# Sorting in Linear Time

Data Structures and Algorithms

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# Comparison Sorts

The only test that all the algorithms we have considered so far is comparison

The only information we obtain about an input sequence $\langle a_1, a_2, \ldots, a_n \rangle$ is given by $a_i < a_j, a_i \leq a_j, a_i = a_j, a_i \geq a_j$ or $a_i > a_j$

We will assume that all the input numbers are different.

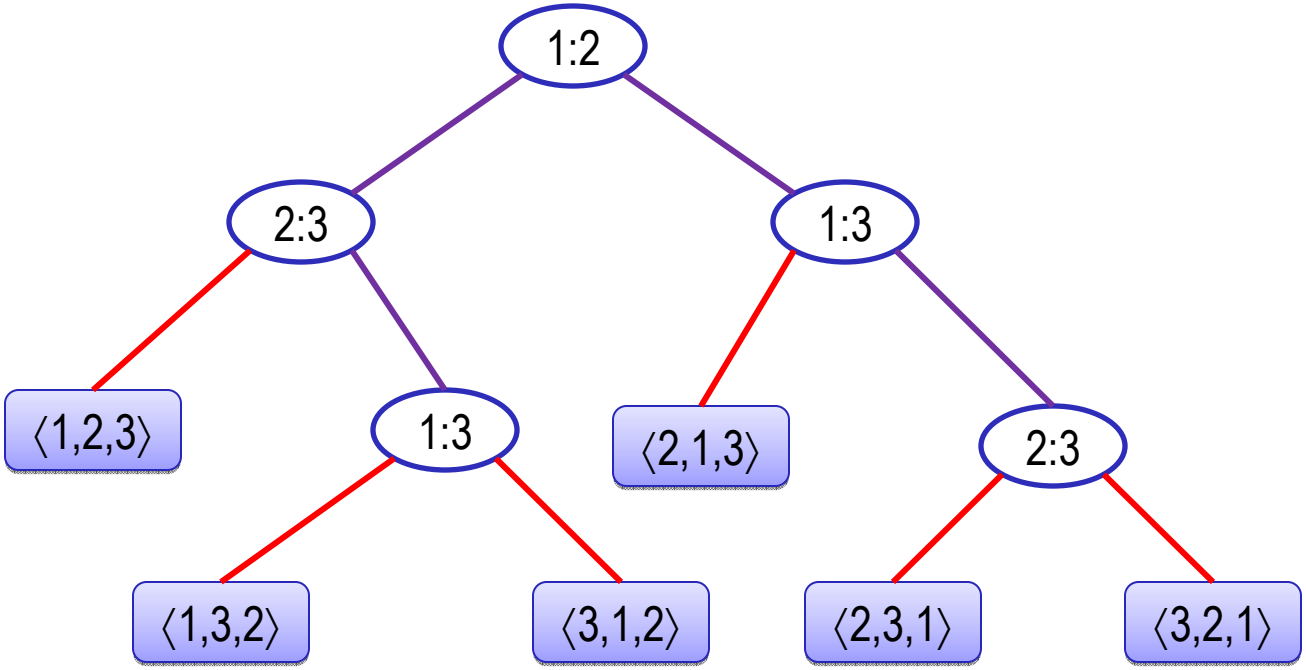
Then the it suffices to use only one comparison $a_i \leq a_j$

# The Decision Tree Model

Observe that for different input permutations, even if they differ only in the order of elements, our algorithm must perform different actions

Therefore, we view its work as recognizing a permutation

It is convenient to represent this process as a decision tree



# Decision Trees

Let the input sequence contain n elements

We assume they are numbers from 1 to n

Therefore, the input sequence is a permutation $\langle \pi(1), \pi(2), \ldots, \pi(n) \rangle$

Each internal node is labeled by i : j for some i and j in the range $1 \leq i,j \leq n$

Execution of the algorithm is a path from the root to a leaf

At each internal node a comparison $a_i \leq a_j$ is made

Depending on the outcome either comparisons on the left or the right subtree

When we come to a leaf , the algorithm recognizes a particular ordering

# Lower Bound

**Theorem**

Any comparison sort algorithm requires $\Omega(mn)$ comparisons in the worst case

**Proof**

There are n! permutations of n elements.

The algorithm should recognize all of them

Therefore the decision tree contains $k \geq n!$ leaves

The complete binary tree has $2^h$ leaves where h is the height of the tree, the length of the longest root-to-leaf path

$$n! \leq k \leq 2^h$$

$$h \geq \log(n!) = \Omega(n \log n)$$

Finally, note that the height of the tree is running time in the worst case

# Counting Sort

Suppose we are allowed to do more than just comparisons

We also assume that the input numbers are in the range 0 to k

We give an algorithm that works in $\Theta(n)$ time provided $k = O(n)$

The idea is: for each input element x to count how many elements are less than x, and use this information to place x to the right place in the output sequence

We use A[1..n] the input array
B[1..n] the output array
C[0..k] auxiliary storage

Note that we do not assume that all elements are different

# Counting Sort: Algorithm

Counting-Sort(A,B,k)

```
for i=0 to k do
   set C[i]:=0
for j=1 to n do
   set C[A[j]]:=C[A[j]]+1   /* C[i] contains the
                            /* number of elements equal to i
for i=1 to k do
   set C[i]:=C[i]+C[i-1]    /* C[i] contains the
          /* number of elemnts less than or equal to i
for j=n downto 1 do
   set B[C[A[j]]]:=A[j]
   set C[A[j]]:=C[A[j]]-1
endfor
```

# Counting Sort: Soundness and Running Time

**Theorem**

Counting Sort correctly sorts a sequence of $n$ elements in $O(n)$ time, provided $k = O(n)$

**Proof**

Running time:

Each of the 4 for loops is iterated at most $\max\{n, k\}$ times

Soundness:

Trace the content of arrays

QED

# Stable Sorting

For the next algorithm it is important how Counting Sort and other sorting algorithms deal with equal elements

A sorting algorithm is called stable if it preserves the order of equal elements in the input sequence, that is,

if $a_i = a_j$ and i < j in the input sentence, then $a_i$ goes first in the output sequence

**Lemma**

Counting Sort is stable

**Proof**

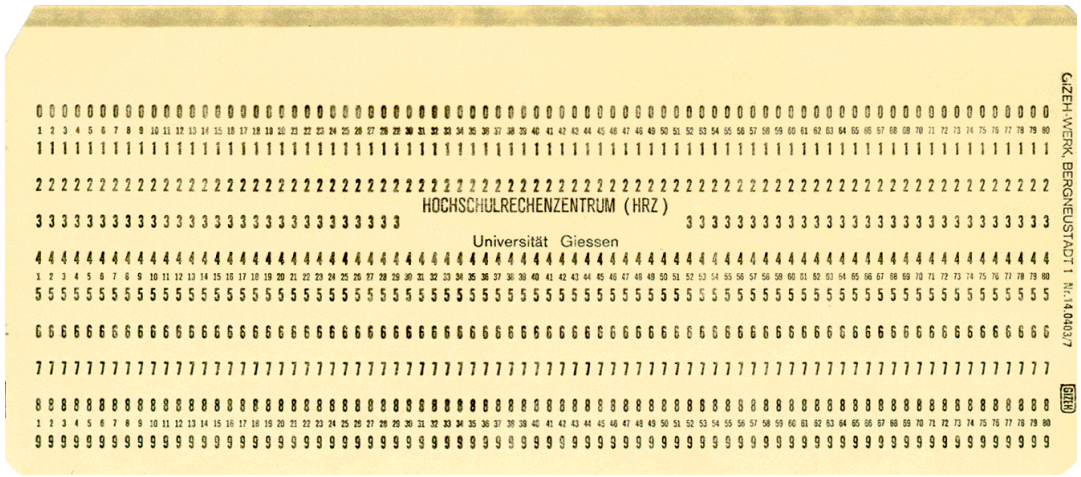
Suppose that A[i] = A[j] and i < j.

Then in the last for loop we first output A[j] then A[i]

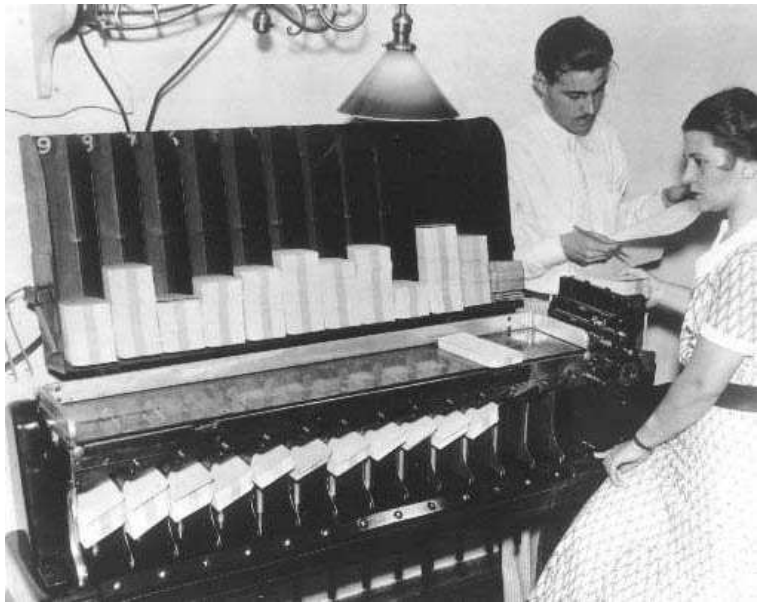
Moreover, between the outputs C[A[i]] = C[A[j]] is decremented.

Therefore A[i] is placed into B with smaller index than A[j]

# Radix Sort




Punch card



Radix sorting machine

# Radix Sort: The Idea

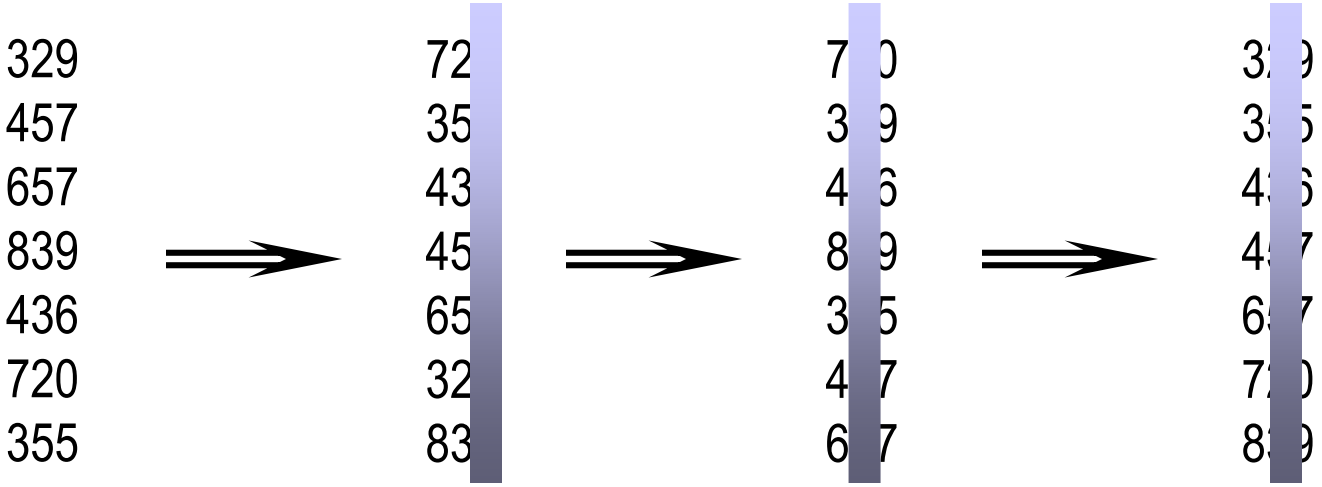
If the elements of the input sequence are d-digit integers, we can first sort the according the most significant digit, then the second most significant digit, etc.

This, however, requires a lot of copying and auxiliary storage

Radix Sort:

Sort in place according to the least significant digit

But use a stable sorting algorithm!!



# Radix Sort: Algorithm

Radix-Sort(A,d)

```
for i=0 to d do
    use a stable sorting algorithm to sort array A on
   digit i
endfor
```

**Theorem**

Radix Sort correctly sorts a sequence of $n$ d-digit numbers in which each digit can take up to $k$ different values in $O(d(n+k))$ time.

# Radix Sort: Analysis

Running time:

RadixSort considers $d$ digits in turn

each pass takes $O(n + k)$ time (when using, say, Counting Sort)

Soundness:

By induction on the number $d$ of digits

Base Case: $d = 1$ Counting Sort just sorts everything

Induction Step: Suppose algorithm works correctly for $d - 1$

Radix sort on d-digit numbers is equivalent to $d$ runs of radix sort on smaller d-1 – numbers, followed by sorting on digit $d$.

By Induction Hypothesis Radix Sort sorts correctly on lower $d - 1$ digits

# Radix Sort: Analysis (cntd)

Before the last sort on digit d, all the numbers are properly sorted accordingly their last d – 1 digits.

When sort on digit d, consider two elements a and b with d-th digits $a_d$ and $b_d$ respectively

(1) If $a_d < b_d$ then the algorithm will put a before b, which is correct

(2) If $a_d > b_d$ the algorithm will put b before a, which is again correct

(3) If $a_d = b_d$ the algorithm will leave a and b in the same order as before, because it is stable.

This order is again correct, for the relative order of a and b depends in this case on the lower d – 1 digits.

QED

# Bucket Sort: The Idea

Counting Sort and Radix Sort achieve significant speed up against comparison algorithms because they use certain assumptions about the input numbers:

They are small integers, or integers of bounded size.

Bucket Sort uses an assumption about the distribution of these numbers:

They are taken uniformly at random from [0; 1)

Then we:

Split [0;1) into n equal intervals (buckets)

Put every input element into the corresponding bucket

Sort each bucket

Concatenate the buckets

# Bucket Sort: Algorithm

A[1..n] the input array

B[1..n] heads of buckets

i-th bucket is organized as a list with a pointer B[i] to the top of the list

Bucket-Sort(A,d)

```
set n:=length(A)
for i=0 to n do
   insert A[i] into list B[⌊n·A[i]⌋]
endfor
for i=0 to n do
   sort list B[i] with insertion sort
concatenate the lists  B[1],B[2],...,B[n]
```

# Bucket Sort: Soundness

**Theorem**

Bucket Sort correctly sorts a sequence of numbers from the interval [0;1)

**Proof**.

Obvious

# Bucket Sort: Running Time

**Theorem**

The expected running time of Bucket Sort is O(n)

**Proof**.

Clearly, the first for loop takes $\Theta(n)$ time to complete

Each iteration of the second for loop contributes $O(n_i^2)$ time where $n_i$ is the number of elements in the i-th bucket.

Therefore

$$T(n) = \Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)$$

# Bucket Sort: Running Time

**Proof** (cntd).

Take the expectation of both sides

$$E[T(n)] = E\left[\Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)\right]$$

$$= \Theta(n) + E\left[\sum_{i=0}^{n-1} O(n_i^2)\right]$$

$$= \Theta(n) + \sum_{i=0}^{n-1} O(E[n_i^2])$$

We show that $E[n_i^2] = 2 - \frac{1}{n}$

# Bucket Sort: Running Time

**Proof** (cntd).

Define indicator random variables

$X_{ij} = 1$ if and only if A[j] falls into bucket i, otherwise $X_{ij} = 0$

Thus $n_i = \sum_{j=1}^{n} X_{ij}$

We get $E[n_i^2] = E\left[\left(\sum_{i=0}^{n-1} X_{ij}\right)^2\right]$

$$= E\left[\sum_{j=1}^{n}\sum_{k=1}^{n} X_{ij}X_{ik}\right] = E\left[\sum_{i=1}^{n} X_{ij}^2 + \sum_{j=1}^{n}\sum_{1\le k\le n, k\ne j} X_{ij}X_{ik}\right]$$

$$= \sum_{i=1}^{n} E[X_{ij}^2] + \sum_{j=1}^{n}\sum_{1\le k\le n, k\ne j} E[X_{ij}X_{ik}]$$

# Bucket Sort: Running Time

**Proof** (cntd).

Then $E[X_{ij}^2] = 1 \cdot \frac{1}{n} + 0 \cdot (1 - \frac{1}{n}) = \frac{1}{n}$

and, since $X_{ij}$ and $X_{ik}$ are independent

$$E[X_{ij} X_{ik}] = E[X_{ij}] \cdot E[X_{ik}] = \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$$

Finally $$E[n_i^2] = \sum_{j=1}^{n} \frac{1}{n} + \sum_{j=1}^{n} \sum_{1 \le k \le n,\, j \ne k} \frac{1}{n^2}$$

$$= n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n^2}$$

$$= 1 + \frac{n-1}{n} = 2 - \frac{1}{n}$$

# Bucket Sort: Running Time

**Proof** (cntd).

For the running time we now have

$$E[T(n)] = \Theta(n) + \sum_{i=0}^{n-1} O(E[n_i^2])$$

$$= \Theta(n) + \sum_{i=0}^{n-1} O(2 - \frac{1}{n})$$

$$= \Theta(n) + O(n) = \Theta(n)$$

QED