**Definiton** (Bernoulli Trials) A sequence of independent experiments, each with 2 possible outcomes, success or failure (head or tail for a coin). The probability for success is $p$ and the probability for failure is $1-p=q$.

Example: Flip a coin *n* times. What is the probability of having *k heads?*

*Ans.* $C(n,k).2^{-n}$.

**Definition** (Random variable) A random variable is a function $f: S \to R$ which assigns a value for every element in *S. We* usually would refer to the function f(.) using capital letter such as X.

Example: Toss a coin 5 times and let X the random variable be the number of heads in those coin tosses. Thus HHHTT → 3; HTTHT → 2 etc. Here the atomic element of the sample space is an outcome of 5 tosses.

**Definition** (Expected Value): For a random variable *X*, the expectation of *X* is its average value.

Formally, $E[X] = \Sigma_{s \in S} X(s).P[\{s\}]$

Example: Roll a die, what is the expected value?

S={1,2,3,4,5,6}; X(s=i) = i;

E[X] = 1.1/6+2.1/6+3.1/6+4.1/6+5.1/6+6.1/6=3.5.

*This is the expected value I will get if I throw the die enough times.*

# Linearity of Expectation:

**Lemma 1**: For a constant $\alpha$, we have $E[\alpha X] = \alpha E[X]$.

*Proof*: $E[\alpha X] = \Sigma_{s\in S}\, \alpha X(s)P[\{s\}] = \alpha\, \Sigma_{s\in S}\, X(s)P[\{s\}] = \alpha E[X]$.

**Lemma 2:** For any random variable $X$ and $Y$, $E[X+Y] = E[X] + E[Y]$.

*Proof*: $E[X+Y] = \Sigma_{s\in S}(X(s)+Y(s))P[\{s\}] = \Sigma_{s\in S}\,(X(s)P[\{s\} + Y(s)P[\{s\}])$

$= E[X] + E[Y]$.

Note that the two random variables need not be independent.

***Theorem 1*** (Linearity of expectation): $E[\alpha_1 X_1+\alpha_2 X_2+ \ldots +\alpha_n X_n] = \alpha_1 E[X_1]+ \ldots. + \alpha_n E[X_n]$.

Example: Roll two dice. What is the expected total value?

Ans. Sample space $S=\{(1,1), (1,2), \ldots,(1,6), \ldots,(6,6)\};$

$X_1((I,j)) = i$, outcome of first throw; $X_2((I,j))=j$, outcome of second throw.

$Y=X_1+X_2$ is a random variable for the total.

$E[Y]=E[X_1+X_2]= E[X_1]+E[X_2] = 3.5 + 3.5 =7$.

## Linearity of Expectation:

**Indicator Variables**: Let $E$ be an event of $S$. The indicator variable of $E$ is the mapping $I_E: S \rightarrow \{0,1\}$, such that for $s \in S$, we have $I_E(s) = 1$ if $s \in E$, otherwise ($s \notin E$), $I_E = 0$.

**Observation**: $E[I_E] = 0. P[s \notin E] + 1. P[s \in E] = P[s \in E]$.

Example: A coin is tossed 10 times. What is the expected number of heads?

Ans: Let $X_i$ be an indiactor function which is 1 if the $i^{th}$ toss is head, otherwise $X_i=0$. Let $Y= X_1+X_2+ \ldots +X_{10}$ be the random variable. Now $E[Y] = E[X_1]+E[X_2]+\ldots+E[X_{10}] = 10.1/2=5$.