CMPT 307-08-2 Assignment 5

(From lecture on June 3, 2008)

Deadline: June 10, 5:30pm

Problem 5.1. Show that expected running time of **Randomized-Quicksort** is $\Omega(n \log n)$. In fact it's enough to show that $E[X] = \Omega(n \log n)$.
Hint: From the lecture notes we note that

- $E[X] = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1}$; and
- $H_n \geq \ln n$.

Use these two facts to show that for some $c > 0$ and n_0, $E[X] \geq c.n \ln n$, for all $n \geq n_0$.

Problem 5.2. Show by mathematical induction that for any n and events $A_1, A_2, \ldots, A_n$ we have the equality:

$$\begin{aligned}
&P(A_1 \cap A_2 \cap \cdots \cap A_n) = \\
&\qquad P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_2 \cap A_1) \cdots \\
&\qquad P(A_n|A_{n-1} \cap \cdots \cap A_2 \cap A_1)
\end{aligned}$$

Problem 5.3. Prove that in the array $\mathcal{P}$ in procedure **Permute-By-Sorting**, the probability that all elements (priorities) are unique is exactly

$$\prod_{i=1}^{n} (1 - \frac{i-1}{n^3})$$

Then prove that this formula is greater than $1 - 1/n$.
Hints:

- Define the events

  $$E_i \text{ is the event that } \mathcal{P}[i] \text{ is different from } \mathcal{P}[1], \ldots, \mathcal{P}[i-1]$$

  In fact, we are looking for probability $P(E_1 \cap \cdots \cap E_n)$. Use the same technique as on the lecture, to compute this probability.

- For the second part, first show that the product is larger than $(1 - 1/n^2)^n$ and then use Binomial Theorem.

Problem 5.4. Consider the following procedure for generating a uniform random permutation:
Permute-By-Cyclic$(A[1 \ldots n])$

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1: offset ← Random(1, n)
2: for i ← 1 to n do
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$dest \leftarrow i + offset$
if $dest > n$ **then**
$dest \leftarrow dest - n$
end if
$B[dest] \leftarrow A[i]$
end for
return B

Show that each element $A[i]$ has a $1/n$ probability of being permuted to any particular position in B. Is the resulting permutation (of the procedure) uniformly random?