Surfaces

JC Dill

CurvSurf/surfaces.doc Last update:28Feb96 18feb99(cv to pc) 26oct00

Hermite Form

The algebraic form of a bicubic surface can be written as

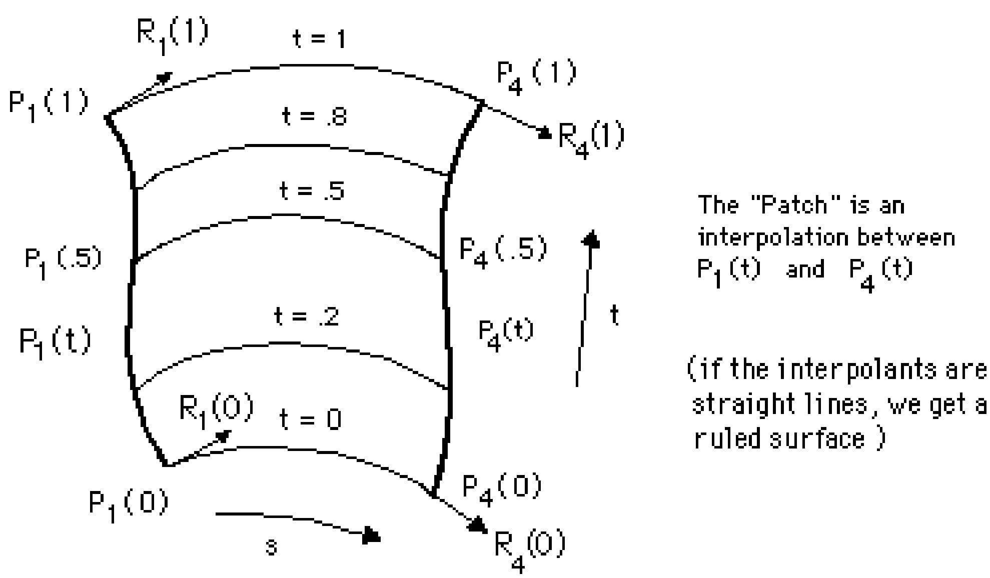
$$
\begin{aligned}
X(s,t) = \; & a_{3,3}s^3t^3 + a_{3,2}s^3t^2 + a_{3,1}s^3t + a_{3,0}s^3 \\
& + a_{2,3}s^2t^3 + a_{2,2}s^2t^2 + a_{2,1}s^2t + a_{2,0}s^2t \\
& + a_{1,3}st^3 + a_{1,2}st^2 + a_{1,1}st + a_{1,0}s \\
& + a_{0,3}t^3 + a_{0,3}t^2 + a_{0,3}t + a_{0,3}
\end{aligned}
$$

Recall the Hermite cubic is $X(s) = S\,M_h\,G_{hx}$. Now if we let G_{hx} be variable we can write

$G_{hx} = G_{hx}(t)$ so that

$$
\begin{aligned}
X(s,t) &= SM_h G_{h_x}(t) \\
&= SM_h \begin{bmatrix} P_1(t) \\ P_4(t) \\ R_1(t) \\ R_4(t) \end{bmatrix}
\end{aligned}
$$

That is, for any specific value of t, e.g. t1, we have new start and end points $P_1(t_1)$ and $P_4(t_1)$. So, at t=0 we use the values $P_1(0)$, $P_4(0)$, $R_1(0)$ and $R_4(0)$.



We want $P_1(t)$, $P_4(t)$, $R_1(t)$ and $R_4(t)$ to be variable. So let's write them in cubic Hermite form

$$P = TM_h G_h \qquad (1)$$

$$P_{1x} = TM_h \begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \\ q_{14} \end{bmatrix} \qquad (2) \qquad\qquad P_{4x} = TM_h \begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \\ q_{24} \end{bmatrix} \qquad (3)$$

$$P_{1x} = TM_h \begin{bmatrix} q_{31} \\ q_{32} \\ q_{33} \\ q_{34} \end{bmatrix} \qquad (4) \qquad\qquad P_{4x} = TM_h \begin{bmatrix} q_{41} \\ q_{42} \\ q_{43} \\ q_{44} \end{bmatrix} \qquad (5)$$

Combining (2), (3), (4) and (5) into a single matrix equation, we get

$$\begin{bmatrix} P_{1x}(t) & P_{4x}(t) & R_{1x}(t) & R_{4x}(t) \end{bmatrix} = TM_h \begin{bmatrix} q_{11} & q_{21} & q_{31} & q_{41} \\ q_{12} & q_{21} & q_{32} & q_{42} \\ q_{13} & q_{21} & q_{33} & q_{43} \\ q_{14} & q_{21} & q_{34} & q_{44} \end{bmatrix}$$

or, using $(A\ B\ C)^T = C^T B^T A^T$, we rewrite so $[P_1\ P_4\ R_1\ R_4]$ is a column vector to match the form of eq. (1):

$$\begin{bmatrix} P_1 \\ P4 \\ R_1 \\ R_4 \end{bmatrix} = TM_h \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix} M_h^T T^T = Q_x M_h^T T^T$$

Substituting into (1) we have:

$$X(s,t) = SM_h Q_x M_h^T T^T$$
$$Y(s,t) = SM_h Q_y M_h^T T^T$$
$$Z(s,t) = SM_h Q_z M_h^T T^T$$

How do we interpret the Q_x, Q_y and Q_z (ie the q_{ij}) geometrically? From eq. (2):

$P_1(0) = (0\ 0\ 0\ 1)\ M_h\ (q_{11}\ q_{12}\ q_{13}\ q_{14})^T$
$= (1\ 0\ 0\ 0)(q_{11}\ q_{12}\ q_{13}\ q_{14})^T$ after the matrix multiply,
$= q_{11}$
= the start of $P_1(t)$.

But by (1), $X(s,t) = SM_h[P_1(t)\ P_4(t)\ R_1(t)\ R_4(t)]^T$

So $X(0,t) = (1\ 0\ 0\ 0)\ [P_1(t)\ P_4(t)\ R_1(t)\ R_4(t)\ ]^T$
$= P_1(t)$

Thus, $X(0,0) = P_1(0)$

and so $q_{11} = X(0,0) = X_{00}$.

Similarly, $P_1(1) = (1\ 1\ 1\ 1)\ M_h\ [q_{11}\ q_{11}\ q_{11}\ q_{11}]^T$
$= (0\ 1\ 0\ 0)\ [q_{11}\ q_{11}\ q_{11}\ q_{11}]^T$ after matrix multiply,
$= q_{12}$
= end of $P_1(t)$.

But by (1),

$X(0,t) = (1\ 0\ 0\ 0)\ [P_1(t)\ P_4(t)\ R_1(t)\ R_4(t)\ ]^T$
$= P_1(t)$

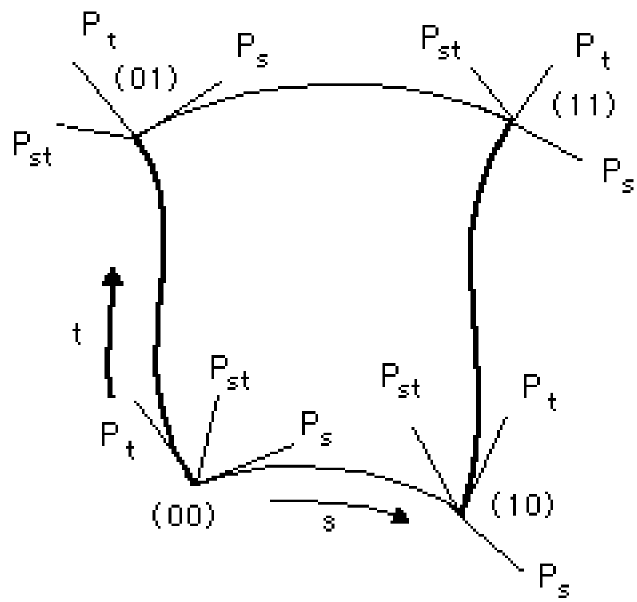
so that $X(0,1) = P_1(1)$

and $q_{12} = X(0,1) = X_{01}$.

Repeating this for the other q_{ij}, we get

$$Q = \begin{bmatrix} X_{00} & X_{00} & X_{t00} & X_{t01} \\ X_{00} & X_{00} & X_{t10} & X_{t11} \\ X_{s00} & X_{s01} & X_{st00} & X_{st01} \\ X_{s10} & X_{s11} & X_{st10} & X_{st11} \end{bmatrix} = \begin{bmatrix} \text{position} & \text{slope} \\ \text{of corners} & \text{along } t \\ \text{slope} & \\ \text{along } s & \text{"twist"} \end{bmatrix}$$

where $X_t = \dfrac{\partial X}{\partial t},\ X_s = \dfrac{\partial X}{\partial s},\quad X_{st} = \dfrac{\partial^2 X}{\partial s \partial t}$

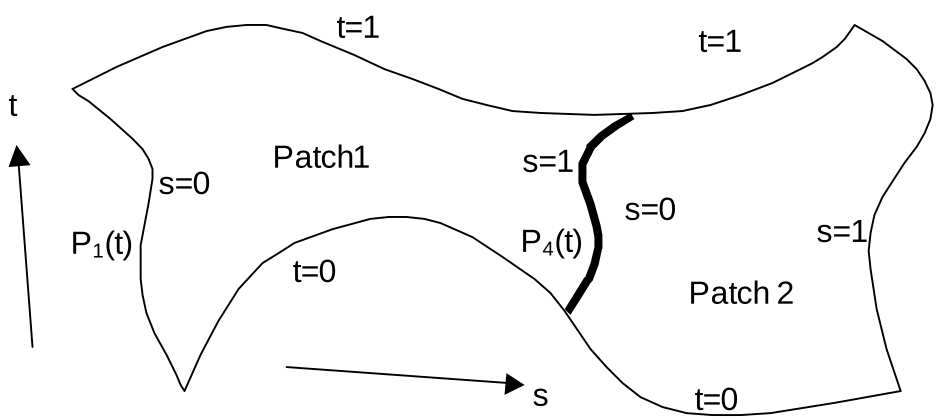

P t
(01)
P s
P st
P st
P t
(11)
P s
t
P st
P t
P s
(00)
s
P st
P t
(10)
P s

Joining Patches

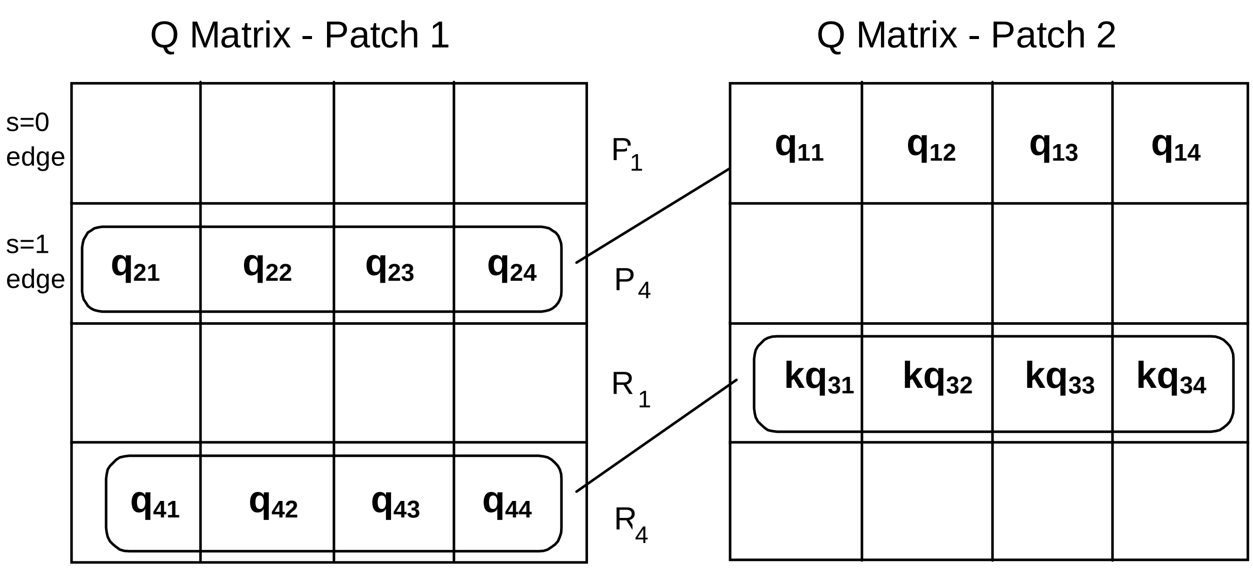
To create a complete surface, we often (almost always for real surfaces) need to use several patches. This raises the question of how to join patches. An Hermite cubic has $C^{(0)}$ -postional- and $G^{(1)}$ -slope- continuity. We want $G^{(1)}$ continuity from patch to patch for an Hermite bicubic. To get this, we need:

- curves along common edge to be the same
- tangent vectors across the common edge to be in the same direction.

Consider the following diagram showing two adjacent patches:



To match position and tangent, consider the Q matrices for the two patches:



Empty cells are unconstrained and can have arbitrary values.

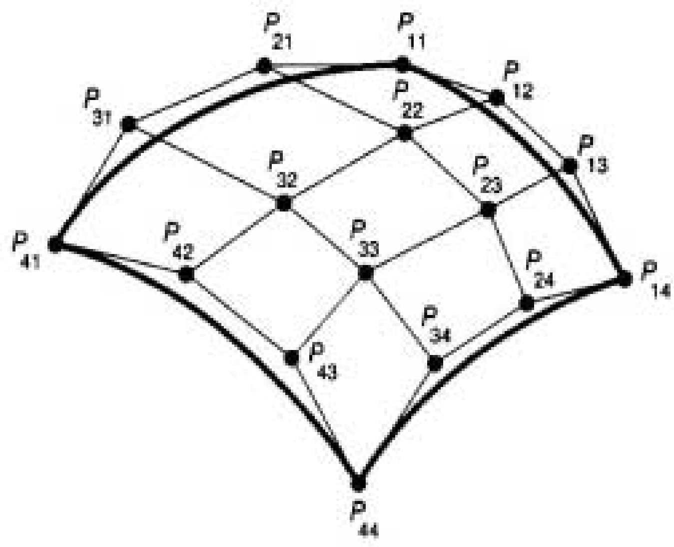
Q matrix constraints for $C^{(0)}$ and $G^{(1)}$

Bezier Patches

The formulation for Bezier patches is similar:

$$\boldsymbol{X}(s,t) = \boldsymbol{S}\,\boldsymbol{M_b}\,\boldsymbol{P_x}\,\boldsymbol{M_b}^T\,\boldsymbol{T}^T$$

Instead of the mix of position, slope and twist of the Q matrix for Hermite surfaces, P has 16 control points, four for each of four curves.



16 Control Points for Bezier Bicubic Patch

We can also write this, for a general n x m patch, as

$$\mathbf{P}(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{m}\mathbf{P}_{i,j}B_{i,n}(s)B_{j,m}(t)$$

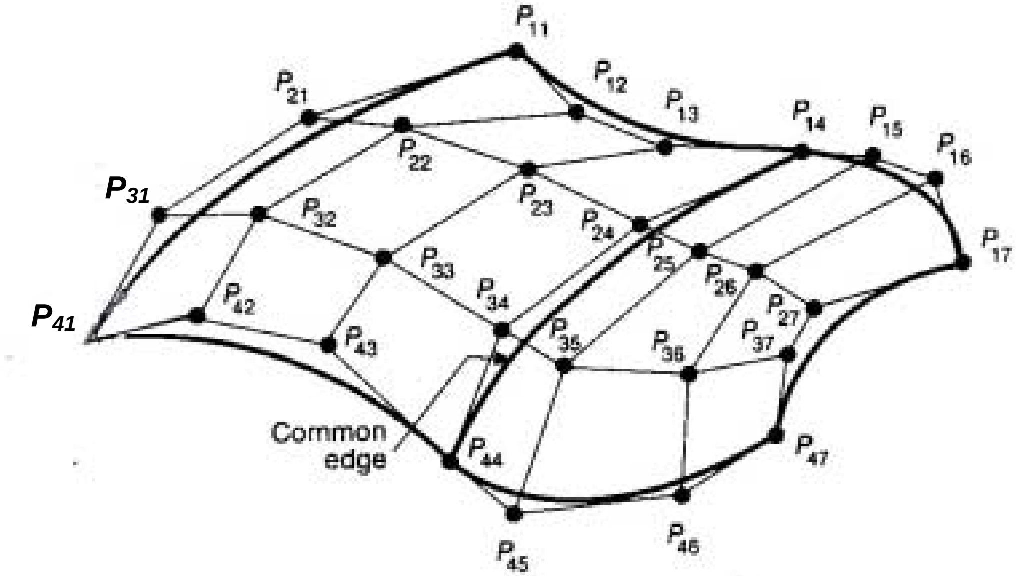
Problems:

- same problems as Bezier Curves
- degree is dependent on number of control points
- no local modification property.

Joining Two Bezier Patches

To join Bezier bicubic patches with $G^{(1)}$ continuity, we need

- common control points at shared edge
- control points on either side colinear (with constant ratios of lengths of colinear segments)



Two Bezier Patches Joined along Edge P_{14},P_{24}, P_{34} and P_{44}.

B-Spline Surface / NURB Surface

B-Spline:

$$\mathbf{P}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}\mathbf{P}_{i,j}N_{i,k}(u)N_{j,l}(v)$$

- $N_{i,k}$ and $N_{j,l}$ are same blending functions as used for B-Spline curves.
- B-Spline reduces to a Bezier curve if orders k and l equal (n+1) and (m+1) respectively. Usual to use order 4 for blending functions to represent surface of order 3
- local modification property
- defined by knot values

NURBS:

$$\mathbf{P}(u,v)=\frac{\sum_{i=0}^{n}\sum_{j=0}^{m}h_{i,j}\mathbf{P}_{i,j}N_{i,k}(u)N_{j,l}(v)}{\sum_{i=0}^{n}\sum_{j=0}^{m}h_{i,j}N_{i,k}(u)N_{j,l}(v)}$$

(introduce homogeneous coordinates for the control points)

- same as B-Spline but with homogeneous coordinates
- when $h_{i,j}$ are equal to 1, denominator is 1 and equation is the same as B-Spline surface
- Big advantages: can represent quadric surfaces exactly: spherical, hyperboloic, paraboloidal, etc.