xy}Q$. By Lemma 12(4) there is $d \in A_{\mathcal{R},v}$ such that $(a',b',d^\alpha) \in Q$ and $(a,b,c) \sqsubseteq_{as} (a',b',d^\alpha)$ in Q. Therefore $d^\alpha \in D$.

It suffices to prove the statement for relations of the form $R'''_{C,w,\gamma\delta}$, because relations $\mathcal{S}'''_{W_{w,\gamma,\delta},Y}$ are pp-definable through $R'''_{C,w,\gamma\delta}$, and we can use Lemma 51(2). Since P^{xy} are as-closed, every relation $R^\dagger_{C,w,\gamma\delta}$ is also as-closed in $\mathsf{umax}(R_{C,w,\gamma\delta})$. We prove by induction that this property is preserved as (2,3)-minimality and block-minimality is being established. The observation about the relations $R^\dagger_{C,w,\gamma\delta}$ establishes the base case. For the induction step, let $R^\ddagger_{C,w,\gamma\delta}$ and $\mathcal{S}^\ddagger_{W_{w,\gamma\delta},Y}$ denote the current state of the corresponding relations, and they are as-closed. There are two cases for the induction step.

In the first case we make a step to enforce (2,3)-minimality, that is, for some $x, y, z \in V$ we check whether or not some $(a,b) \in R^{\ddagger\{x,y\}}$ can be extended by $c \in B_z$ such that $(a,c) \in R^{\ddagger\{x,z\}}$ and $(b,c) \in R^{\ddagger\{y,z\}}$. Let $(a,b) \in R^{\ddagger\{x,y\}}$ be such that there are $c \in B_z$ with $(a,c) \in R^{\ddagger\{x,z\}}$, $(b,c) \in R^{\ddagger\{y,z\}}$, and let

$(a', b') \in R^{\ddagger\{x,y\}}$ such that $(a, b) \sqsubseteq_{as} (a', b')$ in $R^{\ddagger\{x,y\}}$. Then there is $c' \in B_z$ such that $(a', c') \in R^{\{x,z\}}$, $(b', c') \in R^{\{y,z\}}$. Similar to part (1) let

$$Q(x, y, z) = \mathsf{Sg}(R^{\ddagger\{x,y\}})(x, y) \wedge R^{\{x,v\},\mathcal{R}}(x, v) \wedge R^{\{y,v\},\mathcal{R}}(y, v),$$

We have $\mathsf{Sg}(R^{\ddagger\{x,y\}}) = \mathrm{pr}_{xy} Q$ by (S1) for $\mathcal{R}$, and $(a, b, c), (a', b', c') \in Q$. By Lemma 12(4) we may assume that $(a, b, c) \sqsubseteq_{as} (a', b', c')$ in Q. Since $R^{\ddagger\{x,z\}}, R^{\ddagger\{x,z\}}$ are as-closed, we have $(a', c') \in R^{\ddagger\{x,z\}}, (b', c') \in R^{\ddagger\{x,z\}}$

In the second case we solve a subproblem of the form $\mathcal{P}^{\ddagger}_{W_{w,\gamma\delta}}/_{\overline{\mu}^Y}$. Let $U = W_{w,\gamma\delta}$, let $\mathcal{S}^{\ddagger}_{W_{w,\gamma\delta},Y}$ be the corresponding set of solutions. Take $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ and $(u, \eta, \theta) \in \mathcal{W}(\overline{\beta})$, let $U' = \mathbf{s} \cap U \cap W_{u,\eta\theta}$. Suppose $\mathbf{a} \in R^{\ddagger}_{C,u,\eta\theta}$ be such that there is a solution $\varphi \in \mathcal{S}^{\ddagger}_{W_{w,\gamma\delta,\overline{\beta}},Y}$ extending $\mathrm{pr}_{U'}\mathbf{a}$, and let $\mathbf{b} \in R^{\ddagger}_{C,u,\eta\theta}$ is such that $\mathbf{a} \sqsubseteq_{as} \mathbf{b}$ in $R^{\ddagger}_{C,u,\eta\theta}$. We need to show that $\mathrm{pr}_{U'}\mathbf{b}$ is extendible to a solution from $\mathcal{S}^{\ddagger}_{W_{w,\gamma\delta},Y}$. Let $\psi \in \mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma\delta},Y}$ be a solution extending $\mathrm{pr}_{U'}\mathbf{b}$. Since $\mathrm{pr}_{U'}\mathbf{a} \sqsubseteq_{as} \mathrm{pr}_{U'}\mathbf{b}$ by Lemma 12(4) we may assume that $\varphi \sqsubseteq_{as} \psi$ in $\mathcal{S}'^{\mathcal{R}}_{W_{w,\gamma\delta,\overline{\beta}},Y}$. Since by the induction hypothesis $\mathcal{S}^{\ddagger}_{W_{w,\gamma\delta,\overline{\beta}},Y}$ is as-closed, the result follows. □

6.3.2 Step 2.

In this step we tighten the 'near-strategy' $\mathcal{R}''$ in a way similar to that from Section 6.2. We start with showing that the domains of all variables in $W_{v,\alpha\beta}$ have to be tightened.

Lemma 64 *For every $w \in W = W_{v,\alpha\beta_v}$ there is a congruence $\alpha_w \in \mathsf{Con}(\mathbb{A}_w)$ with $\alpha_w \prec \beta_w$, and such that $\mathcal{S}_{vw} \cap (B_v \times B_w)$ is aligned with respect to (α, α_w), that is, for any $(a_1, a_2), (b_1, b_2) \in \mathcal{S}_{vw} \cap (B_v \times B_w)$, $a_1 \stackrel{\alpha}{\equiv} b_1$ if and only if $a_2 \stackrel{\alpha_w}{\equiv} b_2$.*

Proof: It suffices to show that the link congruences $\mathsf{lk}_1, \mathsf{lk}_2$ of $Q = \mathcal{S}_{vw}$ viewed as a subdirect product of $\mathbb{A}_v \times \mathbb{A}_w$ are such that $\beta_v \wedge \mathsf{lk}_1 \le \alpha$ and $\beta_w \wedge \mathsf{lk}_2 < \beta_w$. Since $w \in W$ there are $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_w)$ such that $\gamma \prec \delta \le \beta_w$ and (α, β_v) and (γ, δ) cannot be separated. By Lemma 32 it follows that $\beta_v \wedge \mathsf{lk}_1 \le \alpha$ and $\mathsf{lk}_2 \wedge \delta \le \gamma$. We set $\alpha_w = \beta_w \wedge \mathsf{lk}_2 < \beta_w$. Since $B_w/_{\alpha_w}$ is isomorphic to $B_v/_{\alpha}$, $\alpha_w \prec \beta_w$. □

Let $\beta'_v = \alpha$, $\beta'_w = \alpha_w$ for $w \in W = W_{v,\alpha\beta_v}$, and $\beta'_w = \beta_w$ for $w \in V - W$. Lemma 64 implies that there is an isomorphism $\nu_w : B_v/_{\beta'_v} \to B_w/_{\beta'_w}$. Choose an as-maximal β'_v-block B, an element of D from Step 1 and set $B'_v = B$, $B'_w = \nu_w(B)$ for $w \in V - W$, and $B'_w = B_w$ for $w \in V - W$. Let $\mathcal{P}^*$ be the problem instance obtained from $\mathcal{P}''$ as follows: first restrict the domain of

$w \in W$ in $\mathcal{P}''$ to B'_w, then establish the (2,3)-minimality of the resulting problem, and finally, establish the minimality of all problems of the form $\mathcal{P}''_{W_{w,\gamma\delta}}/_{\overline{\mu}^Y}$ for $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$, where Y is a set specified in the definition of block-minimality for $\mathcal{P}$.

Let $\mathcal{R}^*$ be the following collection of relations;

(T1) $\mathcal{R}^* = \{R^*_{C,w,\gamma\delta} \mid C = \langle \mathbf{s}, R\rangle \in \mathcal{C}, (w,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')\}$;

(T2) for every $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, $(u,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')$, $R^*_{C,u,\gamma\delta} = \mathrm{pr}_{\mathbf{s}\cap W_{u,\gamma\delta}} R^*$, where R^* is the constraint relation of $\mathcal{P}^*$ obtained from R.

Lemma 65 *(1) For every constraint $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, and every $\mathbf{a} \in \mathsf{umax}(R'')$ such that $\mathbf{a}[u] \in \nu_u(D)$ for $u \in \mathbf{s} \cap W$ there is a tuple $\mathbf{b} \in \mathsf{umax}(R'')$ such that $\mathrm{pr}_{\mathbf{s}-W}\mathbf{b} = \mathrm{pr}_{\mathbf{s}-W}\mathbf{a}$ and $\mathbf{b}[u] \in B'_u$ for $u \in \mathbf{s} \cap W$.*

(2) Let $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')$, $W' = W_{w,\gamma\delta}$, and $\mathbf{a} \in \mathsf{umax}(R''_{C,w,\gamma\delta})$ such that $\mathbf{a}[u] \in \nu_u(D)$ for $u \in \mathbf{s} \cap W' \cap W$. Then there is $\mathbf{b} \in \mathsf{umax}(R''_{C,w,\gamma\delta}))$ such that $\mathrm{pr}_{(\mathbf{s}\cap W')-W}\mathbf{b} = \mathrm{pr}_{(\mathbf{s}\cap W')-W}\mathbf{a}$ and $\mathbf{b}[u] \in B'_u$ for $u \in \mathbf{s} \cap W' \cap W$.

(3) Let $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta}')$ and $W' = W_{w,\gamma\delta}$. Let $\mathcal{S}''_{W'}$ be the set of solutions of $\mathcal{P}''_{W'}/_{\overline{\mu}^Y}$, where $Y = \varnothing$ if $(w,\gamma,\delta) \notin \mathcal{W}'$, and is one of the sets specified in the definition of block-minimality otherwise. For every solution $\varphi \in \mathsf{umax}(\mathcal{S}''_{W'})$ such that $\varphi[u] \in \nu_u(D)$ for $u \in W' \cap W$ there is a solution $\psi \in \mathsf{umax}(\mathcal{S}''_{W'})$ such that $\psi(u) = \varphi(u)$ for $u \in W' - W$ and $\psi(u) \in B'(u)$ for $u \in W' \cap W$.

Proof: (1) Let $U_1 = \mathbf{s} \cap W$ and $U_2 = \mathbf{s} - W$. If $U_1 = \varnothing$ there is nothing to prove; assume $U_1 \neq \varnothing$. It suffices to consider $Q = R''/_{\overline{\alpha}'}$ where $\alpha'_u = \beta'_u$ if $u \in U_1$ and $\alpha'_u = \underline{0}_u$ otherwise. So, we assume $\beta'_u = \underline{0}_u$ for all $u \in U_1$. Then for any $u_1, u_2 \in U_1$ and any $\mathbf{d} \in R$ such that $\mathbf{d}[u_1] \in B_{u_1}, \mathbf{d}[u_2] \in B_{u_2}$ we have $\mathbf{d}[u_2] = \nu_{u_2} \circ \nu_{u_1}^{-1}(\mathbf{d}[u_1])$. Therefore we may assume that $|U_1| = 1$, say, $U_1 = \{u\}$.

Considering R'' as a subalgebra of $\mathbb{A}_u \times \mathrm{pr}_{\mathbf{s}-\{u\}}R$, the result follows by Lemma 54. Indeed, since there is a $\alpha_u\beta_u$-collapsing polynomial f of R, that is, $f(\overline{\beta}_{\mathbf{s}-\{u\}}) \subseteq \underline{0}_{\mathbf{s}-\{u\}}$, there are no $\eta, \theta \in \mathsf{Con}(\mathrm{pr}_{\mathbf{s}-\{u\}}R)$ with $\eta \prec \theta \leq \overline{\beta}_{\mathbf{s}-\{u\}}$ such that (α_u, β_u) cannot be separated from (η, θ).

(2) and (3) are proved in essentially the same way. □

To show that $\mathcal{P}^*$ has the desirable properties, in particular, it is nonempty, we consider a collection of unary and binary relations similar to Q^x, Q^{xy} from Step 1. For $x, y \in V$ let T^x, T^{xy} denote the following sets:

$$
\begin{aligned}
T^x &= \{a \in \mathsf{amax}(A''_{\mathcal{R},x}) \mid (a,c) \in \mathsf{amax}(R''^{\{x,v\},\mathcal{R}}) \text{ for some } c \in B\};\\
T^{xy} &= \{(a,b) \in \mathsf{amax}(R''^{\{x,y\},\mathcal{R}}) \mid (a,c) \in \mathsf{amax}(R''^{\{x,v\},\mathcal{R}}),\\
&\qquad (b,c) \in \mathsf{amax}(R''^{\{y,v\},\mathcal{R}}) \text{ for some } c \in B\}.
\end{aligned}
$$

A tuple $\mathbf{a}$ over a set of variables $U \subseteq V$ is said to be *T-compatible* if for any $x, y \in U$, $(\mathbf{a}[x], \mathbf{a}[y]) \in T^{xy}$. The following lemma provides the main structural result necessary for proving that $\mathcal{R}^*$ is a $\overline{\beta}^*$-strategy.

Lemma 66 *Let S be one of the relations R, $R_{C,w,\gamma\delta}$, $\mathcal{S}_{W_{w,\gamma,\delta},Y}$, and S'', S^* the corresponding relations R'', $R''_{C,w,\gamma\delta}$, or $\mathcal{S}''_{W_{w,\gamma,\delta},Y}$, and R^*, $R^*_{C,w,\gamma\delta}$, or $\mathcal{S}^*_{W_{w,\gamma,\delta},Y}$, respectively, for some $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ and $(w,\gamma,\delta) \in \mathcal{W}(\overline{\beta})$, where Y is a set from the definition of block-minimality; and let U be its set of coordinate positions.*

(1) Every Q-compatible $\mathbf{a} \in \mathsf{amax}(S)$ such that $\mathbf{a}[u] \in B'_u$ for $u \in U \cap W$ is also T-compatible.

(2) S^* contains all the as-maximal T-compatible tuples from S''.

(3) If $S^* = R^*_{C,w,\gamma\delta}$, then $\mathsf{umax}(S^*)$ is as-closed in $\mathsf{umax}(R_{C,w,\gamma\delta} \cap \overline{B}')$.

Proof: (1) By Corollary 61, if, say $x \in W$ and $(a,b) \in Q^{xy}$ are such that $a \in B'_x$, then $(a,b) \in T^{xy}$, which can be proved as in the proof of Proposition 60(1). Therefore, it suffices to prove that if $x, y \notin W$, then $T^{xy} = Q^{xy}$. This can be done in the same way as in the proof of Proposition 60. Let $(a,b) \in Q^{xy}$, we need to find $c \in B'_v$ such that $(a,c) \in Q^{xv}$, $(b,c) \in Q^{yv}$. Let

$$Q(x,y,v) = R^{\{x,y\},\mathcal{R}}(x,y) \wedge R^{\{x,v\},\mathcal{R}}(x,v) \wedge R^{\{y,v\},\mathcal{R}}(y,v),$$

which is a subalgebra of $\mathbb{A}_x \times \mathbb{A}_y \times \mathbb{A}_v$. By (2,3)-consistency of $\mathcal{P}''$ $Q^{xy} \subseteq \mathrm{pr}_{xy} Q, Q^{xv} \subseteq \mathrm{pr}_{xv} Q, Q^{yv} \subseteq \mathrm{pr}_{xv} Q$. Let $Q' = Q/_\alpha$. Since each of $R^{\{x,y\},\mathcal{R}}$, $R^{\{x,v\},\mathcal{R}}$, $R^{\{y,v\},\mathcal{R}}$ is polynomially closed in the corresponding constraint relation $R^{\{x,y\}}$, $R^{\{x,v\}}$, or $R^{\{y,v\}}$ of $\mathcal{P}$, Q is polynomiall closed in

$$R^{\{x,y\}}(x,y) \wedge R^{\{x,v\}}(x,v) \wedge R^{\{y,v\}}(y,v),$$

as well, and so is Q' in

$$R^{\{x,y\}}(x,y) \wedge R^{\{x,v\}}/_\alpha(x,v) \wedge R^{\{y,v\}}/_\alpha(y,v).$$

Let $Q^\dagger = \mathrm{pr}_{xy}(Q' \cap (B_x \times B_y \times D))$, By Lemma 54 either $\mathsf{umax}(Q^\dagger) \times D \subseteq Q'$ or there is $\eta \in \mathsf{Con}(R^{\{x,y\}})$ such that $\mathsf{umax}(Q' \cap (B_x \times B_y \times D))$ is the graph of a mapping $\tau : \mathsf{umax}(Q^\dagger) \to D$. In the former case we are done, because then $(a,b,B'_v) \in Q'$, and therefore $(a,b,c) \in Q$ for some $c \in B'_v$. The latter case is not possible, because by Lemma 54 there is $\theta \in \mathsf{Con}(R^{\{x,y\}})$ such that $\eta \prec \theta \le \beta_x \times \beta_y$, and (α, β_v) and (η, θ) cannot be separated. This however is not the case, since $x, y \notin W$.

(2) Similar to Proposition 60 and Corollary 61 it suffices to prove that for any $X \subseteq U$, any T-compatible $\mathbf{a} \in \mathsf{amax}(\mathrm{pr}_X S'')$ can be extended to a T-compatible $\mathbf{b} \in \mathsf{amax}(S'')$. In fact, since S'' contains all the Q-compatible tuples, and therefore all the T-compatible tuples from S, it suffices to prove the statement for S, rather than for S''. We show that for any $w \in U - X$ tuple $\mathbf{a}$ can be extended to a T-compatible tuple $\mathbf{c} \in \mathsf{amax}(\mathrm{pr}_{X\cup\{w\}} S)$. By Proposition 60 there is a Q-compatible $\mathbf{b}' \in S$ with $\mathbf{a} = \mathrm{pr}_X \mathbf{b}'$. If $X \cap W \neq \varnothing$ or $(X \cup \{w\}) \cap W = \varnothing$, we can set $\mathbf{c} = \mathrm{pr}_{X\cup\{w\}} \mathbf{b}'$.

Suppose that $X \cap W = \varnothing$ and $w \in W$. Then we proceed similar to part (1). Let $X = \{x_1, \dots, x_k\}$, let

$$Q(x_1, \dots, x_k, w) = \mathrm{pr}_X S(x_1, \dots, x_k) \wedge \bigwedge_{i=1}^{k} R^{\{x_i,w\},\mathcal{R}}(x_i, w),$$

By (2,3)-consistency of $\mathcal{P}''$ and Proposition 60 $Q^{x_i w} \subseteq \mathrm{pr}_{x_i w} Q$, and by Corollary 61 $S' \subseteq \mathrm{pr}_X Q$, where S' is the set of all T-compatible (equivalently, Q-compatible) tuples from $\mathrm{pr}_X S$. Let $Q' = Q/_\alpha$. Since each of $R^{\{x_i,w\},\mathcal{R}}$ is polynomially closed in $R^{\{x_i,w\}}$ and $\mathrm{pr}_X S(x_1, \dots, x_k)$ is polynomially closed in itself, by Lemma 51(2) Q is polynomially closed in

$$\mathrm{pr}_X S(x_1, \dots, x_k) \wedge \bigwedge_{i=1}^{k} R^{\{x_i,w\}}(x_i, w),$$

as well, and so is Q' in

$$\mathrm{pr}_X S(x_1, \dots, x_k) \wedge \bigwedge_{i=1}^{k} R^{\{x_i,w\}}/_{\alpha_w}(x_i, w).$$

Let $Q^\dagger = \mathrm{pr}_X \left(Q' \cap \left(\prod_{i=1}^{k} B_{x_i} \times D\right)\right)$. Now we can finish the proof in the same way as in part (1).

(3) Let $U = \mathbf{s} \cap W_{w,\delta\gamma}$. Observe first, that every tuple from $\mathsf{umax}(R^*_{C,w,\gamma\delta})$ is P-compatible (see Section 6.3.1). If we prove that $R^*_{C,w,\gamma\delta}$ contains every P-compatible tuple $\mathbf{a}$ from $R''_{C,w,\gamma\delta}$ such that $\mathbf{a}[u] \in B'_u$ for every $u \in U \cap W$, by Lemma 63(2) the result follows. As in the proof of Lemma 63 we proceed by induction on the restriction of the problem $\mathcal{P}''$ being converted to a (2,3)-minimal and block-minimal instance.

Let R'' and $R''_{C,w,\gamma\delta}$ denote the relations associated with the instance $\mathcal{P}''$. Let $R^\dagger$ and $R^\dagger_{C,w,\gamma\delta}$ denote the relations obtained from R'' and $R''_{C,w,\gamma\delta}$ in the first step of converting $\mathcal{P}''$ to $\mathcal{P}^*$, that is, restricting the domains. By Lemma 65 relations

$R^\dagger, R^\dagger_{C,w,\gamma\delta}$ contain all the necessary P-compatible relations. This provides the base case. For the induction step we again consider two cases. We denote the current constraint relations by $R^\ddagger_{C,w,\gamma\delta}$ and the ones from the (2,3)-strategy by $R^{\ddagger X}$.

In the first case we enforce (2,3)-minimality for $x, y, z \in V$. Let $(a, b) \in R^{\ddagger\{x,y\}}$ be a P-compatible tuple. Then there is $c_1, c_2 \in B_z$ such that $(a, c_1) \in R''^{\{x,z\}}$, $(b, c_2) \in R''^{\{y,z\}}$ are P-compatible. As in the proof of item (1) of this lemma, we can argue that $c_1 = c_2$ can be assumed. If $z \notin W$, the pairs $(a, c_1), (b, c_2)$ are as required. Otherwise, c_1, c_2 can be chosen from B'_z by Lemma 65.

In the second case let $(u, \eta, \theta) \in \mathcal{W}(\overline{\beta})$ and $X = W_{u,\eta\theta}$; we solve a problem of the form $\mathcal{P}''_X/_{\overline{\mu}^Y}$, let $\mathcal{S}^\ddagger_X$ be the set of solutions of this problem. Let also $U' = \mathbf{s} \cap X$. We need to show that for any P-compatible $\mathbf{a} \in \mathsf{umax}(R^\ddagger_{C,w,\gamma\delta})$ with $\mathbf{a}[u] \in B'_u$ for $u \in U$ the tuple $\mathrm{pr}_{U'}\mathbf{a}$ can be extended to a P-compatible solution $\varphi \in \mathcal{S}^\ddagger_X$. Since $\mathbf{a} \in \mathsf{umax}(R''_{C,w,\gamma\delta})$, the tuple $\mathrm{pr}_{U'}\mathbf{a}$ can be extended to a u-maximal solution $\varphi \in \mathcal{S}''_X$. If $U' \cap W \neq \varnothing$ or $X \cap W = \varnothing$, solution φ is as required. Otherwise by Lemma 65 φ can be chosen P-compatible and such that $\varphi(x) \in B'_x$ for $x \in X \cap W$; that is $\varphi \in \mathcal{S}^\ddagger_X$ by the induction hypothesis. □

Now we are ready to prove that $\mathcal{R}^*$ is a $\overline{\beta}'$-strategy.

Theorem 67 *$\mathcal{R}^*$ is a $\overline{\beta}'$-strategy with respect to $\overline{B}'$.*

Proof: (S1) follows directly from the construction, since the relations $R^{X,\mathcal{R}^*}$ result from establishing (2,3)-minimality of $\mathcal{P}^*$, and they are nonempty by Lemma 66(1). Conditions (S2) and (S3) are also by construction. Condition (S4) also holds by construction, as all the relations of the form $R^*_{C,w,\gamma\delta}$ are subalgebras. Also, each of them contains a Q-compatible element, which is as-maximal in $R_{C,w,\gamma\delta}$, implying that $\mathsf{umax}(R^*_{C,w,\gamma\delta}) \subseteq \mathsf{umax}(R_{C,w,\gamma\delta})$.

For (S5) the existence of $A_{\mathcal{R}^*,w}$ for $w \in V$ follows from the construction, and the as-closeness of $\mathsf{umax}(A_{\mathcal{R}^*,w})$ follows from Lemma 66(3). Condition (S6) follows from Lemma 66(3) and (S6) for $\mathcal{R}$ as well. Finally, condition (S7) holds by Lemma 55. □

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Index

Notation