Constraint Satisfaction Problems over semilattice block Mal'tsev algebras$^{(*)}$

Andrei A. Bulatov

School of Computing Science, Simon Fraser University, Burnaby, Canada

Abstract

The Dichotomy Conjecture for the Constraint Satisfaction Problem (CSP) was recently settled, independently by Zhuk and the author. The proofs of this conjecture are rather sophisticated and require deep understanding of the algebraic structure of CSPs. This paper is a precursor of the author's proof of the Dichotomy Conjecture, and represents its main ideas in a simpler and clearer form in a more restricted class of the CSP.

There are two well-known types of algorithms for solving CSPs: local propagation and generating a basis of the solution space. For several years the focus of the CSP research has been on 'hybrid' algorithms that somehow combine the two approaches. In this paper we present a new method of such hybridization that allows us to solve certain CSPs that has been out of reach for a quite a while, and eventually leads to resolving the Dichotomy Conjecture.

We apply this method to CSPs parametrized by a universal algebra, an approach that has been very popular in the last decade or so. Specifically, we consider a fairly restricted class of algebras we will call semilattice block Mal'tsev. An algebra $\mathbb{A}$ is semilattice block Mal'tsev if it has a binary operation f, a ternary operation m, and a congruence σ such that the quotient $\mathbb{A}/_\sigma$ with operation f is a semilattice, f is a projection on every block of σ, and every block of σ is a Mal'tsev algebra with Mal'tsev operation m. This means that the domain in such a CSP is partitioned into blocks such that if the problem is considered on the quotient set $\mathbb{A}/_\sigma$, it can be solved by a simple constraint propagation algorithm. On the other hand, if the problem is restricted on

$^{(*)}$A conference version of this paper appeared at LICS 2017

individual σ-blocks, it can be solved by generating a basis of the solution space. We show that the two methods can be combined in a highly nontrivial way, and therefore the constraint satisfaction problem over a semilattice block Mal'tsev algebra is solvable in polynomial time.

Keywords: constraint satisfaction problem, semilattice block Mal'tsev algebras, dichotomy conjecture, block-minimality

1. Introduction

In a Constraint Satisfaction Problem (CSP, for short) we need to decide whether or not a given set of constraints on values that can be assigned simultaneously to a given set of variables can be satisfied. While the general CSP is NP-complete, its versions restricted by specifying a constraint language, a set of allowed constraints, are sometimes solvable in polynomial time. For a constraint language Γ the corresponding restricted CSP is denoted $\mathrm{CSP}(\Gamma)$ and called a nonuniform CSP. The study of the complexity of nonuniform CSPs was initiated by Schaefer [33]. In that paper Schaefer determined the complexity of $\mathrm{CSP}(\Gamma)$ for constraint languages on a 2-element set. The complexity of $\mathrm{CSP}(\Gamma)$ for constraint languages over finite sets has been attracting much attention since then. This research is guided by the Dichotomy Conjecture proposed by Feder and Vardi [20, 21] that states that every CSP of the form $\mathrm{CSP}(\Gamma)$ for a constraint language Γ on a finite set is either solvable in polynomial time or is NP-complete. The Dichotomy Conjecture has been restated and made more precise in different languages, see, e.g. [12, 31]. Also, several powerful approaches to the problem have been developed, through algebra, logic, and graph theory. So far the most successful method of studying the complexity of the CSP has been the algebraic approach introduced by Jeavons et al. [11, 12, 14, 25]. This approach relates the complexity of $\mathrm{CSP}(\Gamma)$ to the properties of a certain universal algebra $\mathbb{A}_\Gamma$ associated with Γ. In particular it allows one to expand $\mathrm{CSP}(\Gamma)$ to the problem $\mathrm{CSP}(\mathbb{A}_\Gamma)$ that depends only on the associated algebra, without changing its complexity. It therefore suffices to restrict ourselves to the study of the complexity of problems of the form $\mathrm{CSP}(\mathbb{A})$, where $\mathbb{A}$ is a finite universal algebra.

The dichotomy conjecture has been confirmed in a number of cases: for constraint languages on 2- and 3-element sets [7, 33] (a dichotomy result was also announced for languages over 4-, 5-, and 7-element sets [27, 34, 35]), for constraint languages containing all unary relations [1, 8, 9], and several others, see, e.g. [2, 3, 24]. Finally, Zhuk [36, 37] and Bulatov [16, 17] confirmed the general Dichotomy Conjecture.

One of the most remarkable phenomena discovered in the CSP research is that, generally, there are only two types of algorithms applicable to CSPs solvable in polynomial time. The first one has long been known to researchers in Artificial Intelligence as constraint propagation [19]. Algorithms of the other type resemble Gaussian elimination in the sense that they construct a small generating set of the set of all solutions [10, 24]. The scope of both types of algorithms is precisely known [2, 24].

Dichotomy results, however, cannot be proved using only algorithms of a single 'pure' type. In all such results, see, e.g. [1, 7, 8, 9] a certain mix of the two types of algorithms is needed. In some cases, for instance, [7] such a hybrid algorithm is somewhat ad hoc; in other cases, [1, 8, 9] it is based on intricate decompositions of the problem instance. In this paper we present a different approach to mixed types algorithms. It is a precursor and a much simplified version of the general algorithm from [16, 17]. We believe that this algorithm is worth attention in its own right, because it avoids the technicalities and complications of the general algorithm from [16, 17], while retaining most of the main ideas. It therefore is accessible and can be read as an introduction to [16, 17]. The first new feature of our algorithm is that it decomposes a CSP instance into subproblems that unlike local propagation are not necessarily small — some of these 'subproblems' may even contain all the variables of the original instance. Then it solves the problem by establishing some 'extreme' consistency by recursively solving the subproblems identified in the first stage. Later we give a more detailed description of the algorithm.

We follow the line of research pioneered in [28, 29, 30, 32]. In these works the researchers tried to tackle somewhat limited cases of the CSP, in which a combination of local consistency properties and Gaussian elimination type fragments is very explicit. To provide the context for our results we explain those cases in detail.

Suppose that a constraint language Γ is such that it is possible to partition its domain

A into blocks with the property that the restriction of $\mathrm{CSP}(\Gamma)$ on each block of the partition can be solved by an algorithm of one type; while if we collapse each block into a single element, the resulting quotient problem can be solved by an algorithm of another type. What can be said about $\mathrm{CSP}(\Gamma)$ itself? For instance, consider constraint language $\Gamma = \{R\}$ on $A = \{0, 1, 2\}$ where the ternary relation R is given by (triples in R are written vertically)

$$R = \left(\begin{array}{cccc|cccc|cc|cc} 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \end{array} \right).$$

If A is partitioned into $B = \{0, 1\}$ and $C = \{2\}$, then the restriction of R on the blocks B, C is one of the relations above separated by vertical lines (we can choose between B and C for different coordinate positions), and the corresponding CSP can be solved by Gaussian elimination. Indeed, the only nontrivial relation obtained this way is the first one, that is, $R \cap B^3$, and it is given by a linear equation $x + y + z = 0$. The quotient relation R' then looks like

$$R' = \left(\begin{array}{cccc} B & C & C & C \\ B & B & C & B \\ B & B & B & C \end{array} \right),$$

and it follows from [33] that $\mathrm{CSP}(R')$ can be solved by a local propagation algorithm, as R' can be represented by a Horn clause. Solving $\mathrm{CSP}(\Gamma)$ itself is not so easy, see, [7], and similar but more complicated cases have not been known to be polynomial time solvable for a long time.

To make constructions like the one above more precise we use the algebraic representation of nonuniform CSPs, in which a constraint language is replaced with its (universal) algebra of polymorphisms. This allows us to exploit structural properties of algebras to design a hybrid algorithm. So, starting from $\mathrm{CSP}(\Gamma)$, where Γ is a constraint language on a set A, we first consider the corresponding algebra $\mathbb{A}_\Gamma$ with base set A such that $\mathrm{CSP}(\mathbb{A}_\Gamma)$ is polynomial time reducible to $\mathrm{CSP}(\Gamma)$. A partition of $\mathbb{A}_\Gamma$ is given by a congruence of $\mathbb{A}_\Gamma$, that is, an invariant equivalence relation. Recall that due to the results of [12] the algebra $\mathbb{A}_\Gamma$ can be assumed idempotent, this makes restrictions on congruence blocks possible. Now, suppose that an idempotent algebra $\mathbb{A}$ is such that it has a congruence σ with the property that the CSP of its quotient $\mathbb{A}/_\sigma$ can be solved by the small generating set algorithm, say, it is Mal'tsev, while for every σ-block $\mathbb{B}$ (a subalgebra of $\mathbb{A}$) the CSP over $\mathbb{B}$ can be solved by a local propagation algorithm; or the other way round, see Figure 1. How can one solve the CSP over $\mathbb{A}$ itself? Maroti in [29] considered the first case, when $\mathbb{A}/_\sigma$ can be solved by the small generating set algorithm. This case turns out to be easier because of the property of the σ-blocks we can exploit. Suppose for simplicity that every σ-block $\mathbb{B}$ is a semilattice, as shown in Figure 1. Then every CSP instance on $\mathbb{B}$ has some sort of a canonical solution that assigns the maximal element of the semilattice (that is an element $a \in \mathbb{B}$ such that $ab = a$ for all $b \in \mathbb{B}$) to every variable. It then can be shown that if we find a solution $\varphi : V \to \mathbb{A}/_\sigma$ where V is the set of variables of the instance on $\mathbb{A}/_\sigma$, and then assign the maximal elements of the σ-block $\varphi(v)$ to v, we obtain a solution of the original instance.

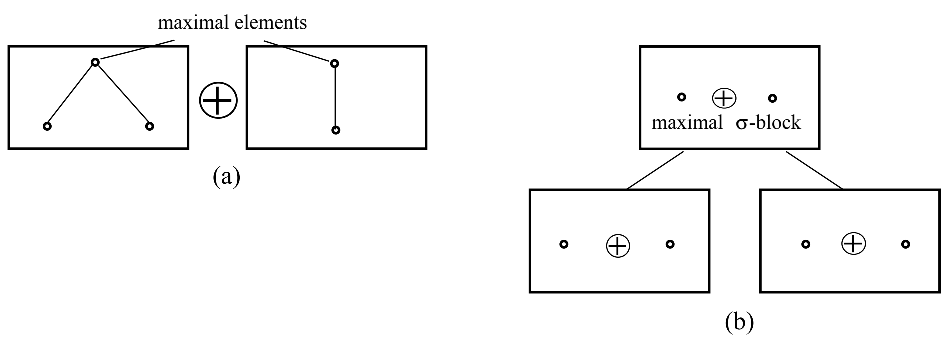

Figure 1: (a) Algebra $\mathbb{A}$ such that $\mathbb{A}/_\sigma$ is Mal'tsev; (b) an SBM algebra. Rectangles represent σ-blocks, dots represent elements, lines show the semilattice structure, and $\oplus$ represents a Mal'tsev operation acting on elements or σ-blocks.

The case when $\mathbb{A}/_\sigma$ is a semilattice, while every σ-block is Mal'tsev is much more difficult. We will call such algebras *semilattice block Mal'tsev* algebras (SBM algebras, for short). More precisely, we consider idempotent algebras $\mathbb{A}$ with the following property: There are a binary operation f and a ternary operation m, and a congruence σ of $\mathbb{A}$ such that $\mathbb{A}/_\sigma$ is a semilattice with a semilattice operation f, and every σ-block B is a Mal'tsev algebra with Mal'tsev operation m, and $f|_B$ is a projection. The main difficulty with this kind of algebras is that the only solution of a CSP over a semilattice we can reliably find is the canonical one assigning the maximal available element to each variable. However, if we restrict our instance only to the maximal σ-block $\mathbb{B}$, it may have no solution there, even though the original instance has a solution, which simply does not belong to the maximal block. If this is the case, it has been unclear for nearly 10 years how the domain can be reduced so that the maximal block is eliminated.

The problem has been resolved in some special cases. Firstly, Maróti in [30] showed that it suffices to consider SBM algebras of a certain restricted type. We will use this result in this paper. Marcovič and McKenzie suggested an algorithm that solves the CSP over SBM algebras $\mathbb{A}$ when $\mathbb{A}/_\sigma$ is a chain, that is, $ab \in \{a, b\}$ for any $a, b \in \mathbb{A}/_\sigma$. In this case their algorithm is capable of eliminating the maximal block using the fact that if a semilattice is a chain, any of its subsets is a subalgebra. Finally, very recently Payne in [32] suggested an algorithm that works for a more general class of algebras than SBM, but algebras in this class have to satisfy an extra condition that in SBM algebras manifests itself as the existence of certain well behaving mappings between σ-blocks. In particular, this condition guarantees that the instance restricted to the maximal σ-block has a solution whenever the original problem has a solution.

In this paper we continue the effort started in [28, 30, 32] and present an algorithm that solves the CSP over an arbitrary SBM algebra.

Theorem 1. *If $\mathbb{A}$ is a SBM algebra then $\mathrm{CSP}(\mathbb{A})$ is solvable in polynomial time.*

The algorithm is based upon a new local consistency notion that we call *block-minimality* (although in our case it is necessarily not quite local, since it has to deal with Mal'tsev algebras). A slightly generalized version of block-minimality is one of the two main ingredients of the general CSP algoritm [16, 17]. More specifically, our algorithm first separates the set V of variables of a CSP instance into overlapping subsets, coherent sets, and considers subproblems on these sets of variables. For block-minimality these subproblems have to be minimal, that is, every tuple from every constraint relation has to be a part of a solution. This can be achieved by solving the problem many times with additional constraints. However, this is not very straightforward, because coherent sets may contain all the variables from V. To overcome this problem we show that the subproblems restricted to coherent sets are either over a Mal'tsev domain and therefore can be solved efficiently, or they split up into a collection of disjoint instances, each of which has a strictly smaller domain. In the latter case we can recurse on these smaller instances. Finally, we prove that any block-minimal instance has a solution.

The results of this paper can easily be made more general by removing some of the restrictions on the basic operations of SBM algebras. However, the goal of the paper is to illustrate the work of the block-minimality condition in its pure form, and so we stop short of giving more general but also more technically involved proofs just restricting ourselves to demonstrating the general idea.

In Section 2 we recall the basic definitions on CSP and the algebraic approach. A somewhat simplified outline of the solution algorithm and block-minimality is given in Section 3. More advanced facts from algebra and a study of certain properties of SBM algebras are given in Section 4. In Section 5 we strengthen the results of [5] about the structure of relations over Mal'tsev algebras and extend them to SBM algebras[1]. In Section 6 we extend these notions to CSP instances. Finally, in Section 7 we prove the main result and present a solution algorithm.

2. Preliminaries

2.1. Multisorted Constraint Satisfaction Problem

By $[n]$ we denote the set $\{1, \ldots, n\}$. Let $A_1, \ldots, A_n$ be finite sets. Tuples from $A_1 \times \ldots \times A_n$ are denoted in boldface, say, $\mathbf{a}$, and their entries by $\mathbf{a}[1], \ldots, \mathbf{a}[n]$. A *relation* R over $A_1, \ldots, A_n$ is a subset of $A_1 \times \cdots \times A_n$. We refer to n as the *arity* of the tuple $\mathbf{a}$ and the relation R. Let $I = (i_1, \ldots, i_k)$ be an (ordered) multiset, a subset of $[n]$. Then let $\mathrm{pr}_I \mathbf{a} = (\mathbf{a}[i_1], \ldots, \mathbf{a}[i_k])$ and $\mathrm{pr}_I R = \{\mathrm{pr}_I \mathbf{a} \mid \mathbf{a} \in R\}$. Relation R is said to be a *subdirect product* of $A_1, \ldots, A_n$ if $\mathrm{pr}_i R = A_i$ for $i \in [n]$. In some

[1]Kearnes and Szendrei in [26] developed a technique based on so-called critical relations that resembles in certain aspects what can be achieved through coherent sets. However, [26] only concerns congruence modular algebras, and so cannot be used for SBM algebras.

cases it will be convenient to consider tuples and relations whose entries are indexed by sets other than subsets of $[n]$, most often those will be sets of variables. Then we either assume the index set is somehow ordered, or consider tuples as functions from the index set to the domain and relations as sets of such functions.

Let $\mathcal{A}$ be a set of sets, in this paper $\mathcal{A}$ is usually the set of universes of finite algebras derived from an SBM algebra; we clarify 'derived' later. An instance of a *(Multisorted) Constraint Satisfaction Problem* (CSP) over $\mathcal{A}$ is given by $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$, where V is a set of *variables*, $\mathcal{A}$ is a collection of *domains* $A_v \in \mathcal{A}$, $v \in V$, and $\mathcal{C}$ is a set of *constraints*; every constraint $\langle \mathbf{s}, R\rangle$ is a pair consisting of an ordered multiset $\mathbf{s} = (v_1, \dots, v_k)$, a subset of V, called the *constraint scope*, and R, a relation over $A_{v_1}, \dots, A_{v_k}$, called the *constraint relation*.

2.2. Algebraic structure of the CSP

For a detailed introduction to CSP and the algebraic approach to its structure the reader is referred to a very recent and very nice survey by Barto et al. [4]. Basics of universal algebra can be learned from the textbook [18] and monograph [23].

A (*universal*) *algebra* is a pair $\mathbb{A} = (A; F)$, where A is a set (always finite in this paper) called the *universe* of $\mathbb{A}$, and F is a set of *basic operations*, multi-ary operations on A. Algebras $\mathbb{A} = (A, F^{\mathbb{A}})$ and $\mathbb{B} = (B, F^{\mathbb{B}})$ are said to be *similar* if their basic operations are indexed by elements of the same set F in such a way that operations from $F^{\mathbb{A}}$ and $F^{\mathbb{B}}$ indexed by the same element have the same arity. Operations that can be obtained from the basic operations of $\mathbb{A}$ or a class $\mathfrak{A}$ of similar algebras by means of compositions are said to be *term* operations of $\mathbb{A}$ or, respectively, $\mathfrak{A}$.

The CSP is related to algebras through the notion of polymorphism. Let R be a relation on a set A and f be a k-ary operation on the same set. Operation f is said to be a *polymorphism* of R if for any $\mathbf{a}_1, \dots, \mathbf{a}_k \in R$ the tuple $f(\mathbf{a}_1, \dots, \mathbf{a}_k)$ also belongs to R. More generally, let R be a subset of $A_1 \times \dots \times A_\ell$ and f be an operation symbol such that $f^{\mathbb{A}_i}$ is a k-ary operation on A_i for $i \in [\ell]$. Then f is a polymorphism of R if for any $\mathbf{a}_1, \dots, \mathbf{a}_k \in R$ the tuple $f(\mathbf{a}_1, \dots, \mathbf{a}_k)$ belongs to R, where $f(\mathbf{a}_1, \dots, \mathbf{a}_k) = (f^{\mathbb{A}_1}(\mathbf{a}_1[1], \dots, \mathbf{a}_k[1]), \dots, f^{\mathbb{A}_\ell}(\mathbf{a}_1[\ell], \dots, \mathbf{a}_k[\ell]))$. Let Γ be a constraint language, that is, a set of relations, on a set A. Then $\mathsf{Pol}(\Gamma)$ denotes the set of all operations f on A such that f is a polymorphism of every relation from Γ; also $\mathbb{A}_\Gamma = (A, \mathsf{Pol}(\Gamma))$ is the corresponding algebra. Similarly, let $\mathcal{A}$ be a collection of sets and Γ a constraint language over $\mathcal{A}$, that is, a set of relations $R \subseteq A_1 \times \cdots \times A_\ell$, $A_1, \ldots, A_\ell \in \mathcal{A}$. Then $F = \mathsf{Pol}(\Gamma)$ is the set of all operation symbols f along with their interpretations on sets from $\mathcal{A}$ such that f is a polymorphism of all relations from Γ. The corresponding set of algebras is denoted by $\mathfrak{A}_\Gamma$, that is, for every $A \in \mathcal{A}$ the set $\mathfrak{A}_\Gamma$ contains algebra $\mathbb{A} = (A, F^{\mathbb{A}})$, where $F^{\mathbb{A}} = \{f^{\mathbb{A}} \mid f \in F\}$.

Any class of similar algebras also gives rise to a CSP. Let $\mathfrak{A}$ be a class of similar finite algebras and $\mathcal{A}$ the set of universes of algebras from $\mathfrak{A}$. Then $\mathrm{CSP}(\mathfrak{A})$ is the class of instances $(V, \mathcal{A}, \mathcal{C})$ of CSPs over $\mathcal{A}$ such that every constraint relation R from $\langle \mathbf{s}, R\rangle \in \mathcal{C}$, $\mathbf{s} = (v_1, \ldots, v_k)$, is a subalgebra of $A_{v_1} \times \cdots \times A_{v_k}$, where A_v, $v \in V$, are viewed as algebras from $\mathfrak{A}$.

In this paper we will use two special types of operations.

Example 2. A binary operation f on A is said to be *semilattice* if $f(a,a) = a$, $f(a,b) = f(b,a)$, and $f(f(a,b),c) = f(a, f(b,c))$ for any $a,b,c \in A$. Similarly, f is a semilattice operation on a class $\mathfrak{A}$ of similar algebras, if it is a term operation of that class and $f^{\mathbb{A}}$ is a semilattice operation for every $\mathbb{A} \in \mathfrak{A}$. We will treat a semilattice operation as multiplication and denote it by $\cdot$ or omit the sign altogether. A semilattice operation defines an order on its domain: $a \le b$ if and only if $ab = b$. This means that there is always the greatest element of such a semilattice order — the product of all the elements of A. We will denote this element by $\max(\mathbb{A})$.

Example 3. A ternary operation m is said to be *Mal'tsev* if it satisfies the equations $m(a,b,b) = m(b,b,a) = a$ for any $a,b \in A$. A term operation m of a class $\mathfrak{A}$ is Mal'tsev if $m^{\mathbb{A}}$ is Mal'tsev for every $\mathbb{A} \in \mathfrak{A}$. An algebra with a Mal'tsev term operation is said to be *Mal'tsev*.

If $\mathfrak{A}$ has a Mal'tsev term operation, the algorithm from [10] constructs a compact representation of the set of solutions of any instance from $\mathrm{CSP}(\mathfrak{A})$, thus solving the problem in polynomial time.

A *subalgebra* of an algebra $\mathbb{A} = (A, F)$ is a subset $B \subseteq A$ equipped with the restrictions of operations from F on B and such that $f(a_1, \ldots, a_k) \in B$ for every

$f \in F$ and $a_1, \ldots, a_k \in B$. An equivalence relation on A invariant with respect to the basic operations of $\mathbb{A}$ is said to be a *congruence* of $\mathbb{A}$. If a, b are related by a congruence α, we write $a \stackrel{\alpha}{\equiv} b$; the α-block containing a is denoted a^α. The *quotient algebra* $\mathbb{A}/_\alpha$ has the universe $A/_\alpha$ and basic operations f^α, $f \in F$, such that for any $a_1, \ldots, a_k \in A$ operation f^α is given by $f^\alpha(a_1^\alpha, \ldots, a_k^\alpha) = (f(a_1, \ldots, a_k))^\alpha$. We will omit the superscript in f^α whenever this does not lead to a confusion. Algebra $\mathbb{A}$ is said to be *idempotent* if $f(a, \ldots, a) = a$ for any $f \in F$ and any $a \in A$. A useful property of idempotent algebras is that every class of any of its congruences is a subalgebra. In particular, every 1-element subset of A is a subalgebra. Algebras $\mathbb{A}, \mathbb{A}'$ with the same universe are called *term equivalent* if they have the same set of term operations. If $\mathbb{A} = (A, F)$, $\mathbb{A}' = (A, F')$ and F' is a subset of the set of term operations of $\mathbb{A}$, then $\mathbb{A}'$ is said to be a *reduct* of $\mathbb{A}$.

Definition 4. *Idempotent algebra $\mathbb{A}$ is said to be* semilattice block Mal'tsev *if there are a binary term operation f and a ternary term operation m, and a congruence $\sigma_\mathbb{A}$ of $\mathbb{A}$ such that $\mathbb{A}/_{\sigma_\mathbb{A}}$ is term equivalent to a semilattice with a semilattice operation f, operation m is a Mal'tsev operation on every $\sigma_\mathbb{A}$-block B, and $f|_B$ is the first projection, that is, $f|_B(x, y) = x$.*

2.3. Partial solutions and local consistency

Let $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ be a CSP instance. Let $W \subseteq V$. By $\mathcal{P}_W$ we denote the instance $(W, \mathcal{A}^W, \mathcal{C}^W)$ defined as follows: $A_v^W = A_v$ for each $v \in W$; for every constraint $C = \langle \mathbf{s}, R\rangle$, $C \in \mathcal{C}$, the set $\mathcal{C}^W$ includes the constraint $C^W = \langle \mathbf{s}', R'\rangle$, where $\mathbf{s}' = \mathbf{s} \cap W$ and $R' = \mathrm{pr}_{\mathbf{s}'} R$. A solution of $\mathcal{P}_W$ is called a *partial solution* of $\mathcal{P}$ on W. The set of all such solutions is denoted by $\mathcal{S}_W$. If $W = \{v\}$ or $W = \{u, v\}$, we simplify notation to $\mathcal{P}_v, \mathcal{S}_v$ and $\mathcal{P}_{uv}, \mathcal{S}_{uv}$, respectively.

Instance $\mathcal{P}$ is called *minimal* if every tuple $\mathbf{a} \in R$ for any constraint $\langle \mathbf{s}, R\rangle \in \mathcal{C}$ can be extended to a solution of $\mathcal{P}$; that is, there is $\varphi \in \mathcal{S}$ such that $\varphi(v) = \mathbf{a}[v]$ for $v \in \mathbf{s}$. Instance $\mathcal{P}$ is called k-*minimal* if $\mathcal{P}_W$ is minimal for all k-element $W \subseteq V$. For any fixed k every instance can be reduced to a k-minimal instance in polynomial time by a standard algorithm [13]: cycle over all k element subsets $W \subseteq V$, solve the problem

$\mathcal{P}_W$, and for every constraint $\langle \mathbf{s}, R\rangle$ exclude from R all tuples inconsistent with $\mathcal{S}_W$. If $\mathcal{P} \in \mathrm{CSP}(\mathfrak{A})$ for some class $\mathfrak{A}$ of similar algebras closed under subalgebras, the resulting problem also belongs to $\mathrm{CSP}(\mathfrak{A})$. In particular, from now on we will assume that all the instances we deal with are 1-minimal. For such problems we can also *tighten* the instance reducing the domains A_v, $v \in V$, to the sets $\mathcal{S}_v$. Every constraint relation will therefore be assumed to be a subdirect product of the respective domains. If $\mathfrak{A}$ consists of idempotent algebras, then any problem from $\mathrm{CSP}(\mathfrak{A})$ can be reduced to a minimal one by solving polynomially many instances of $\mathrm{CSP}(\mathfrak{A})$. First of all, *constant relations*, $R_a = \{(a)\}$, $a \in \mathbb{A} \in \mathfrak{A}$, are subalgebras of $\mathbb{A}$ and therefore can be used in constraints. Then the algorithm proceeds as follows: cycle over all constraints $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ and all $\mathbf{a} \in R$; replace C with the collection of unary constraints $\langle (\mathbf{s}[i]), R_{\mathbf{a}[\mathbf{s}[i]]}\rangle$; solve the resulting instance $\mathcal{P}_{C,\mathbf{a}}$; remove $\mathbf{a}$ from R if $\mathcal{P}_{C,\mathbf{a}}$ has no solutions. This procedure, however, obviously amounts to solving instances from $\mathrm{CSP}(\mathfrak{A})$, and therefore there is no guarantee this can be done in polynomial time.

Example 5. If a class $\mathfrak{A}$ of similar algebras has a semilattice term operation then $\mathrm{CSP}(\mathfrak{A})$ can be solved by establishing 1-minimality. More precisely, if $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ is a 1-minimal instance from $\mathrm{CSP}(\mathfrak{A})$, where A_v is the domain of $v \in V$, then the mapping $\varphi(v) = \max(A_v)$ is a solution of $\mathcal{P}$.

2.4. Congruences and polynomials

The set (lattice) of congruences of an algebra $\mathbb{A}$ will be denoted by $\mathsf{Con}(\mathbb{A})$. So, $\mathsf{Con}(\mathbb{A})$ is equipped with two binary operations of *join*, $\vee$, and *meet*, $\wedge$. The smallest congruence of $\mathbb{A}$, the equality relation, is denoted by $\underline{0}_{\mathbb{A}}$, and the greatest congruence, the total relation, is denoted by $\underline{1}_{\mathbb{A}}$. Let R be a subdirect product of $\mathbb{A}_1, \ldots, \mathbb{A}_k$, and $\alpha_i \in \mathsf{Con}(\mathbb{A}_i)$, $i \in [k]$. Then by $\overline{\alpha}_R$, or simply $\overline{\alpha}$ if R is clear from the context, we denote the congruence $\alpha_1 \times \cdots \times \alpha_k$ of R given by $\mathbf{a} \stackrel{\overline{\alpha}}{\equiv} \mathbf{b}$ if and only if $\mathbf{a}[i] \stackrel{\alpha_i}{\equiv} \mathbf{b}[i]$ for all $i \in [k]$. Also, if $I = \{i_1, \ldots, i_\ell\} \subseteq [k]$ then by $\overline{\alpha}_I$ we denote the congruence $\alpha_{i_1} \times \cdots \times \alpha_{i_\ell}$ of $\mathrm{pr}_I R$.

Let $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ be an instance of $\mathrm{CSP}(\mathfrak{A})$ and α_v a congruence of $\mathbb{A}_v \in \mathfrak{A}$ for each $v \in V$. By $\mathcal{P}_{\overline{\alpha}}$ we denote the instance $(V, \mathcal{A}^{\overline{\alpha}}, \mathcal{C}^{\overline{\alpha}})$, in which $\mathbb{A}_v^{\overline{\alpha}} = \mathbb{A}_v/_{\alpha_v}$, and a constraint $\langle \mathbf{s}, R' \rangle$, $\mathbf{s} = (v_1, \ldots, v_k)$, belongs to $\mathcal{C}^{\overline{\alpha}}$ if and only if a constraint $\langle \mathbf{s}, R \rangle$, where

$$R' = R/_{\overline{\alpha}} = \{\mathbf{a}^{\overline{\alpha}} = (\mathbf{a}[1]^{\alpha_{v_1}}, \ldots, \mathbf{a}[k]^{\alpha_{v_k}}) \mid \mathbf{a} \in R\},$$

belongs to $\mathcal{C}$.

A pair of congruences $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$ is said to be a *prime interval*, denoted $\alpha \prec \beta$, if $\alpha \leq \beta$ and $\alpha < \gamma < \beta$ for no congruence $\gamma \in \mathsf{Con}(\mathbb{A})$. Then $\alpha \preceq \beta$ means that $\alpha \prec \beta$ or $\alpha = \beta$. For an operation f on $\mathbb{A}$ we write $f(\beta) \subseteq \alpha$ if, for any $a, b \in \mathbb{A}$ with $a \overset{\beta}{\equiv} b$, $f(a) \overset{\alpha}{\equiv} f(b)$.

Polynomials of $\mathbb{A}$ are formed from term operations as follows. Let $f(x_1, \ldots, x_k, y_1, \ldots, y_\ell)$ be a term operation of $\mathbb{A}$ and $a_1, \ldots, a_\ell \in \mathbb{A}$. Then the operation $g(x_1, \ldots, x_k) = f(x_1, \ldots, x_k, a_1, \ldots, a_\ell)$ is said to be a *polynomial* of $\mathbb{A}$. Note that although a polynomial does not have to be a polymorphism of invariant relations of $\mathbb{A}$, unary polynomials and congruences of $\mathbb{A}$ are in a special relationship: an equivalence relation α is a congruence of $\mathbb{A}$ if and only if it is preserved by every unary polynomial f, that is, $f(\alpha) \subseteq \alpha$. As usual, by an *idempotent unary polynomial* we mean a polynomial $f(x)$ such that $f \circ f = f$ or, equivalently, such that $f(x) = x$ for any x from its range.

Let R be a subdirect product of $\mathbb{A}_1, \ldots, \mathbb{A}_k$. Similar to tuples from R, polynomials of R are also denoted in boldface, say, $\mathbf{f}$. A polynomial $\mathbf{f}$ can be represented as $\mathbf{f}(x_1, \ldots, x_k) = g(x_1, \ldots, x_k, \mathbf{a}^1, \ldots, \mathbf{a}^\ell)$ where g is a term operation of R and $\mathbf{a}^1, \ldots, \mathbf{a}^l \in R$. Then the polynomial $g(x_1, \ldots, x_k, \mathbf{a}^1[i], \ldots, \mathbf{a}^\ell[i])$ of $\mathbb{A}_i$ is denoted by f_i, and for $I = \{i_1, \ldots, i_s\} \subseteq [n]$, $\mathbf{f}_I$ denotes the polynomial $g(x_1, \ldots, x_k, \mathrm{pr}_I \mathbf{a}^1, \ldots, \mathrm{pr}_I \mathbf{a}^\ell)$ of $\mathrm{pr}_I R$. For any i, and any polynomial f of $\mathbb{A}_i$, there is a polynomial $\mathbf{g}$ of R such that $g_i = f$. We shall call $\mathbf{g}$ an *extension* of f to a polynomial of R. Finally, for $I \subseteq [k]$, and $\mathbf{a} \in \prod_{i \in I} \mathbb{A}_i$ and $\mathbf{b} \in \prod_{i \in [k]-I} \mathbb{A}_i$, $(\mathbf{a}, \mathbf{b})$ denotes the tuple $\mathbf{c}$ such that $\mathbf{c}[i] = \mathbf{a}[i]$ for $i \in I$ and $\mathbf{c}[i] = \mathbf{b}[i]$ if $i \in [k] - I$. To distinguish such concatenation of tuples from pairs of tuples, we will denote pairs of tuples by $\langle \mathbf{a}, \mathbf{b} \rangle$.

The proposition below lists the main basic properties of relations over Mal'tsev algebras.

Proposition 6 (Folklore). *Let R be a subdirect product of Mal'tsev algebras $\mathbb{A}_1 \times$*

$\cdots \times \mathbb{A}_k$ *and* $I \subseteq [k]$. *Then the following properties hold*

(1) R *is* rectangular*, that is if* $\mathbf{a}, \mathbf{b} \in \mathrm{pr}_I R, \mathbf{c}, \mathbf{d} \in \mathrm{pr}_{[k]-I} R$ *and* $(\mathbf{a}, \mathbf{c}), (\mathbf{a}, \mathbf{d})$, $(\mathbf{b}, \mathbf{c}) \in R$*, then* $(\mathbf{b}, \mathbf{d}) \in R$.

(2) The relation $\nu_I = \{\langle \mathbf{a}, \mathbf{b} \rangle \in (\mathrm{pr}_I R)^2 \mid$ *there is* $\mathbf{c} \in \mathrm{pr}_{[k]-I} R$ *such that* $(\mathbf{a}, \mathbf{c})$, $(\mathbf{b}, \mathbf{c}) \in R\}$ *is a congruence of* $\mathrm{pr}_I R$.

3. Outline of the algorithm

Our solution algorithm works by establishing some sort of minimality condition and repeatedly alternates two phases. The first phase is based on the results of Maroti [30] that allow us to reduce an instance over SBM algebras to one over SBM algebras with a *minimal* element. If $\mathbb{A}$ is an SBM algebra then there is a congruence σ such that $\mathbb{A}/_\sigma$ is a semilattice. This means that $\mathbb{A}/_\sigma$ has a maximal or *absorbing* element a such that $ax = xa = a$ for any $x \in \mathbb{A}/_\sigma$. This element will be in the focus of our argument. We will also show with help of [30], Corollary 13, that it can always be assumed that $\mathbb{A}/_\sigma$ has a minimal or *neutral* element b such that $bx = xb = x$ for any $x \in \mathbb{A}/_\sigma$. In fact, one can assume an even stronger condition: that b is a 1-element σ-block.

For the second phase we introduce the *block-minimality* condition defined with the help of congruences and polynomials of an algebra. Let R be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ and $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_i)$, $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_j)$ such that $\alpha \prec \beta$, $\gamma \prec \delta$ for some $i, j \in [n]$. Interval (α, β) *can be separated* from (γ, δ) if there is a unary polynomial $\mathbf{f}$ of R such that $f_i(\beta) \not\subseteq \alpha$ while $f_j(\delta) \subseteq \gamma$. We are mostly interested in the situation when prime intervals cannot be separated.

Suppose that $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ is a 3-minimal instance and the domain $\mathbb{A}_v$ of $v \in V$ is an SBM algebra and σ_v is such that $\mathbb{A}_v/_{\sigma_v}$ is a semilattice. Let θ_v denote the congruence of $\mathbb{A}_v$ such that the maximal element of $\mathbb{A}_v/_{\sigma_v}$ is one block of θ_v, and all other θ_v-blocks are singletons. We show, Lemma 10, that this is indeed a congruence. For every $v \in V$ and $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$ with $\alpha \prec \beta \le \theta_v$ let $W_{v\alpha\beta} \subseteq V$ denote the set of variables w such that (α, β) and (γ, δ) for some $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_w)$ with $\gamma \prec \delta \le \theta_w$ cannot be separated from each other in the binary relation $\mathcal{S}_{vw}$. We call such sets of variables *coherent sets*. Instance $\mathcal{P}$ is said to be *block-minimal* if for every $v \in V$ and

$\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$ with $\alpha \prec \beta \leq \theta_v$ the problem $\mathcal{P}_{W_{v\alpha\beta}}$ is minimal.

The result now follows from the following two statements. First, Proposition 22 claims that any instance $\mathcal{P}$ over SBM algebras can be efficiently reduced to an equivalent block-minimal instance by solving polynomially many SBM instances over domains of smaller size. The second statement, Theorem 23, claims that any block-minimal SBM instance has a solution.

The key to the proof of Proposition 22 is Lemma 20 stating that every problem $\mathcal{P}_{W_{v\alpha\beta}}$ is a disjoint union of problems over smaller domains, or its domains are Mal'tsev algebras. More precisely, in the first case there is k such that for every $w \in W_{v\alpha\beta}$ the domain $\mathbb{A}_w$ can be partitioned into a disjoint union $\mathbb{A}_w^{(1)} \cup \dots \cup \mathbb{A}_w^{(k)}$ in such a way that for any constraint $\langle (v_1, \dots, v_\ell), R\rangle$ of $\mathcal{P}_{W_{v\alpha\beta}}$, every tuple $\mathbf{a} \in R$ belongs to $\mathbb{A}_{v_1}^{(j)} \times \dots \times \mathbb{A}_{v_k}^{(j)}$ for some $j \in [k]$. This property follows from the existence of a minimal element in every domain and the fact that certain prime intervals in congruence lattices of the domains of $\mathcal{P}_{W_{v\alpha\beta}}$ cannot be separated from each other, Lemma 20. It means, of course, that it suffices to solve k problems $\mathcal{P}_{W_{v\alpha\beta}}^{(j)}$ whose domains are $\mathbb{A}_w^{(j)}$.

We prove Theorem 23 by induction, showing that for every $\overline{\beta} = (\beta_v)_{v\in V}$ with $\beta_v \in \mathsf{Con}(\mathbb{A}_v)$ with $\beta_v \leq \theta_v$ there is a collection of solutions $\varphi_{v\alpha\beta}$ of $\mathcal{P}_{W_{v\alpha\beta}}$ such that whenever $u \in W_{v\alpha\beta} \cap W_{w\gamma\delta}$ we have $\varphi_{v\alpha\beta}(u) \stackrel{\beta_u}{\equiv} \varphi_{w\gamma\delta}(u)$. Observe that this condition implies that the collection $\{\varphi_{v\alpha\beta}\}$ gives a solution of $\mathcal{P}_{\overline{\beta}}$. If every β_w equals θ_w then such a collection exists because the maximal element of $\mathbb{A}_w/_{\beta_w}$ is a singleton, and we always can choose mappings $\varphi_{v\alpha\beta}$ to be such that $\varphi_{v\alpha\beta}(w)/_{\theta_w}$ is the maximal element of $\mathbb{A}_w/_{\sigma_w}$. On the other hand, if β_w is the equality relation for every $w \in V$ then solutions $\varphi_{v\alpha\beta}$ agree with each other and provide a solution of $\mathcal{P}$. Thus, showing that the existence of solutions $\varphi_{v\alpha\beta}$ for some $\overline{\beta}$ implies the existence of such solutions for smaller congruences $\overline{\beta}'$ is the crux of our argument.

4. Semilattice block Mal'tsev algebras and minimal elements

4.1. Minimal sets and polynomials

We will use several basic concepts of the tame congruence theory, [23].

An (α, β)-*minimal set* is a minimal (under inclusion) set U such that $U = f(\mathbb{A})$ for a unary polynomial of $\mathbb{A}$ satisfying $f(\beta) \not\subseteq \alpha$. Sets B, C are said to be *polynomially isomorphic* in $\mathbb{A}$ if there are unary polynomials f, g such that $f(B) = C$, $g(C) = B$, and $f \circ g, g \circ f$ are identity mappings on C and B, respectively.

Lemma 7 (Theorem 2.8, [23]). *Let $\mathbb{A}$ be any finite algebra, $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$, $\alpha \prec \beta$. Then the following hold.*

(1) Any (α, β)-minimal sets U, V are polynomially isomorphic.

(2) For any (α, β)-minimal set U and any unary polynomial f, if $f(\beta|_U) \not\subseteq \alpha$ then $f(U)$ is an (α, β)-minimal set, U and $f(U)$ are polynomially isomorphic, and f witnesses this fact.

(3) For any (α, β)-minimal set U there is a unary polynomial f such that $f(\mathbb{A}) = U$, $f(\beta) \not\subseteq \alpha$, and f is idempotent, in particular, f is the identity mapping on U.

(4) For any unary polynomial f such that $f(\beta) \not\subseteq \alpha$ there is an (α, β)-minimal set U such that f witnesses that U and $f(U)$ are polynomially isomorphic. In particular, $f(U)$ is an (α, β)-minimal set.

Minimal sets of a Mal'tsev algebra form a particularly dense collection. The following lemma is well known, see, e.g., Exercise 8.8(1) from [23].

Lemma 8. *Let $\mathbb{A}$ be a finite Mal'tsev algebra and $\alpha \prec \beta$ for $\alpha, \beta \in \mathsf{Con}(\mathbb{A})$. Then for any $a, b \in \mathbb{A}$ with $(a, b) \in \beta - \alpha$, there is an (α, β)-minimal set U such that $a^\alpha \cap U \neq \varnothing$ and $b^\alpha \cap U \neq \varnothing$.*

4.2. Semilattice block Mal'tsev algebras

Since the fewer basic operations an algebra has, the richer the corresponding constraint language is, we assume that the algebras we are dealing with have only two basic operations, just enough to guarantee the required properties. Therefore we assume that our semilattice block Mal'tsev algebras have only two basic operations: a binary operation $\cdot$ that we will often omit, and a ternary operation m satisfying the conditions specified earlier. For elements $a, b \in \mathbb{A}$ such that $ab = ba = b$ we write $a \leq b$.

Lemma 9. *Let $\mathbb{A}$ be an SBM algebra. By choosing a reduct of $\mathbb{A}$ we may assume that*

(a) *Operation $\cdot$ satisfies the property: for any $a, b \in \mathbb{A}$, $a \leq ab$.*

(b) *Operation m can be chosen such that for any $a, b, c \in \mathbb{A}$, $m(a, b, c)^{\sigma_{\mathbb{A}}} = (abc)^{\sigma_{\mathbb{A}}}$.*

Proof: (1) Follows from Proposition 10 of [15] .

(2) Consider the operation $m'(x, y, z) = m(x, y, z)xyz$. If B is a $\sigma_{\mathbb{A}}$-block, then, since $ab = a$ for any $a, b \in B$, operation m' is Mal'tsev on B. Also, as $\mathbb{A}/_{\sigma_{\mathbb{A}}}$ is term equivalent to a semilattice, $d = m(a, b, c)^{\sigma_{\mathbb{A}}}$ belongs to the subsemilattice of $\mathbb{A}/_{\sigma_{\mathbb{A}}}$ generated by $a^{\sigma_{\mathbb{A}}}, b^{\sigma_{\mathbb{A}}}, c^{\sigma_{\mathbb{A}}}$. Therefore $m'(a, b, c)^{\sigma_{\mathbb{A}}} = d(abc)^{\sigma_{\mathbb{A}}} = (abc)^{\sigma_{\mathbb{A}}}$, and we can choose m' for m. $\square$

Next we show some useful properties of SBM algebras. Let $\mathbb{A}$ be an SBM algebra and $\max(\mathbb{A})$ the maximal block of σ, that is, $\max(\mathbb{A}) \cdot a, a \cdot \max(\mathbb{A}) \subseteq \max(\mathbb{A})$ for all $a \in \mathbb{A}$.

Lemma 10. *(1) The equivalence relation $\theta_{\mathbb{A}}$ whose blocks are $\max(\mathbb{A})$, and all the remaining elements form singleton blocks, is a congruence.*

(2) Let R be a subdirect product of SBM algebras $\mathbb{A}_1, \dots, \mathbb{A}_n$, and let the equivalence relation θ_R be such that its blocks are $\max(R) = R \cap (\max(\mathbb{A}_1) \times \cdots \times \max(\mathbb{A}_n))$, and all the remaining elements form singleton blocks. Then θ_R is a congruence.

Proof: (1) It suffices to observe that for any $a \in \max(\mathbb{A})$ we have $ax, xa, m(a, x, y), m(x, a, y), m(x, y, a) \in \max(\mathbb{A})$ for any x, y, and therefore all non-constant polynomials of $\mathbb{A}$ preserve $\max(\mathbb{A})$.

(2) is similar to (1). $\square$

When dealing with a relation over algebras $\mathbb{A}_1, \dots, \mathbb{A}_n$ or a CSP with domains $\mathbb{A}_v$ we will simplify the notation $\theta_{\mathbb{A}_i}, \theta_{\mathbb{A}_v}$ to θ_i, θ_v.

Lemma 11. *Every (α, β)-minimal set, for $\alpha \prec \beta \leq \theta_{\mathbb{A}}$, is a subset of $\max(\mathbb{A})$.*

Proof: Let U be a (α, β)-minimal set and f an idempotent polynomial with $f(\mathbb{A}) = U$ and $f(\beta) \not\subseteq \alpha$. Since $\beta \leq \theta_{\mathbb{A}}$, $c, d \in U \cap \max(\mathbb{A})$ for some $(c, d) \in \beta - \alpha$, as otherwise we would have $f(\beta) \subseteq \alpha$. Take $a \in \max(\mathbb{A})$ and set $g(x) = f(x)a$. For any $b \in U \cap \max(\mathbb{A})$ we have $g(b) = f(b)a = ba = b$. Therefore $g(\beta) \not\subseteq \alpha$ and

$g(\mathbb{A}) \subseteq \max(\mathbb{A})$. Finally, $f(\max(\mathbb{A})) \subseteq \max(\mathbb{A})$, therefore $f \circ g(\mathbb{A}) \subseteq U \cap \max(\mathbb{A})$ and $f \circ g(x) = x$ for $x \in U \cap \max(\mathbb{A})$. As U is minimal, $U = U \cap \max(\mathbb{A})$. □

4.3. Maroti's reduction

In this section we describe a reduction introduced by Maroti in [30] that allows us to reduce CSPs over SBM algebras to CSPs over SBM algebras of a certain restricted type. More precisely, it allows us to assume that every domain $\mathbb{A}$ is either a Mal'tsev algebra with m as a Mal'tsev operation, or it contains a *minimal element* a, that is, an element such that $ab = ba = b$ for all $b \in \mathbb{A}$. Moreover, as is easily seen, such element is unique and forms a $\sigma_{\mathbb{A}}$-block, which is also the smallest element of the semilattice $\mathbb{A}/_{\sigma_{\mathbb{A}}}$.

Let f be an idempotent unary polynomial of algebra $\mathbb{A}$ and A the universe of $\mathbb{A}$. The *retract* $f(\mathbb{A})$ of $\mathbb{A}$ is the algebra with universe $f(A)$, whose basic operations are of the form $f \circ g$, given by $f \circ g(x_1, \ldots, x_n) = f(g(x_1, \ldots, x_n))$ for $x_1, \ldots, x_n \in f(A)$, where g is a basic operation of $\mathbb{A}$.

Lemma 12. *A retract of an SBM algebra through an idempotent polynomial is an SBM algebra.*

Proof: Let f be an idempotent polynomial. Let $g_1(x, y) = f(xy)$, $m_1(x, y, z) = f(m(x, y, z))$ be the basic operations of the retract, $\mathbb{A}_1 = f(\mathbb{A})$, and $\sigma_1 = \sigma_{\mathbb{A}}|_{\mathbb{A}_1}$. Firstly, note that σ_1 is a congruence of $\mathbb{A}_1$ and $\mathbb{A}_1$ is an idempotent algebra. Since $\mathbb{A}/_{\sigma_{\mathbb{A}}}$ is term equivalent to a semilattice and any retract of a semilattice by a semilattice polynomial is a semilattice, so is $\mathbb{A}_1/_{\sigma_1}$. Finally,

$$m_1(x, y, y) = f(m(x, y, y)) = f(x) = x$$
$$m_1(y, y, x) = f(m(y, y, x)) = f(x) = x,$$

for any $x, y \in \mathbb{A}_1$ with $x \stackrel{\sigma_1}{\equiv} y$. □

The results of [30] imply the following. Let $\mathfrak{A}$ be a class of similar finite algebras closed under subalgebras and retracts via idempotent unary polynomials. Suppose that $\mathfrak{A}$ has a term operation f satisfying the following conditions for some $\mathbb{B} \in \mathfrak{A}$:

(1) $f(x, f(x, y)) = f(x, y)$ for any $x, y \in \mathbb{B}$;

(2) for each $a \in \mathbb{B}$ the mapping $x \mapsto f(a, x)$ is not surjective;

(3) the set C of $a \in \mathbb{B}$ such that $x \mapsto f(x, a)$ is surjective generates a proper subalgebra of $\mathbb{B}$.

Then $\mathrm{CSP}(\mathfrak{A})$ is polynomial time reducible to $\mathrm{CSP}(\mathfrak{A} - \{\mathbb{B}\})$.

By Lemma 9 the operation $\cdot$ of the class of SBM algebras from $\mathfrak{A}$ satisfies condition (1). If the operation $a \cdot x$ is surjective for some a, then $a \leq x$ for all $x \in \mathbb{B}$. Therefore the only case when condition (2) is not satisfied is when $\mathbb{B}$ has a minimal element. Moreover, if $a \in \mathbb{B}$ is such that ax is a surjective polynomial, it also satifies the condition $ax = x$ for $x \in \mathbb{B}$. Indeed, if $ab \neq b$ for some b, the surjectivity of $g(x) = ax$ implies that $a(ab) \neq ab$, a contradiction with Lemma 9. Finally, condition (3) is satisfied whenever $\mathbb{B}$ is not a Mal'tsev algebra, because if $h(x) = xa$ is surjective then $a^{\sigma_{\mathbb{B}}}$ is the minimal element of $\mathbb{B}/_{\sigma_{\mathbb{B}}}$. Therefore, (3) holds unless $\sigma_{\mathbb{B}}$ is the total relation, in which case $\mathbb{B}$ is a Mal'tsev algebra by definition. Therefore, choosing $\mathbb{B}$ to be a maximal (in terms of cardinality) algebra from $\mathfrak{A}$ satisfying conditions (1)–(3) we may only consider instances of $\mathrm{CSP}(\mathfrak{A})$, in which every domain has a minimal element or is a Mal'tsev algebra.

Corollary 13. *Every instance $\mathcal{P} \in \mathrm{CSP}(\mathfrak{A})$ can be reduced in polynomial time to polynomially many instances over algebras each of which either is Mal'tsev or has a minimal element a such that $ax = x$ for all $x \in \mathbb{A}$.*

Throughout the rest of the paper $\mathfrak{A}$ is a finite class of finite SBM algebras closed under taking subalgebras, quotient algebras, and retracts through unary idempotent polynomials.

5. Separating congruences

In this section we develop a method that will lead to some way to decompose CSPs over SBM algebras. First, we introduce and study the notion of separation of prime intervals. Let R be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ and $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_i)$, $\gamma, \delta \in \mathsf{Con}(\mathbb{A}_j)$, for some $i, j \in [n]$, such that $\alpha \prec \beta$, $\gamma \prec \delta$. Recall that interval (α, β) can be separated from (γ,δ) if there is a unary polynomial $\mathbf{f}$ of R such that $f_i(\beta)\not\subseteq\alpha$ while $f_j(\delta)\subseteq\gamma$. If $\mathbf{f}$ satisfies this property we will also say that $\mathbf{f}$ *separates* (α,β) from (γ,δ). In the definition above it is possible that $i=j$ or that $n=1$; in this cases the argument in some proofs may be slightly different. To avoid such complications we will always assume that $i\neq j$, as the following lemma allows us to do.

Lemma 14. *Let Q be the binary equality relation on $\mathbb{A}$. Prime interval (α,β), $\alpha\prec\beta\le\theta_{\mathbb{A}}$, can be separated from (γ,δ), $\gamma\prec\delta\le\theta_{\mathbb{A}}$, as intervals in $\mathsf{Con}(\mathbb{A})$ if and only if (α,β) can be separated from (γ,δ) in Q (as intervals in the congruence lattices of the factors of a binary relation).*

Proof: Note that for any polynomial $\mathbf{f}$ of Q its action on the first and second factors of Q is the same polynomial of $\mathbb{A}$. By definition $\alpha\prec\beta$ can be separated from $\gamma\prec\delta$ in $\mathsf{Con}(\mathbb{A})$ if and only if there is a unary polynomial f of $\mathbb{A}$, $f(\beta)\not\subseteq\alpha$ while $f(\delta)\subseteq\gamma$. This condition can be expressed as follows: there is a unary polynomial $\mathbf{f}$ of Q, $f_1(\beta)\not\subseteq\alpha$ while $f_2(\delta)\subseteq\gamma$, which precisely means that (α,β) can be separated from (γ,δ) in Q $\square$

In Section 5.1 we study the sets of intervals that cannot be separated from each other. These sets will later give us some sort of decomposition of CSP instances. Collapsing polynomials introduced in Section 5.2 yeild one of the main ingredients of the solution algorithm. Section 5.3 provides a sufficient condition for separation of intervals and a related notion of decomposition, which is the second ingredient.

5.1. Basic properties of separation

Let again R be a subdirect product of SBM algebras $\mathbb{A}_1\times\ldots\times\mathbb{A}_n$, $i,j\in[n]$, and $\alpha,\beta\in\mathsf{Con}(\mathbb{A}_i)$, $\gamma,\delta\in\mathsf{Con}(\mathbb{A}_j)$ with $\alpha\prec\beta\le\theta_i$, $\gamma\prec\delta\le\theta_j$.

First, we show that separating polynomials can be chosen to satisfy certain simple conditions.

Lemma 15. *If (α,β) can be separated from (γ,δ) then there is a polynomial $\mathbf{f}$ that separates (α,β) from (γ,δ) and such that $f_\ell(\mathbb{A}_\ell)\subseteq\max(\mathbb{A}_\ell)$ for every $\ell\in[n]$.*

Proof: Let $\mathbf{g}$ separate (α, β) from (γ, δ). Choose a tuple $\mathbf{a} \in \max(R)$ and consider the polynomial $\mathbf{f}(x) = \mathbf{g}(x) \cdot \mathbf{a}$. As is easily seen, $f_\ell(\mathbb{A}_\ell) \subseteq \max(\mathbb{A}_\ell)$ for $\ell \in [n]$. Since $g_j(\delta) \subseteq \gamma$, we have $f_j(\delta) \subseteq \gamma$. Finally, as $g_i(\beta) \not\subseteq \alpha$, there are $(a', b') \in \beta - \alpha$ such that for $a = g_i(a')$, $b = g_i(b')$ we have $(a, b) \in \beta - \alpha$. Since $\beta \leq \sigma_{\mathbb{A}_i}$ and all the nontrivial (that is, different from an α-block) β-blocks are inside $\max(\mathbb{A}_i)$, it also holds that $a', b' \in \max(\mathbb{A}_i)$. Then

$$f_i(a') = g_i(a')\mathbf{a}[i] = a\mathbf{a}[i] = a \neq b = b\mathbf{a}[i] = g_i(b')\mathbf{a}[i] = f_i(b').$$

□

From now on we assume that all polynomials separating intervals satisfy the conditions of Lemma 15.

Lemma 16. *If (α, β) can be separated from (γ, δ) then, for any (α, β)-minimal set U, there is an idempotent unary polynomial $\mathbf{g}$ such that $g_i(\mathbb{A}_i) = U$, and $\mathbf{g}$ separates (α, β) from (γ, δ).*

Proof: Let $\mathbf{f}$ separate (α, β) from (γ, δ). Then by Lemma 7(4) $f_i(\mathbb{A}_i)$ contains an (α, β)-minimal set U', and by Lemma 7(3) there is an idempotent polynomial h_i with $h_i(\mathbb{A}_i) = U'$. The polynomial h_i can be extended to a polynomial $\mathbf{h}$ of R. Then $\mathbf{f}' = \mathbf{h} \circ \mathbf{f}$ separates (α, β) from (γ, δ) and $f'_i(\mathbb{A}_i) = U'$.

Since $f'_i(\beta) \not\subseteq \alpha$, by Lemma 7(4) there is an (α, β)-minimal set U'' such that f'_i witnesses that U'' and $f'_i(U'')$ are polynomially isomorphic. This means that $f'_i(U'')$ is an (α, β)-minimal set, and as $f'_i(\mathbb{A}_i) = U'$ we obtain $f'_i(U'') = U'$. By Lemma 7(1) there exists an idempotent polynomial h'_i with $h'_i(U') = U''$. As above, the polynomial h'_i can be extended to a polynomial $\mathbf{h}'$ of R. For a certain k, $(\mathbf{f}' \circ \mathbf{h}')^k$ is idempotent, separates i from j, and $(f'_i \circ h'_i)^k(\mathbb{A}_i) = U''$. Now the lemma follows easily from Lemma 7(1). □

Let $\mathcal{I}_R$ be the set of triples (i, α, β) such that $i \in [n]$, $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_i)$ and $\alpha \prec \beta \leq \theta_i$. The relation 'cannot be separated in R' on $\mathcal{I}_R$ is clearly reflexive and transitive. Now, we prove it is also symmetric

Lemma 17. *If (α, β) can be separated from (γ, δ) then (γ, δ) can be separated from (α, β).*

Proof: Let $U_1, \ldots, U_k$ be all the (α, β)-minimal sets. By Lemma 16, for every U_ℓ, there is an idempotent unary polynomial $\mathbf{g}^{(\ell)}$ separating (α, β) from (γ, δ) and such that $g_i^{(\ell)}(\mathbb{A}_i) = U_\ell$. Take a δ-block B that contains more than one γ-block, a tuple $\mathbf{a} \in R$ such that $\mathbf{a}[j] \in B$, and set $\mathbf{a}^{(\ell)} = \mathbf{g}^{(\ell)}(\mathbf{a})$. By Lemmas 11 and 15 $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(k)}$ can be assumed to be from $\max(R)$ and $U_1, \ldots, U_k \subseteq \max(\mathbb{A}_i)$, and $B \subseteq \max(\mathbb{A}_j)$. The operation $\mathbf{h}^{(\ell)}(x) = m(x, \mathbf{g}^{(\ell)}(x), \mathbf{a}^{(\ell)})$ satisfies the following conditions

- $h_i^{(\ell)}(x) = m(x, g_i^{(\ell)}(x), \mathbf{a}^{(\ell)}[i]) = m(x, x, \mathbf{a}^{(\ell)}[i]) = \mathbf{a}^{(\ell)}[i]$ for all $x \in U_\ell$;
- $h_j^{(\ell)}(x) = m(x, g_j^{(\ell)}(x), \mathbf{a}^{(\ell)}[j]) \overset{\gamma}{\equiv} m(x, \mathbf{a}^{(\ell)}[j], \mathbf{a}^{(\ell)}[j]) = x$ for all $x \in B$;
- $\mathbf{h}^{(\ell)}(R) \subseteq \max(R)$.

We are going to compose the polynomials $\mathbf{h}^{(\ell)}$ such that the composition collapses β. To this end take a sequence $1 = \ell_1, \ell_2, \ldots$ such that U_{ℓ_2} is a subset of the range of $\overline{h}^{(1)} = h_i^{(\ell_1)}$, and, for $s > 2$, U_{ℓ_s} is a subset of the range of $\overline{h}^{(s-1)} = h_i^{(\ell_{s-1})} \circ \ldots \circ h_i^{(\ell_1)}$. Since $|\overline{h}^{(s)}(\mathbb{A}_i)| < |\overline{h}^{(s-1)}(\mathbb{A}_i)|$, there is r such that $|\overline{h}^{(r)}(\mathbb{A}_i)|$ contains no (α, β)-minimal sets. Therefore, setting $\mathbf{h}(x) = \mathbf{h}^{(\ell_r)}(\mathbf{h}^{(\ell_{r-1})}(\ldots \mathbf{h}^{(\ell_1)}(x) \ldots))$ we have that h_i collapses all the (α, β)-minimal sets, and h_j acts identically on $B/_\gamma$. Thus, $\mathbf{h}$ separates (γ, δ) from (α, β). $\square$

Lemma 17 together with the observation before it shows that the relation 'cannot be separated' is an equivalence relation on $\mathcal{I}_R$.

5.2. Collapsing polynomials

Intuitively, a collapsing polynomial for some prime interval $\alpha \prec \beta$ in an algebra or a subdirect product of algebras is a polynomial that collapses all prime intervals that can be separated from $\alpha \prec \beta$ and only such prime intervals.

Let R be a subdirect product of SBM algebras $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, and $(i, \alpha, \beta) \in \mathcal{I}_R$. A unary idempotent polynomial $\mathbf{f}$ of R is called (α, β)-*collapsing* if the following conditions hold:

(C1) for any $(j, \gamma, \delta) \in \mathcal{I}_R$, it holds $f_j(\delta) \subseteq \gamma$, unless (α, β) and (γ, δ) cannot be separated;

(C2) for any $(j,\gamma,\delta) \in \mathcal{I}_R$ such that (α,β), (γ,δ) cannot be separated, the set $f_j(\mathbb{A}_j)$ is a (γ,δ)-minimal set.

First, we show that (α,β)-collapsing polynomials exist even if we impose some additional requirements.

Lemma 18. *Let R be a subdirect product of SBM algebras $\mathbb{A}_1\times\cdots\times\mathbb{A}_n$ and $(i,\alpha,\beta) \in \mathcal{I}_R$, and let $\mathbf{a} \in \max(R)$ be such that $\mathbf{a}[i]$ belongs to a β-block containing more than one α-block and $b \in \mathbb{A}_i$ with $(\mathbf{a}[i], b) \in \beta - \alpha$. Then there is an (α,β)-collapsing polynomial $\mathbf{f}$ of R such that $\mathbf{f}(\mathbf{a}) = \mathbf{a}$ and $f_i(b) \stackrel{\alpha}{\equiv} b$.*

Proof: First, we find an (α,β)-collapsing polynomial. For every $(j,\gamma,\delta) \in \mathcal{I}_R$ such that (α,β) can be separated from (γ,δ) there is an idempotent polynomial $\mathbf{g}^{j\gamma\delta}$ such that $g_j^{j\gamma\delta}(\delta) \subseteq \gamma$, but $g_i^{j\gamma\delta}(\beta) \not\subseteq \alpha$. Moreover, we may assume by Lemma 16 that for every $\mathbf{g}^{j\gamma\delta}$, $g_i^{j\gamma\delta}(\mathbb{A}_i) = U$ for the same (α,β)-minimal set U. Composing all such polynomials we obtain a polynomial $\mathbf{h}$ such that $h_i(\mathbb{A}_i) = U$, and so $h_i(\beta) \not\subseteq \alpha$, and $h_j(\delta) \subseteq \gamma$ for any j,γ,δ as above. By iterating $\mathbf{h}$ can be assumed idempotent. Choose $\mathbf{h}$ to have the smallest image among unary idempotent polynomials such that $h_i(\mathbb{A}_i)$ is an (α,β)-minimal set and $h_j(\delta) \subseteq \gamma$ for any $(j,\gamma,\delta) \in \mathcal{I}_R$ such that (α,β) can be separated from (γ,δ).

Suppose now that for some $(j,\gamma,\delta) \in \mathcal{I}_R$ such that the interval (α,β) cannot be separated from (γ,δ) the set $U' = h_j(\mathbb{A}_j)$ is not a (γ,δ)-minimal set. Then, since $h_j(\delta) \not\subseteq \gamma$, the set U' contains a (γ,δ)-minimal set U''. Let g be an idempotent polynomial of $\mathbb{A}_j$ with $g(\mathbb{A}_j) = U''$ and $\mathbf{g}$ its extension to a polynomial of R. Then $\mathbf{h}' = \mathbf{g} \circ \mathbf{h}$ satisfies the following conditions:

– $h'_j(\mathbb{A}_j) = U''$ and $h'_j(\delta) \not\subseteq \gamma$;

– $h'_i(\beta) \not\subseteq \alpha$, because (α,β) cannot be separated from (γ,δ);

– $|\mathbf{h}'(R)| < |\mathbf{h}(R)|$.

Iterating $\mathbf{h}'$ it can be assumed idempotent. Then the last property contradicts the choice of $\mathbf{h}$. Therefore $\mathbf{h}$ is (α,β)-collapsing.

By Lemma 8 there is an (α,β)-minimal set U such that $\mathbf{a}[i]^\alpha \cap U, b^\alpha \cap U \neq \varnothing$. Moreover, an (α,β)-collapsing polynomial $\mathbf{h}$ can be chosen such that $h_i(\mathbb{A}_i) = U$.

Then set $\mathbf{f}(x) = m(\mathbf{h}(x), \mathbf{h}(\mathbf{a}), \mathbf{a})$. For the polynomial $\mathbf{f}$ we have:

– $\mathbf{f}(\mathbf{a}) = m(\mathbf{h}(\mathbf{a}), \mathbf{h}(\mathbf{a}), \mathbf{a}) = \mathbf{a}$;

– $c = f_i(b) = m(h_i(b), h_i(\mathbf{a}[i]), \mathbf{a}[i]) \stackrel{\alpha}{\equiv} m(h_i(b), \mathbf{a}[i], \mathbf{a}[i]) = h_i(b) \stackrel{\alpha}{\equiv} b$, because, since $\mathbf{h}$ is idempotent, $h_i(\mathbf{a}[i]) \stackrel{\alpha}{\equiv} \mathbf{a}[i]$ and $h_i(b) \stackrel{\alpha}{\equiv} b$;

– for any $(j, \gamma, \delta) \in \mathcal{I}_R$ such that and $(\alpha, \beta), (\gamma, \delta)$ can be separated, $f_j(\delta) \subseteq \gamma$.

By iterating $\mathbf{f}$ we obtain an idempotent polynomial $\mathbf{f}'$ that satisfies all the conditions above. Indeed, the first and third conditions are straightforward, while the second one follows from the equality $f_i(c) \stackrel{\alpha}{\equiv} c$. Finally, for any $(j, \gamma, \delta) \in \mathcal{I}_R$ such that $(\alpha, \beta), (\gamma, \delta)$ cannot be separated we have $f'_j(\delta) \not\subseteq \gamma$, because $f'_i(\beta) \not\subseteq \alpha$. Also, $f'_j(\mathbb{A}_j)$ is a (γ, δ)-minimal set, because $h_j(\mathbb{A}_j)$ is a one.

Thus, $\mathbf{f}'$ satisfies all the required conditions. The lemma is proved. □

5.3. Splits and alignments

In this section we present a sufficient condition for two prime intervals to be separated. As we shall see using this condition certain projections of a relation can be partitioned into a small number of subdirect products of smaller algebras.

Let R be a subdirect product of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $\alpha_i, \beta_i \in \mathsf{Con}(\mathbb{A}_i)$, $i \in [n]$, such that $\alpha_i \prec \beta_i \le \theta_{\mathbb{A}_i}$. An element $a \in \mathbb{A}_i$, $i \in [n]$, is called *$\alpha_i\beta_i$-split* if there is a β_i-block B and $b, c \in B$ such that $ab \stackrel{\alpha_i}{\not\equiv} ac$. Note that no element from $\max(\mathbb{A}_i)$ is $\alpha_i\beta_i$-split, while the minimal element is $\alpha_i\beta_i$-split. Indeed, if $a \in \mathbb{A}_i$ is a minimal element satisfying the conditions of Corollary 13, then $ax = x$ for any $x \in \mathbb{A}_i$ and clearly satifies the definition of an $\alpha_i\beta_i$-split element. We say that $i, j \in [n]$ are *$\overline{\alpha}\overline{\beta}$-aligned* if for any $\mathbf{a} \in R$ such that $\mathbf{a}[i]$ is $\alpha_i\beta_i$-split then $\mathbf{a}[j]$ is $\alpha_j\beta_j$-split as well, and the other way round.

Lemma 19. *If i, j are not $\overline{\alpha}\overline{\beta}$-aligned then (α_i, β_i) can be separated from (α_j, β_j).*

Proof: It suffices to consider the case $n = 2$, $i = 1$, $j = 2$. Let $(a, b) \in R$ be such that a is $\alpha_i\beta_i$-split, while b is not $\alpha_j\beta_j$-split. Consider operation $\mathbf{f}((x_1, x_2)) = (a, b) \cdot (x_1, x_2)$. We claim that $f_1(\beta_1) \not\subseteq \alpha_1$ while $f_2(\beta_2) \subseteq \alpha_2$.

For any β_2-block B_2 and any $a', b' \in B_2$ we have $f_2(a') = ba' \stackrel{\alpha_2}{\equiv} bb' = f_2(b')$, as b is not $\alpha_2\beta_2$-split. Thus $f_2(\beta_2) \subseteq \alpha_2$. On the other hand, since a is $\alpha_1\beta_1$-split, there is a β_1-block B_1 and $a'', b'' \in B_1$ such that $f_1(a'') = aa'' \stackrel{\alpha_1}{\not\equiv} ab'' = f_1(b'')$. Therefore $f_1(\beta_1) \not\subseteq \alpha_1$. $\Box$

6. From relations to instances

Here we apply the results of the previous section to CSP instances. In particular, we introduce coherent sets of an instance and show that if an instance has solutions on every coherent set, which are consistent in some weak sense, then the entire instance has a solution.

Let $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ be a 3-minimal instance of $\mathrm{CSP}(\mathfrak{A})$. We assume that the domain $\mathbb{A}_v$ of each variable $v \in V$ is the set of solutions $\mathcal{S}_v$, and so the constraint relations are subdirect products of the domains.

Since separation of prime intervals depends only on binary projections of a relation, it can be defined for 3-minimal instances as well. More precisely, let $\mathcal{I}_{\mathcal{P}}$ (or just $\mathcal{I}$ if $\mathcal{P}$ is clear from the context) be the set of all triples (v, α, β), where $v \in V$, $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$ are such that $\alpha \prec \beta \leq \theta_v$. Let $(v, \alpha, \beta), (w, \gamma, \delta) \in \mathcal{I}$; we say that (α, β) cannot separated from (γ, δ) if this is the case for $\mathcal{S}_{vw}$. Due to 3-minimality — we can consider sets of solutions on 3 variables — this relation is transitive. It is also reflexive and symmetric by Lemma 17.

Next we define two partitions of a CSP instance $\mathcal{P}$. The first one, link partition allows us to reduce solving subinstances of $\mathcal{P}$ to instances over smaller domains. The second one provides a sufficient condition to have a link partition and is defined through alignment properties.

Let again $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ be a 3-minimal instance of $\mathrm{CSP}(\mathfrak{A})$. Partitions $A_{v1} \cup \ldots \cup A_{vk_v} = \mathbb{A}_v$ for $v \in V$ are called a *link partition* if the following conditions hold:

(A) For any $v, w \in V$, $k_v = k_w \geq 2$, and there is a bijection $\varphi_{vw} : [k_v] \to [k_w]$ such that for any $(a, b) \in \mathcal{S}_{vw}$ and any $j \in [k_v]$, $a \in A_{vj}$ if and only if $b \in A_{w\varphi_{vw}(j)}$; and

(B) any partitions $A'_{v1} \cup \cdots \cup A'_{v\ell_v} = \mathbb{A}_v$, $v \in V$, such that for every $v \in V$ and any $i \in [\ell_v]$, the set A'_{vi} is a subset of A_{vj} for some $j \in [k_v]$, does not satisfy condition (A).

Observe that, since $\mathcal{P}$ is 3-minimal, the mappings φ_{vw} are consistent, that is, for any $u, v, w \in V$ it holds that $\varphi_{vw} \circ \varphi_{uv} = \varphi_{uw}$. Without loss of generality we will assume that φ_{vw} is an identity mapping.

As is easily seen, the partition $A_{v1} \cup \ldots \cup A_{vk_v} = \mathbb{A}_v$ defines a congruence of $\mathbb{A}_v$. In particular, each of A_{vi} is a subalgebra of $\mathbb{A}_v$.

Let $\alpha_v, \beta_v \in \mathsf{Con}(\mathbb{A}_v)$ for $v \in V$ be such that $\alpha_v \prec \beta_v \leq \theta_v$. Variables $v, w \in V$ are *$\overline{\alpha}\overline{\beta}$-aligned* if they are $\overline{\alpha}\overline{\beta}$-aligned in $\mathcal{S}_{vw}$. In the following lemma we assume that every domain $\mathbb{A}_v$ of $\mathcal{P}$ either has a minimal element, or $\sigma_{\mathbb{A}_v}$ is the full congruence, and so $\mathbb{A}_v$ is a Mal'tsev algebra.

Lemma 20. *(1) If variables $v, w \in V$ of an instance $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ are $\overline{\alpha}\overline{\beta}$-aligned and $\mathbb{A}_v$ has a minimal element then $\mathbb{A}_w$ also has a minimal element.*
(2) If every domain of an instance $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ has a minimal element and any two variables $v, w \in V$ are $\overline{\alpha}\overline{\beta}$-aligned, then $\mathcal{P}$ has a link partition.

Proof: For every $v \in V$ let L_v denote the set of $\alpha_v\beta_v$-split elements of $\mathbb{A}_v$ and let N_v denote the set of $\alpha_v\beta_v$-non-split elements. As we observed before Lemma 19, both sets are nonempty if $\mathbb{A}_v$ has a minimal element, and $L_v = \varnothing$ if $\mathbb{A}_v$ is a Mal'tsev algebra.

(1) If $\mathbb{A}_w$ is a Mal'tsev algebra then v, w cannot be $\overline{\alpha}\overline{\beta}$-aligned since $L_w = \varnothing$, while $L_v, N_v \neq \varnothing$, and $\mathcal{S}_{vw}$ is a subdirect product.

(2) For any $v, w \in V$ and any pair $(a, b) \in \mathcal{S}_{vw}$, $a \in L_v$ if and only if $b \in L_w$. This provides some nontrivial partitions satisfying condition (A), and we may choose the finest such partition. Therefore $\mathcal{S}_{vw}$ is link-partitioned, as well as R for any constraint $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$. □

7. The algorithm

In the first part of this section we introduce the property of block-minimality, the key property of CSP instances for our algorithm. We also prove that block-minimality can be efficiently established. Then in the second part we show that block-minimality is sufficient for the existence of a solution, Theorem 23, which is the main result of this section, and provides a polynomial time algorithm for CSPs over SBM algebras.

Let $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ be a 3-minimal instance such that for every its domain $\mathbb{A}_v$ either σ_v is the full congruence, and so $\mathbb{A}_v$ is a Mal'tsev algebra with Mal'tsev operation m, or $\mathbb{A}_v$ has a minimal element.

Recall that $\mathcal{I}_\mathcal{P}$ or just $\mathcal{I}$ denotes the set of all triples (v, α, β), where $v \in V$, $\alpha, \beta \in \mathsf{Con}(\mathbb{A}_v)$ are such that $\alpha \prec \beta \leq \theta_v$. For a triple $(v, \alpha, \beta) \in \mathcal{I}$ by $\mathcal{I}(v, \alpha, \beta)$ we denote the set of all triples $(w, \gamma, \delta) \in \mathcal{I}$ such that (α, β) cannot be separated from (γ, δ). Also, by $W_{v\alpha\beta}$ we denote the set $\{w \mid (w, \gamma, \delta) \in \mathcal{I}(v, \alpha, \beta)\}$. Sets of the form $W_{v\alpha\beta}$ are called *coherent sets*.

The next lemma gives one of the key properties of coherent sets and collapsing polynomials.

Lemma 21. *Let $(v, \alpha, \beta) \in \mathcal{I}$ and $w \notin W_{v\alpha\beta}$, and let* $\mathbf{f}$ *be an (α, β)-collapsing polynomial of $\mathcal{S}_{vw}$. Then $f_w(\underline{1}_w) \subseteq \underline{0}_w$.*

Proof: Since w does not belong the coherent set $W_{v\alpha\beta}$ for any prime interval $\underline{0}_w \leq \gamma \prec \delta \leq \theta_w$ we have $f_w(\delta) \subseteq \gamma$. This means that $f_w(\theta_w) \subseteq \underline{0}_w$. However, as the range of f_w is a subset of $\max(\mathbb{A}_w)$, we also have $f_w(\underline{1}_w) \subseteq \theta_w$. Finally, as f_w is idempotent, it also implies $f_w(\underline{1}_w) \subseteq \underline{0}_w$. The result follows. $\square$

Instance $\mathcal{P}$ is said to be *block-minimal* if for any $(v, \alpha, \beta) \in \mathcal{I}$ the instance $\mathcal{P}_{W_{v\alpha\beta}}$ is minimal.

In the next section we prove, Theorem 23, that every block-minimal instance has a solution. To show that Theorem 23 gives rise to a polynomial-time algorithm for $\mathrm{CSP}(\mathfrak{A})$ we need to show how block-minimality can be established. We prove that establishing block-minimality can be reduced to solving polynomially many smaller instances of $\mathrm{CSP}(\mathfrak{A})$.

Proposition 22. *Transforming an instance $\mathcal{P} = (V, \mathcal{A}, \mathcal{C}) \in \mathrm{CSP}(\mathfrak{A})$ to a block-minimal instance can be reduced to solving polynomially many instances $\mathcal{P}' = (V', \mathcal{A}', \mathcal{C}') \in \mathrm{CSP}(\mathfrak{A})$ such that $V' \subseteq V$ and either $\mathbb{A}'_v$ is a Mal'tsev algebra for all $v \in V'$, or $|\mathbb{A}'_v| < |\mathbb{A}_v|$ for all $v \in V'$.*

Since the cardinalities of algebras in $\mathfrak{A}$ are bounded, the depth of recursion when establishing block-minimality is also bounded. Therefore, together with Theorem 23 this proposition gives a polynomial time algorithm for $\mathrm{CSP}(\mathfrak{A})$.

Proof: Using the standard propagation algorithm and Maroti's reduction (Section 4.3) we may assume that $\mathcal{P}$ is 3-minimal and every $\mathbb{A}_v$ is either Mal'tsev or has a minimal element. Take $(v, \alpha, \beta) \in \mathcal{I}$ as in the definition of block-minimality. We need to show how to make problems $\mathcal{P}_{W_{v\alpha\beta}}$ minimal. If every $\mathbb{A}_w$ for $w \in W_{v\alpha\beta}$ is Mal'tsev, $\mathcal{P}_{W_{v\alpha\beta}}$ can be made minimal using the algorithm from [10]. If $\mathbb{A}_w$ has a minimal element for some $w \in W_{v\alpha\beta}$ then set $\alpha_v = \alpha, \beta_v = \beta$, and for each $w \in W_{v\alpha\beta}$ choose α_w, β_w in such a way that $(w, \alpha_w, \beta_w) \in \mathcal{I}(v, \alpha, \beta)$. Then by Lemmas 20 and 19 $\mathcal{P}_{W_{v\alpha\beta}}$ is link partitioned, that is, it is a disjoint union of instances $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_m$, where $\mathcal{P}_i = (W_{v\alpha\beta}, \mathcal{A}^i, \mathcal{C}^i)$ are such that $\mathbb{A}_w = \mathbb{A}_w^1 \cup \dots \cup \mathbb{A}_w^m$ is a disjoint union. We then transform them to minimal instances separately.

If at any stage there is a tuple from a constraint relation that does not extend to a solution of a certain subinstance, we tighten the original problem $\mathcal{P}$ by excluding all such tuples and start all over again. Observing that the set of tuples from a constraint relation that can be extended to a solution of the subinstance is a subalgebra, the resulting instance belongs to $\mathrm{CSP}(\mathfrak{A})$ as well. □

7.2. Block-minimality and solutions of the CSP

We now prove that block-minimality is a sufficient condition to have a solution.

Theorem 23. *Every block-minimal instance $\mathcal{P} \in \mathrm{CSP}(\mathfrak{A})$ with nonempty constraint relations has a solution.*

Proof: Let $\mathcal{P} = (V, \mathcal{A}, \mathcal{C})$ be a 3-minimal and block-minimal instance from $\mathrm{CSP}(\mathfrak{A})$, and such that every domain $\mathbb{A}_v$ is either a Mal'tsev algebra or has a minimal element. We make use of the following construction. Let $\gamma_v \in \mathsf{Con}(\mathbb{A}_v)$, $\gamma_v \le \theta_v$ for $v \in V$. A collection of mappings $\mathcal{M} = \{\varphi_{v\alpha\beta} \mid (v, \alpha, \beta) \in \mathcal{I}\}$ is called an $\overline{\gamma}$*-ensemble* for $\mathcal{P}$ if

(1) for every $(v, \alpha, \beta) \in \mathcal{I}$ the mapping $\varphi_{v\alpha\beta}$ is a solution of $\mathcal{P}_{W_{v\alpha\beta}}$; and

(2) for every $(v, \alpha, \beta), (w, \gamma, \delta) \in \mathcal{I}$, and any $u \in W_{v\alpha\beta} \cap W_{w\gamma\delta}$, it holds $\varphi_{v\alpha\beta}(u) \overset{\gamma_u}{\equiv} \varphi_{w\gamma\delta}(u)$;

(3) for any $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ the tuple $\mathbf{a}$ where $\mathbf{a}[u] = \varphi_{v\alpha\beta}(u)^{\gamma_v}$ for $u \in \mathbf{s}$ and any $(v, \alpha, \beta) \in \mathcal{I}$ with $u \in W_{v\alpha\beta}$, belongs to $R/_{\overline{\gamma}_{\mathbf{s}}}$.

We prove that for any $\gamma_v \in \mathsf{Con}(\mathbb{A}_v)$, $\gamma_v \leq \theta_v$ for $v \in V$ the instance $\mathcal{P}$ has a $\overline{\gamma}$-ensemble.

If $\gamma_v = \theta_v$ for each $v \in V$ then any collection of solutions $\varphi_{v\alpha\beta}$ of $\mathcal{P}_{W_{v\alpha\beta}}$ such that $\varphi_{v\alpha\beta}(u) \in \max(\mathbb{A}_u)$ for all $(v, \alpha, \beta) \in \mathcal{I}$, and $u \in W_{v\alpha\beta}$, satisfies the conditions of a $\overline{\gamma}$-ensemble. Moreover the block-minimality of $\mathcal{P}$ guarantees that each $\mathcal{P}_{W_{v\alpha\beta}}$ is minimal, therefore has some solution of this kind.

If $\gamma_v = \underline{0}_v$ for $v \in V$ then for any $(v, \alpha, \beta), (w, \gamma, \delta) \in \mathcal{I}$ condition (2) implies $\varphi_{v\alpha\beta}(u) = \varphi_{w\gamma\delta}(u)$ for $u \in W_{v\alpha\beta} \cap W_{w\gamma\delta}$. Let us denote this value by $\psi(u)$. Then condition (3) implies that ψ is a solution of $\mathcal{P}$.

Finally, the inductive step follows from Lemma 24. $\Box$

Lemma 24. *Let $\mathcal{P} = (V, \mathcal{A}, \mathcal{C}) \in \mathrm{CSP}(\mathfrak{A})$ be a 3-minimal and block-minimal instance such that every $\mathbb{A}_v$, $v \in V$, either is Mal'tsev or has a minimal element. Let $v \in V$ and $\beta_w, \gamma_w \in \mathsf{Con}(\mathbb{A}_w)$, $w \in V$, be such that $\beta_w \preceq \gamma_w \leq \theta_w$, $\beta_v \prec \gamma_v$ and $\beta_w = \gamma_w$ for $w \neq v$. If there is a $\overline{\gamma}$-ensemble for $\mathcal{P}$ then there is a $\overline{\beta}$-ensemble for $\mathcal{P}$.*

Proof: Let $\mathcal{M} = \{\varphi_{w\gamma\delta} \mid (w, \gamma, \delta) \in \mathcal{I}\}$ be a $\overline{\gamma}$-ensemble and $\xi(u) = \varphi_{w\gamma\delta}(u)^{\gamma_u}$ for $u \in W_{w\gamma\delta}$. By condition (2) for $\overline{\gamma}$-ensembles this definition is consistent. If $\xi(v)$ is a γ_v-block that is equal to a β_v-block, then $\mathcal{M}$ is also a $\overline{\beta}$-ensemble, and there is nothing to prove.

Otherwise to simplify notation we use β for β_v and γ for γ_v. Let B be the β-block containing $\varphi_{v\beta\gamma}(v)$. We show that for every $(w, \delta, \eta) \in \mathcal{I}$ with $v \in W_{w\delta\eta}$ a solution $\varphi'_{w\delta\eta}$ can be found such that $\varphi'_{w\delta\eta}(v) \in B$ and $\varphi'_{w\delta\eta}(u) \stackrel{\gamma_u}{\equiv} \varphi_{w\delta\eta}(u)$ for $u \in W_{w\delta\eta}$. Then, setting $\varphi'_{w\delta\eta} = \varphi_{w\delta\eta}$ for $(w, \delta, \eta) \in \mathcal{I}$ such that $v \notin W_{w\delta\eta}$ and $\mathcal{M}' = \{\varphi'_{w\delta\eta} \mid (w, \delta, \eta) \in \mathcal{I}\}$ we conclude that $\mathcal{M}'$ is a $\overline{\beta}$-ensemble.

Let $(w, \delta, \eta) \in \mathcal{I}$ be such that $v \in W_{w\delta\eta}$, and let $W = W_{v\beta\gamma}$, $U = W_{w\delta\eta}$, $\varphi = \varphi_{v\beta\gamma}|_{W\cap U}$, $\psi = \varphi_{w\delta\eta}$. Note that in this notation $\mathcal{S}_W$, $\mathcal{S}_U$, and $\mathcal{S}_{W\cap U}$ are the sets of solutions of $\mathcal{P}_{W_{v\beta\gamma}}$, $\mathcal{P}_{W_{w\delta\eta}}$, and $\mathcal{P}_{W_{v\beta\gamma}\cap W_{w\delta\eta}}$. It will often be convenient for us to treat these sets as relations rather than sets of solutions of a CSP. Then $\mathrm{pr}_{W\cap U}\mathcal{S}_W, \mathrm{pr}_{W\cap U}\mathcal{S}_U \subseteq \mathcal{S}_{W\cap U}$, and so $\varphi, \mathrm{pr}_{W\cap U}\psi \in \mathcal{S}_{W\cap U}$.

Let $\mathbf{f}$ be a (β,γ)-collapsing polynomial of $\mathcal{S}_U$. By Lemma 18 it can be selected such that $\psi = \mathbf{f}(\psi)$ and $B \cap f_v(\mathbb{A}_v) \neq \varnothing$. Let $\pi = \mathbf{f}_{W\cap U}(\varphi)$. We show that the mapping φ' on U given by $\varphi'(u) = \pi(u)$ for $u \in W \cap U$, and $\varphi'(u) = \psi(u)$ for $u \in U - W$ is a solution from $\mathcal{S}_U$. Since $\varphi(v) \in B$ and $B \cap f_v(\mathbb{A}_v) \neq \varnothing$, that is, $f_v(B) \subseteq B$ as $\mathbf{f}$ is idempotent, we have $\pi(v) = f_v(\varphi(v)) \in B$. Also, as for every $u \in (W \cap U) - \{v\}$, we have

$$\varphi'(u) = \pi(u) = f_u(\varphi(u)) \stackrel{\beta_u}{\equiv} f_u(\psi(u)) = \psi(u).$$

Therefore, φ' satisfies condition (2) of $\overline{\beta}$-ensembles for w,δ,η.

Now we prove that φ' is a solution from $\mathcal{S}_U$. Let $C = \langle \mathbf{s}, R\rangle$ be a constraint from $\mathcal{P}_U$, $W' = \mathbf{s} \cap W$ and $\mathbf{a} = \mathrm{pr}_{W'}\varphi$. Then, since φ is a solution from $\mathcal{S}_{W\cap U}$, there is $\mathbf{b} \in R$ with $\mathbf{a} = \mathrm{pr}_{W'}\mathbf{b}$. Let $\mathbf{c} = \mathbf{f}_{\mathbf{s}}(\mathbf{b})$, clearly, $\mathbf{c} \in R$. For the tuple $\mathbf{c}$ we have:

– $\mathbf{c}[u] = f_u(\mathbf{a}[u]) = f_u(\varphi(u)) = \varphi'(u)$ for $u \in W'$;

– $\mathbf{c}[u] = f_u(\mathbf{b}[u]) = \psi(u)$ for $u \in \mathbf{s} - W'$, because in this case $f_u(\underline{1}_u) \subseteq \underline{0}_u$ by Lemma 21, and therefore, as $f_u(\psi(u)) = \psi(u)$, we have $f_u(\mathbb{A}_u) = \{\psi(u)\}$.

Thus, $\mathbf{c} = \mathrm{pr}_{\mathbf{s}}\varphi'$, and thus φ' is a solution from $\mathcal{S}_{W\cap U}$.

So far we have defined mappings $\varphi'_{w\delta\eta}$, proved that they are solutions of the respective subinstances, that is, condition (1), and that they are consistent modulo $\overline{\beta}$, that is, condition (2). It remains to verify condition (3). Let $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ and $\xi(u) = \varphi_{w\delta\eta}(u)^{\beta_u}$ $(u \neq v)$, $\xi'(u) = \varphi'_{w\delta\eta}(u)^{\beta_u}$ for $u \in V$ and any $(w,\delta,\eta) \in \mathcal{I}$, such that $u \in W_{w\delta\eta}$. We need to show that $\mathrm{pr}_{\mathbf{s}}\xi' \in R' = R/_{\overline{\beta}_{\mathbf{s}}}$.

We use a simplified version of the argument above. Let $W' = W \cap \mathbf{s}$. If $v \notin \mathbf{s}$, the result follows from condition (3) for $\overline{\gamma}$. Suppose $v \in W'$ and let $\mathbf{f}$ be a (β,γ)-collapsing polynomial of R'. Also, let $\mathbf{a} = \mathrm{pr}_{\mathbf{s}}\xi$, $\mathbf{b}' = \mathrm{pr}_{W'}\varphi/_{\overline{\beta}_{W'}}$, where $\varphi = \varphi_{v\beta\gamma}$ as before, and $\mathbf{b} \in R'$ such that $\mathbf{b}' = \mathrm{pr}_{W'}\mathbf{b}$. By Lemma 18 $\mathbf{f}$ can be selected such that $\mathbf{a} \in \mathbf{f}(R')$ and $\mathbf{b}[v] \in f_v(\mathbb{A}_v/_{\beta_v})$. Let $\mathbf{c} = \mathbf{f}_{\mathbf{s}}(\mathbf{b})$. We have

– $\mathbf{c}[v] = \mathbf{b}'[v]$;

– $\mathbf{c}[u] = f_u(\mathbf{b}'[u]) = f_u(\mathbf{a}[u]) = \mathbf{a}[u]$ for $u \in W' - \{v\}$, as $\varphi(u) \in \xi(u) = \xi'(u)$;

– $\mathbf{c}[u] = f_u(\mathbf{b}[u]) = f_u(\mathbf{a}[u]) = \mathbf{a}[u]$ for $u \in \mathbf{s} - W'$, as in this case $f_u(\underline{1}_u) \subseteq \beta_u$

by Lemma 21, and therefore, since $f_u(\mathbf{a}[u]) = \mathbf{a}[u]$, we have $f_u(\mathbb{A}_u/_{\beta_u}) = \{\mathbf{a}[u]\}$.

Therefore $\mathbf{c} \in R'$, and as $\mathbf{c} = \mathrm{pr}_{\mathbf{a}}\xi'$, the result follows. □

References

[1] Libor Barto. The dichotomy for conservative constraint satisfaction problems revisited. In *LICS*, pages 301–310, 2011.

[2] Libor Barto and Marcin Kozik. Constraint satisfaction problems solvable by local consistency methods. *J. ACM*, 61(1):3, 2014.

[3] Libor Barto, Marcin Kozik, and Todd Niven. The CSP dichotomy holds for digraphs with no sources and no sinks (A positive answer to a conjecture of Bang-Jensen and Hell). *SIAM J. Comput.*, 38(5):1782–1802, 2009.

[4] Libor Barto, Andrei A. Krokhin, and Ross Willard. Polymorphisms, and how to use them. In *The Constraint Satisfaction Problem: Complexity and approximability [Result of a Dagstuhl Seminar]*, pages 1–44, 2017.

[5] Andrei A. Bulatov. Three-element Mal'tsev algebras. *Acta Sci. Math (Szeged)*, 71(3-4):469–500, 2002.

[6] Andrei A. Bulatov. A graph of a relational structure and constraint satisfaction problems. In *LICS*, pages 448–457, 2004.

[7] Andrei A. Bulatov. A dichotomy theorem for constraint satisfaction problems on a 3-element set. *J. ACM*, 53(1):66–120, 2006.

[8] Andrei A. Bulatov. Complexity of conservative constraint satisfaction problems. *ACM Trans. Comput. Log.*, 12(4):24, 2011.

[9] Andrei A. Bulatov. Conservative constraint satisfaction re-revisited. *J. Comput. Syst. Sci.*, 82(2):347–356, 2016.

[10] Andrei A. Bulatov and Víctor Dalmau. A simple algorithm for Mal'tsev constraints. *SIAM J. Comput.*, 36(1):16–27, 2006.

[11] Andrei A. Bulatov and Peter Jeavons. An algebraic approach to multi-sorted constraints. In *CP*, pages 183–198, 2003.

[12] Andrei A. Bulatov, Peter Jeavons, and Andrei A. Krokhin. Classifying the complexity of constraints using finite algebras. *SIAM J. Comput.*, 34(3):720–742, 2005.

[13] Andrei A. Bulatov, Andrei A. Krokhin, and Benoit Larose. Dualities for constraint satisfaction problems. In *Complexity of Constraints - An Overview of Current Research Themes [Result of a Dagstuhl Seminar]*, pages 93–124, 2008.

[14] Andrei A. Bulatov and Matthew Valeriote. Recent results on the algebraic approach to the CSP. In *Complexity of Constraints - An Overview of Current Research Themes [Result of a Dagstuhl Seminar]*, pages 68–92, 2008.

[15] Andrei A. Bulatov. Graphs of finite algebras, edges, and connectivity. *CoRR*, abs/1601.07403, 2016.

[16] Andrei A. Bulatov. A dichotomy theorem for nonuniform CSPs. *CoRR*, abs/1703.03021, 2017.

[17] Andrei A. Bulatov. A dichotomy theorem for nonuniform CSPs. In *FOCS*, 2017.

[18] Stanley Burris and H.P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981.

[19] Rina Dechter. *Constraint processing*. Elsevier Morgan Kaufmann, 2003.

[20] Tomas Feder and Moshe Y. Vardi. Monotone monadic SNP and constraint satisfaction. In *STOC*, pages 612–622, 1993.

[21] Tomas Feder and Moshe Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: A study through Datalog and group theory. *SIAM J. of Comput.*, 28:57–104, 1998.

[22] Georg Gratzer. *General lattice theory*, Birkhäuser Verlag, Basel, 1998.

[23] David Hobby and Ralph N. McKenzie. *The structure of finite algebras*, volume 76 of *Contemporary Mathematics*. American Mathematical Society, Providence, R.I., 1988.

[24] Pawel M. Idziak, Petar Marković, Ralph McKenzie, Matthew Valeriote, and Ross Willard. Tractability and learnability arising from algebras with few subpowers. *SIAM J. Comput.*, 39(7):3023–3037, 2010.

[25] Peter Jeavons, David A. Cohen, and Marc Gyssens. Closure properties of constraints. *J. ACM*, 44(4):527–548, 1997.

[26] Keith A. Kearnes and Ágnes Szendrei. Clones of algebras with parallelogram terms. *Internat. J. Algebra Comput.*, 22(1), 2012

[27] Petar Marković. The complexity of CSPs on a 4-element set. Personal communication, 2011.

[28] Petar Marković and Ralph McKenzie. Block algorithm. Personal communication, 2009.

[29] Miklós Maróti. Malcev on top. Manuscript, available at http://www.math.u-szeged.hu/∼ mmaroti/pdf/200x%20Maltsev%20on%20top.pdf, 2011.

[30] Miklós Maróti. Tree on top of Malcev. Manuscript, available at http://www.math.u-szeged.hu/∼ mmaroti/pdf/200x%20Tree%20on%20top%20of%20Maltsev.pdf, 2011.

[31] Jaroslav Nesetril, Mark H. Siggers, and László Zádori. A combinatorial constraint satisfaction problem dichotomy classification conjecture. *Eur. J. Comb.*, 31(1):280–296, 2010.

[32] Ian Payne. A CSP algorithm for some subvarieties of Maltsev products. *A workshop on constraint satisfaction, structure theory and computation in algebra, Boulder CO*, 2016. (oral communication)

[33] Thomas J. Schaefer. The complexity of satisfiability problems. In *STOC*, pages 216–226, 1978.

[34] Dmitriy Zhuk, The proof of the CSP Dichotomy Conjecture for 5-element domain, In *Arbeitstagung Allgemeine Algebra AAA'91 Brno, Czech Republic*, 2016

[35] Dmitriy Zhuk, On the CSP Dichotomy Conjecture, In *Arbeitstagung Allgemeine Algebra AAA'92 Prague, Czech Republic*, page 32, 2016.

[36] Dmitriy Zhuk, The Proof of CSP Dichotomy Conjecture, *CoRR*, abs/1704.01914, 2017.

[37] Dmitriy Zhuk, The Proof of CSP Dichotomy Conjecture, In *FOCS*, 2017.