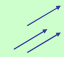
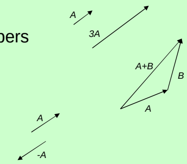
2D Transformations

Vector, Affine & Euclidean Space
Geometrical Interpretations
Basic Transformations
Homogeneous Transformations
Window to Viewport Transformations

Vector Space

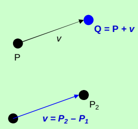


- Any quantity with **Direction** and **Magnitude** (e.g. velocity, force)
- Operations:
  - Multiplication by real numbers
  - Addition (head-to-tail rule)
  - Inverse
  - Zero Vector (0)



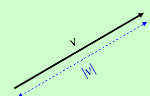
Affine Space

- Extension of vector space
- Includes **Points**: Locations in the space
- Operations:
  - Point Vector addition:
  - Point Point subtraction:

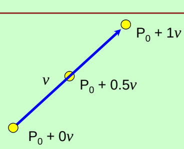


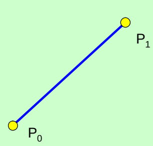
Euclidean Space

- Extension of vector space
- Includes real numbers: size/distance



Geometrical Interpretations – Line Segment





Parametric form of line

$P(a) = P_0 + av$ 0 = a = 1

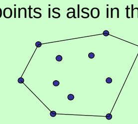
$v = P_1 - P_0$

$P(a) = P_0 + a(P_1 - P_0)$

$P(a) = aP_1 + (1 - a)P_0$

Geometric Interpretations - Convex Hull

Convex Hull of a set of points: Any point on the line connecting two points is also in the Hull

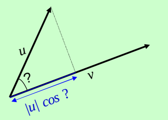


$P = a_1P_1 + a_2P_2 + a_3P_3 + \dots + a_nP_n$ $a_1 + a_2 + a_3 + \dots + a_n = 1$

Can you visually examine it?!

Geometrical Interpretations - Projection

- Dot Product: *u . v*
- *u . v* = | *u*| |*v*| *cos* ?
- *u.u* = |*u*| |*u*|



Orthogonal Projection of *u* onto *v*

Enough Mathematics?!

Coordinate System

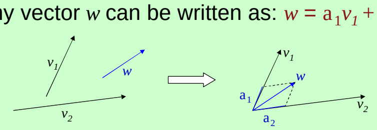
Dfn: vectors *u* and *v* are *linearly independent*

iff ∀a∈R *u* ? a*v*

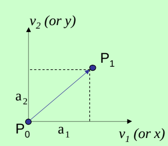
(i.e. not in the same direction)

if v_1 and v_2 be two linearly independent vectors in a 2D vector space,

any vector *w* can be written as: $w = a_1 v_1 + a_2 v_2$



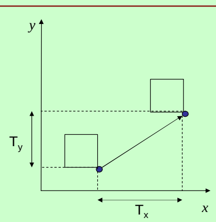
Coordinate System (2)



$P_1 = P_0 + (a_1 v_1 + a_2 v_2)$

$P_1 = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Basic Transformations - Translation

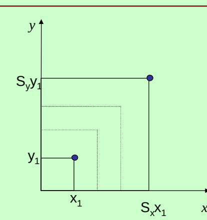


$x_2 = T_x + x_1$

$y_2 = T_y + y_1$

$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

Basic Transformations - Scaling

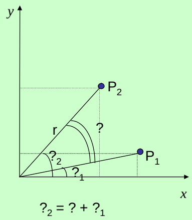


$x_2 = S_x x_1$

$y_2 = S_y y_1$

$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

Basic Transformations - Rotation



$?_2 = ? + ?_1$

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} r \cos ?_1 \\ r \sin ?_1 \end{pmatrix}$$

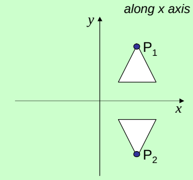
$$P_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} r \cos ?_2 \\ r \sin ?_2 \end{pmatrix}$$

$$= \begin{pmatrix} r \cos ? \cos ?_1 - r\sin ? \sin ?_1 \\ r \sin ? \cos ?_1 + r\cos ? \sin ?_1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} \cos ? & -\sin ? \\ \sin ? & \cos ? \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Basic Transformations - Reflection

along x axis (vertical)

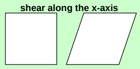


$x_2 = x_1$
$y_2 = -1 \times y_1$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Basic Transformations - Shear

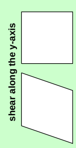
shear along the x-axis



$x_2 = x_1 + a\, y_1$
$y_2 = y_1$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

shear along the y-axis



$x_2 = x_1$
$y_2 = y_1 + ßx_1$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ ß & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

So In General …

if:

$x_2 = a_1x_1 + a_2y_1$
$y_2 = a_3x_1 + a_4y_1$

$$\Rightarrow \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

But, what if we have an independent factor (constant)?

Translation: $x_2 = x_1 + C$ (= x_1+ $0y_1$ + C)

independent to x and y

Homogeneous Coordinates

1- We can express translations by addition rather than multiplication
   Problem: All basic transforms can be expressed by multiplication except translation.

2- Homogeneous Coordinates: Adding a third coordinate *W* to points : (x, y, *W*)
   $(x_1, y_1, W_1) = (x_2, y_2, W_2) \Leftrightarrow \exists\, t : x_2 = tx_1,\ y_2 = ty_1,\ W_2 = tW_1$
   e.g. (3, 5, 2) = (6, 10, 4)

   the Cartesian coordinates: (x/*W* , y/*W*, 1)

Homogeneous Coordinates (2)

$x_2 = x_1 + 0 \times y_1 + T_x \times 1$

$y_2 = 0 \times x_1 + y_1 + T_y \times 1$

$W = 1$

$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \quad \text{translation}$$

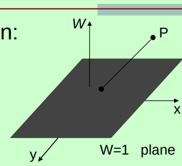
$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \quad \text{scaling}$$

$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos? & -\sin? & 0 \\ \sin? & \cos? & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \quad \text{rotation}$$

Homogeneous Coordinates (3)

The geometrical explanation:

All triples of the form (**t**x, **t**y, **t***W*)

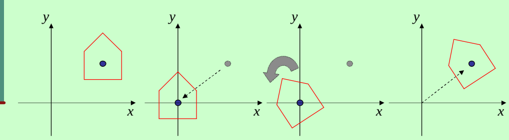


- (0, 0, 0): is not allowed
- (x, y, 0): points at infinity

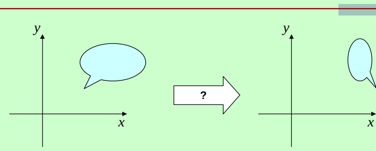
Composition of Transformations

Example: Rotation about some arbitrary point P_1 rather than the origin:

1. Translate such that P1 is at the origin
2. Rotate
3. Translate such that the point at the origin returns to P1



Composition of Transformations (2)



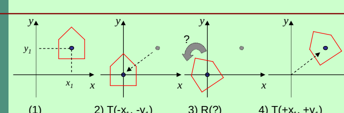
Translate to origin

Reflect horizontally (along y axis : $x_2 = -x_1$)

Scale X

Translate back

Composition of Transformations (3)



$P_2 = T(-x_1, -y_1).P_1$

$P_3 = R(?).P_2$

$P_4 = T(x_1, y_1).P_3$

$\Rightarrow$ $P' = P_4 = T(x_1, y_1).R(?).T(-x_1, -y_1).P_1$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos ? & -\sin ? & 0 \\ \sin ? & \cos ? & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

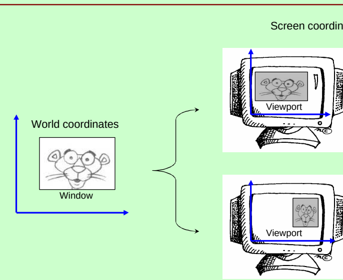
Composition of Transformations (4)

P' = P4 = T(x1, y1).R(?).T(-x1, -y1).P1

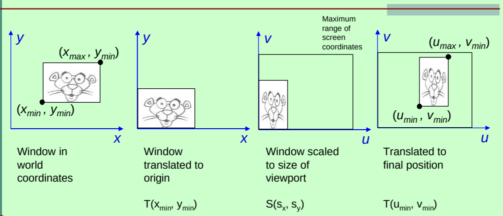
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos ? & -\sin ? & 0 \\ \sin ? & \cos ? & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos ? & -\sin ? & x_1(1-\cos ?) + y_1 \sin ? \\ \sin ? & \cos ? & y_1(1-\cos ?) - x_1 \sin ? \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Window-To-Viewport Transformation



Window-To-Viewport Transformation (2)



$$s_x = (u_{max} - u_{min}) / (x_{max} - x_{min})$$
$$s_y = (v_{max} - v_{min}) / (y_{max} - y_{min})$$

Window-To-Viewport Transformation (3)

$$M_{wv} = T(u_{min}, v_{min}) \cdot S(s_x, s_y) \cdot T(-x_{min}, -y_{min})$$

$$= \begin{pmatrix} 1 & 0 & u_{min} \\ 0 & 1 & v_{min} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_{min} \\ 0 & 1 & -y_{min} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} sx & 0 & -x_{min} \cdot s_x + u_{min} \\ 0 & s_y & -y_{min} \cdot s_y + v_{min} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad P_{viewport} = M_{wv} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} (x - x_{min}).s_x + u_{min} \\ (y - y_{min}).s_y + v_{min} \\ 1 \end{pmatrix}$$

Types of Transformations

$$\begin{pmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- Rigid-body Transforms:
  - preserves angles and lengths
  - object transformed is not distorted
  - translation? scale? rotate? shear?
  - product of rigid-body transformations?
  - upper 2x2 matrix must be orthogonal ($A.A^T = I$)
- Euclidean
  - similarity: preserves angles
  - translation, rotation, uniform scale
- Isometric
  - preserves distance
  - reflection, rigid-body

Types of Transformations (2)

- Affine Transforms
  - preserve parallel lines
  - may not preserve length nor angles
  - translation, scale, rotate, shear
  - product of affine transformations?
- Nonlinear
  - lines become curves
  - twists, bends warps, morphs