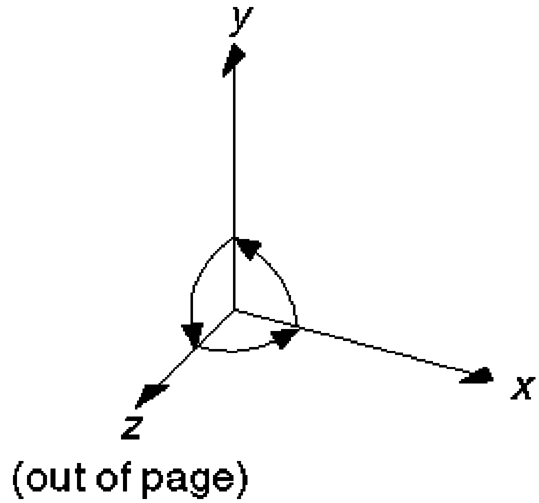
Coordinate Systems

Right handed coordinate system

- positive rotations are counter-clockwise

| Axis of rotation | Direction of positive rotation |
|---|---|
| x | y to z |
| y | z to x |
| z | x to y |

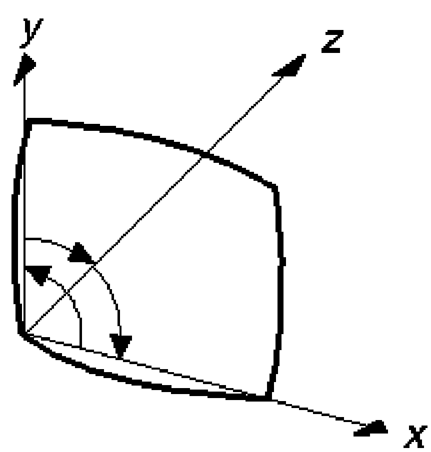



- glOrtho(-1.0, 1.0, -1.0, 1.0, 1.0, -1.0)
- glOrtho(left, right, bottom, top, near, far)

Left handed coordinate system

- positive rotations are clockwise

| Axis of rotation | Direction of positive rotation |
|---|---|
| x | y to z |
| y | z to x |
| z | x to y |




- glOrtho(-1.0, 1.0, -1.0, 1.0, -1.0, 1.0)

## 3D Transformations

- 2D Transformations used 3x3 T matrix
- 3D Transformations will use 4x4 T matrices

- Homogeneous Coordinates

  [x y z W] <-> $\begin{bmatrix} \frac{x}{W} & \frac{y}{W} & \frac{z}{W} & 1 \end{bmatrix}$ and $W \neq 0$

  $$T = \begin{bmatrix} a & b & c & l \\ d & e & f & m \\ h & i & j & n \\ p & q & r & s \end{bmatrix}$$

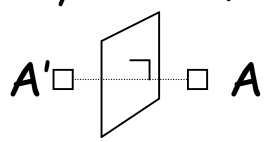
- Translation

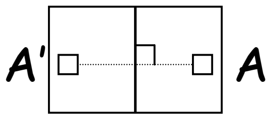
$$T = \begin{bmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x+tx \\ y+ty \\ z+tz \\ 1 \end{bmatrix}$$

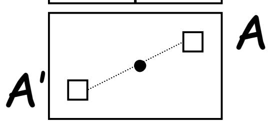
- Scaling

$$S = \begin{bmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x(Sx) \\ y(Sy) \\ z(Sz) \\ 1 \end{bmatrix}$$

Reflection, can reflect with respect to a

1. plane 

2. line 

3. point 


Reflections about the coordinate planes

$$\underset{0zy}{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{0xz}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{0xy}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Reflections about the coordinate axes

$$\underset{0z}{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{0y}{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{0x}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Reflections around a point (the origin)

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Shearing

Along x axis:

$$SH_{yz}= \begin{bmatrix} 1 & 0 & 0 & 0 \\ shy & 1 & 0 & 0 \\ shz & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ x(Shy)+y \\ x(Shz)+z \\ 1 \end{bmatrix}$$

Along y axis:

$$SH_{xz}= \begin{bmatrix} 1 & shx & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & shz & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} y(Shx)+x \\ y \\ y(Shz)+z \\ 1 \end{bmatrix}$$

Along z axis:

$$SH_{xy}= \begin{bmatrix} 1 & 0 & shx & 0 \\ 0 & 1 & shy & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} z(Shx)+x \\ z(Shy)+y \\ z \\ 1 \end{bmatrix}$$

- Rotations
  - About x-axis (with respect to 0x)

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} x \\ y(\cos\theta) - z(\sin\theta) \\ y(\sin\theta) + z(\cos\theta) \\ 1 \end{bmatrix}$$

  - About y-axis (with respect to 0y)

$$R_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} x(\cos\varphi) + z(\sin\varphi) \\ y \\ -x(\sin\varphi) + z(\cos\varphi) \\ 1 \end{bmatrix}$$

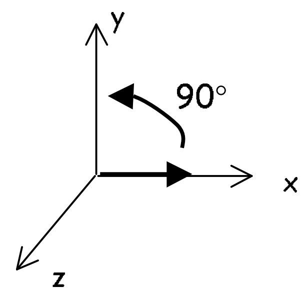
  - About z-axis (with respect to 0z)

$$R_z = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} x(\cos\varphi) - y(\sin\varphi) \\ x(\sin\varphi) + y(\cos\varphi) \\ z \\ 1 \end{bmatrix}$$

Example: rotate vector in x by 90° about z-axis

(remember, positive rotation about z should transform x axis to y axis)

$$T = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$



## Transforming points, lines and planes

Points: what we've been doing so far

Lines: defined by endpoints; just transform the endpoints and join the transformed points

Planes: trickier; if defined by three points, can transform points, but often defined by a plane equation:

$Ax + By + Cz + D = 0$

- how do we transform the plane equation coefficients?

## Transforming the Plane Equation

- first, represent the plane as a column vector of coefficients:

$$N = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

- the plane can be defined by all points

$$P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \text{such that } N \bullet P = 0$$

(dot products gives us the plane equation: $Ax+By+Cz+D=0$)

- now, in matrix form, we use the row vector of coefficients and a column vector for P:

  $N^T \bullet P = 0$

To transform by matrix M:

- to maintain $N^T P = 0$, we must also transform N by some matrix Q:

  $(Q \bullet N)^T \bullet M \bullet P = 0$

  $N^T \bullet Q^T \bullet M \bullet P = 0$

  $N^T \bullet (Q^T \bullet M) \bullet P = 0$

- this will be true if $(Q^T \bullet M)$ is kI (multiple of the identity matrix)
- therefore, if k=1, $Q = (M^{-1})^T$

Example: $x + y + z - 1 = 0$; rotate 90° about z-axis

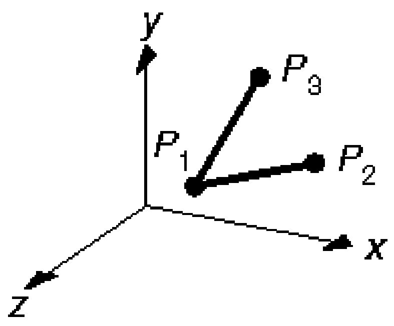
$$R_{z(\varphi)} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_z(90) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(M^{-1})^T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = Q$$

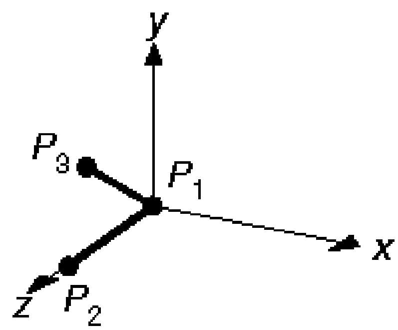
$$QN = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \text{ or } -x + y + z - 1 = 0$$

Composition of 3D Transformations
(plane defined by 3-points)

Example: 3 points defining an arbitrary plane
Want to align plane to 0yz principal plane, without changing length of connecting segments.



(a) Initial position

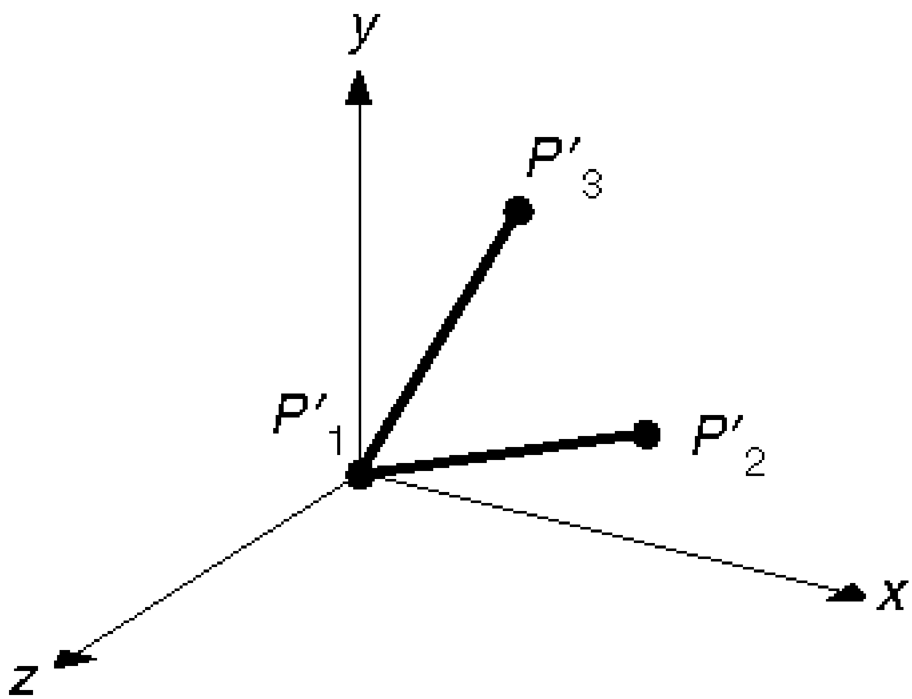


(b) Final position

Break it down into simpler subproblems:

1. Translate $P_1$ to the origin
2. Rotate about the y-axis such that $P_1P_2$ lies in the (y,z) plane
3. Rotate about the x-axis such that $P_1P_2$ lies on the z-axis
4. Rotate about the z-axis such that $P_1P_3$ lies in the (y,z) plane

Step 1. Translate $P_1$ to the origin

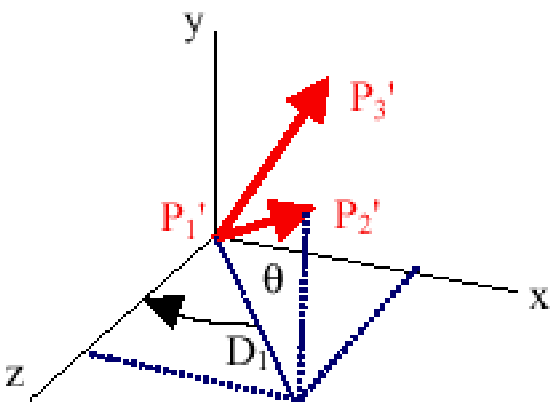


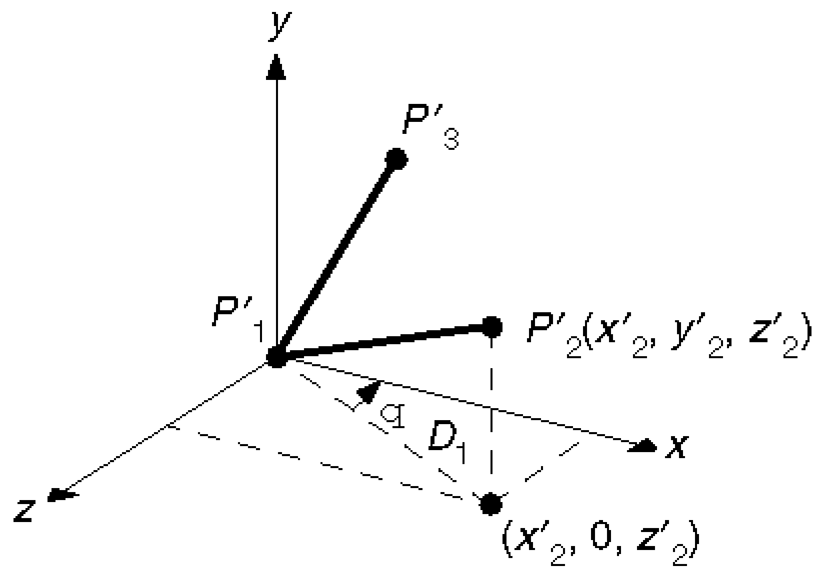

$$T=\begin{bmatrix} 1 & 0 & 0 & -x1 \\ 0 & 1 & 0 & -y1 \\ 0 & 0 & 1 & -z1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply to $P_1$, $P_2$, and $P_3$ gives:

$$P'_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad P'_2 = \begin{bmatrix} x2-x1 \\ y2-y1 \\ z2-z1 \\ 1 \end{bmatrix} \quad P'_3 = \begin{bmatrix} x3-x1 \\ y3-y1 \\ z3-z1 \\ 1 \end{bmatrix}$$

2. Rotate about y-axis (so $P_1P_2$ lies in (x,z) plane)




Rotation is $-(90-\theta) = \theta-90$ (why?)

$$Ry(\theta-90)=\begin{bmatrix} \cos(\theta-90) & 0 & \sin(\theta-90) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta-90) & 0 & \cos(\theta-90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos(\theta-90) = \cos(\theta)\cos(-90) + \sin(\theta)\sin(90)$$
$$= \cos(\theta)\bullet(0) + \sin(\theta)\bullet(1)$$
$$= \sin(\theta) = \frac{Z'2}{D1} = \frac{Z2-Z1}{D1}$$

$$\sin(\theta-90) = \sin(\theta)\cos(-90) - \cos(\theta)\sin(90)$$
$$= \sin(\theta)\bullet(0) - \cos(\theta)\bullet(1)$$
$$= -\cos(\theta) = \frac{-X'2}{D1} = \frac{-(X2-X1)}{D1}$$

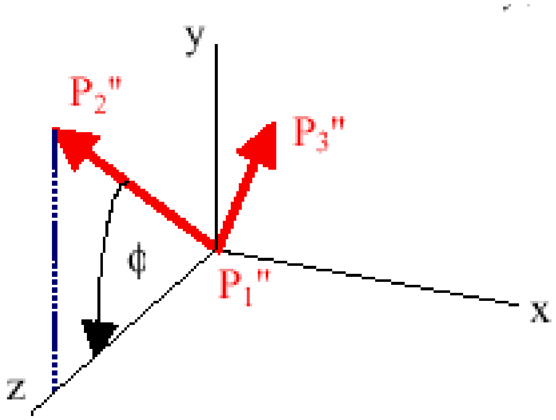
$$D1\sqrt{=(z'2)^2+(x'2)^2}$$

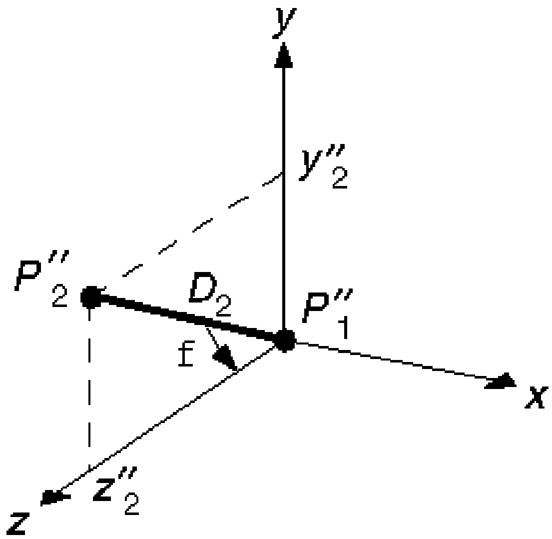
$$Ry=\begin{bmatrix} \frac{z'2}{D1} & 0 & \frac{-x'2}{D1} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x'2}{D1} & 0 & \frac{z'2}{D1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P''_2 = RyP'_2 = \begin{bmatrix} \frac{z'2}{D1} & 0 & \frac{-x'2}{D1} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x'2}{D1} & 0 & \frac{z'2}{D1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'2 \\ y'2 \\ z'2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{z'2x'2}{D1} - \frac{z'2x'2}{D1} \\ y'2 \\ \frac{x'2x'2}{D1} + \frac{z'2z'2}{D1} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ y'2 \\ \frac{D1^2}{D1} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ y2-y1 \\ D1 \\ 1 \end{bmatrix}$$

## 3. Rotate about x-axis (such that $P_1P_2$ lies on z-axis)




$$Rx(\phi)=\begin{bmatrix}\cos(\phi) & -\sin(\phi) & 0 & 0\\ \sin(\phi) & \cos(\phi) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\cos(\phi) = \frac{z''2}{D2} \qquad \sin(\phi) = \frac{y''2}{D2}$$

$$D2 = |P''_1 P''_2| = |P_1P_2| = \sqrt{(x2 - x1)^2 + (y2 - y1)^2 + (z2 - z1)^2}$$

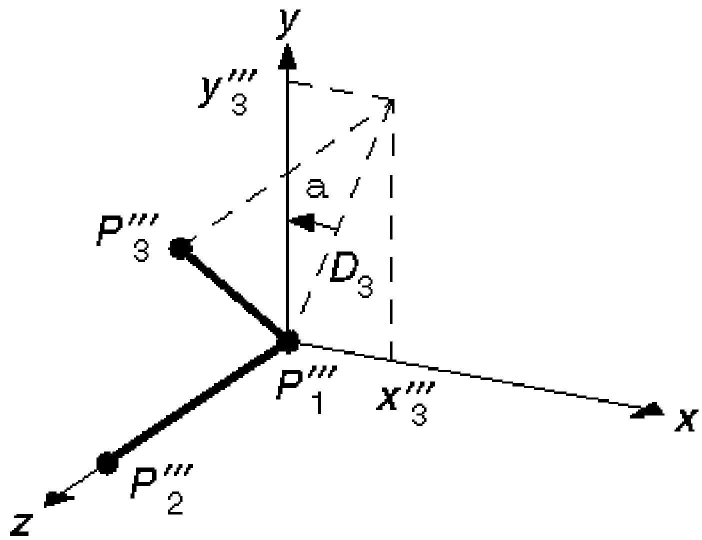
$$P'''2 = Rx(\phi)\ P''2$$

$$= Rx(\phi)\ Ry(\theta-90)\ P'2$$

$$= Rx(\phi)\ Ry(\theta-90)\ T\ P2$$

$$= \begin{bmatrix}\frac{z''2}{D2} & \frac{-y''2}{D2} & 0 & 0\\ \frac{y''2}{D2} & \frac{z''2}{D2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ y'2\\ D1\\ 1\end{bmatrix} = \begin{bmatrix}\frac{-y''2y'2}{D2}\\ \frac{z''2y'2}{D2}\\ D1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ D1\\ 1\end{bmatrix}$$

4. Rotate about z-axis (such that $P_1P_3$ lies in (y,z) plane)




Rotation is $\alpha$

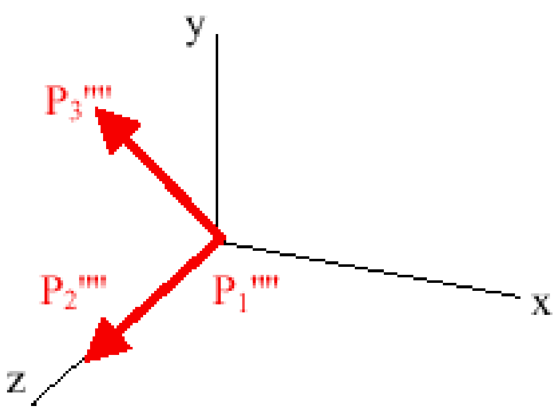
$$Rz(\alpha)=\begin{bmatrix} \cos(\alpha) & 0 & \sin(\alpha) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\alpha) & 0 & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\cos(\alpha) = \frac{y''3}{D3}$ $\sin(\alpha)$ $\frac{x''2}{D3}$ =

$$D3 = \sqrt{(x''3)^2 + (y''3)^2}$$

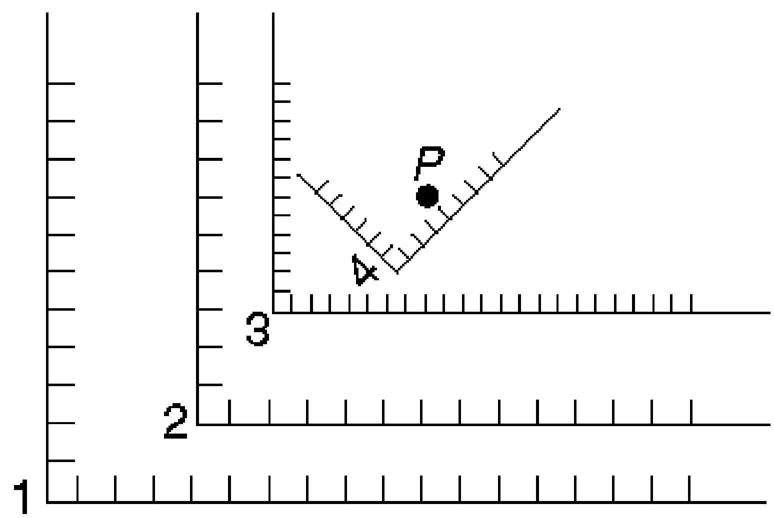
# Composite Matrix

$Rz(\alpha) \cdot Rx(\phi) \cdot Ry(\theta\text{-}90) \cdot T(-x1, -y1, -z1) = R \cdot T$



Transformations as a Change in Coordinate Systems

- We have been moving (transforming) points from one position to another *within the same coordinate system*
- It is also helpful to change coordinate system so each object may have its own coordinate system




Let $M_{i->j}$ be the transformation that converts points in coordinate system i to coordinate system j

- Let $P_i$ denote points in coordinate system i
- So,

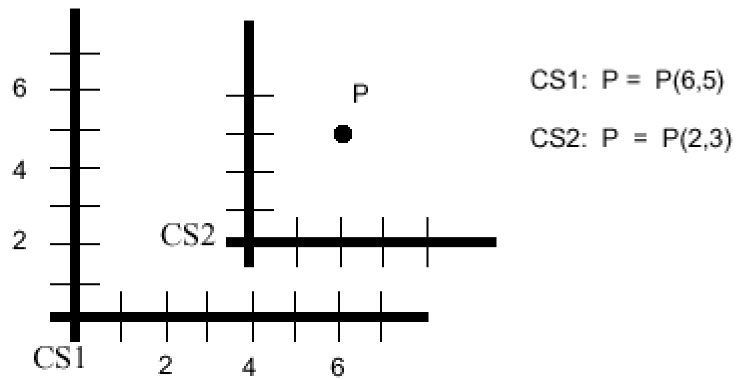
$P_j = M_{i->j} P_i$

and if $P_k = M_{j->k} P_j$

then $P_k = M_{i->k} P_i$

Example #1

- Given coordinates of P(6,5) in coordinate system 1 (CS1), find the coordinates of P in CS2.




Notation:

(a) $P^{(i)}$ = point in coordinate system i

(b) $M_{1\text{->}2}$ coverts representation of point in CS1 to representation of point in CS2

(c) Alternate interpretation:
$M_{1\text{->}2}$ transforms axes of (2) into axes of (1)

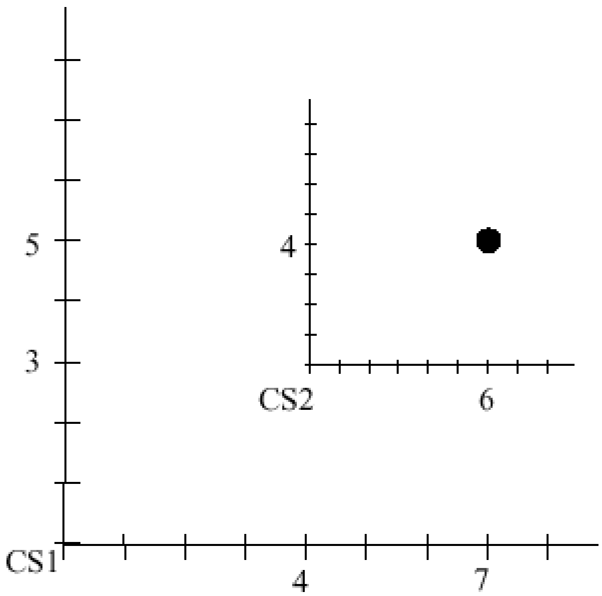
$M_{1\text{->}2}$ = T(-4, -2)

$P^{(1)} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$ $P^{(2)} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

$$P^{(2)} = M_{1\text{->}2} P^{(1)} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Example #2

Similar to previous example but here the “scaling” changes as well as the location



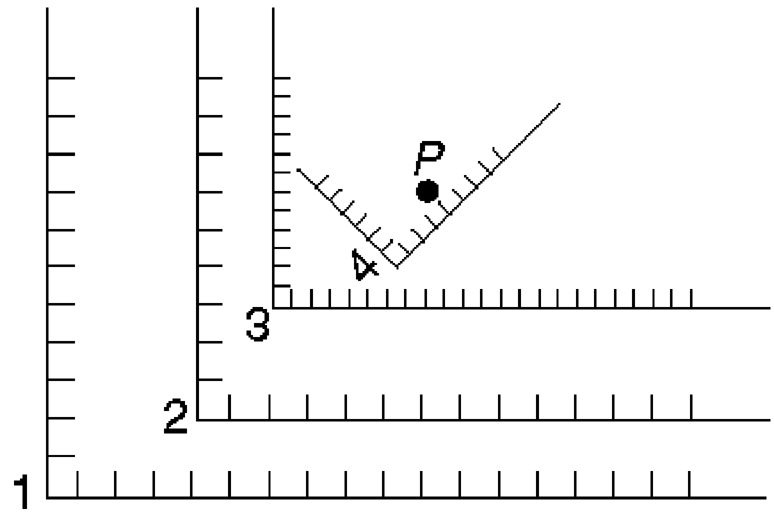

M1->2 = S(2,2) T(-4, -3)

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -8 \\ 0 & 2 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

P(2) = M1->2 P(1)

$$= \begin{bmatrix} 2 & 0 & -8 \\ 0 & 2 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

# Example #3




- has 4 coordinate systems
  - $M_{2\to 1}$ = Translate(4,2)
  - $M_{3\to 2}$ = Translate (2,3)Scale (0.5, 0.5)
  - $M_{4\to 3}$ = Translate (6.7, 1.8)Rotate(-45°)
  - $M_{4\to 1}$ = $M_{2\to 1}\ M_{3\to 2}\ M_{4\to 3}$
    =T(4,2)T(2,3)S(0.5,0.5)T(6.7,1.8)R(-45)

- Note that the reverse is possible and that

  $M_{i\text{->}j} = M_{j\text{->}i}^{-1}$

- Also, in changing left to right-handed CS and right to left…

$$M_{R\text{->}L} = M_{L\text{->}R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$