Lemma 1. $\log(n!) \in \Theta(n \log n)$

Proof. Obviously, $\log(n!) \in O(n \log n)$. We need to show that $\log(n!) \in \Omega(n \log n)$, i.e., that there are constant $c > 0$ and n_0 such that $\log(n!) \geq cn \log n$ for every $n \geq n_0$.

Induction hypothesis: $\log((n-1)!) \geq c(n-1)\log(n-1)$.

We have:

$$\begin{aligned} \log(n!) &= \log(n.(n-1)!) = \log n + \log((n-1)!) \\ &\geq \log n + c(n-1)\log(n-1) \end{aligned} \tag{1}$$

Now, we need to show that $\log(n-1) \geq \log n - \epsilon$ for a very small epsilon. A constant ϵ is not enough, as we would not get the claim, we need something of order $1/n$. For this we will use the Taylor expansion of the exponential function e^x:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Hence, for any $x > 0$, we have $e^x > 1 + x$. Substituting $1/(n-1)$ for x, we have $e^{1/(n-1)} > 1 + \frac{1}{n-1} = \frac{n}{n-1}$. The logarithm on both sides, we get

$$\begin{aligned} &1/(n-1).\log e > \log(n/(n-1)) = \log n - \log(n-1)\,, \quad \text{and hence,} \\ &\quad \log(n-1) > \log n - \log e/(n-1)\,, \end{aligned}$$

which is what we wanted. Now, let's plug it back to (1):

$$\begin{aligned} \log(n!) &\geq \log n + c(n-1)\log(n-1) \\ &> \log n + c(n-1)(\log n - \log e/(n-1)) = \log n + c(n-1)\log n - c\log e \\ &= cn\log n + (1-c)\log n - c\log e \end{aligned}$$

Now, it is enough to show that $(1-c)\log n - c\log e \geq 0$. Let's set $c = 1/2$. Then it's enough to show that $\log n \geq \log e$, i.e., $n \geq e$. This is true, for all $n \geq 3$. Hence, for $c = 1/2$ and $n \geq 3$, it follows that

$$\log(n!) \geq cn\log n\,,$$

which finishes the induction step.

It is easy to check the base case, which we can set in this case to $n = 2$ (since, induction step can be done for any $n \geq 3$). For $n = 2$, we have $\log(2!) = 1 \geq 1/2.2\log 2 = 1$, and hence $\log(n!) \geq cn\log n$ is true in the base case as well.

To summarize we have $\log(n!) \geq 1/2n\log n$, for every $n \geq n_0 = 2$. Done! $\square$