Red-Black Trees

Data Structures and Algorithms

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Red-Black Trees

All binary search tree operations take O(h) time, where h is the height of the tree

Therefore, it is important to `balance’ the tree so that its height is as small as possible

There are many ways to achieve this

One of them: Red-Black trees

Every node of such a tree contains one extra bit, its color

Another agreement: the Nil pointer is treated as a leaf, an extra node

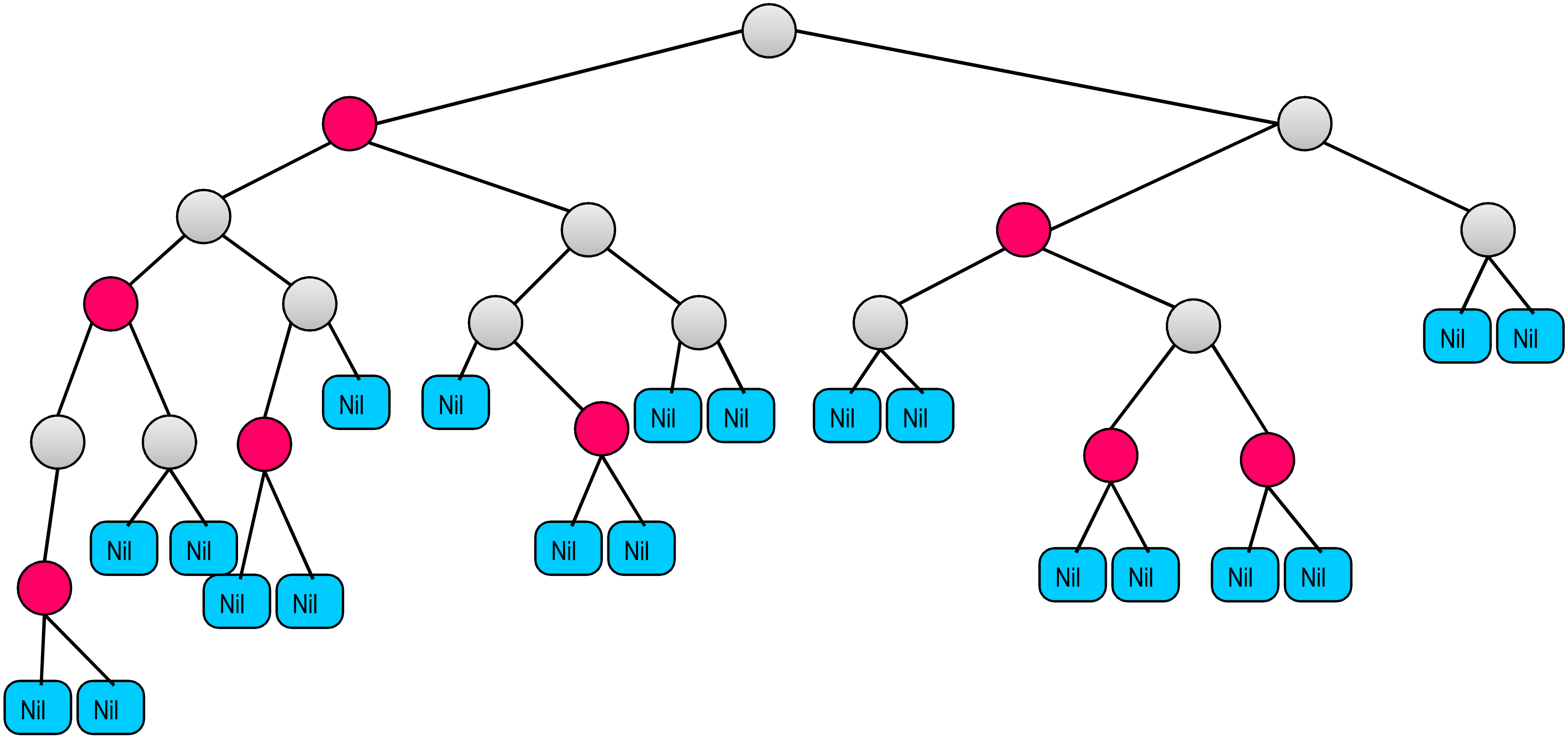
The rest of the nodes are called internal

Red-Black Properties

A binary search tree is a red-black tree if it satisfies the following red-black properties:

- Every node is either red or black
- The root is black
- Every leaf (Nil) is black
- If a node is red, then both its children are black
- For each node, all paths from the node to descendant leaves contain the same number of black nodes

Example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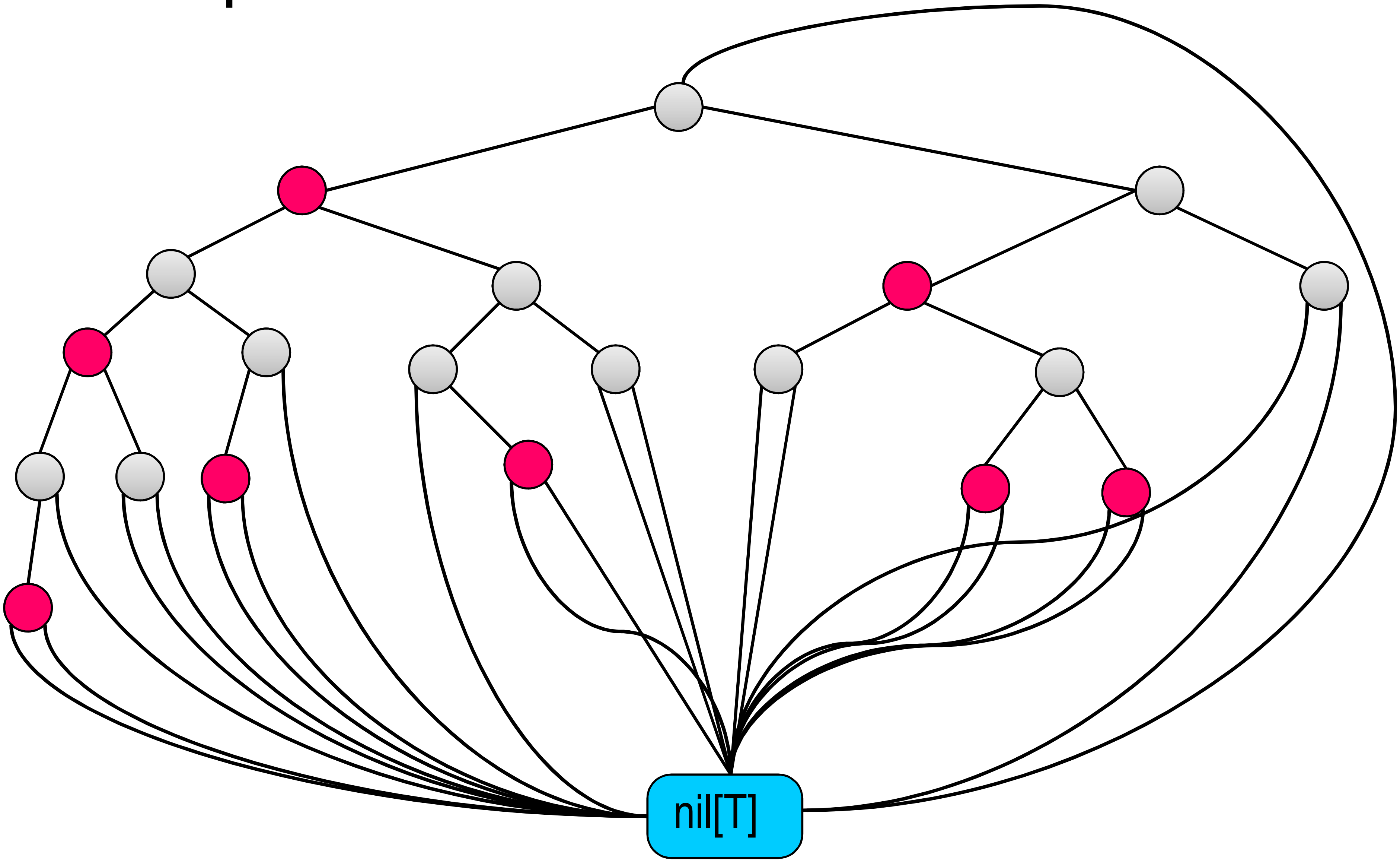
Example



Black Height

The number of black nodes on paths from node x to its descendant leaves in a red-black tree is called its black height, denoted bh(T)

Lemma

A red-black tree with n internal nodes has height at most 2·log(n + 1)

Proof

We show first that the subtree rooted at x contains at least $2^{bh(x)} - 1$ nodes

Induction on bh(x)

Base Case: If bh(x) = 0, then x is a leaf, nil[T]

In this case, $2^{bh(x)} - 1 = 2^0 - 1 = 0$ internal nodes

Black Height (cntd)

Inductive hypothesis: the claim is true for any y with height less than that of x

Inductive Case: Let bh(x) > 0 and has two children

The black height of the children is either bh(x) or bh(x) – 1, depending on its color

Since the children have smaller height, we can apply the induction hypothesis

Thus the subtree rooted at x contains at least

$$(2^{bh(x)-1}-1)+(2^{bh(x)-1}-1)+1=2^{bh(x)}-1$$

nodes

Black Height (cntd)

Suppose h is the height of the tree

By the red-black property at least half of nodes on every root-to-leaf path are black (not including the root)

Therefore the black height of the root is at least h/2

Thus

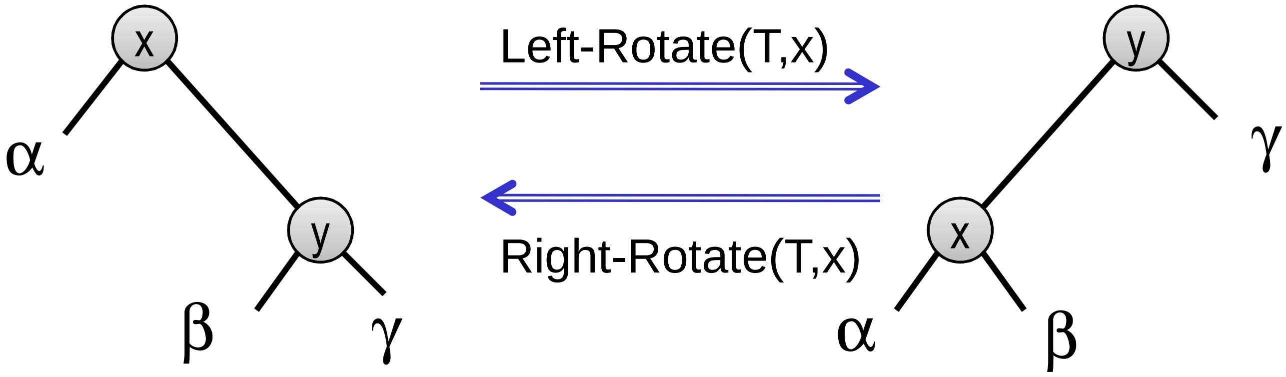
$$n \geq 2^{h/2} - 1$$

$$\log(n+1) \geq h/2$$

QED

Rotations

Sometimes we will need to rearrange pointers inside an RB-tree

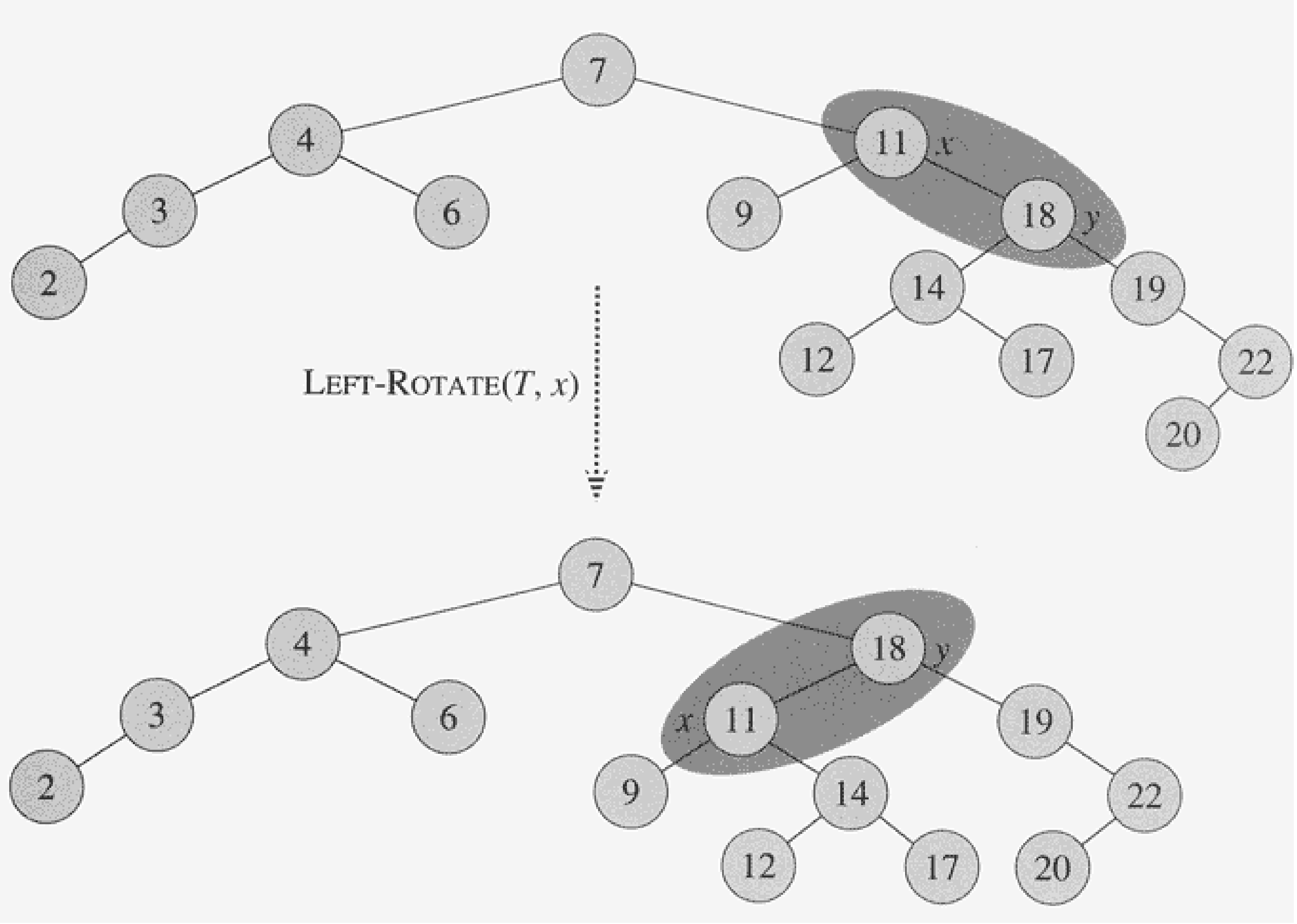


Rotations: Pseudocode

Left-Rotate(T,x)

```
set y:=right[x]
set right[x]:=left[y]
set parent[left[y]]:=x
set parent[y]:=parent[x]
if parent[x]=nil[T] then
    set root[T]:=y
else if x=left[parent[x]] then
         set left[parent[x]]:=y
     else
         right[parent[x]]:=y
set left[y]:=x
set parent[x]:=y
```

Rotations: Example



Insertion

Insertion for RB-trees is done in the same way as for ordinary binary search trees.

Except:

we should be careful about Nil links

the new node is colored red

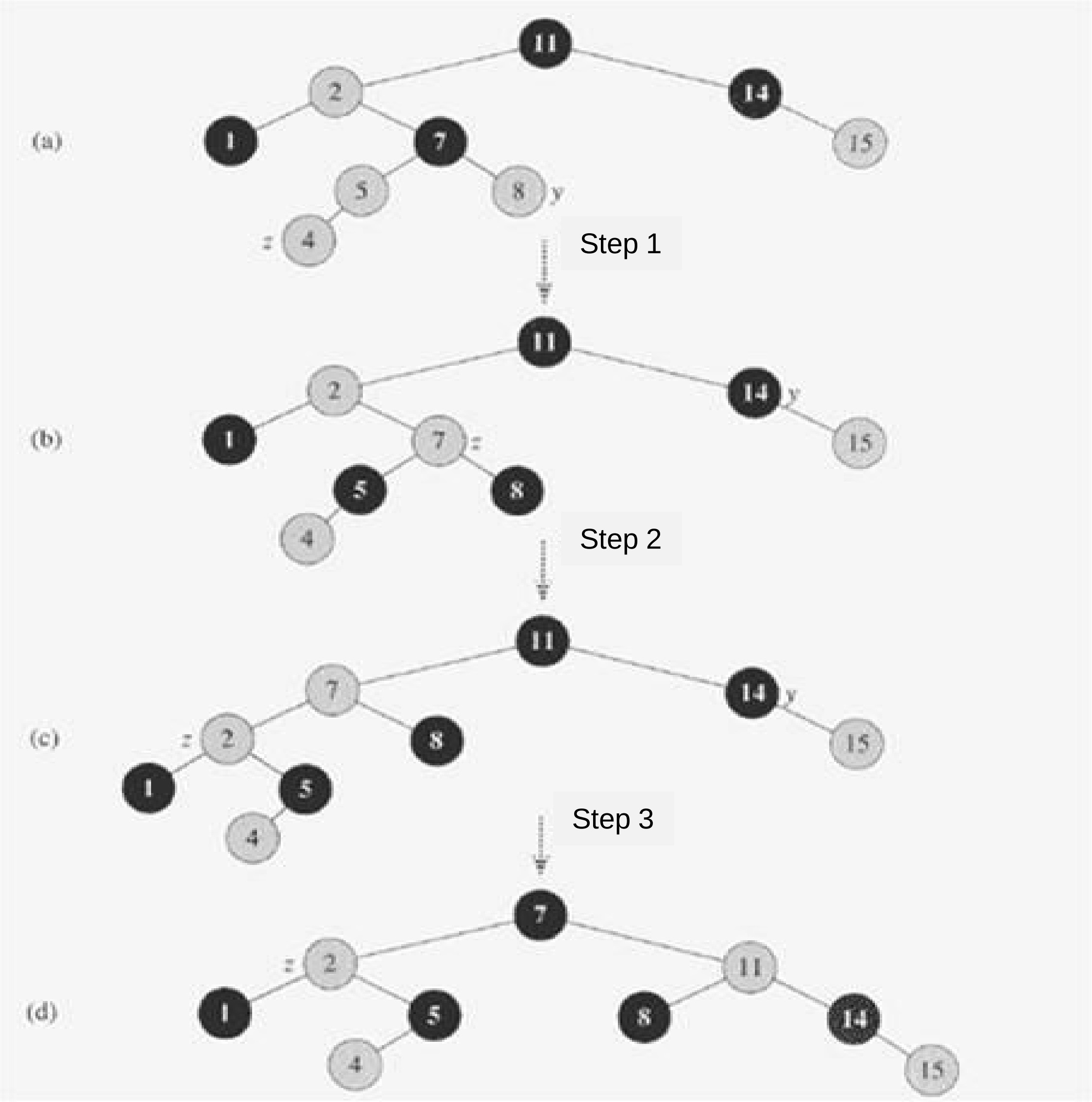
the resulting tree may not be an RB-tree, we need to fix it

Insertion: Pseudocode

RB-Insert(T,z)

```
set y:=Nil[T],   x:=root[T]
while x≠Nil[T] do
   set y:=x
   if key[z]<key[x] then set x:=left[x]
                    else set x:=right[x]
endwhile
set parent[z]:=y
if y=Nil[T] then set root[T]:=z
         else if key[z]<key[y] then set left[y]:=z
                               else set right[y]:=z
set left[z]:=Nil[T]  right[z]:=Nil[T]  color[z]:=RED
RB-Insert-FixUp(T,z)
```

Insertion: FixUp



FixUp: Pseudocode

```
RB-Insert-FixUp(T,z)
while color[parent[z]]=RED do
  if parent[z]=left[parent[parent[z]]] then do
    set y:=right[parent[parent[z]]]
    if color[y]=RED then do
      set color[parent[z]]:=BLACK  color[y]:=BLACK
      set color[parent[parent[z]]]:=RED
      set z:=parent[parent[z]]
    else do
      if z=right[parent[z]] then do
        set z:=parent[z]    Left-Rotate(T,z)
      set color[parent[z]]:=BLACK
      set color[parent[parent[z]]]:=RED
    else   (same as  then  with left and right swopped)
color[root[T]]:=BLACK
```

Soundness

Lemma

Given an RB-tree, algorithm RB-Insert produces an RB-tree

Proof

We show that the main loop preserves the following loop invariant:

(a) Node z is red

(b) If parent[z] is the root then parent[z] is black

(c) If there is a violation of the red-black property, there is at most one violation, and it is of either property 2 or property 4.

If there is a violation of property 2, it occurs because z is the root and red

If there is a violation of property 4, it occurs because both z and parent[z] are red

Soundness: Initialization

(a) When RB-Insert-FixUp is called, z is a red node

(b) If p[z] is the root, it hasn't changed yet, and so is black

(c) Properties 1, 3, and 5 are true

If property 2 is violated, then the red root is just added, that is the root is z. In this case the tree does not have internal vertices. Property 4 is not violated in this case

If property 4 is violated, then, since the children of z are black sentinels and the tree had no prior violations, the violation must be because both z and parent[z] are red.

There are no other violations of red-black properties.

Soundness: Termination

When the big loop of RB-Insert-FixUp terminates parent[z] is black (if z is the root then parent[z] is the sentinel)

Therefore there is no violation of property 4.

If property 2 is violated, then the last line of RB-Insert-FixUp restores it.

Soundness: Maintenance

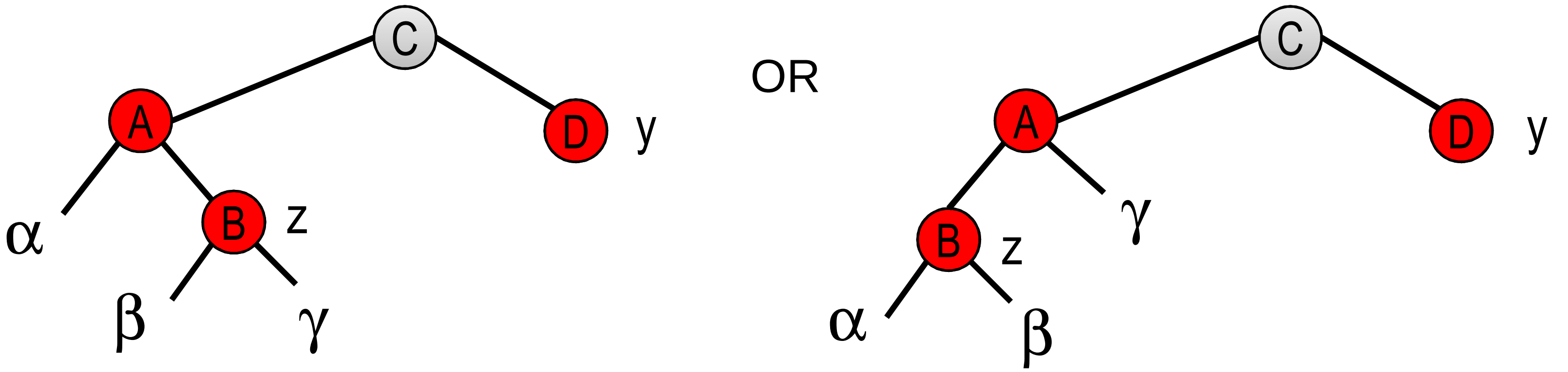
There are several cases to consider. We only consider those in which parent[z] is the left child of parent[parent[z]]

This is the situation given in the pseudocode

If parent[parent[z]] does not exist then parent[z] is the root, and by part (b) is black.

Therefore if the loop does not terminate here, parent[parent[z]] exists

Case 1. z’s uncle y is red

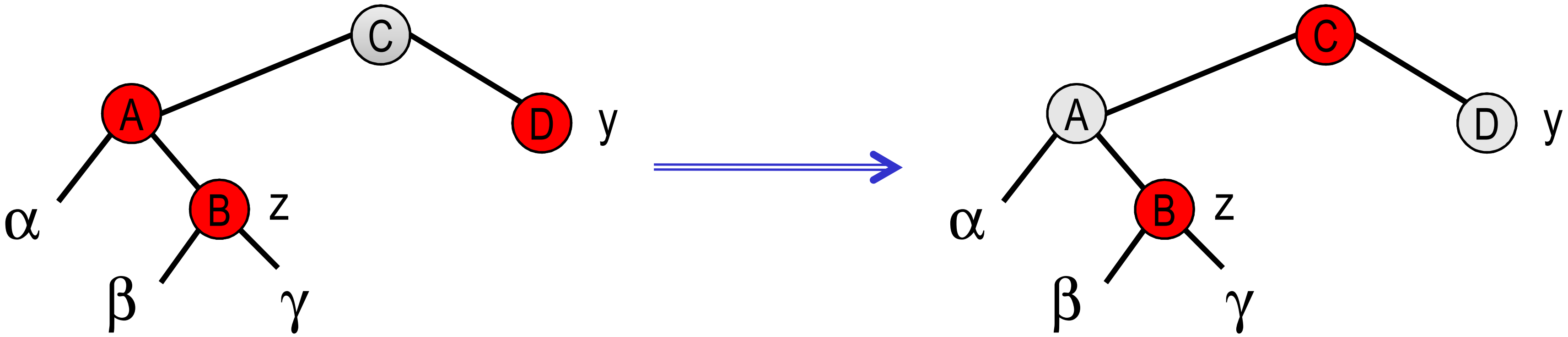


Soundness: Maintenance (cntd)

Both y and parent[z] are red, and parent[parent[z]] is black

We color y and parent[z] black, and parent[parent[z]] – red

We then set parent[parent[z]] to be the new node z



Show that the loop invariant is preserved

(a) Since we color parent[parent[z]] red, the new z (denoted z’) is red

Soundness: Maintenance (cntd)

(b) We have parent[z’] = parent[parent[parent[z]]], and the color of this node does not change.

If it is the root, it is black

(c) It is easy to see that Case 1 maintains property 5, and does not violate properties 1 and 3

If z’ is the root, then Case 1 corrected the only violation of property 4. Since z’ is red and it is the root, property 2 becomes violated, and it is due to z’

If z’ is not the root, then Case 1 has not created a violation of property 2.

Case 1 corrected the violation of property 4 due to z and parent[z].

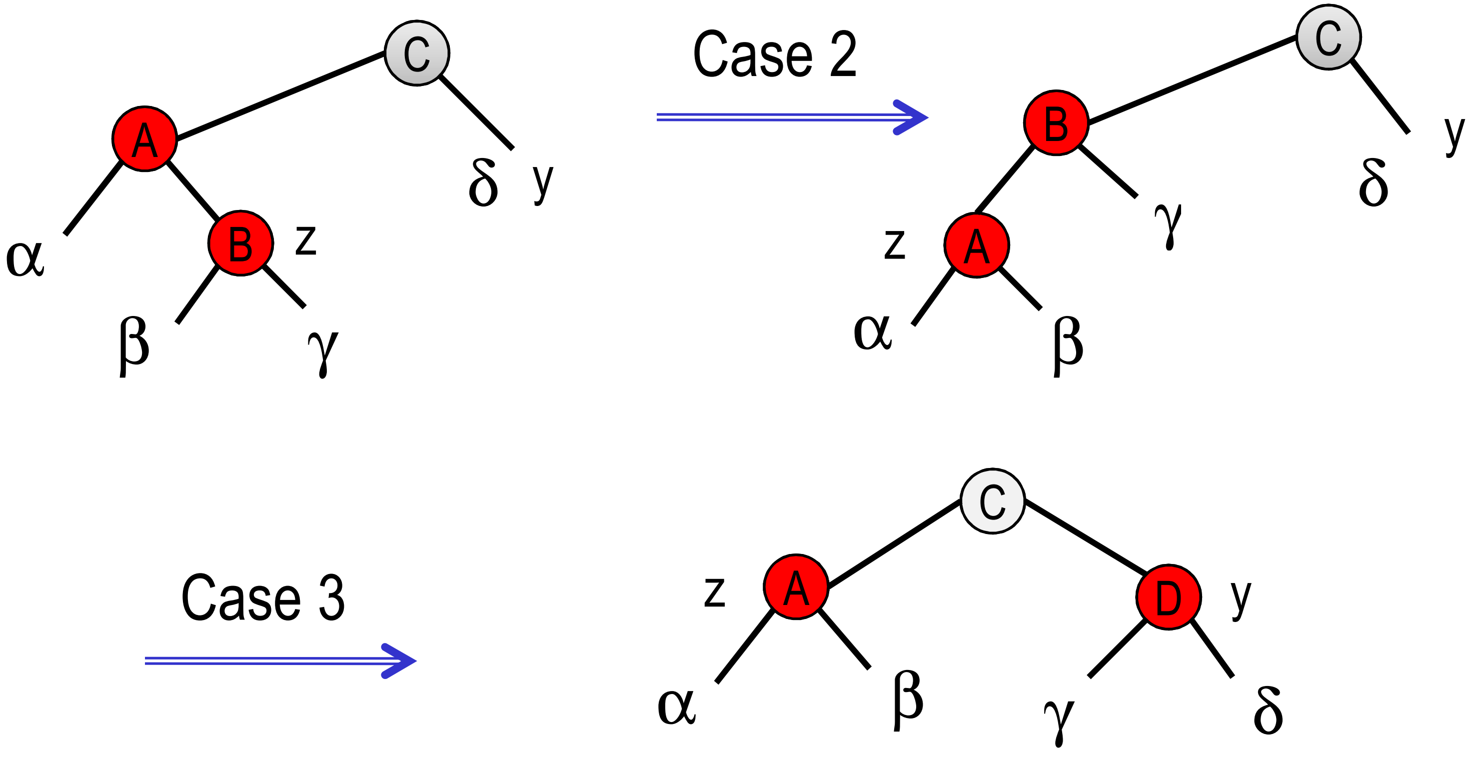
If parent[z’] is black then we are done

Otherwise z’ and parent[z’] create the only violation of property 4

Soundness: Maintenance (cntd)

Case 2. z’s uncle y is black and z is a right child

Case 3. z’s uncle y is black and z is a left child



Soundness: Maintenance (cntd)

Case 2. z’s uncle y is black and z is a right child

Case 3. z’s uncle y is black and z is a left child

In Case 2 the algorithm performs rotation and we get Case 3

Since both nodes being rotated are red no RB properties change

Moreover, parent[parent[z]] exists and it does not change when Case 2 is transformed into Case 3

In Case 3 we perform same rotations and color changes that does not affect property 5

Then since there are no longer 2 red nodes in a row, the while loop is not executed again, for parent[z] is now black

Soundness: Maintenance (cntd)

(a) Case 2 makes z points to parent[z], which is red

Nothing changes in Case 3

(b) Case 2 does not change the color of the root

Case 3 makes parent[z] black, so if it is the root, it is black

(c) As in Case 1, properties 1, 3, and 5 are maintained in Cases 2 and 3

Since node z is not the root, there is no violation of property 2

Cases 2 and 3 do not introduce any violation of property 2, since the only node that becomes red also becomes a child of a black node by the rotation in Case 3

Cases 2 and 3 correct the only violation of property 4 and do not introduce another violation

Running Time

Since the height of an RB-tree with n nodes is O(\log n), RB-Insert takes O(log n) time, except for RB-Insert-FixUp

In RB-Insert-FixUp the while loop repeats only if Case 1 takes place, and in this case z moves up 2 levels.

The total number of iterations is O(log n)

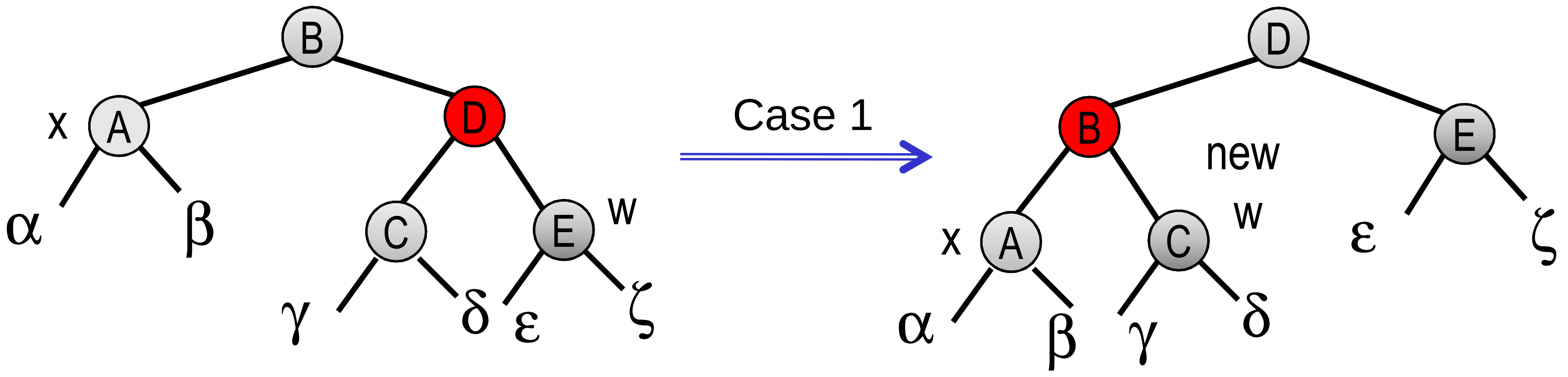
RB-Deletion

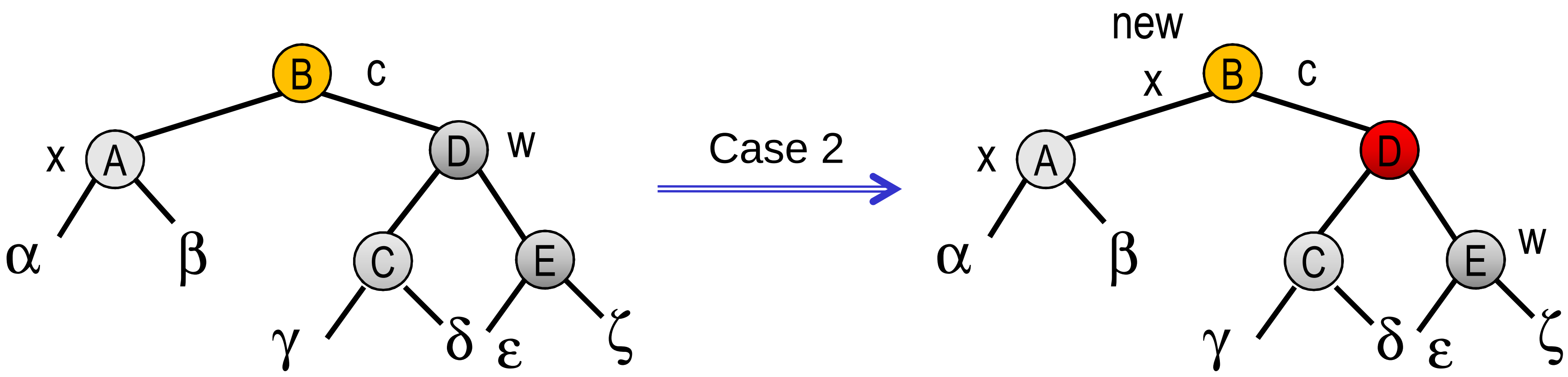
RB-Delete also first runs the regular Deletion algorithm, and then fixes violations of the RB properties

RB-Delete(T,z)

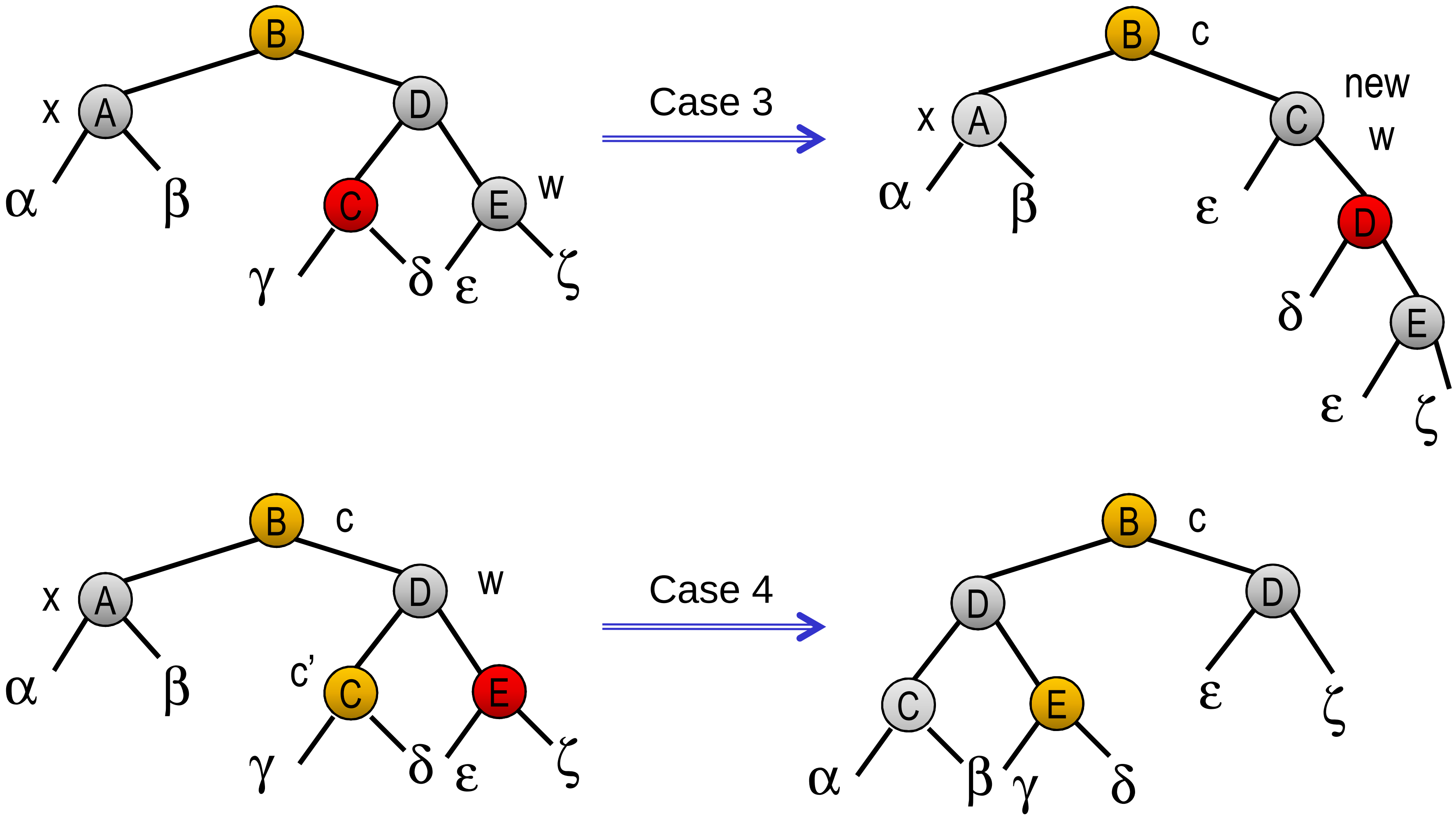
```
if left[z]=Nil[T] or right[z]=Nil[T] then set y:=z
                       else set y:=Tree-Successor(z)
if left[y]≠Nil[T] then set x:=left[y]
                  else set x:=right[y]
if x≠Nil[T] then set parent[x]:=parent[y]
            else set x:=right[x]
if parent[y]=Nil[T] then set root[T]:=x
 else if y=left[parent[y]]:=x then left[parent[y]:=x
                                else right[parent[y]]:=x
if y≠z then do
  set key[z]:=key[y]
  copy y’s data into z
if color[y]=BLACK then  RB-Delete-FixUp(T,x)
return y
```

Deletion: Pictures





Deletion: Pictures



new x = root[T]

RB-Delete-FixUp

RB-Delete-FixUp(T,x)

```
while x≠root[T] and color[x]=BLACK do
  if x=left[parent[x]] then do
    set w:=right[parent[x]]
Case 1
    if color[w]=RED then do
      set color[w]:=BLACK   color[parent[x]]:=RED
      Left-Rotate(T,parent[x])   set w:=right[parent[x]]
Case 2
    if color[left[w]]=BLACK and color[right[w]]=BLACK then
      set color[w]:=RED  x:=parent[x]
    else if color[right[w]]=BLACK then do
Case 3
           set color[left[w]]:=BLACK   color[w]:=RED
           Right-Rotate(T,w)   set w:=right[parent[x]]
Case 4
         set color[w]:=color[parent[x]]
         set color[parent[x]]:=BLACK
         set color[right[w]]:=BLACK
         Left-Rotate(T,parent[x])   set x:=root[T]
  else (same as then  flipping left and right)
set color[x]:=BLACK
```