Constraint Satisfaction Problems

CHAPTER 6

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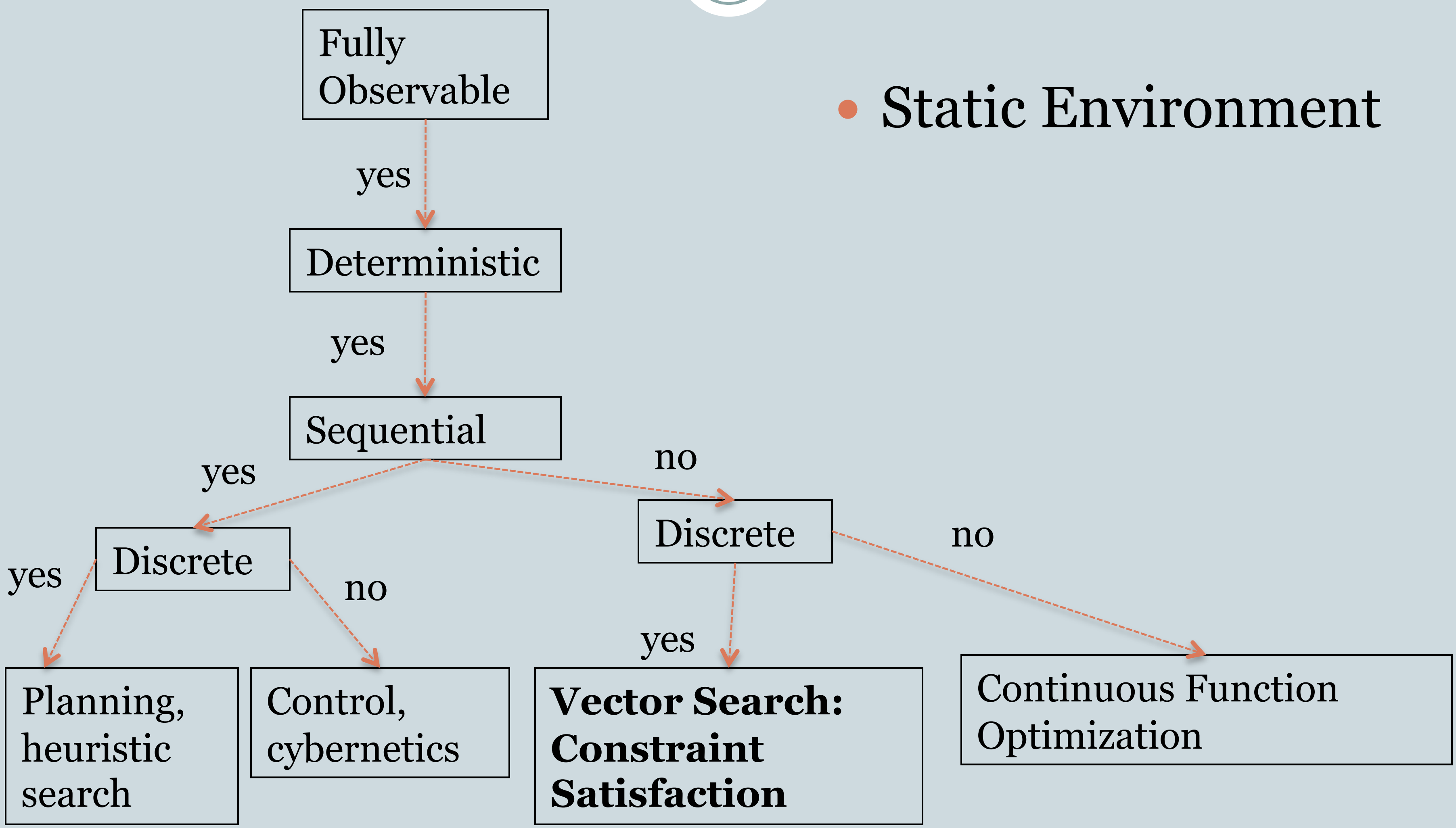
Outline

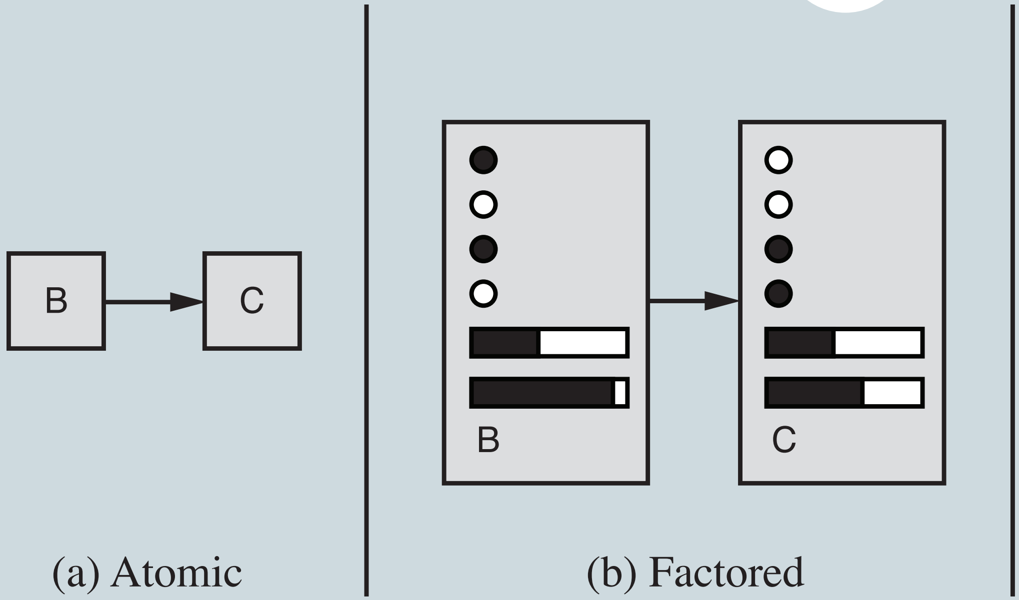
- CSP examples
- Backtracking search for CSPs
- Problem structure and problem decomposition
- Local search for CSPs

Environment Type Discussed In this Lecture

- Static Environment



Agent Architecture Discussed In this Lecture



(a) Atomic (b) Factored

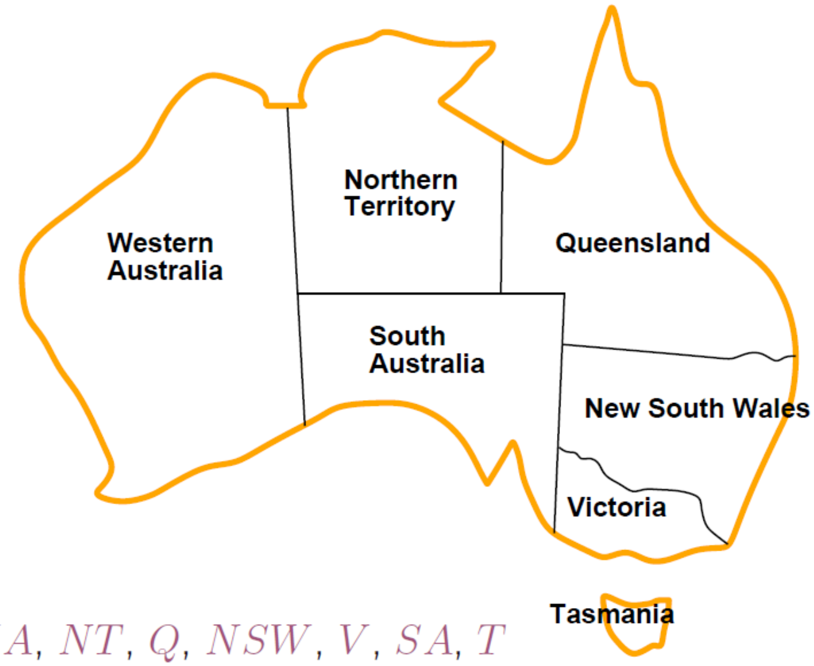
A model is a **structured** representation of the world.

- Graph-Based Search: State is **black box**, no internal structure, atomic.
- Factored Representation: State is list or vector of **facts**.
- CSP: a fact is of the form "Variable = value".

Constraint satisfaction problems (CSPs)

- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying **allowable combinations of values for subsets of variables.**
- Allows useful general-purpose algorithms with more power than standard search algorithms.
- Power close to simulating Turing Machines.

Example: Map-Coloring



Variables WA, NT, Q, NSW, V, SA, T

Domains $D_i = \{red, green, blue\}$

Constraints: adjacent regions must have different colors

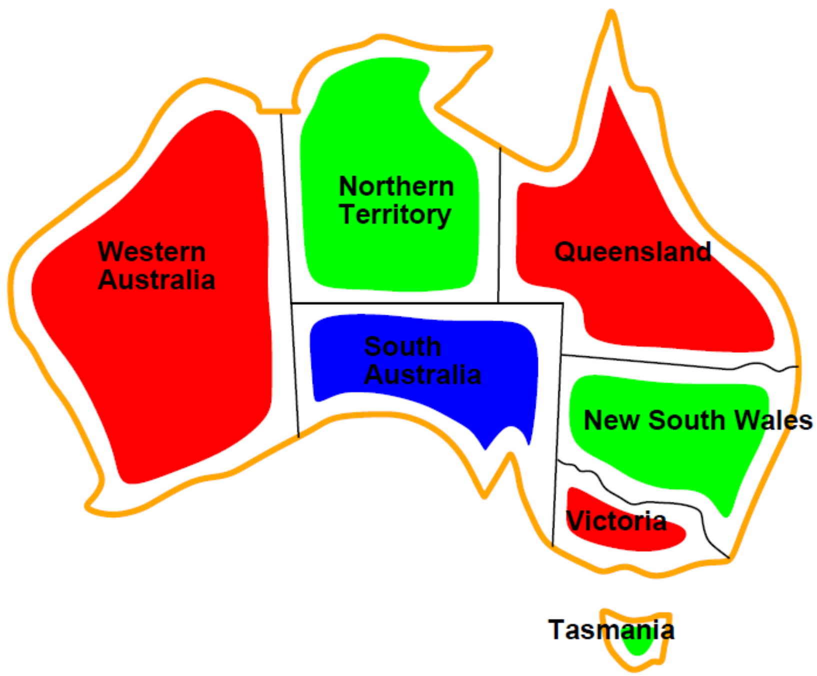
e.g., $WA \neq NT$ (if the language allows this), or

$(WA, NT) \in \{(red, green), (red, blue), (green, red), (green, blue), \ldots\}$

CSPs (continued)

- An assignment is *complete* when every variable is mentioned.
- A *solution* to a CSP is a complete assignment that satisfies all constraints.
- Some CSPs require a solution that maximizes an *objective function*.
- Constraints with continuous variables are common.
  - Linear Constraints ➔ **linear programming.**
- Examples of Applications:
  - Airline schedules
  - Final Exam Scheduling.
  - Cryptography
  - Sudoku, cross-words.

Example: Map-Coloring contd.



Solutions are assignments satisfying all constraints, e.g.,

$\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$

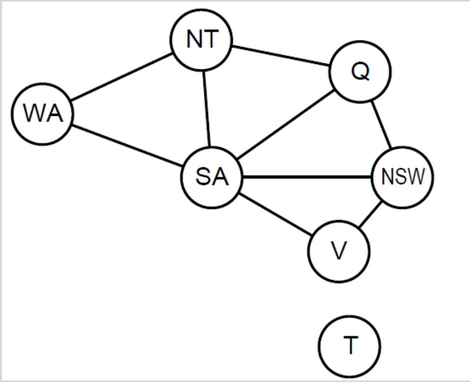
Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA 6= green
- Binary constraints involve pairs of variables,
  - e.g., SA <> WA
- Higher-order constraints involve 3 or more variables
- Preferences (soft constraints), e.g., red is better than green
  often representable by a cost for each variable assignment

→constrained optimization problems

Constraint graph

- Binary CSP: each constraint relates at most two variables
- Constraint graph: nodes are variables, arcs show constraints

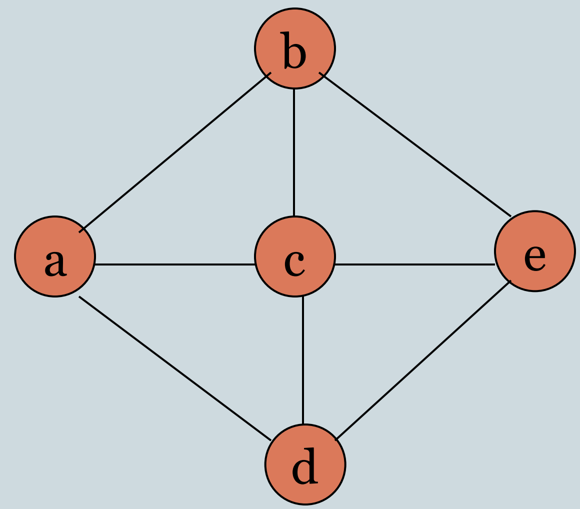


- General-purpose CSP algorithms use the graph structure
- to speed up search. E.g., Tasmania is an independent subproblem!

Graphs and Factored Representations

- UBC AI Space CSP
- Graphs for variables (concepts, facts) capture **local dependencies** between variables (concepts, facts).
- Absence of edges = independence.
- AI systems try to reason locally as much as possible.
- Potential Solution to the Relevance Problem:
  - How does the brain retrieve relevant facts in a given situation, out of the million facts that it knows?
  - Answer: Direct links represent direct relevance.
- Computation in general is a local process operating on a factored state. (What is the state of a program run?)

Problem



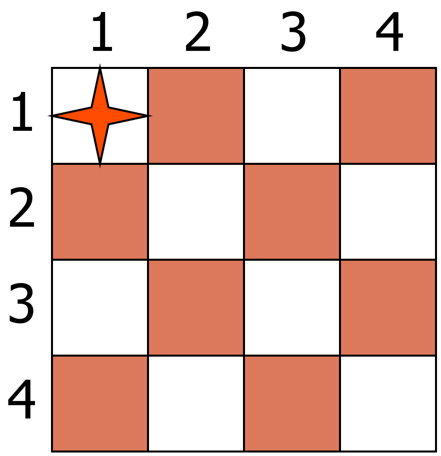
Consider the constraint graph on the right.

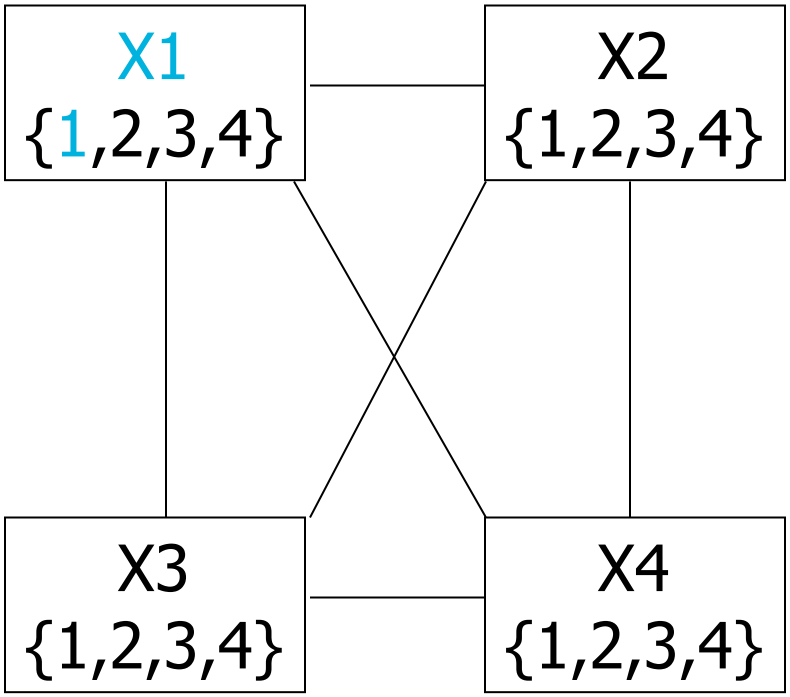
The domain for every variable is [1,2,3,4].

There are 2 unary constraints:
- variable “a” cannot take values 3 and 4.
- variable “b” cannot take value 4.

There are 8 binary constraints stating that variables connected by an edge cannot have the same value.

Example: 4-Queens Problem





[5 Queen's Problem]

Standard search formulation (incremental)

- Let's formulate a state graph problem, then take advantage of special structure later.

- States are defined by the values assigned so far
  - Initial state: the empty assignment, { }
  - Successor function: assign a value to an unassigned variable that does not conflict with current assignment.
    ⇒ fail if no legal assignments (not fixable!)
  - Goal test: the current assignment is complete

- This is the same for all CSPs!

Standard search formulation (incremental)

- Can we use breadth first search?
  - Branching factor at top level?
    - *nd* any of the d values can be assigned to any variable
  - Next level?
    - *(n-1)d*
  - We generate $n!.d^n$ leaves even though there are d^n complete assignments. Why?
  - Commutatively
    - If the order of applications on any given set of actions has no effect on the outcome.

Backtracking search

- Variable assignments are commutative, i.e.,
  - [WA=red then NT =green] same as [NT =green thenWA=red]
- Only need to consider assignments to a single variable at each node
  - ⇨ b=d and there are d^n leaves
- Depth-first search for CSPs with single-variable assignments is called **backtracking search**
- Is this uninformed or informed?
  - Backtracking search is the basic uninformed algorithm for CSPs

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
  return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
     if value is consistent with assignment given CONSTRAINTS[csp] then
        add {var = value} to assignment
        result ← RECURSIVE-BACKTRACKING(assignment, csp)
        if result ≠ failure then return result
        remove {var = value} from assignment
  return failure
```

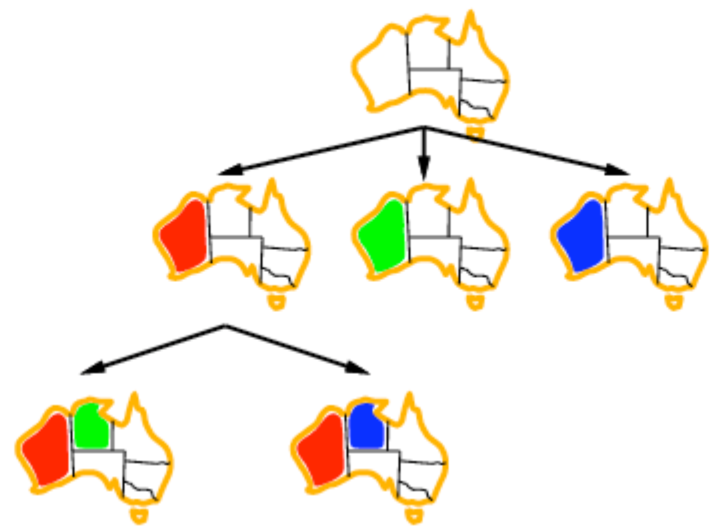
Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - Constraint Learning: Can we keep track of what search has learned?
  - Can we take advantage of problem structure?

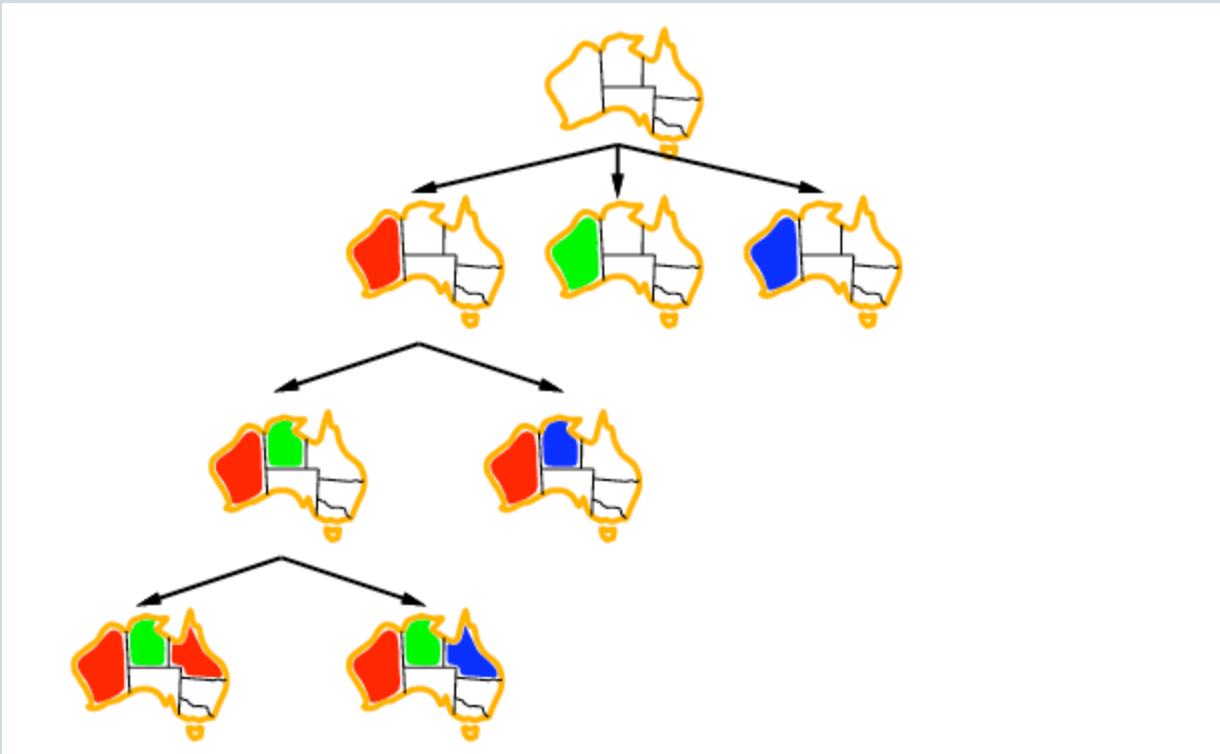
Backtracking example

Backtracking example

Backtracking example



Backtracking example

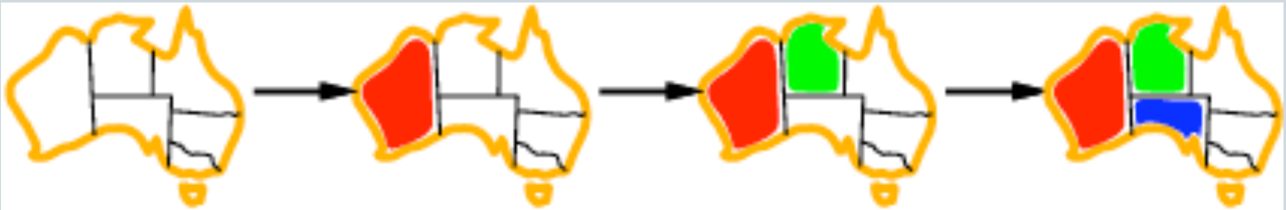


Most constrained variable

- Most constrained variable:

  choose the variable with the fewest legal values

  a.k.a. minimum remaining values (MRV) heuristic

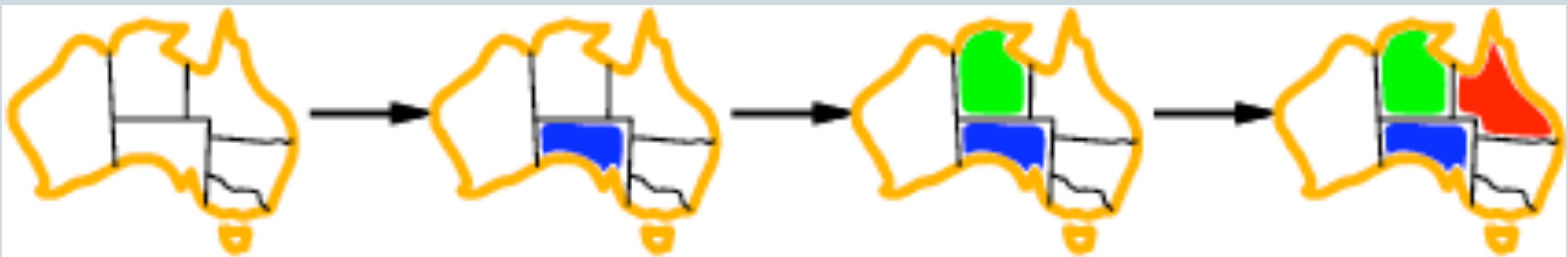


Only picks a variable (Not a value)

Demo for MRV

Most constraining variable

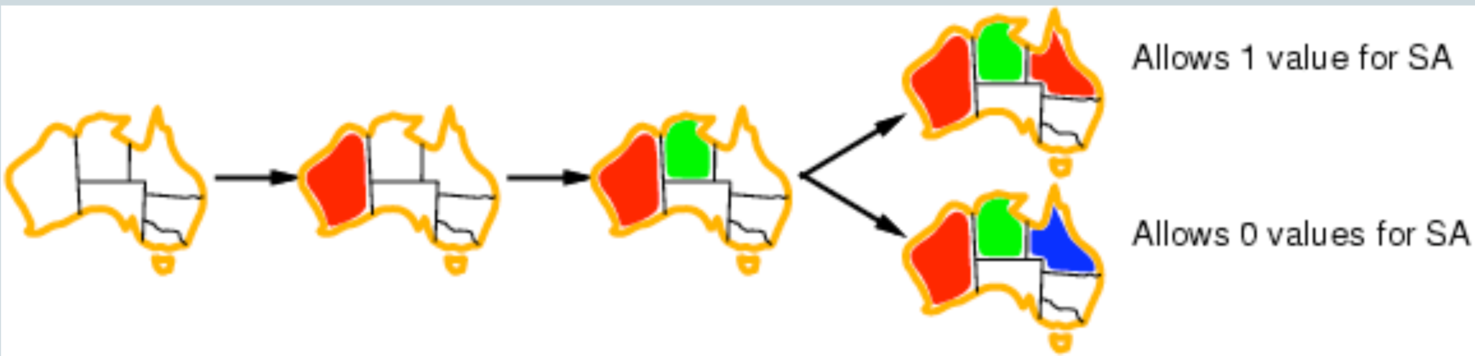
- How to choose between the variable with the fewest legal values?



- Tie-breaker among most constrained variables
  - Degree heuristic: choose the variable with the **most constraints** on remaining variables

Least constraining value

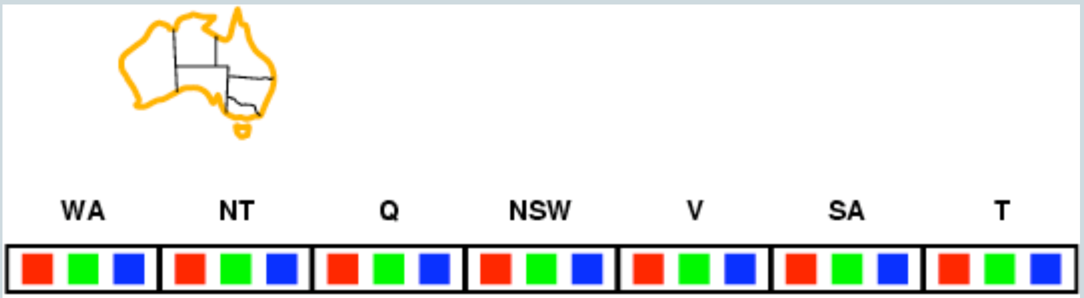
- Given a variable, choose the least constraining value:
- the one that rules out the fewest values in the remaining variables.
- Intuition: choose "most likely" solution.



- Combining these heuristics makes 1000 queens feasible

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values



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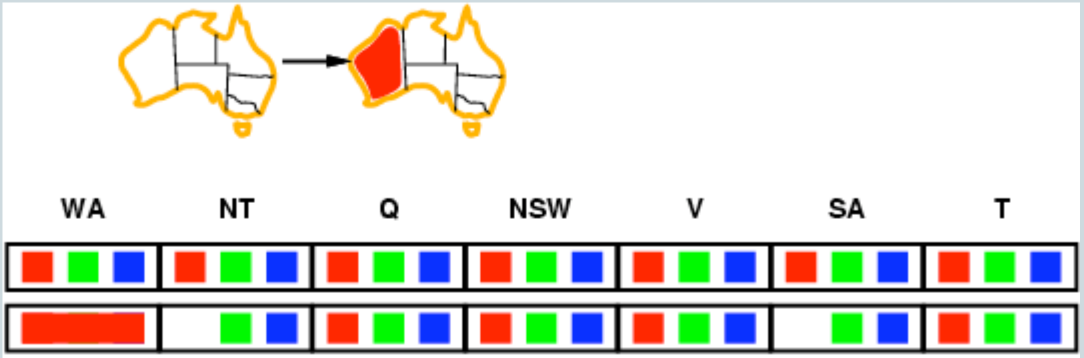

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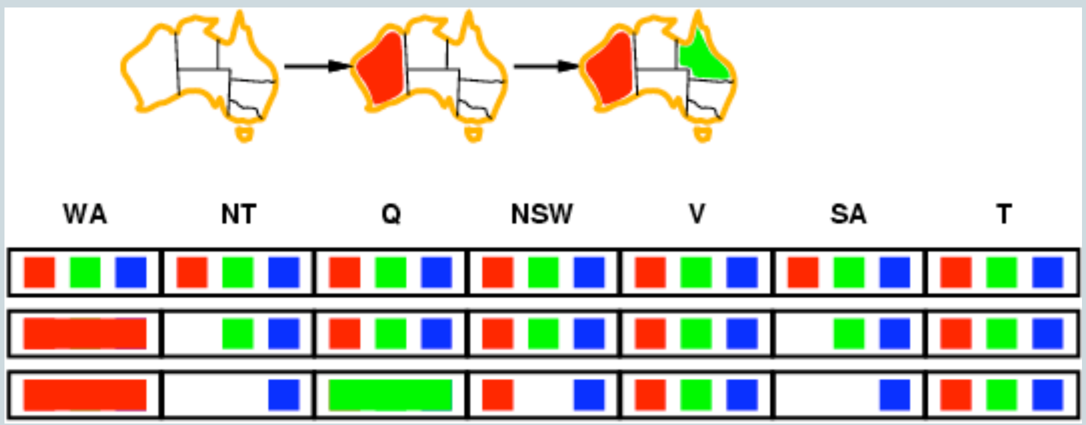

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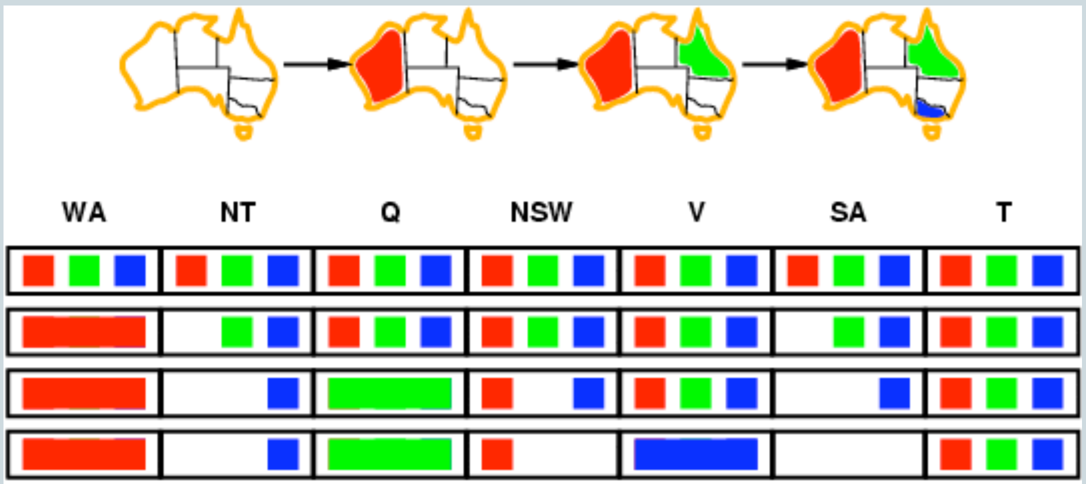
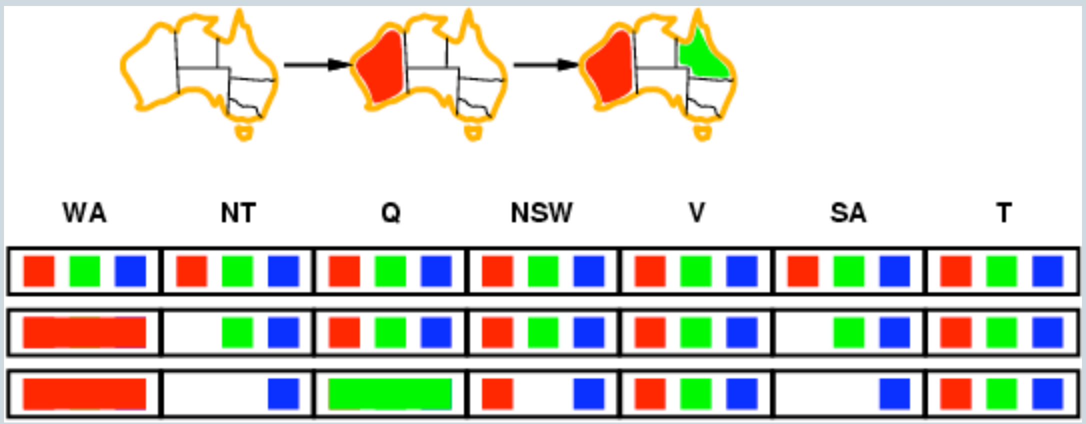


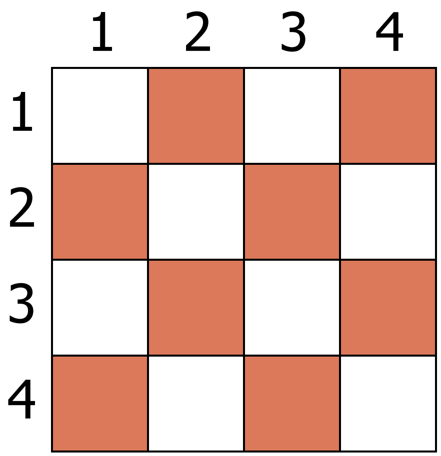
Constraint propagation

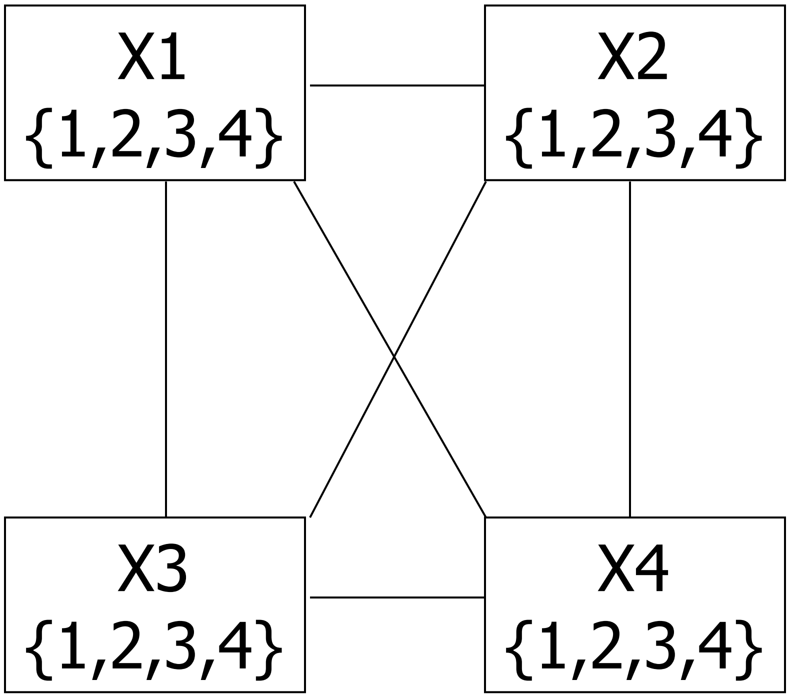
- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally. Has to be faster than searching

Example: 4-Queens Problem





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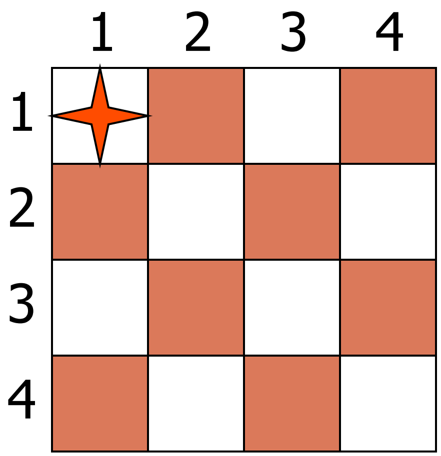

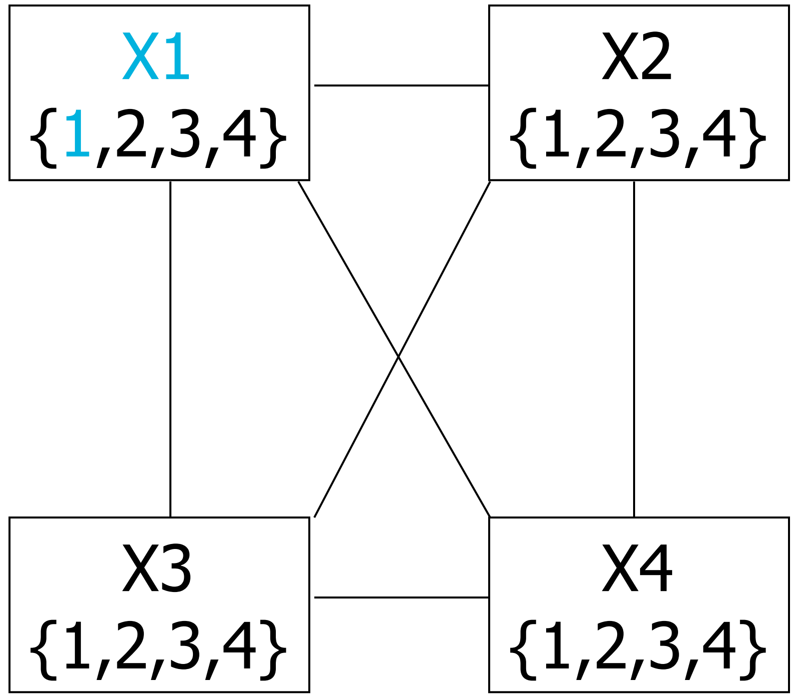

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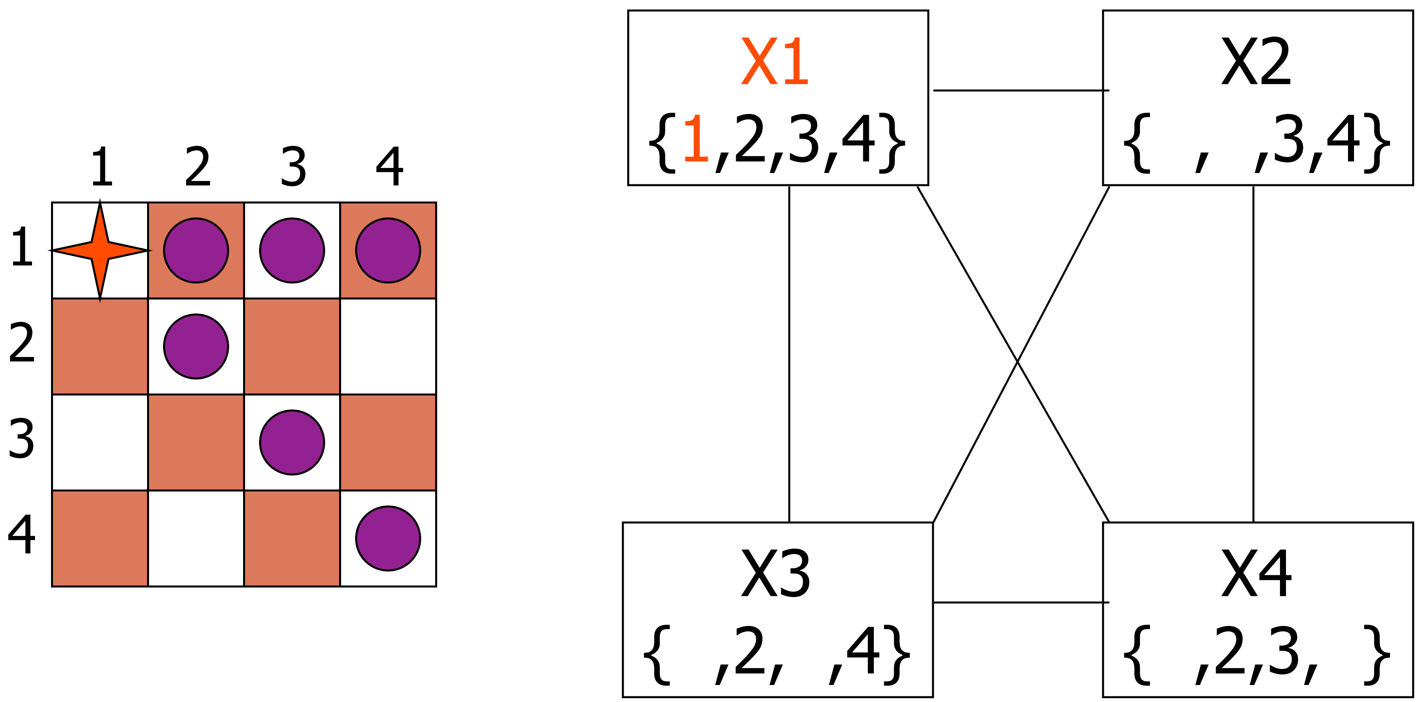

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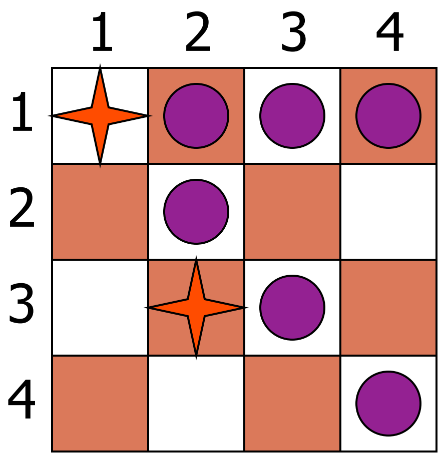

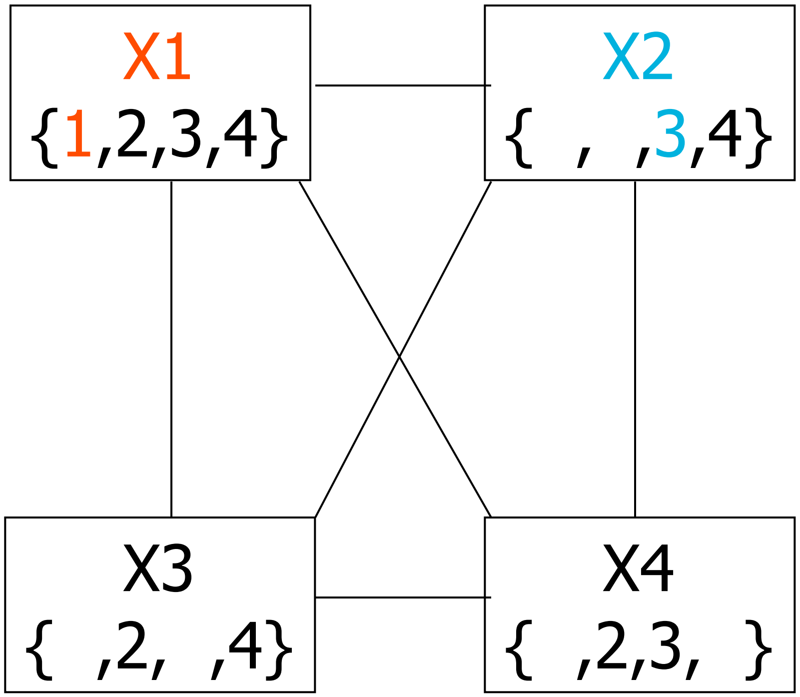

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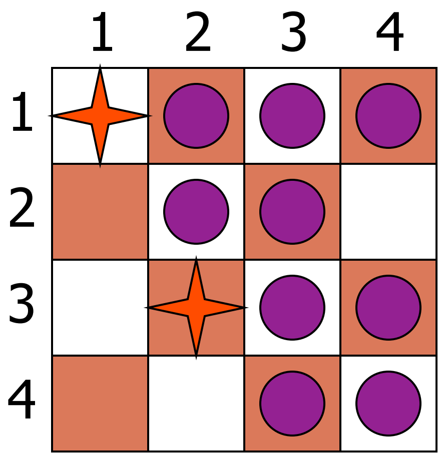

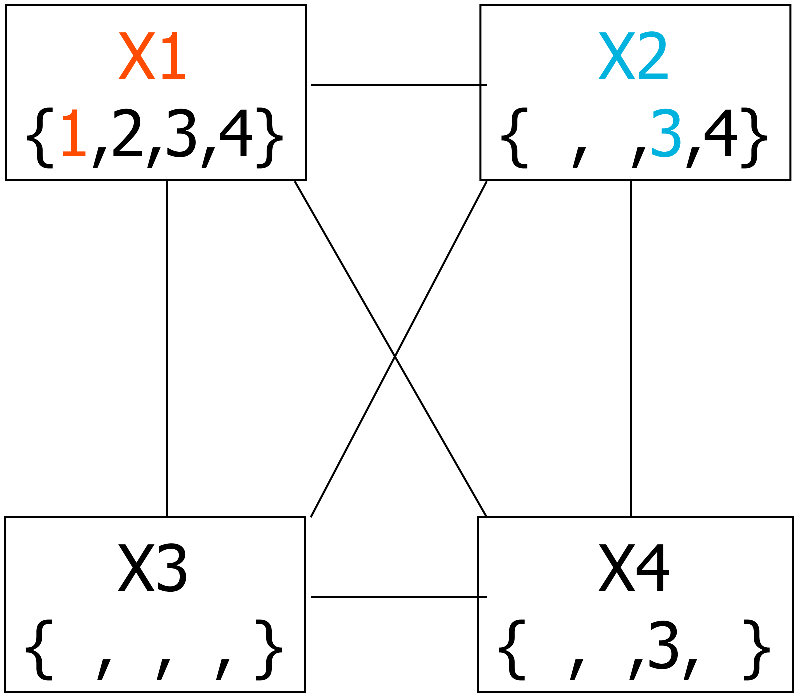

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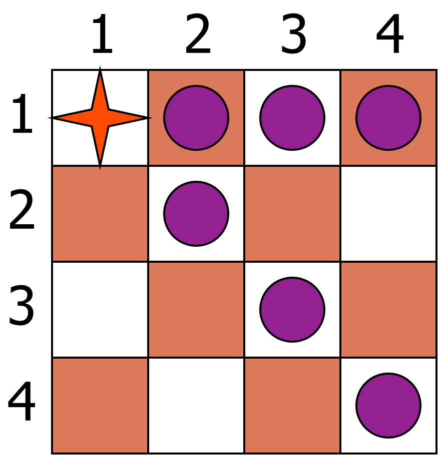

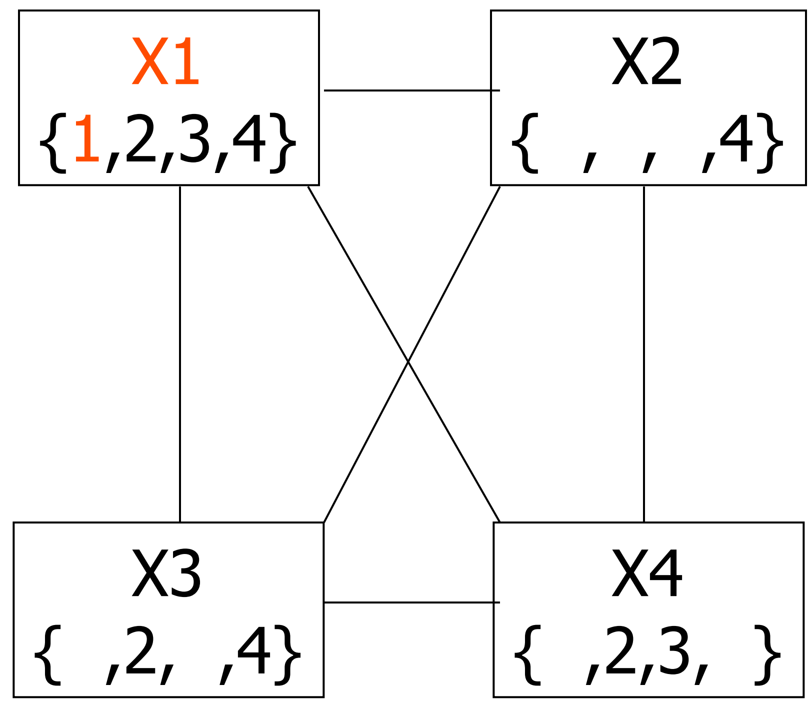

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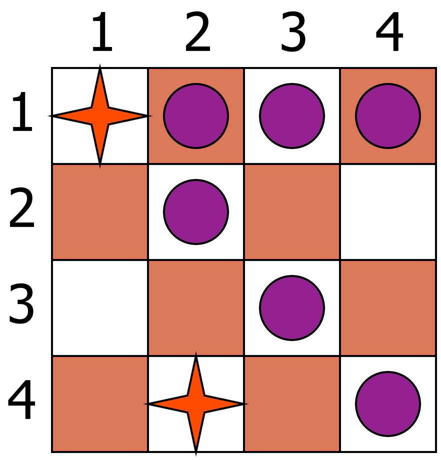

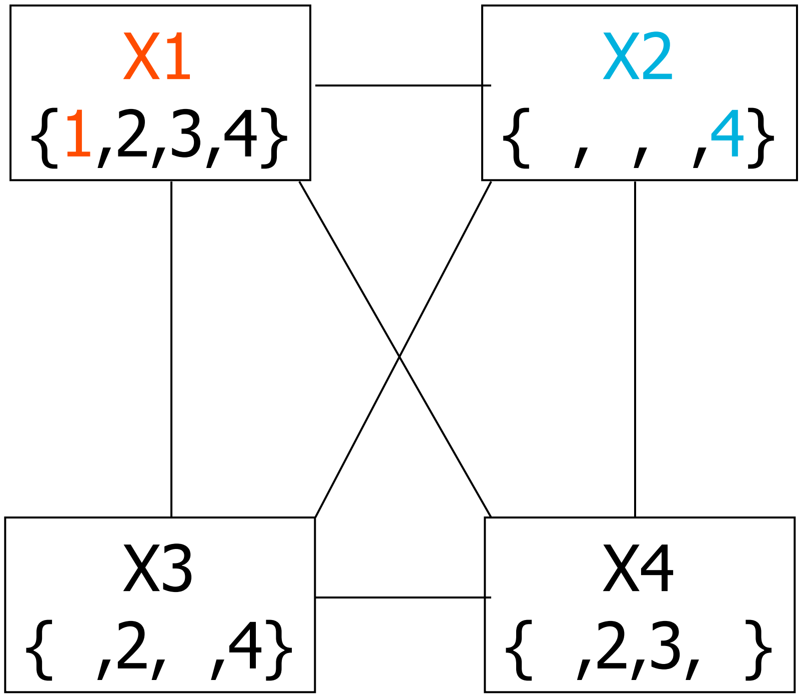

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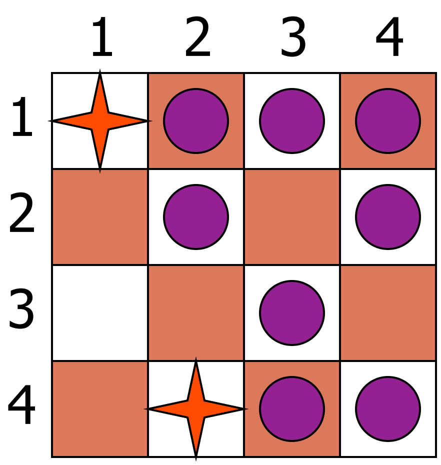

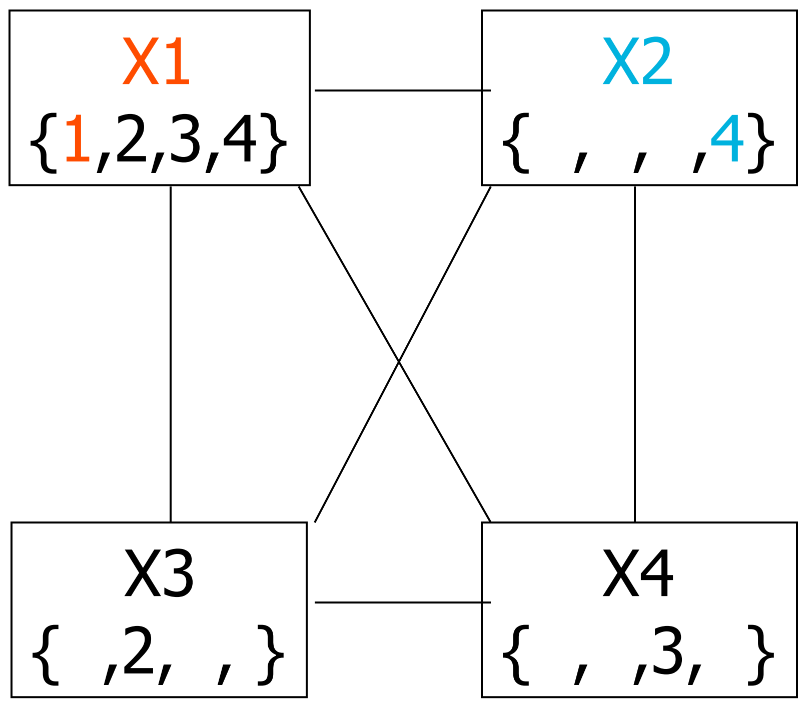

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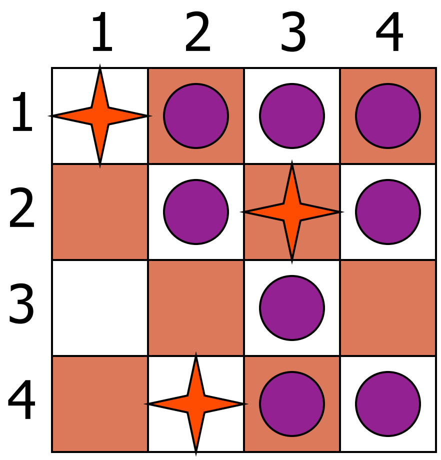

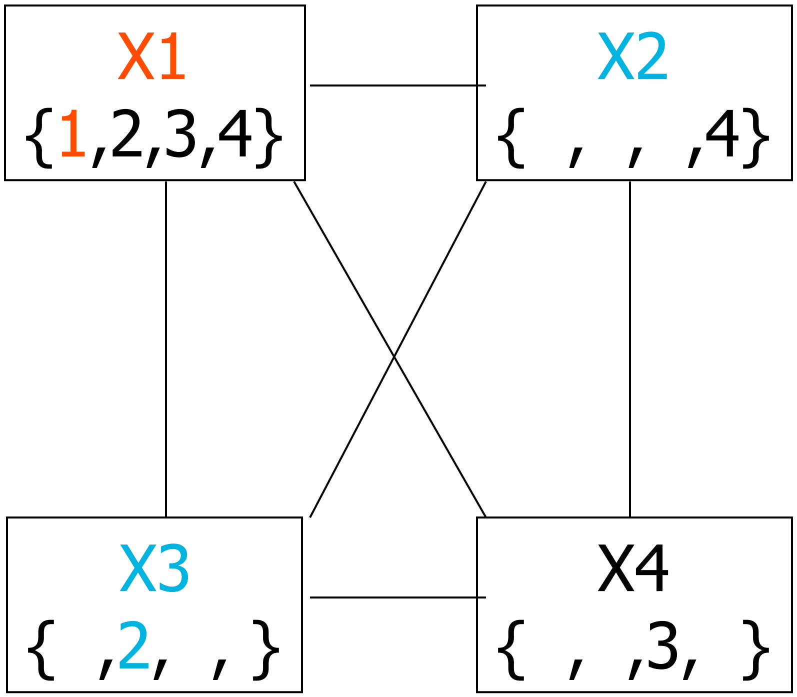

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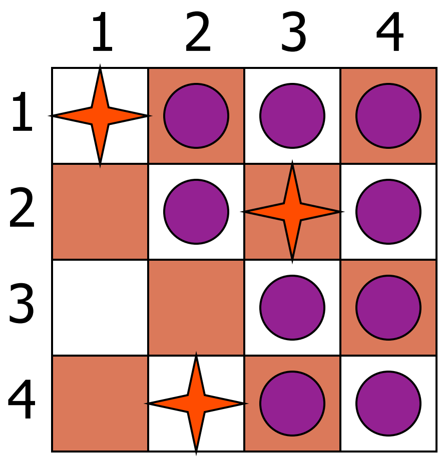
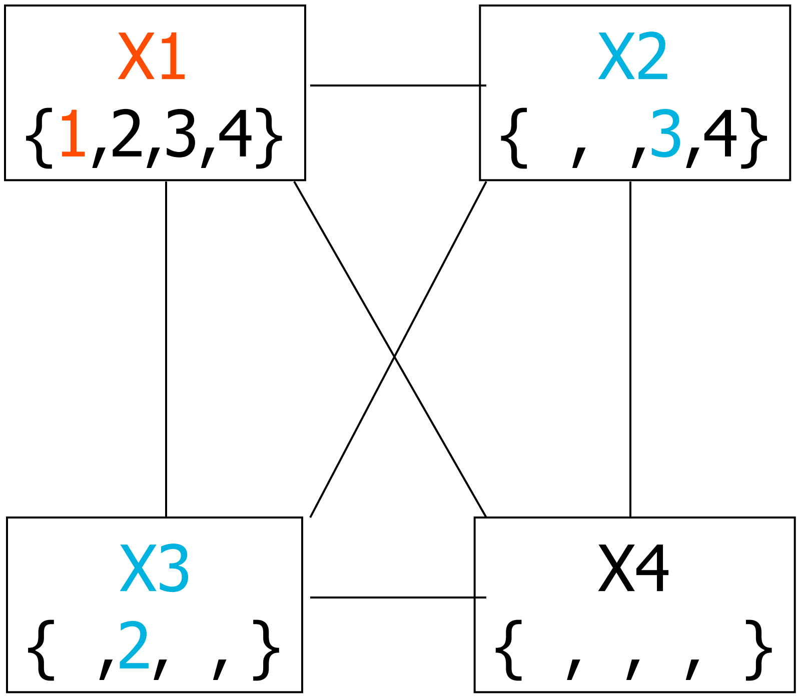




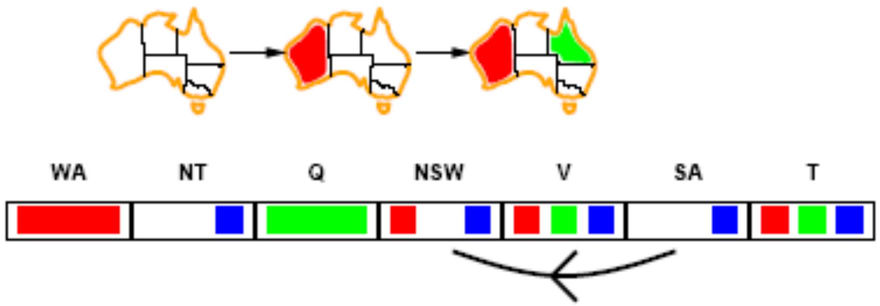
Constraint propagation

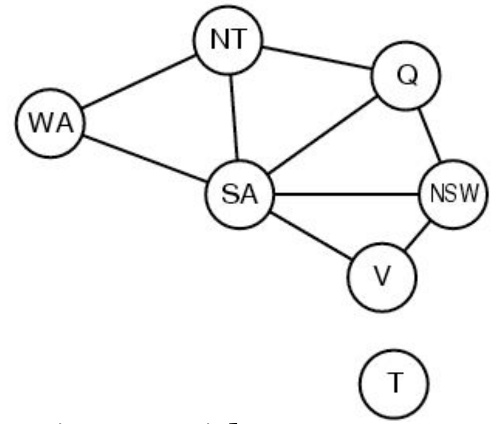
- Techniques like CP and FC are in effect eliminating parts of the search space
  - **Inference** complements search (= simulation).

- Constraint propagation goes further than FC by repeatedly enforcing constraints locally.

- Arc-consistency (AC) is a systematic procedure for Constraint propagation (Macworth 1977 UBC).

Arc consistency





- *An Arc X* $\rightarrow$ *Y* is consistent if

  for *every* value *x* of *X* there is some value *y* consistent with *x*

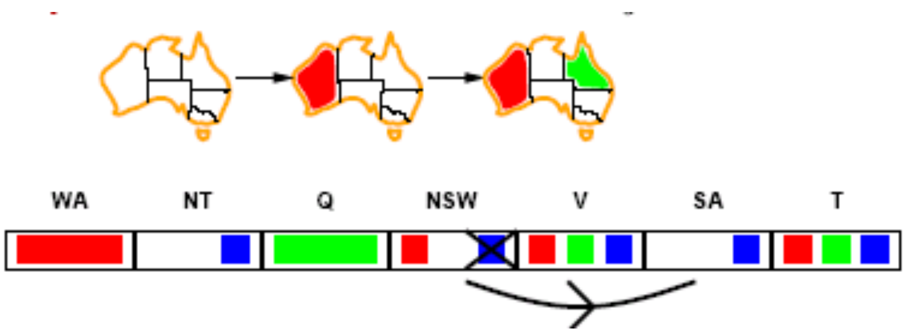
  (note that this is a directed property)

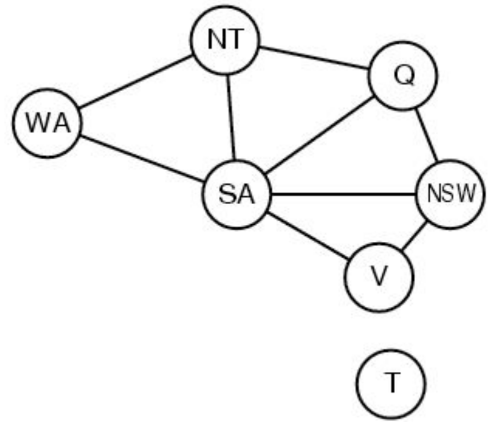
- Consider state of search after WA and Q are assigned:

SA $\rightarrow$ *NSW* is consistent if

SA=blue and *NSW=red*

Arc consistency





- $X \rightarrow Y$ is consistent if

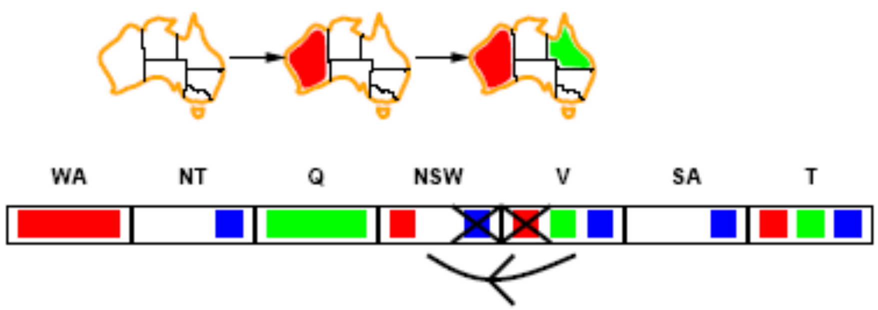
  for *every* value x of X there is some value y consistent with x

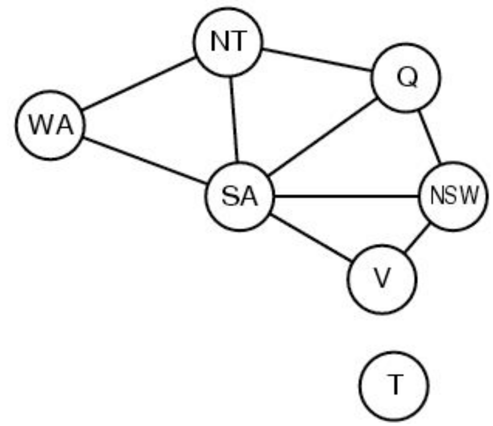
- $NSW \rightarrow SA$ is consistent if

  NSW=red and *SA=blue*

  NSW=blue and SA=???

Arc consistency



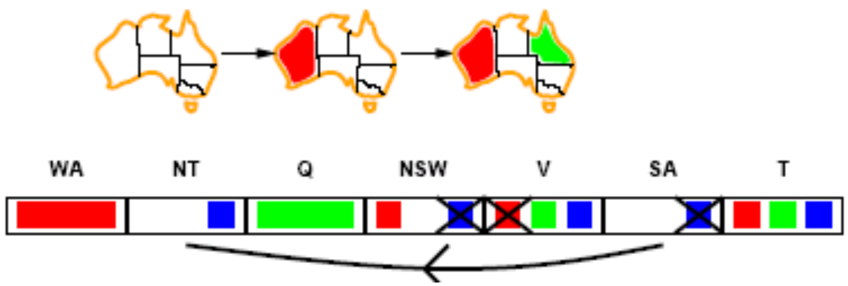


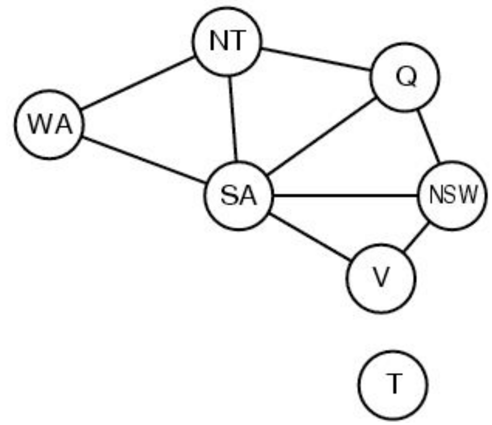
- Can enforce arc-consistency:

  Arc can be made consistent by removing *blue* from *NSW*

- Continue to propagate constraints….
  - Check $V \rightarrow NSW$
  - Not consistent for V = red
  - Remove red from *V*

Arc consistency





- Continue to propagate constraints....
- $SA \rightarrow NT$ is not consistent
  - and cannot be made consistent
- Arc consistency detects failure earlier than FC

Arc consistency checking

- Can be run as a preprocessor or after each assignment
  - Or as preprocessing before search starts
- AC must be run repeatedly until no inconsistency remains
- Trade-off
  - Requires some overhead to do, but generally more effective than direct search
  - In effect it can eliminate large (inconsistent) parts of the state space more effectively than search can
- Need a systematic method for arc-checking
  - If X loses a value, neighbors of X need to be rechecked.

Arc consistency checking

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

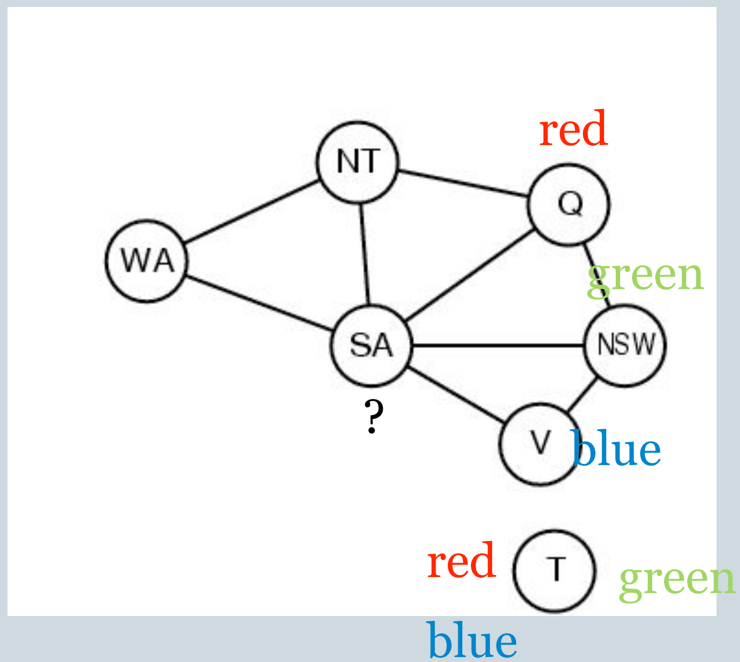
  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS$[X_i]$ **do**
        add (X_k, X_i) to *queue*

---

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff we remove a value
  removed $\leftarrow$ *false*
  for each x **in** DOMAIN$[X_i]$ **do**
    if no value y in DOMAIN$[X_j]$ allows (x,y) to satisfy the constraint between X_i and X_j
      then delete x from DOMAIN$[X_i]$; *removed* $\leftarrow$ *true*
  return *removed*

Back-tracking or back-jumping?

- {Q=red , NSW= green, V= blue, T=red}



Local search for CSPs

- Use complete-state representation
  - Initial state = all variables assigned values
  - Successor states = change 1 (or more) values

- For CSPs
  - allow states with unsatisfied constraints (unlike backtracking)
  - operators **reassign** variable values
  - hill-climbing with n-queens is an example

- Variable selection: randomly select any conflicted variable.
  - Local Stochastic Search Demo
- Value selection: **min-conflicts heuristic**
  - Select new value that results in a minimum number of conflicts with the other variables

Local search for CSP

function MIN-CONFLICTS(*csp, max_steps*) **return** solution or failure

inputs: *csp*, a constraint satisfaction problem

max_steps, the number of steps allowed before giving up

current ← an initial complete assignment for *csp*

for *i* = 1 to *max_steps* **do**

if *current* is a solution for *csp* then return *current*

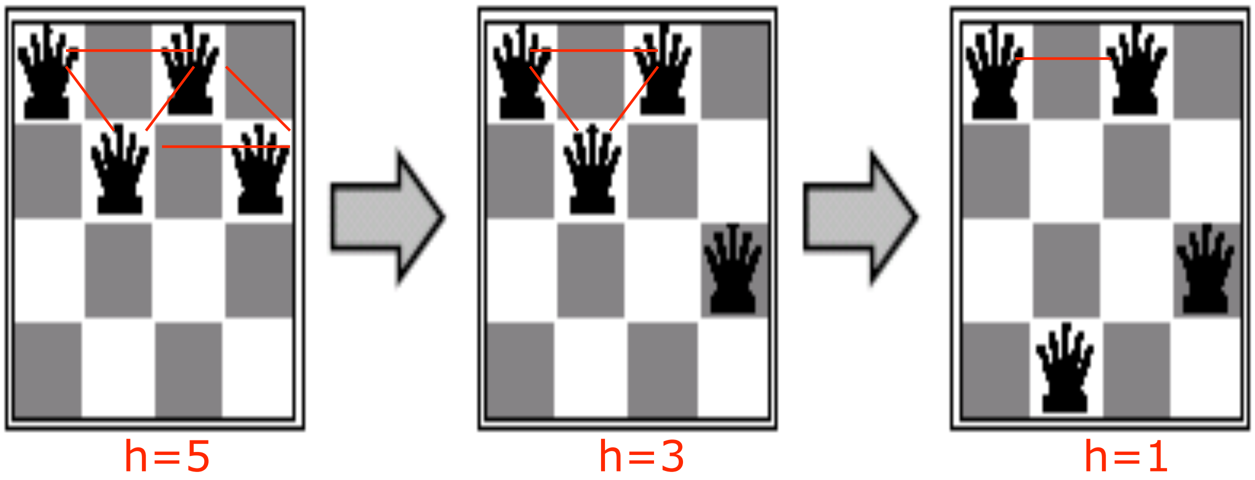
var ← a randomly chosen, conflicted variable from VARIABLES[*csp*]

value ← the value *v* for *var* that minimize CONFLICTS (*var*,*v*,*current*,*csp*)

set *var* = *value* in *current*

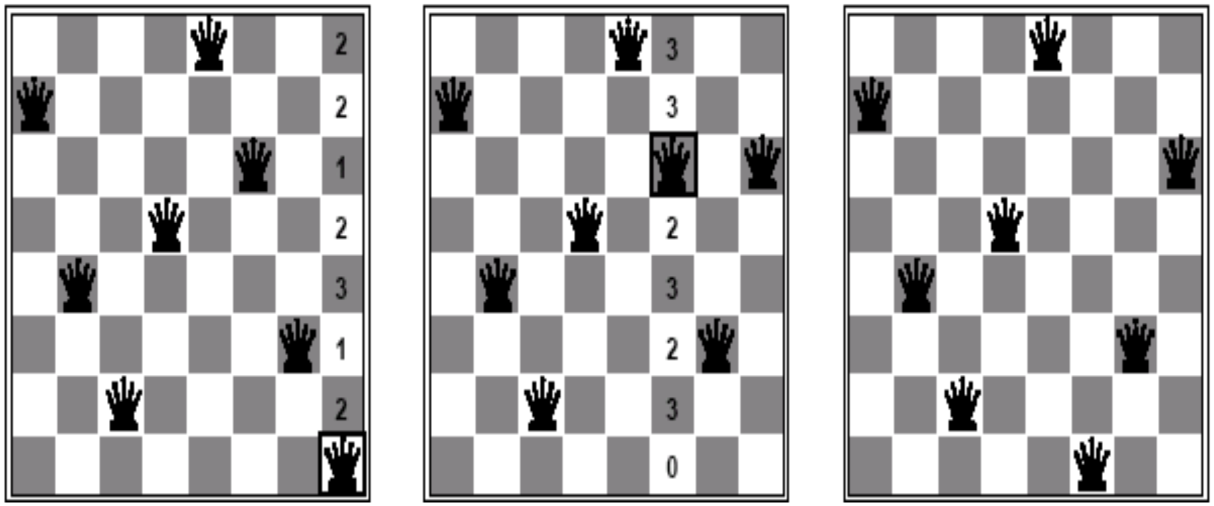
return *failure*

Min-conflicts example 1



Use of min-conflicts heuristic in hill-climbing.

Min-conflicts example 2



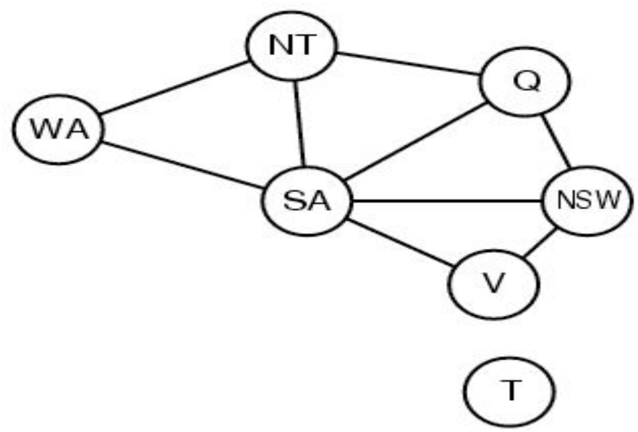
- A two-step solution for an 8-queens problem using min-conflicts heuristic
- At each stage a queen is chosen for reassignment in its column
- The algorithm moves the queen to the min-conflict square breaking ties randomly.

Advantages of local search

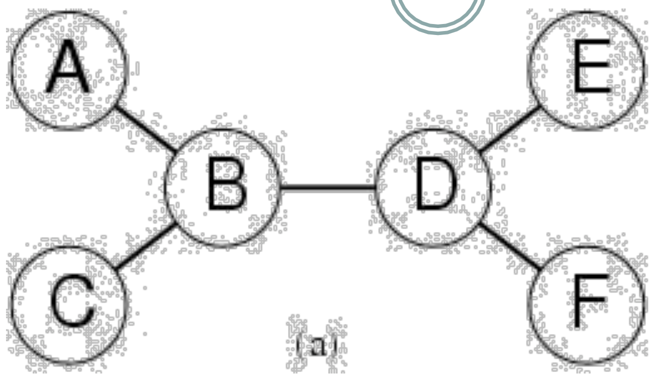
- Local search can be particularly useful in an online setting
  - Airline schedule example
    - E.g., mechanical problems require than 1 plane is taken out of service
    - Can locally search for another "close" solution in state-space
    - Much better (and faster) in practice than finding an entirely new schedule.
- The runtime of min-conflicts is roughly independent of problem size.
  - Can solve the millions-queen problem in roughly 50 steps.
  - Why?
    - n-queens is easy for local search because of the relatively high density of solutions in state-space.

Graph structure and problem complexity

- Divide-and-conquer: Solving **disconnected subproblems.**
  - Suppose each subproblem has *c* variables out of a total of *n*.
  - Worst case solution cost is $O(n/c \ d^c)$, i.e. linear in *n*
    - Instead of $O(d^{\,n})$, exponential in *n*
- E.g. $n= 80$, $c= 20$, $d=2$
  - 2^{80} = 4 billion years at 1 million nodes/sec.
  - $4 * 2^{20}$= .4 second at 1 million nodes/sec



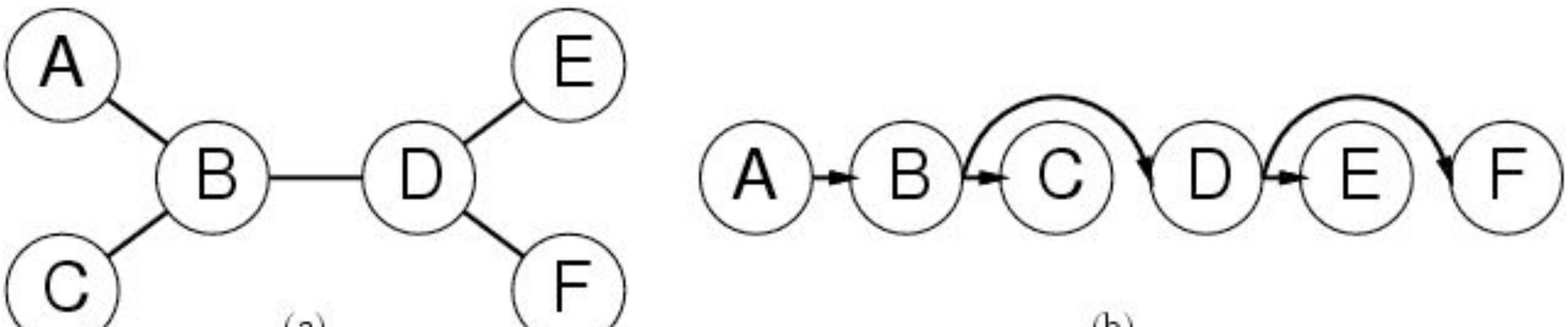
Tree-structured CSPs



- Theorem:
  - if a constraint graph has no loops then the CSP can be solved in $O(nd^2)$ time
  - linear in the number of variables!
- Compare difference with general CSP, where worst case is $O(d^n)$

Algorithm for Solving Tree-structured CSPs

- Choose some variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering.
  - Label variables from X_1 to X_n.
  - Every variable now has 1 parent

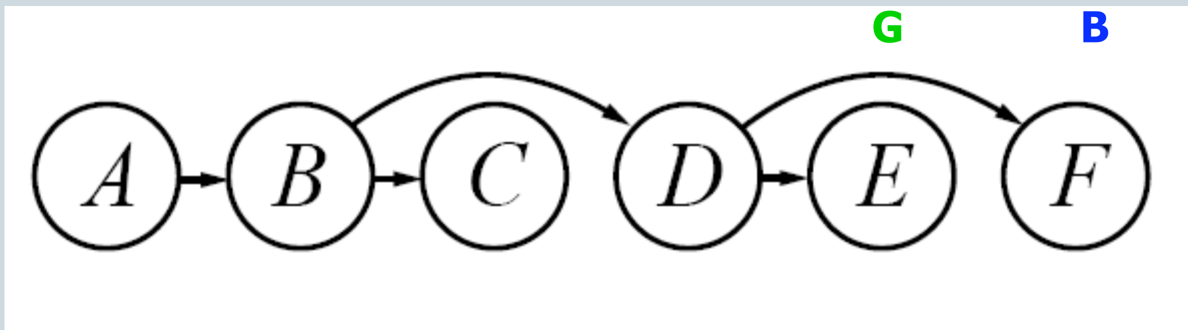


(a) (b)

- Backward Pass
  - For j from n down to 2, apply arc consistency to arc [Parent(X_j), X_j)]
  - Remove values from Parent(X_j) if needed to make graph **directed arc consistent.**

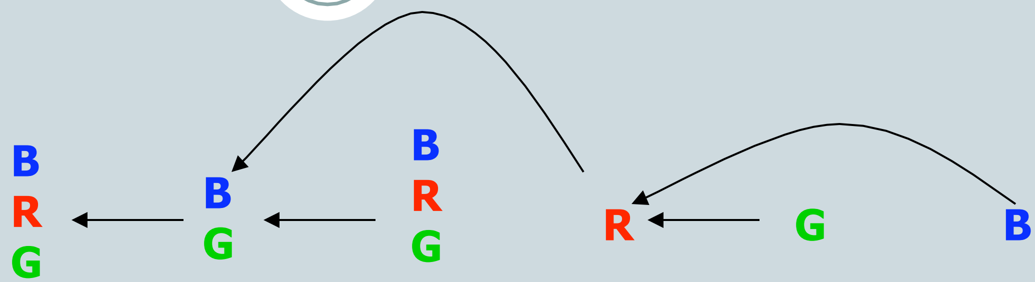
- Forward Pass
  - For j from 1 to n assign X_j consistently with Parent(X_j)

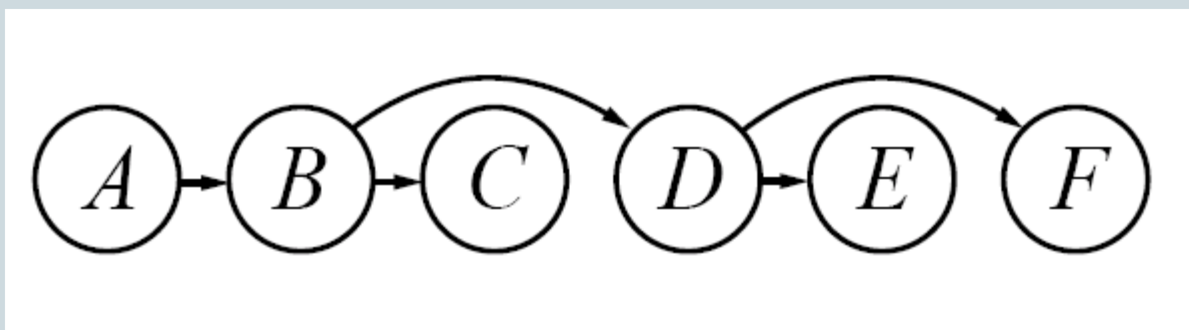
Tree CSP Example



Tree CSP Example

Backward Pass
(constraint
propagation)

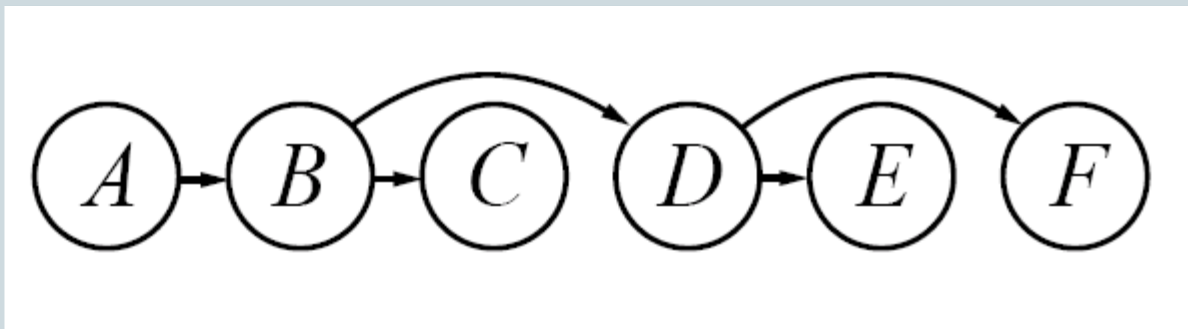




Tree CSP Example

Backward Pass
(constraint
propagation)

B R G	B G	B R G	R	G	B



Forward Pass
(assignment)

B	G	R	R	G	B

Tree CSP complexity

- Backward pass
  - n arc checks
  - Each has complexity d^2 at worst

- Forward pass
  - n variable assignments, $O(nd)$

⇒ Overall complexity is $O(nd^2)$

Algorithm works because if the backward pass succeeds, then every variable by definition has a legal assignment in the forward pass

What about non-tree CSPs?

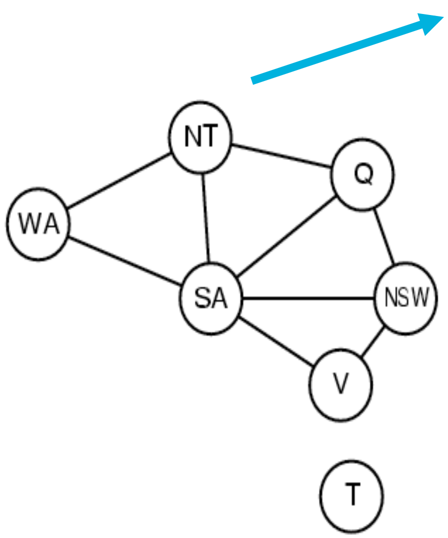
- General idea is to convert the graph to a tree

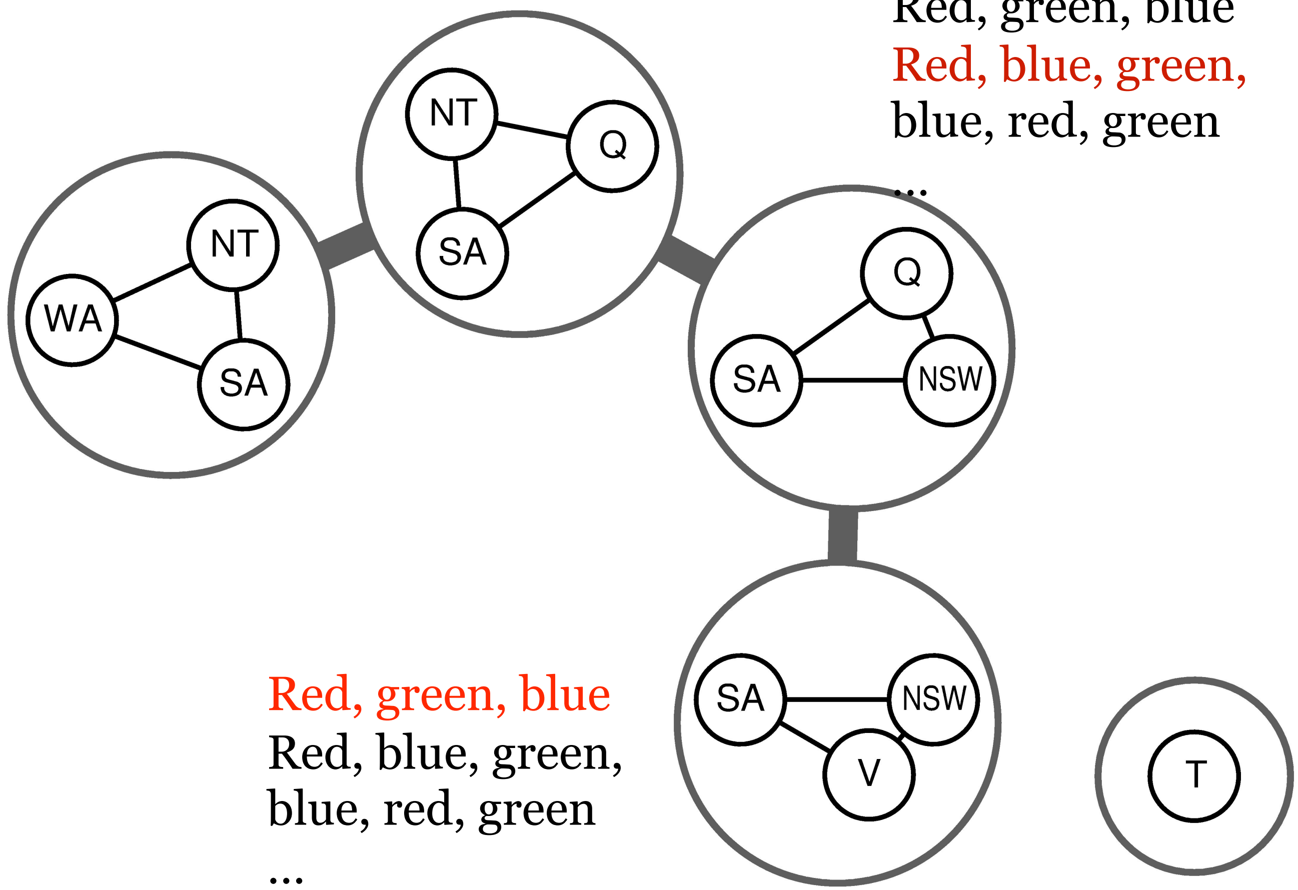
2 general approaches

1. Assign values to specific variables (Cycle Cutset method).
   1. Tries to exploit **context-specific** independence.

2. Construct a **tree-decomposition** of the graph
   - Connected subproblems (subgraphs) becomes nodes in a tree structure.

Tree Decompositions





Red, green, blue
Red, blue, green,
blue, red, green
...

Red, green, blue
Red, blue, green,
blue, red, green
...

Rules for a Tree Decomposition

- Every variable appears in at least one of the subproblems.
- If two variables are connected in the original problem, they must appear together (with the constraint) in at least one subproblem.
- If a variable appears in two subproblems, it must appear in each node on the path between them.

Tree Decomposition Algorithm

- View each subproblem as a "super-variable"
  - Domain = set of solutions for the subproblem
  - Obtained by running a CSP on each subproblem.
  - Maximum-size of subproblem = **treewidth of constraint graph.**

  - E.g., 6 solutions for 3 fully connected variables in map problem
- Now use the tree CSP algorithm to solve the constraints connecting the subproblems
  - Declare a subproblem a root node, create tree
  - Backward and forward passes
- Example of "divide and conquer" strategy

Tree Decomposition

- *Every* graph has a tree decomposition, not just constraint graphs.
- Tree decomposition shows optimal divide-and-conquer strategy.
  - For CSPs, optimization, search, inference.
- Typical result: if treewidth of a problem graph is constant, then the search/optimization/inference problem is solvable by divide and conquer.

Summary

- CSPs
  - special kind of problem: states defined by values of a fixed set of variables, goal test defined by constraints on variable values: **factored representation**.
- Backtracking=depth-first search with one variable assigned per node
- Heuristics
  - Variable ordering and value selection heuristics help significantly
- Constraint propagation does additional work to constrain values and detect inconsistencies
  - Works effectively when combined with heuristics
- Iterative min-conflicts is often effective in practice.
- Graph structure of CSPs determines problem complexity
  - e.g., tree structured CSPs can be solved in linear time.