Answer Set Programming

CMPT 411/721

(based on slides by
Torsten Schaub)

Introduction:

Model-Based Problem Solving

Goal: Declarative problem solving

In declarative problem solving:

- Instead of asking: *"How can the problem be solved?"*
- Ask: *"How can the problem be described?"*

☞ Then use a domain-independent solver to compute a solution

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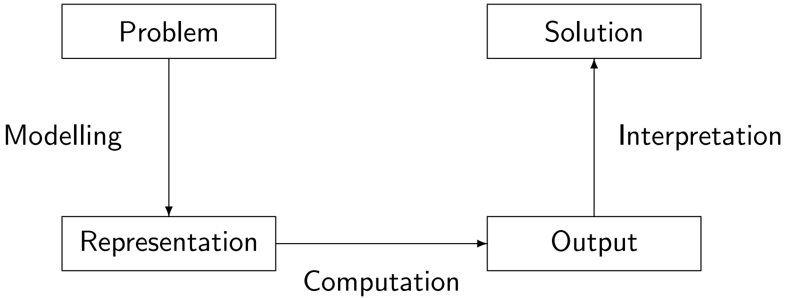
General KR Methodology:



Answer set programming (ASP)

- Has its roots in
  - Knowledge representation and reasoning
    - ☞ In particular nonmonotonic reasoning
  - Deductive databases (particularly Datalog)
  - Constraint solving (in particular, SAT solving)
  - Logic programming (with negation)

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- Allows for solving all search problems within NP (and NP^{NP}) (over finite domains).
- Allows for using powerful off-the-shelf systems (nowadays capable of dealing with millions of variables)

Example: 3–colourability of graphs

C(I)	vertex(1)	$\leftarrow$	edge(1,2)	$\leftarrow$
	vertex(2)	$\leftarrow$	edge(2,3)	$\leftarrow$
	vertex(3)	$\leftarrow$	edge(3,1)	$\leftarrow$
C(P)	coloured(V,r)	$\leftarrow$	not coloured(V,b), not coloured(V,g), vertex(V)	
	coloured(V,b)	$\leftarrow$	not coloured(V,r), not coloured(V,g), vertex(V)	
	coloured(V,g)	$\leftarrow$	not coloured(V,r), not coloured(V,b), vertex(V)	
		$\leftarrow$	edge(V,U), coloured(V,C), coloured(U,C), colour(C)	
Answer set	{ coloured(1,r), coloured(2,b), coloured(3,g), . . . }			

Goal: Find a *minimal* set of literals that *satisfies* the rules.

☞ Such a set of literals is called an *answer set*

Model-Based Problem Solving

Compare:

I Inference-based approach

1. Provide a specification of the problem.
2. A solution is given by a derivation of an appropriate query.

- E.g. resolution in logic, top-down rule-based reasoning, Prolog

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Key Idea: Rules represent *constraints* on the problem.

Applications of ASP

- Combinatorial search problems:
  - auctions, bio-informatics, computer-aided verification, configuration, constraint satisfaction, diagnosis, information integration, planning and scheduling, security analysis, semantic web, wire-routing, zoology and linguistics, ...
- ASP has also been used as a target language into which a high level language can be compiled.
  - E.g.: Action language $\Rightarrow$ ASP

Introduction to ASP

ASP: Idea

- A (normal) rule, r, is of the form

  $$A_0 \leftarrow A_1, \ldots, A_m, \mathit{not}\ A_{m+1}, \ldots, \mathit{not}\ A_n,$$

  - *not* can be read as negation as failure.
  - Variables are treated as standing for all possible instances.

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- Want to determine answer sets of a set of rules, or program.
- An answer set is a minimal set of atoms satisfying the rules.
  - I.e. for rule r above, if X is an answer set, then if $A_1, \ldots, A_m$ are in X and no $A_{m+1}, \ldots, A_n$ is in X then A_0 is in X.

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- E.g. $\{a \leftarrow b, not\ c., \quad b.\}$ has answer set $\{a, b\}$.
  $\{a \leftarrow not\ b., \quad b \leftarrow not\ a.\}$ has answer sets $\{a\}$ and $\{b\}$.

ASP: Atoms and Terms

Atoms

- An atom is the elementary construct for representing knowledge
- An atom in general represents a relation between objects
- Examples: $answer(42)$, $coloured(1, red)$, hot
- An atom can be either true or false

Terms

- Terms are the subatomic components of atoms
- Terms represent objects
    - Examples: 42, red, joe
- Variables are also terms, but are removed from a program by grounding the program

Normal logic programs

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- Notation

$$\begin{array}{rcl} \mathit{head}(r) & = & A_0 \\ \mathit{body}(r) & = & \{A_1, \ldots, A_m, \mathit{not}\ A_{m+1}, \ldots, \mathit{not}\ A_n\} \\ \mathit{body}^+(r) & = & \{A_1, \ldots, A_m\} \\ \mathit{body}^-(r) & = & \{A_{m+1}, \ldots, A_n\} \end{array}$$

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- Notation

$$\begin{aligned} head(r) &= A_0 \\ body(r) &= \{A_1, \ldots, A_m, not\ A_{m+1}, \ldots, not\ A_n\} \\ body^+(r) &= \{A_1, \ldots, A_m\} \\ body^-(r) &= \{A_{m+1}, \ldots, A_n\} \end{aligned}$$

- A program is called positive if $body^-(r) = \emptyset$ for all its rules.
  - ☞ $=$ set of Horn clauses

Examples of Rules

Examples

- a :- b, not c.
- a :- not c, b.
- a.
- a :- b.
- a :- not c.
- ugrad(joe) :- student(joe), not grad(joe).
- ugrad(X) :- student(X), not grad(X).

Notational Conventions

The following notation is used interchangeably in order to stress a particular view:

	if	and	or	negation as failure	classical negation
logic program	$\leftarrow$	,	;	*not*/$\sim$	$\neg$
formula	$\rightarrow$	$\wedge$	$\vee$		$\neg$
source code	`:-`	`,`	`|`	`not`	`-`

Answer Set: Intuitions

- An *answer set* for a program P is a minimal set of atoms X such that, for every rule:

$$A_0 \leftarrow A_1, \ldots, A_m, not\ A_{m+1}, \ldots, not\ A_n,$$

if

$$\{A_1, \ldots, A_m\} \subseteq X$$

and

$$\{A_{m+1}, \ldots, A_n\} \cap X = \emptyset$$

then

$$A_0 \in X.$$

- This is a *nonconstructive* specification.
- Think of rules as specifying *constraints* on an answer set.

Answer sets and models

Classical Logic

- A model in classical logic can be written as a set of atoms X where
  - atoms in X are *true* and
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- The formula $\neg b \rightarrow a$ has models $\{a\}$, $\{b\}$, and $\{a, b\}$,

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ASP

- An answer set X can be regarded as a model where
  - atoms in X are *true* and
  - atoms not in X are *false*.
- Program $\{a \leftarrow \mathit{not}\ b\}$ has answer set $\{a\}$.

☞ The negation-as-failure operator *not* makes a difference!

Answer sets: Basic idea

Consider the set of formulas: $\{q, \; (q \wedge \neg r) \rightarrow p\}$

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Roughly, a set of atoms X is an answer set of a logic program Π if

- X is a (classical) model of Π and
- all atoms in X are justified by some rule in Π

☞ Each atom in X is a fact or is the head of a satisfied rule.

Answer Set: Formal Definition

Positive programs

- A set of atoms X is closed under a positive program Π iff for any $r \in \Pi$: if $body^+(r) \subseteq X$ then $head(r) \in X$.
  - ☞ X corresponds to a model of Π (seen as a formula).

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- The smallest set of atoms which is closed under a positive program Π is denoted by $Cn(\Pi)$.
  - $Cn(\Pi)$ corresponds to the $\subseteq$-smallest model of Π
  - This is the set of consequences obtained by forward chaining.

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Example

$$\{p \leftarrow, \quad q \leftarrow p, \quad r \leftarrow p, q, \quad t \leftarrow s\}$$

has answer set $\{p, q, r\}$.

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Normal programs

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- Given a program Π and a set of atoms X, the reduct, Π^X, of Π relative to X is defined by

$$\Pi^X = \{head(r) \leftarrow body^+(r) \mid r \in \Pi \text{ and } body^-(r) \cap X = \emptyset\}.$$

  - Think of X as being a "guess" of an answer set.
  - The reduct "compiles out" negation as failure, given X.

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Intuition: Every atom in X is justified by an *"applying rule" from Π*

A Closer Look at Π^X

Given a set of atoms X from Π, Π^X is obtained from Π by

1. deleting each rule having a *not* A in its body with $A \in X$ and then
2. deleting all negative atoms of the form *not* A in the bodies of the remaining rules.

- Thus Π^X is Π, but where negative atoms are taken into account.
- Then X is an answer set of Π just if Π^X "generates" X, i.e. $Cn(\Pi^X) = X$.

A first example

$$\Pi = \{\ p \leftarrow p, \quad q \leftarrow not\ p\ \}$$

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$\Pi = \{ p \leftarrow p, \quad q \leftarrow not\ p \}$

X	Π^X	$Cn(\Pi^X)$
$\emptyset$	$p \leftarrow p$ $q \leftarrow$	$\{q\}$
$\{p\}$	$p \leftarrow p$	$\emptyset$
$\{q\}$	$p \leftarrow p$ $q \leftarrow$	$\{q\}$
$\{p, q\}$	$p \leftarrow p$	$\emptyset$

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This program has two answer sets, $\{a, c\}$ and $\{a, d\}$.

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Here are 3 possibilities for X:

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Answer sets: Some properties

- A program may have zero, one, or multiple answer sets.
- If X is an answer set of a logic program Π, then X is a model of Π (seen as formulas of classical logic).
- If X and Y are answer sets of a logic program Π, then $X \not\subset Y$.

Programs with Variables

Let Π be a logic program.

- The *Herbrand Universe* U^{Π} is the set of constants in Π
- The *Herbrand Base* B^{Π} is the set of (variable-free) atoms constructible from U^{Π}
  - ☞ We usually denote this as $\mathcal{A}$, and call it the *alphabet*.

Programs with Variables

- *Ground instances* of $r \in \Pi$:

  Set of variable-free rules obtained by replacing all variables in r by elements from U^{Π}:

  $$ground(r) = \{r\theta \mid \theta : var(r) \rightarrow U^{\Pi}\}$$

  where $var(r)$ stands for the set of all variables occurring in r and θ is a (ground) substitution.

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- *Ground instantiation* of Π:

  $$ground(\Pi) = \{ground(r) \mid r \in \Pi\}$$

An Example

$$\Pi = \{ \;\; r(a,b) \leftarrow, \quad r(b,c) \leftarrow, \quad t(X,Y) \leftarrow r(X,Y) \;\; \}$$

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$$U^{\Pi} = \{a, b, c\}$$

An Example

$$\Pi = \{ \;\; r(a,b) \leftarrow, \quad r(b,c) \leftarrow, \quad t(X,Y) \leftarrow r(X,Y) \; \}$$

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$$ground(\Pi) = \left\{ \begin{array}{lllllll} r(a,b) & \leftarrow & , & & & & \\ r(b,c) & \leftarrow & , & & & & \\ t(a,a) & \leftarrow & r(a,a), & t(b,a) \leftarrow & r(b,a), & t(c,a) \leftarrow & r(c,a), \\ t(a,b) & \leftarrow & r(a,b), & t(b,b) \leftarrow & r(b,b), & t(c,b) \leftarrow & r(c,b), \\ t(a,c) & \leftarrow & r(a,c), & t(b,c) \leftarrow & r(b,c), & t(c,c) \leftarrow & r(c,c) \end{array} \right\}$$

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- *Intelligent Grounding* aims to reduce the ground instantiation.

Answer Sets of Programs with Variables

Let Π be a normal logic program with variables.

We define a set X of (*ground*) atoms as an *answer set* of Π if $Cn(ground(\Pi)^X) = X$.

Programs with Integrity Constraints

Purpose: Integrity constraints eliminate unwanted candidate solutions

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Syntax: An integrity constraint is of the form

$$\leftarrow A_1, \ldots, A_m, not\ A_{m+1}, \ldots, not\ A_n,$$

where $n \geq m \geq 1$, and each A_i $(1 \leq i \leq n)$ is a atom.

Example

$$\leftarrow Edge(X, Y), Col(X, C), Col(Y, C)$$

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Example

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Implementation: For a new symbol x,

map:		$\leftarrow$	$A_1, \ldots, A_m, not\ A_{m+1}, \ldots, not\ A_n$
to:	x	$\leftarrow$	$A_1, \ldots, A_m, not\ A_{m+1}, \ldots, not\ A_n, not\ x$

Computation: Standard Approach

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Global parameters: Logic program Π and its set of atoms $\mathcal{A}$.

- X is a set of atoms known to be true;
- Y is a set of atoms known to be false.
- Initially $X = Y = \emptyset$.

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answerset$_\Pi(X, Y)$:

1. $(X, Y) \leftarrow$ *propagation*$_\Pi(X, Y)$
2. **if** $(X \cap Y) \neq \emptyset$ **then fail**
3. **if** $(X \cup Y) = \mathcal{A}$ **then return**(X)
4. **select** $A \in \mathcal{A} \setminus (X \cup Y)$
5. *answerset*$_\Pi(X \cup \{A\}, Y)$
6. *answerset*$_\Pi(X, Y \cup \{A\})$

Computation: Standard Approach

Comments:

- (X, Y) is supposed to be a 3-valued model such that
  - $X \subseteq Z$ and
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  for an answer set Z of Π.

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  for an answer set Z of Π.
- Key operations:
  - $\mathit{propagation}_{\Pi}(X, Y)$ and
  - "**select** $A \in \mathcal{A} \setminus (X \cup Y)$"
- Worst case complexity: $\mathcal{O}(2^{|\mathcal{A}|})$

☞ More later...