The complexity of homomorphism and constraint satisfaction problems seen from the other side

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The Homomorphism Problem

$\mathcal{C}$, $\mathcal{D}$ classes of relational structures

$\text{HOM}(\mathcal{C}, \mathcal{D})$ is the following problem

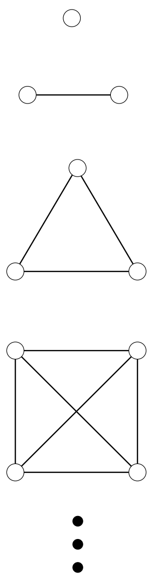
Input:	Structures $A \in \mathcal{C}$, $B \in \mathcal{D}$
Problem:	Decide if there is a homomorphism from A to B

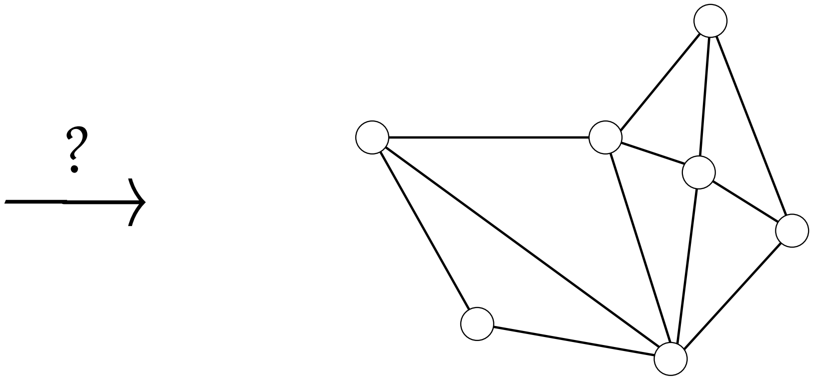
We write $\text{HOM}(\_, \mathcal{D})$ if $\mathcal{C}$ is the class of all structures.

Similarly $\text{HOM}(\mathcal{C}, \_)$.

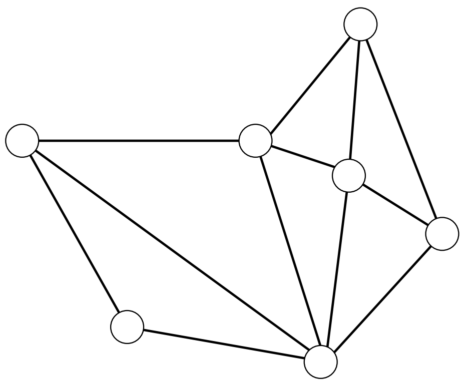
Examples

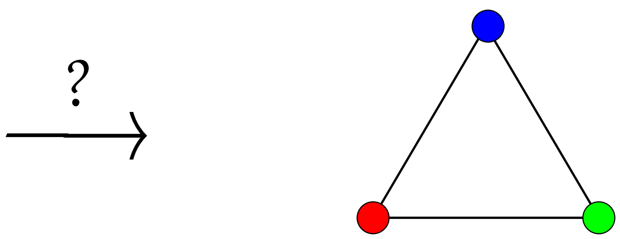
CLIQUE is HOM(Class of all Complete Graphs, _).





3-COLOURABILITY is HOM(_, {Triangle}).





Constraint Satisfaction Problems

Observation (Feder and Vardi 1998)

HOM$(\mathcal{C}, \mathcal{D})$ *is equivalent to the constraint satisfaction problem with instances from* $\mathcal{C}$ *and templates from* $\mathcal{D}$*.*

One Side — the Hell-Nešetřil Theorem

Theorem (Hell and Nešetřil 1990)

Let B *be an (undirected simple) graph. Then* HOM$(_, \{\mathrm{B}\})$ *is in* PTIME *iff* B *is bipartite. Otherwise,* HOM$(_, \{\mathrm{B}\})$ *is* NP*-complete.*

Corollary

Assume that PTIME $\neq$ NP*. Then for every class* $\mathcal{D}$ *of graphs,* HOM$(_, \mathcal{D})$ *is in* PTIME *iff all graphs in* $\mathcal{D}$ *are bipartite.*

Remark

The Hell-Nešetřil Theorem only holds for undirected graphs. No classification result is known for directed graphs or arbitrary relational structures.

The Other Side

Goal

Find a similar classification for problems $\mathrm{HOM}(\mathcal{C}, \_)$.

The Combinatorial Ingredients

Tree-Width

The tree-width of a graph (or relational structure) measures the similarity of the graph (structure) with a tree.

Cores

A core of a graph (or relational structure) G is a minimum subgraph (substructure) w.r.t. inclusion that is a homomorphic image of G.

Fact

The core of a graph is unique up to up to isomorphism.

Tractable Problems

Theorem (Freuder 1990)

Let $\mathcal{C}$ *be a class of structures of bounded tree-width. Then* HOM$(\mathcal{C}, _)$ *is in* PTIME*.*

Theorem (Dalmau, Kolaitis, and Vardi 2002)

Let $\mathcal{C}$ *be a class of structures whose cores have bounded tree-width. Then* HOM$(\mathcal{C}, _)$ *is in* PTIME*.*

Proof uses Ehrenfeucht-Fraisse games for a logic describing homomorphism problems and the fact that winning strategies for these games can be computed in PTIME.

The Main Result

Theorem

Assume that FPT $\neq$ W[1]*.*

Let $\mathcal{C}$ *be a recursively enumerable class of structures of bounded arity.*

Then HOM$(\mathcal{C}, _)$ *is in* PTIME *iff the cores of the structures in* $\mathcal{C}$ *have bounded tree-width.*

Assumptions on the Class $\mathcal{C}$

Theorem

Assume that FPT $\neq$ W[1]*.*
Let $\mathcal{C}$ *be a recursively enumerable class of structures of bounded arity.*
Then HOM$(\mathcal{C}, _)$ *is in* PTIME *iff the cores of the structures in* $\mathcal{C}$ *have bounded tree-width.*

The assumption that $\mathcal{C}$ is recursively enumerable can be omitted if the complexity theoretic assumption FPT $\neq$ W[1] is slightly strengthened to

non-uniform-FPT $\neq$ non-uniform-W[1]

(which is still believed to be true).

Assumptions on the Class $\mathcal{C}$ (cont'd)

Theorem

Assume that FPT $\neq$ W[1].
Let $\mathcal{C}$ *be a recursively enumerable class of structures of* ***bounded arity***.
Then HOM$(\mathcal{C}, _)$ *is in* PTIME *iff the cores of the structures in* $\mathcal{C}$ *have bounded tree-width.*

Bounded arity means that there is an $\mathfrak{r}$ such all relations in all structures in $\mathcal{C}$ are at most $\mathfrak{r}$-ary.

All classes of graphs, directed graphs, or coloured graphs, and all classes of structures of a fixed vocabulary have bounded arity.

The assumption that $\mathcal{C}$ has bounded arity is necessary, the theorem does not hold for classes $\mathcal{C}$ of unbounded arity.

The Complexity Theoretic Assumption

Theorem

Assume that FPT $\neq$ W[1]*.*

Let $\mathcal{C}$ *be a recursively enumerable class of structures of bounded arity.*

Then HOM$(\mathcal{C}, _)$ *is in* PTIME *iff the cores of the structures in* $\mathcal{C}$ *have bounded tree-width.*

FPT $=$ W[1] is equivalent to either of the following three problems being fixed-parameter tractable, i.e., solvable in time $f(k)p(n)$ for some computable function f and polynomial p:

The CLIQUE problem

Input:	Graph G of size n, positive integer k
Problem:	Decide if G has a clique of size k

The Complexity Theoretic Assumption (cont'd)

The k-STEP HALTING problem

Input:	Non-deterministic Turing machine M of size n, positive integer k
Problem:	Decide if M has a k-step halting computation

The WEIGHTED SATISFIABILITY problem

Input:	Boolean formula ϕ in 3-CNF of size n, positive integer k
Problem:	Decide if ϕ has a satisfying assignment setting precisely k variables to TRUE

If FPT $=$ W[1] then 3-SAT is solvable in time $2^{o(n)}$.

The Complexity Theoretic Assumption (cont'd)

The assumption FPT $\neq$ W[1] is necessary:

Proposition

FPT $=$ W[1] *iff there is a recursively enumerable class of graphs* $\mathcal{C}$ *whose cores have unbounded tree-width such that* HOM$(\mathcal{C}, _)$ *is in* PTIME*.*

A Few Words on the Proof

Proof uses graph minor theory and parameterized complexity theory.

Call a recursively enumerable class $\mathcal{C}$ of structures of bounded vocabulary whose cores have unbounded tree-width complicated.

Goal

Prove that for complicated classes $\mathcal{C}$ the problem HOM$(\mathcal{C}, _)$ is W[1]-complete.

Idea

- Let $\mathcal{C}$ be a complicated class and $\mathcal{C}'$ the class of cores of structures in $\mathcal{C}$. Then HOM$(\mathcal{C}, _)$ and HOM$(\mathcal{C}', _)$ are equivalent (w.r.t. their parameterized complexity).
- By Robertson and Seymour's Excluded Grid Theorem, the primal graphs of the structures in $\mathcal{C}'$ contain arbitrarily large grid minors.
- A combinatorial reduction reduces the parameterized CLIQUE problem to the parameterized homomorphism problem for cores with arbitrarily large grid minors.

Is there a Dichotomy?

LOG-CLIQUE is the following problem:

Input:	Graph G with n vertices
Problem:	Decide if G has a clique of size at least $\log n$

Seems unlikely to be either in PTIME or NP-complete.

Proposition

There is a polynomial time computable class $\mathcal{C}$ of graphs such that HOM$(\mathcal{C}, _)$ and LOG-CLIQUE are polynomial time equivalent.

Remark

There is a dichotomy for the parameterized complexity of problems HOM$(\mathcal{C}, _)$ — they are either fixed-parameter tractable or W[1]-complete.

Open Problems

- Classify problems HOM$(\mathcal{C}, _)$ for classes $\mathcal{C}$ of unbounded arity.
- Classify the embedding problems EMB$(\mathcal{C}, _)$
- Classify the corresponding counting problems.