Binary Search Trees

CMPT 225

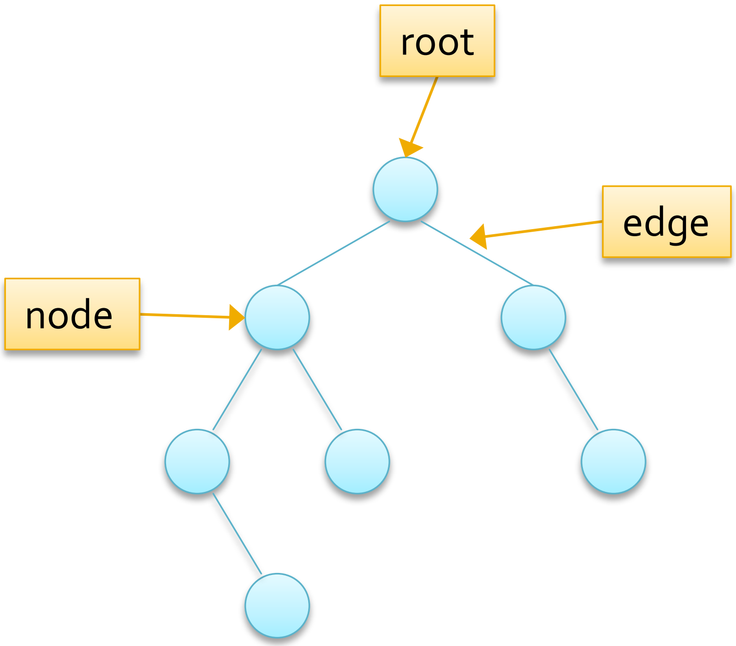
Objectives

- Understand tree terminology
- Understand and implement tree traversals
- Define the binary search tree property
- Implement binary search trees
- Implement the TreeSort algorithm

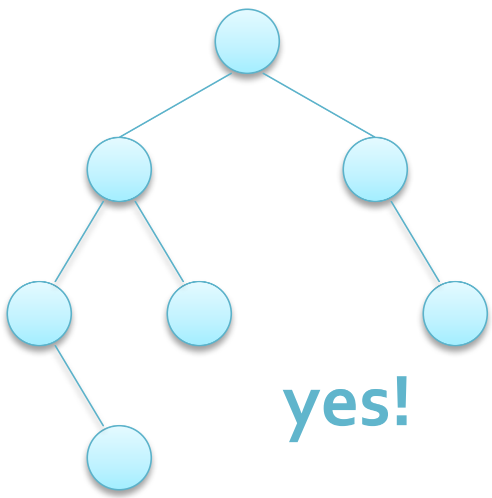
Tree Terminology

Trees

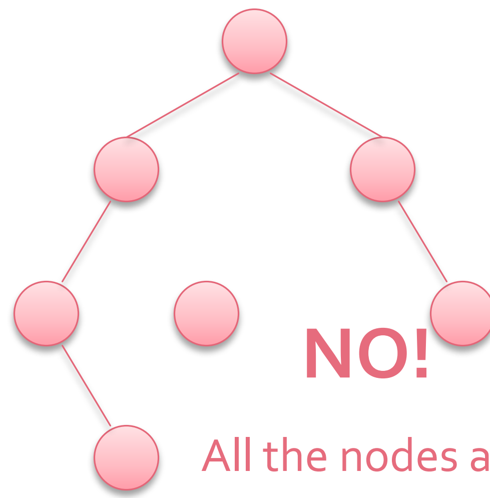
- A set of nodes (or vertices) with a single starting point
  - called the *root*
- Each node is connected by an *edge* to another node
- A tree is a connected graph
  - There is a path to every node in the tree
  - A tree has one less edge than the number of nodes



Is it a Tree?

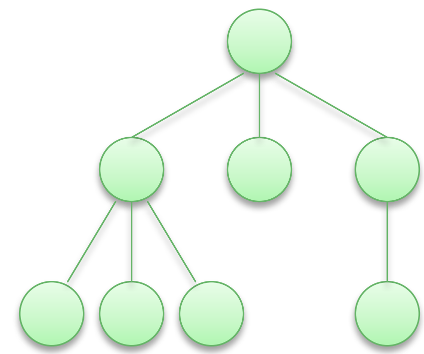


yes!

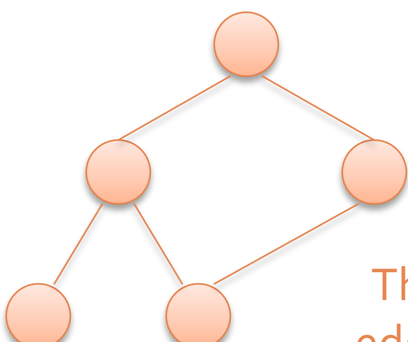


NO!

All the nodes are not connected

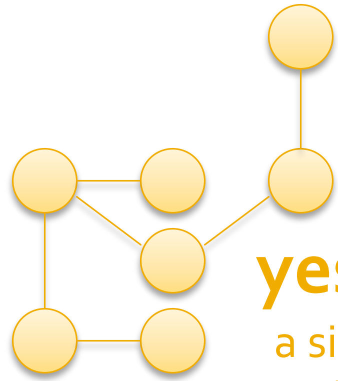


yes! (but not a *binary* tree)



NO!

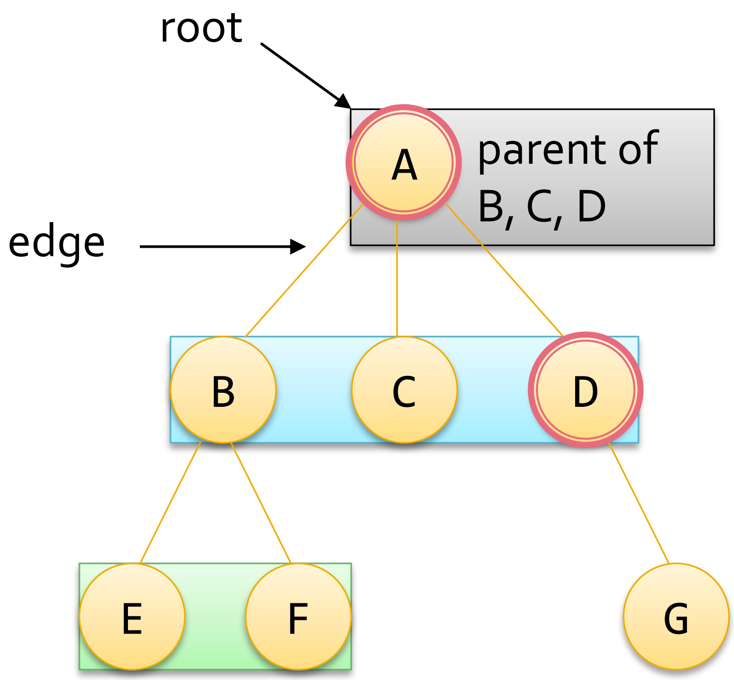
There is an extra edge (5 nodes and 5 edges)



yes! (it's actually a similar graph to the blue one)

Tree Relationships

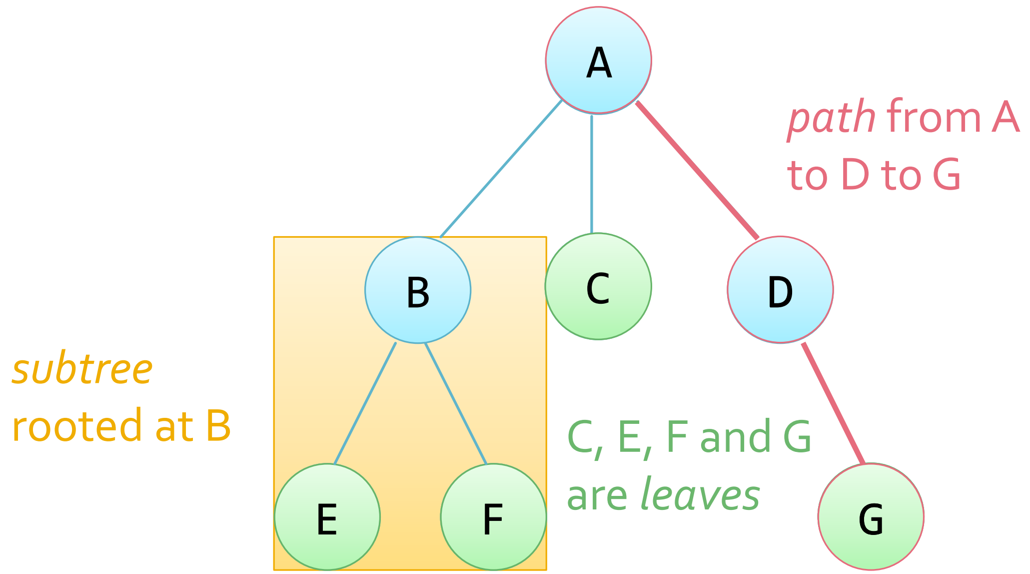
- Node *v* is said to be a *child* of *u*, and *u* the *parent* of *v* if
  - There is an edge between the nodes *u* and *v*, and
  - *u* is above *v* in the tree,
- This relationship can be generalized
  - E and F are *descendants* of A
  - D and A are *ancestors* of G
  - B, C and D are *siblings*
  - F and G are?



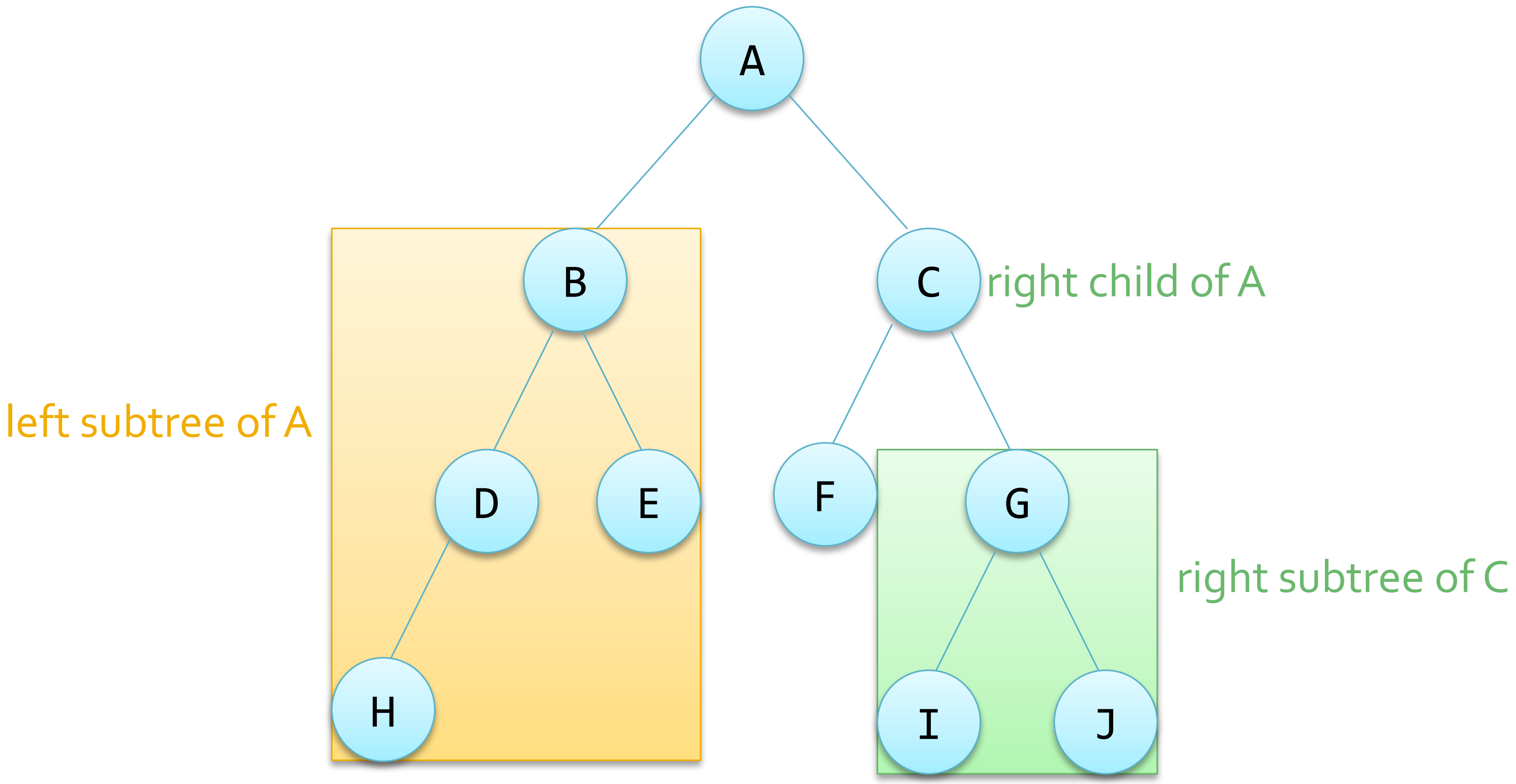
More Tree Terminology

- A *leaf* is a node with no children
- A *path* is a sequence of nodes $v_1 \ldots v_n$
  - where v_i is a parent of v_{i+1} ($1 \leq i \leq n$)
- A *subtree* is any node in the tree along with all of its descendants
- A *binary tree* is a tree with at most two children per node
  - The children are referred to as *left* and *right*
  - We can also refer to left and right subtrees

Tree Terminology Example



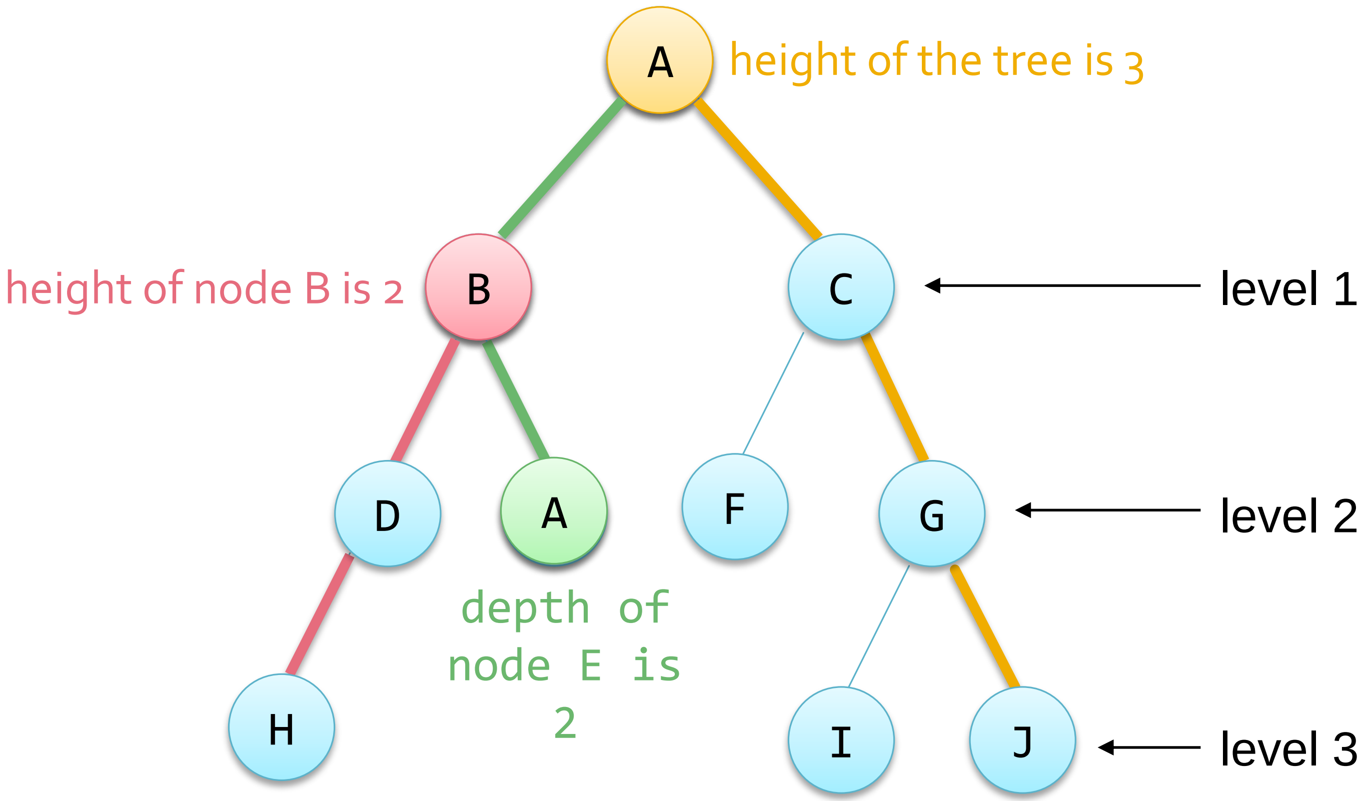
Binary Tree Terminology



Measuring Trees

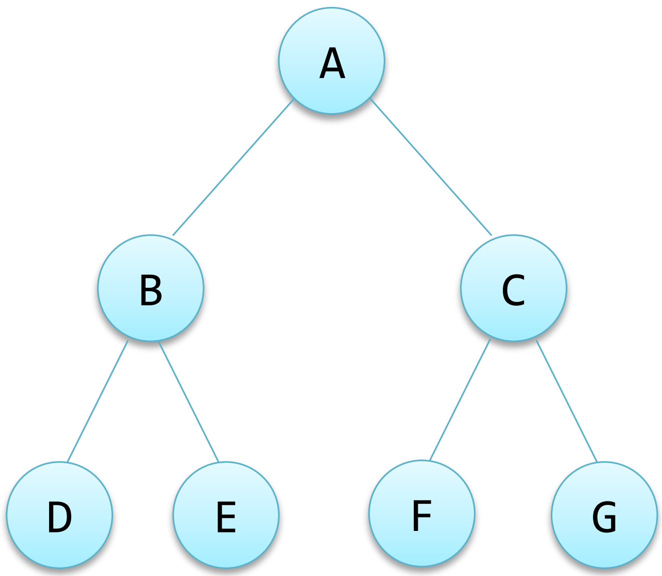
- The *height* of a node *v* is the length of the longest path from *v* to a leaf
  - The height of the tree is the height of the root
- The *depth* of a node *v* is the length of the path from *v* to the root
  - This is also referred to as the *level* of a node
- Note that there is a slightly different formulation of the height of a tree
  - Where the height of a tree is said to be the number of different *levels* of nodes in the tree (including the root)

Height of a Binary Tree



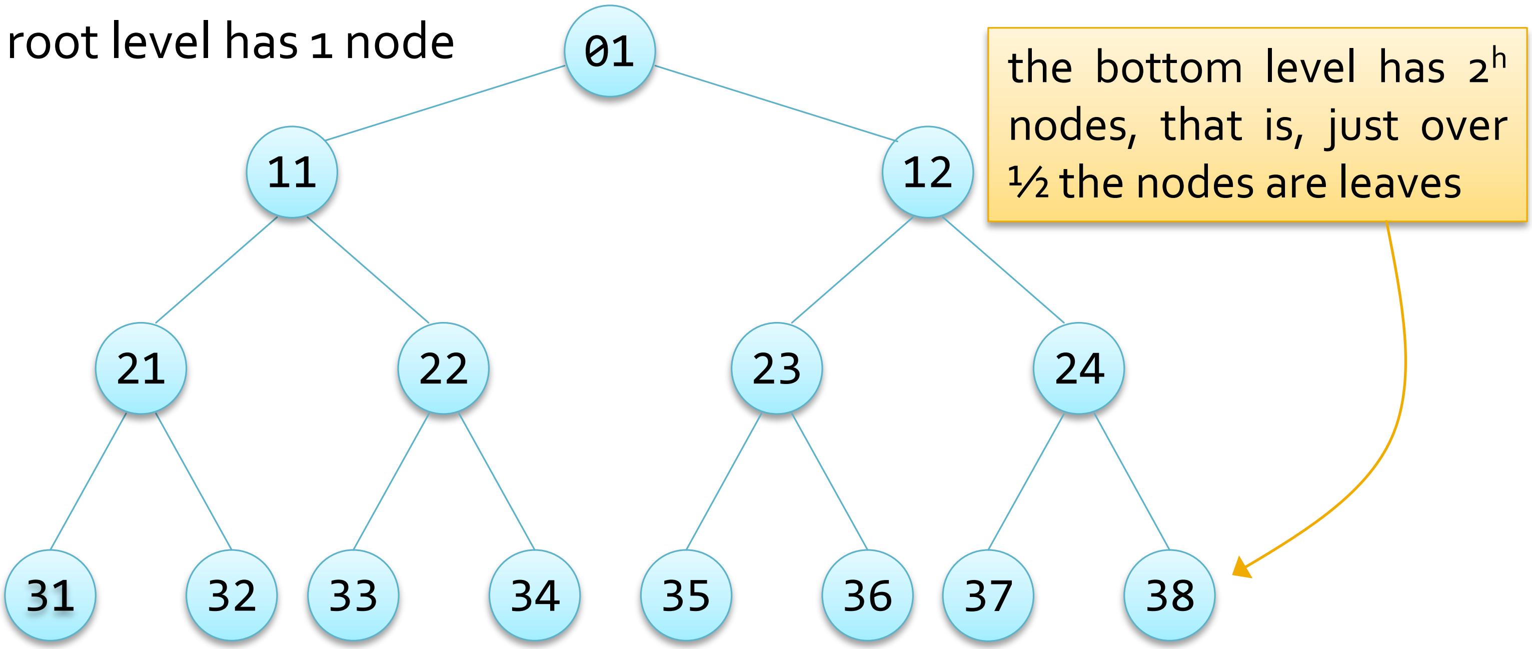
Perfect Binary Trees

- A binary tree is *perfect*, if
  - No node has only one child
  - And all the leaves have the same depth
- A perfect binary tree of height h has
  - $2^{h+1} - 1$ nodes, of which 2^h are leaves
- Perfect trees are also *complete*



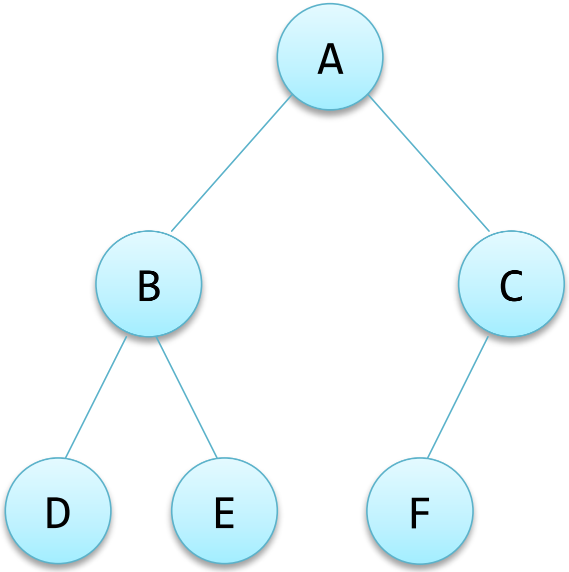
Nodes in a Perfect Tree

- Each level doubles the number of nodes
  - Level 1 has 2 nodes (2^1)
  - Level 2 has 4 nodes (2^2) or 2 times the number in Level 1
- Therefore a tree with h levels has $2^{h+1} - 1$ nodes
  - The root level has 1 node



Complete Binary Trees

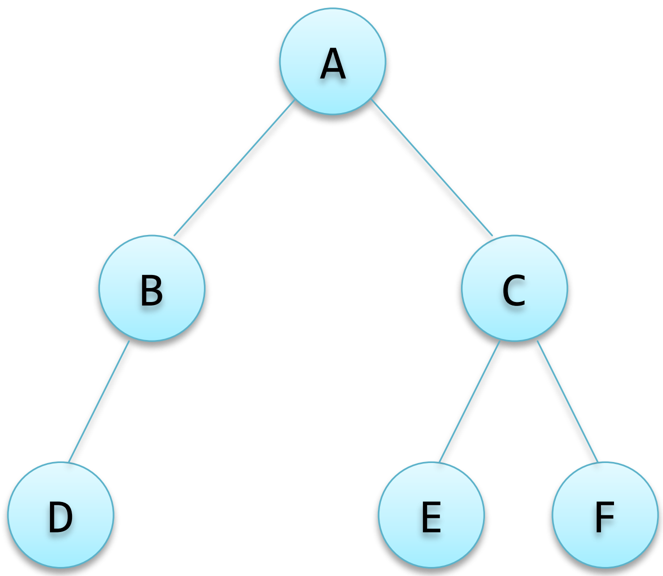
- A binary tree is *complete* if
  - The leaves are on at most two different levels,
  - The second to bottom level is completely filled in and
  - The leaves on the bottom level are as far to the left as possible

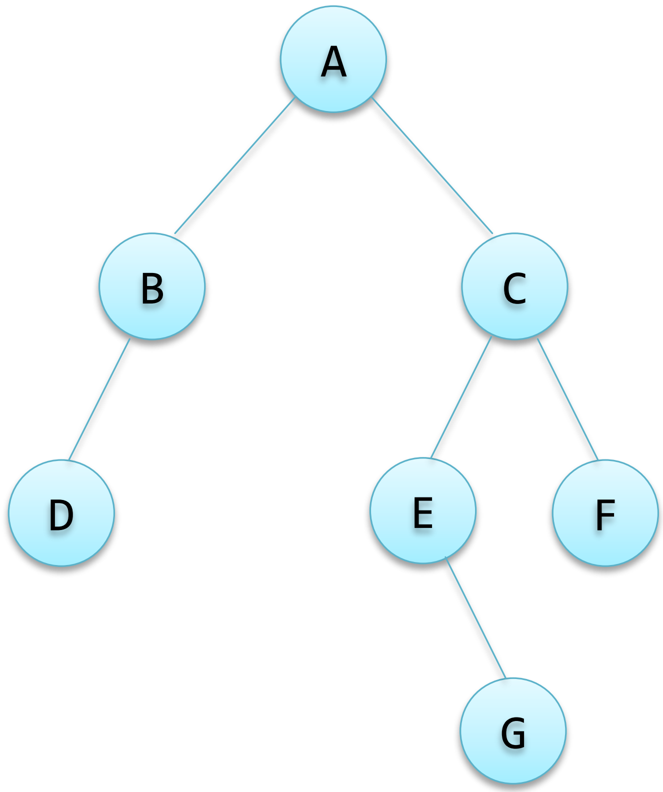


Balanced Binary Trees

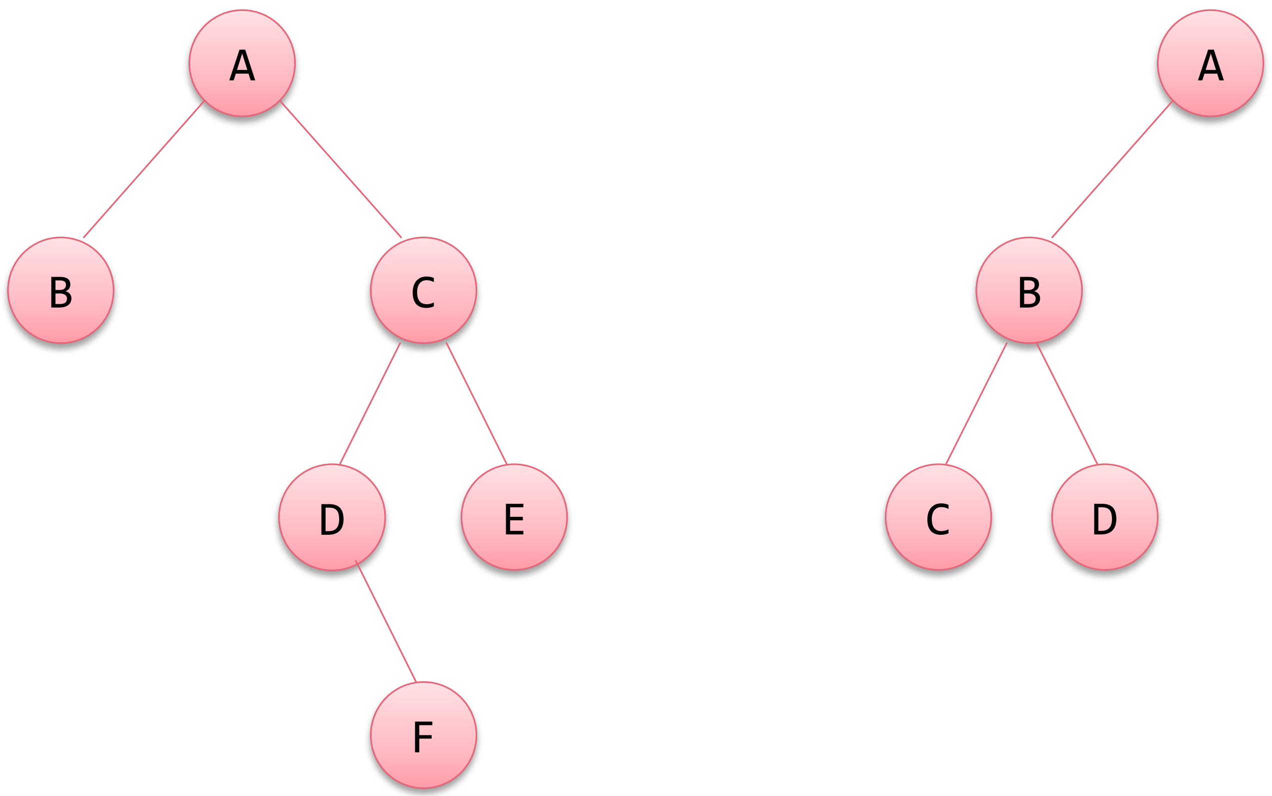
- A binary tree is *balanced* if
  - Leaves are all about the same distance from the root
  - The exact specification varies
- Sometimes trees are balanced by comparing the height of nodes
  - e.g. the height of a node's right subtree is at most one different from the height of its left subtree
- Sometimes a tree's height is compared to the number of nodes
  - e.g. red-black trees

Balanced Binary Trees





Unbalanced Binary Trees



Tree Traversals

Binary Tree Traversals

- A traversal algorithm for a binary tree visits each node in the tree
  - Typically, it will do something while visiting each node!
- Traversal algorithms are naturally recursive
- There are three traversal methods
  - Inorder
  - Preorder
  - Postorder

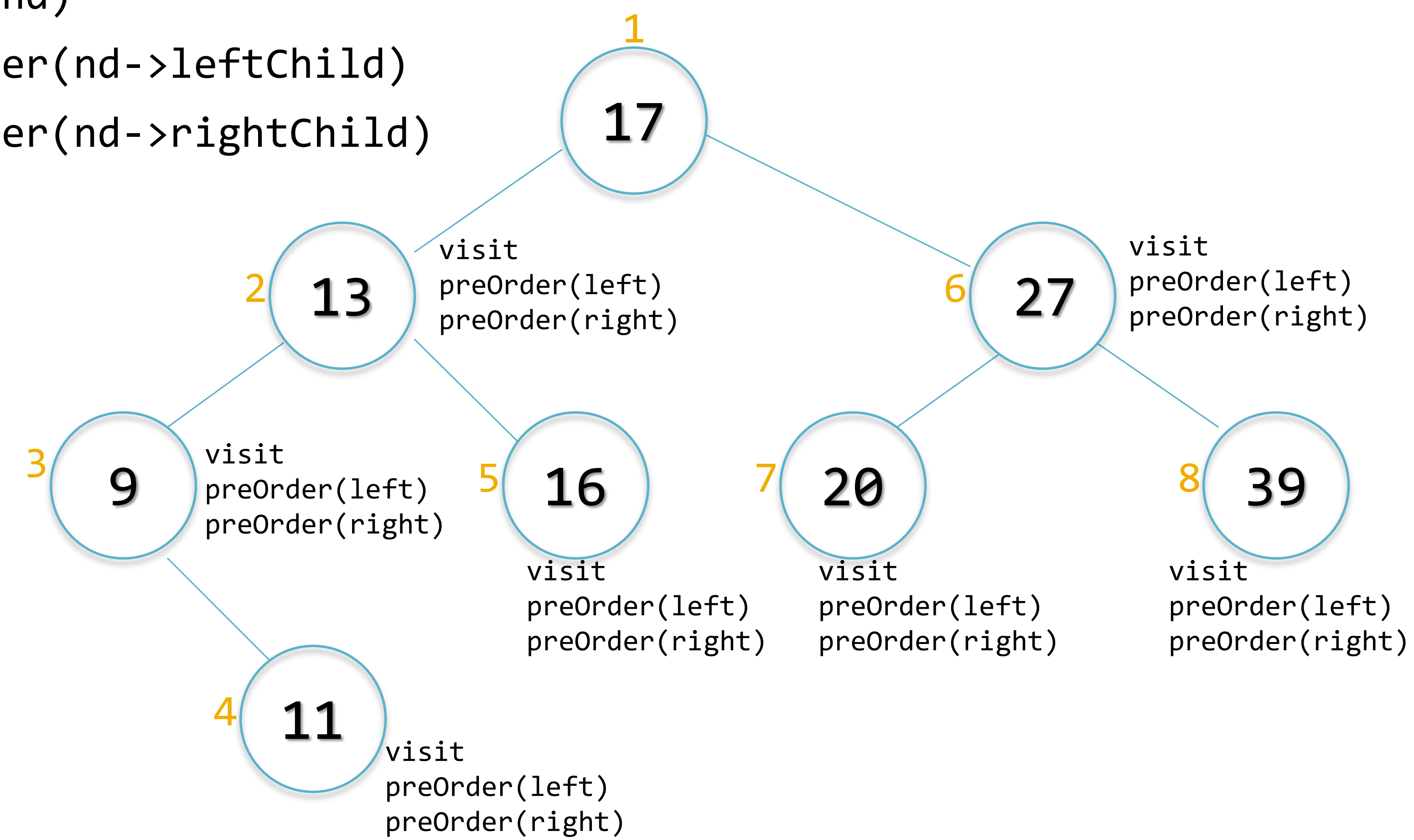
InOrder Traversal Algorithm

```
inOrder(Node* nd) {
    if (nd != NULL) {
        inOrder(nd->leftChild);
        visit(nd);
        inOrder(nd->rightChild);
    }
}
```

The visit function would do whatever the purpose of the traversal is, for example print the data in the node

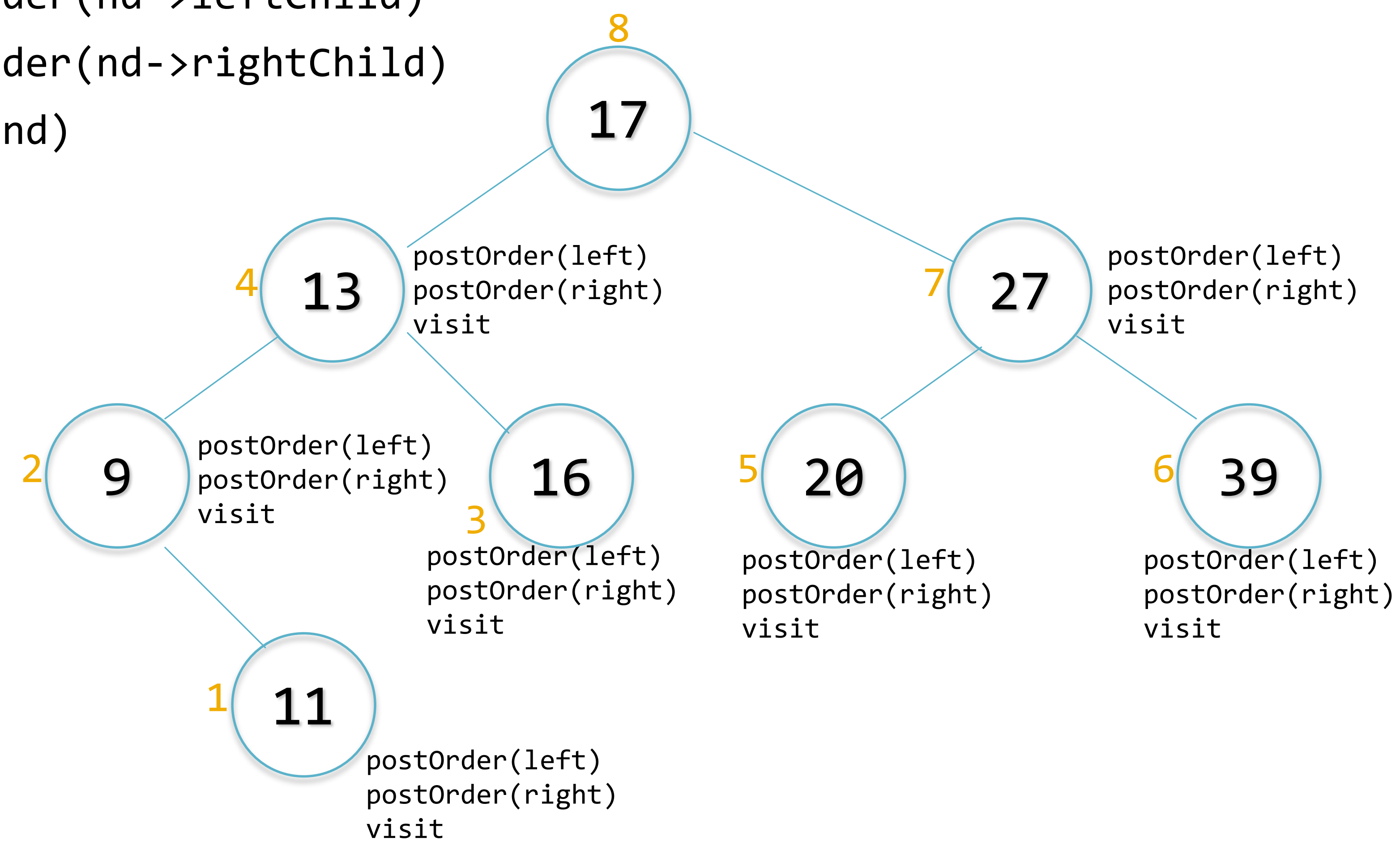
PreOrder Traversal

```
visit(nd)
preOrder(nd->leftChild)
preOrder(nd->rightChild)
```

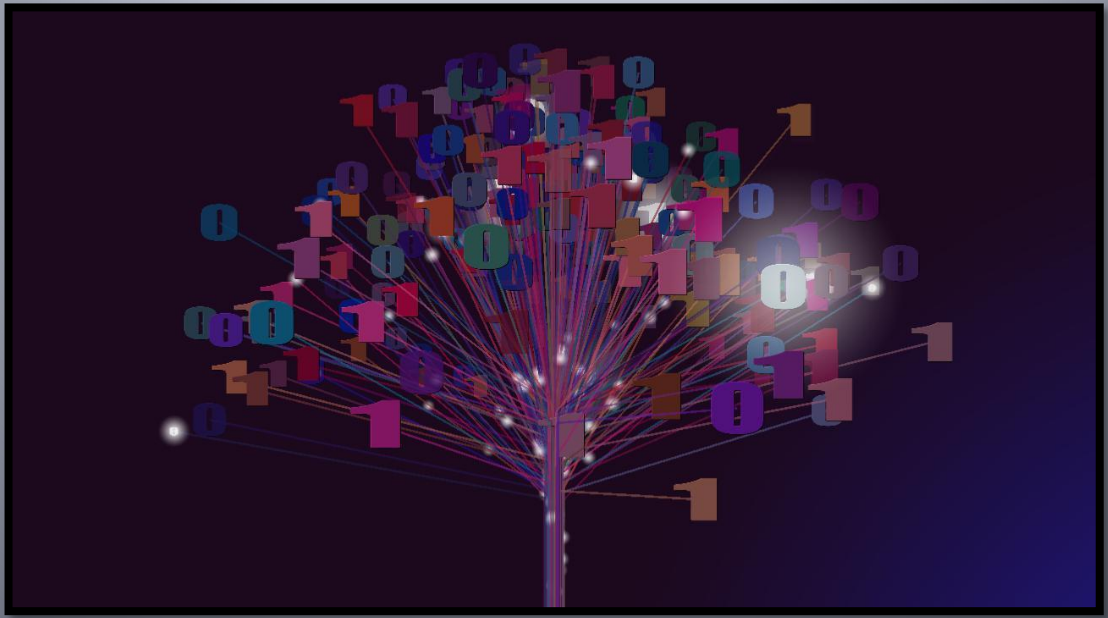


PostOrder Traversal

```
postOrder(nd->leftChild)
postOrder(nd->rightChild)
visit(nd)
```



Binary Search Trees



Binary Tree Implementation

- The binary tree ADT can be implemented using different data structures
  - Reference structures (similar to linked lists)
  - Arrays
- Example implementations
  - Binary search trees (references)
  - Red – black trees (references again)
  - Heaps (arrays) – not a binary *search* tree
  - B trees (arrays again) – not a *binary* search tree

Problem: Accessing Sorted Data

- Consider maintaining data in some order
  - The data is to be frequently searched on the sort key e.g. a dictionary
- Possible solutions might be:
  - A sorted array
    - Access in O(log*n*) using binary search
    - Insertion and deletion in linear time
  - An ordered linked list
    - Access, insertion and deletion in linear time

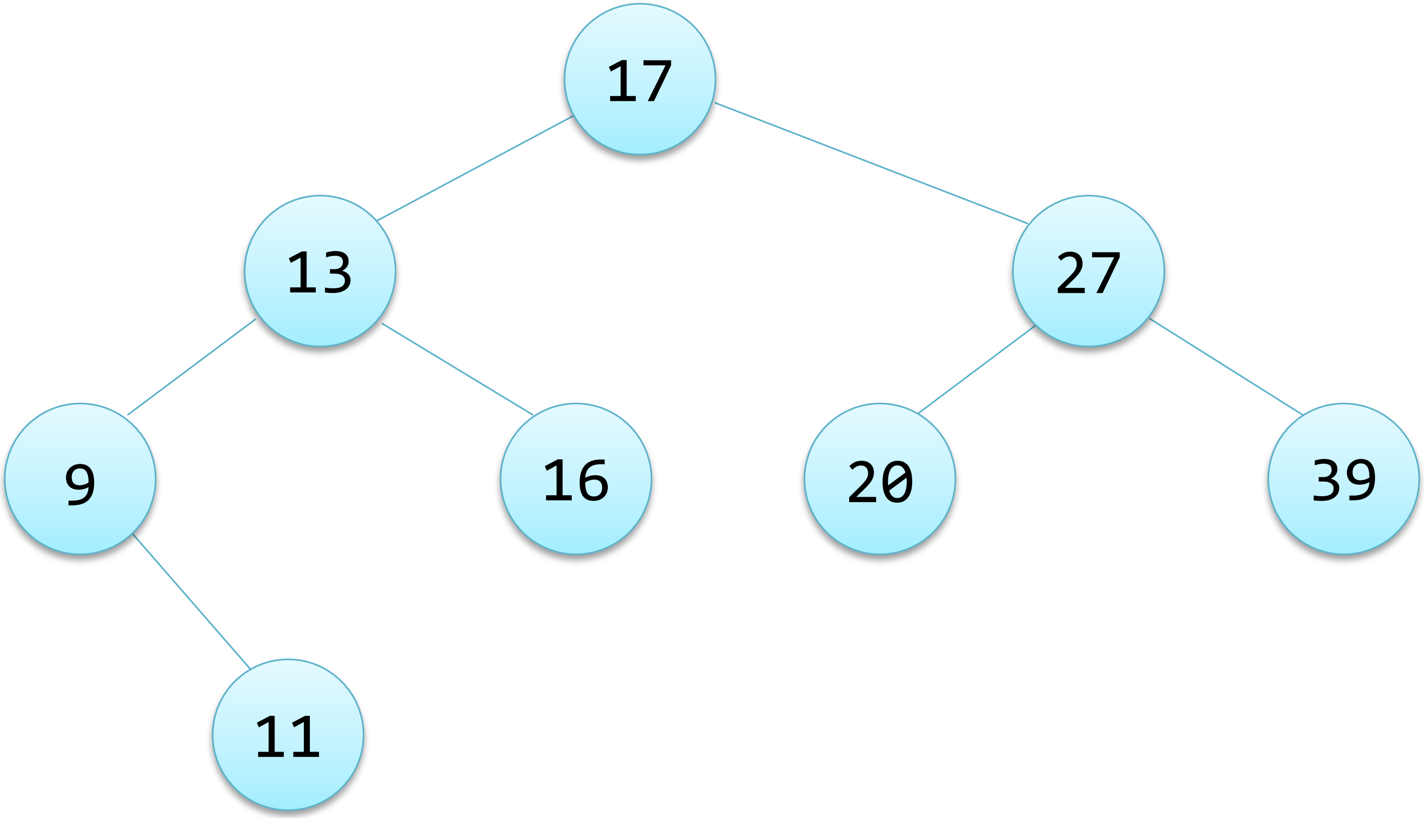
Dictionary Operations

- The data structure should be able to perform all these operations efficiently
  - Create an empty dictionary
  - Insert
  - Delete
  - Look up
- The insert, delete and look up operations should be performed in at most O(log*n*) time

Binary Search Tree Property

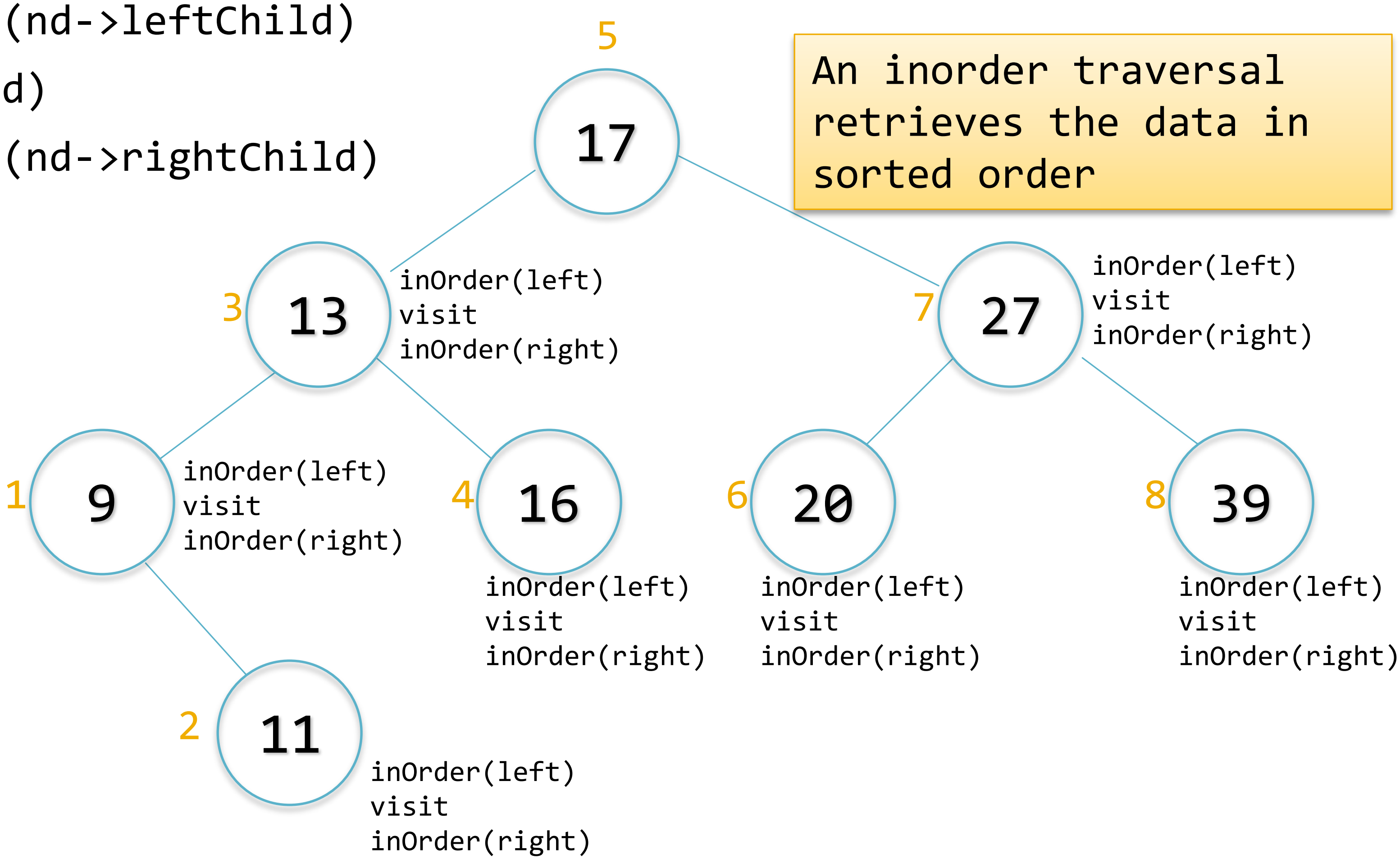
- A binary search tree is a binary tree with a special property
  - For all nodes in the tree:
    - All nodes in a left subtree have labels *less* than the label of the subtree's root
    - All nodes in a right subtree have labels *greater* than or equal to the label of the subtree's root
- Binary search trees are fully ordered

BST Example



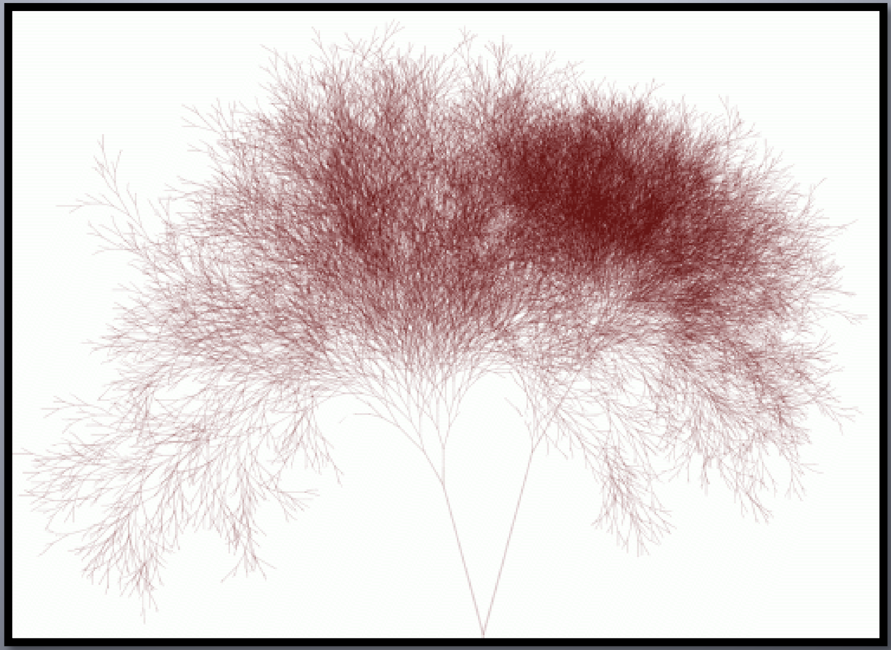
BST InOrder Traversal

```
inOrder(nd->leftChild)
visit(nd)
inOrder(nd->rightChild)
```



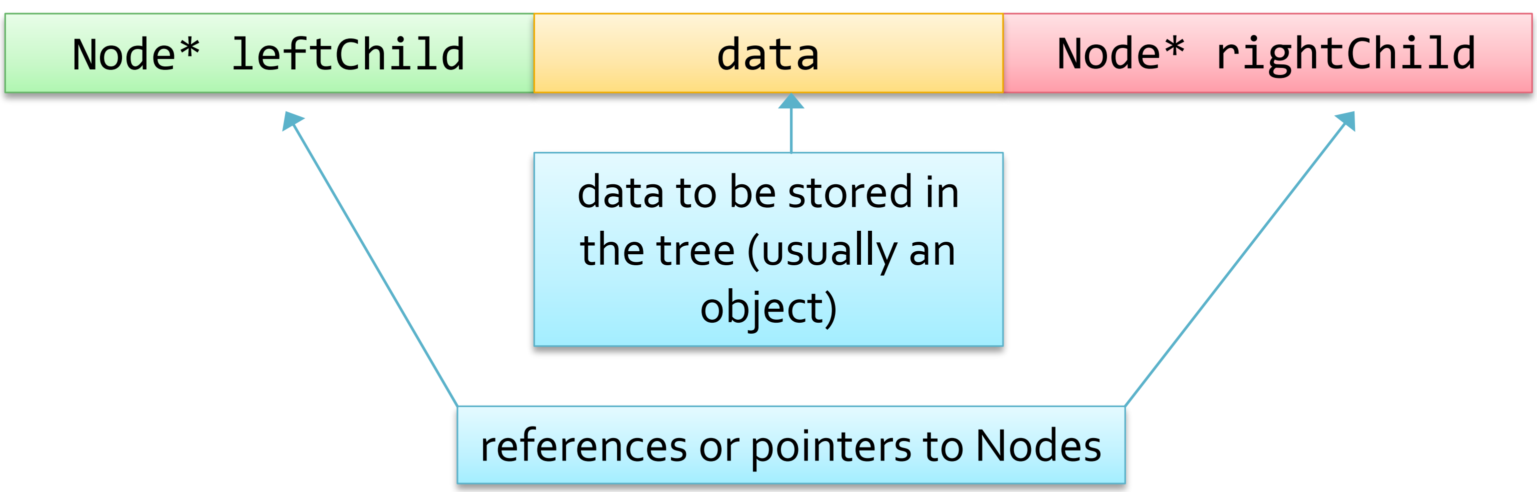
An inorder traversal retrieves the data in sorted order

Binary Search Tree Search



BST Implementation

- Binary search trees can be implemented using a reference structure
- Tree nodes contain data and two pointers to nodes



BST Search

- To find a value in a BST search from the root node:
  - If the target is less than the value in the node search its left subtree
  - If the target is greater than the value in the node search its right subtree
  - Otherwise return true, (or a pointer to the data, or …)
- How many comparisons?
  - One for each node on the path
  - Worst case: height of the tree + 1

BST Search Algorithm

```
bool search(Node* nd, int x){
        if (nd == NULL){
                return false;
        }else if(x == nd->data){
                return true;
        } else if (x < nd->data){
                return search(x, nd->left);
        } else {
                return search(x, nd->right);
        }
}
```

reached the end of this path

note the similarity to binary search

called by a helper method like this: *search(root, target)*

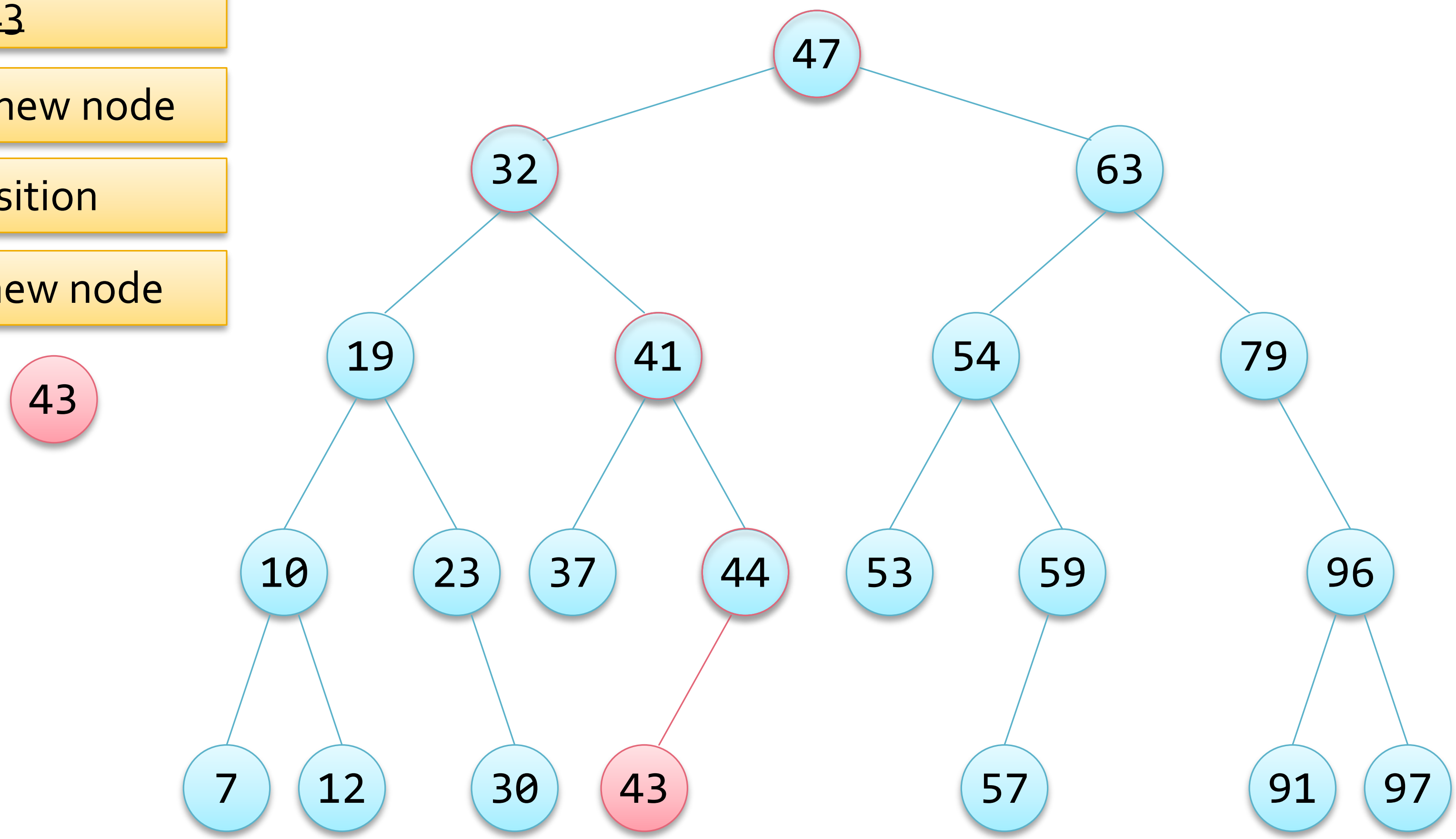
BST Insertion

BST Insertion

- The BST property must hold after insertion
- Therefore the new node must be inserted in the correct position
  - This position is found by performing a search
  - If the search ends at the NULL left child of a node make its left child refer to the new node
  - If the search ends at the NULL right child of a node make its right child refer to the new node
- The cost is about the same as the cost for the search algorithm, O(*height*)

BST Insertion Example

- insert 43
- create new node
- find position
- insert new node



BST Removal

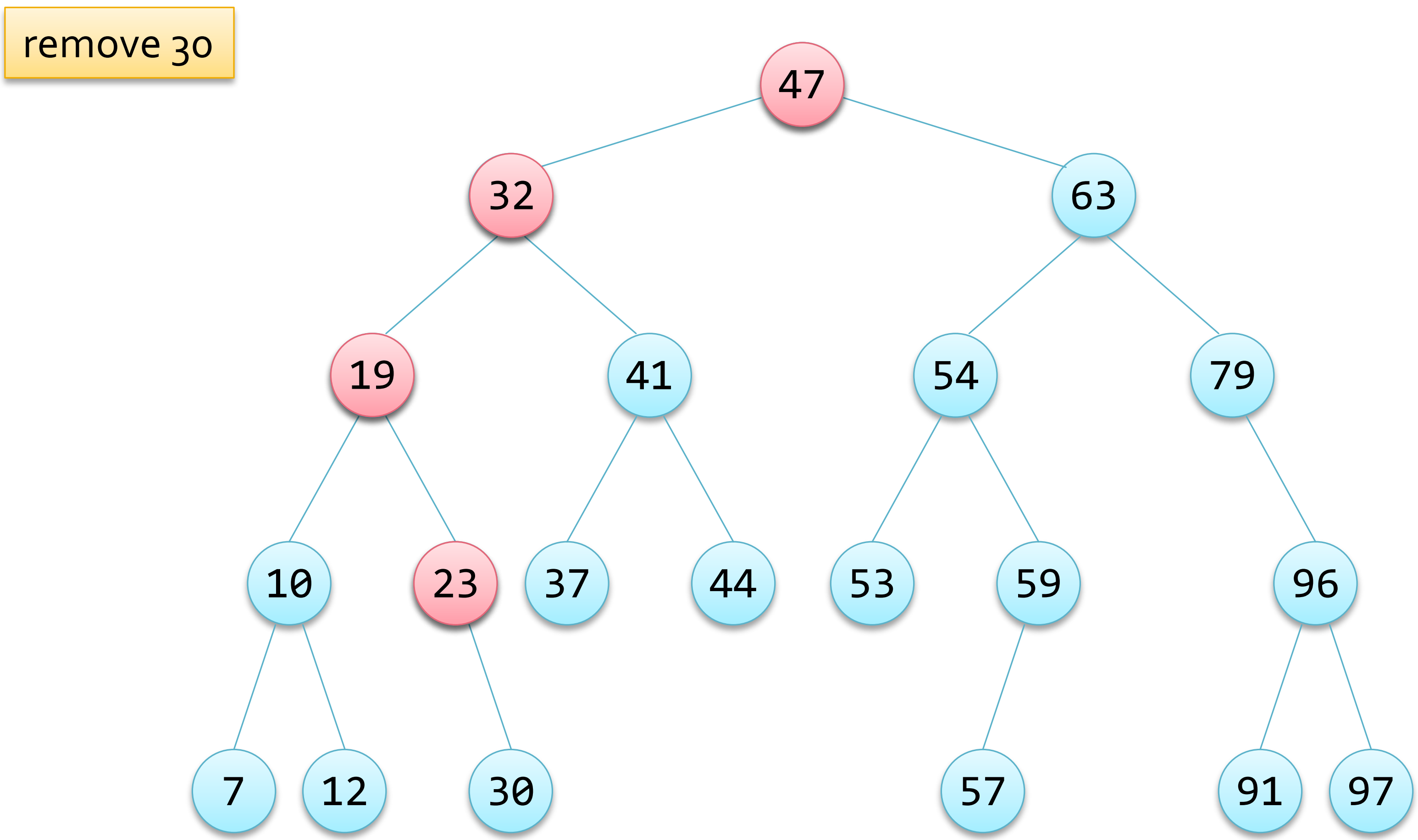
BST Removal

- After removal the BST property must hold
- Removal is not as straightforward as search or insertion
  - With insertion the strategy is to insert a new leaf
  - Which avoids changing the internal structure of the tree
  - This is not possible with removal
    - Since the removed node's position is not chosen by the algorithm
- There are a number of different cases to be considered

BST Removal Cases

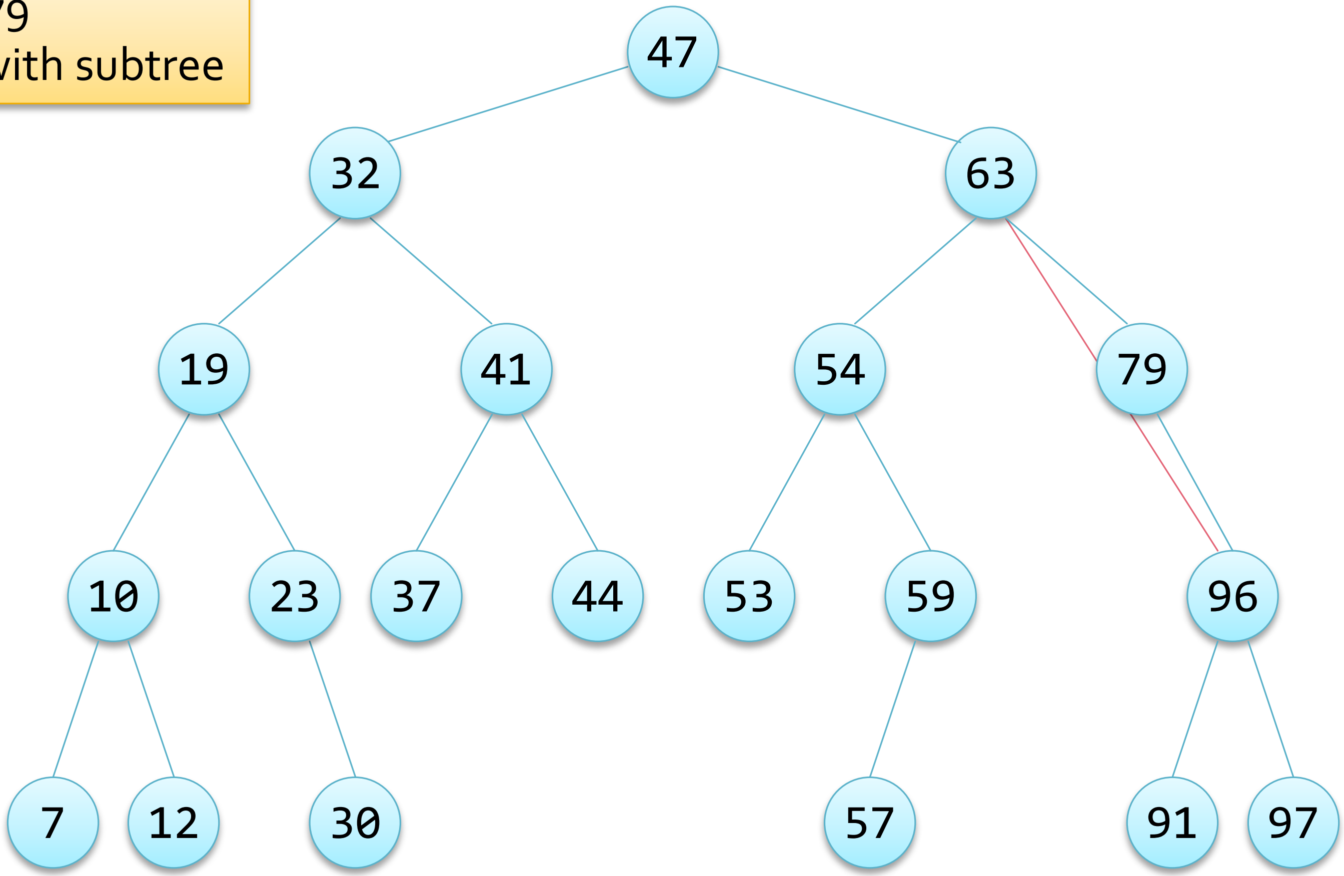
- The node to be removed has no children
  - Remove it (assigning NULL to its parent's reference)
- The node to be removed has one child
  - Replace the node with its subtree
- The node to be removed has two children
  - ...

BST Removal – Target is a Leaf



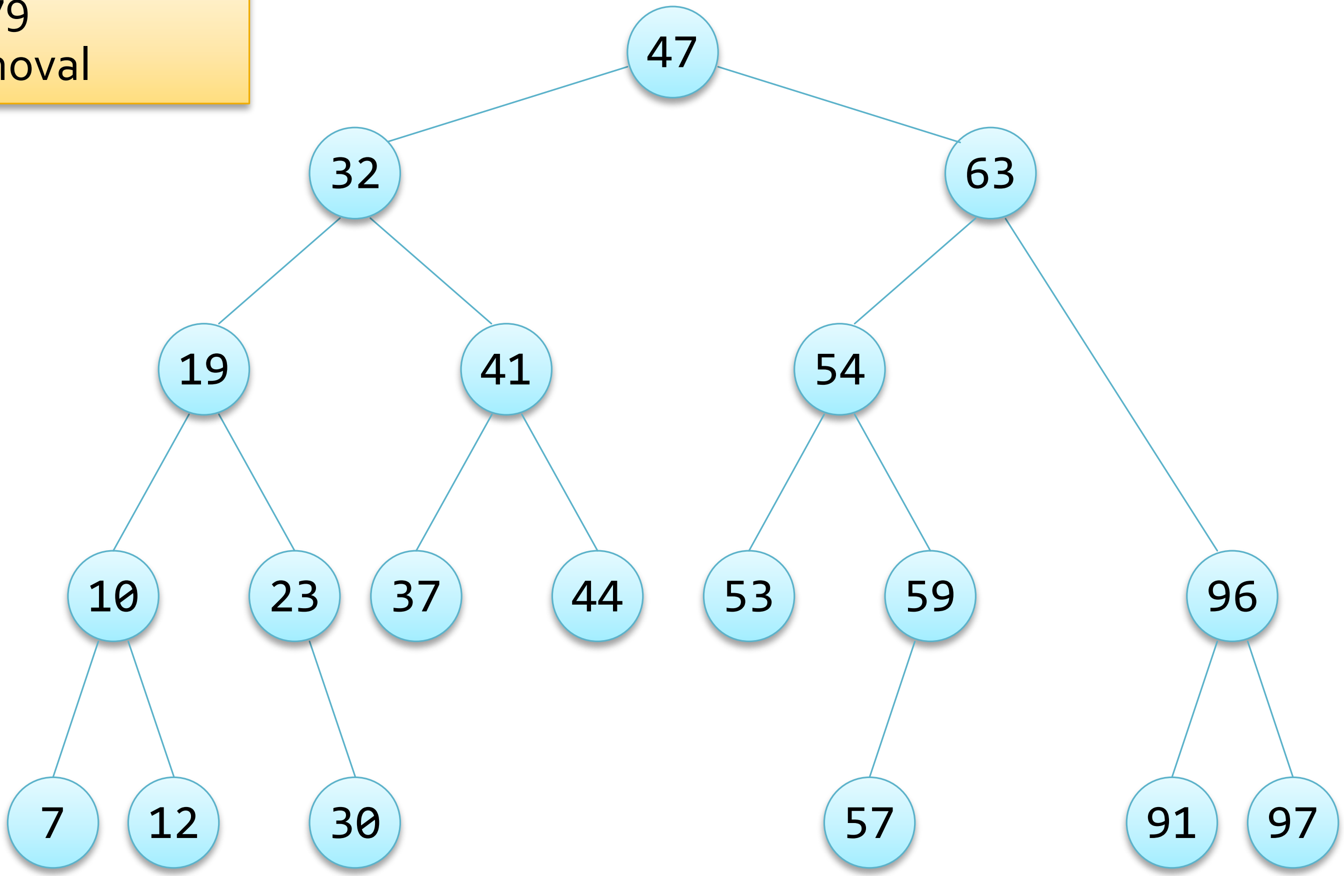
BST Removal – Target Has One Child

remove 79
replace with subtree



BST Removal – Target Has One Child

remove 79
after removal



Looking At the Next Node

- One of the issues with implementing a BST is the necessity to look at both children
  - And, just like a linked list, *look ahead* for insertion and removal
  - And check that a node is null before accessing its member variables
- Consider removing a node with one child in more detail

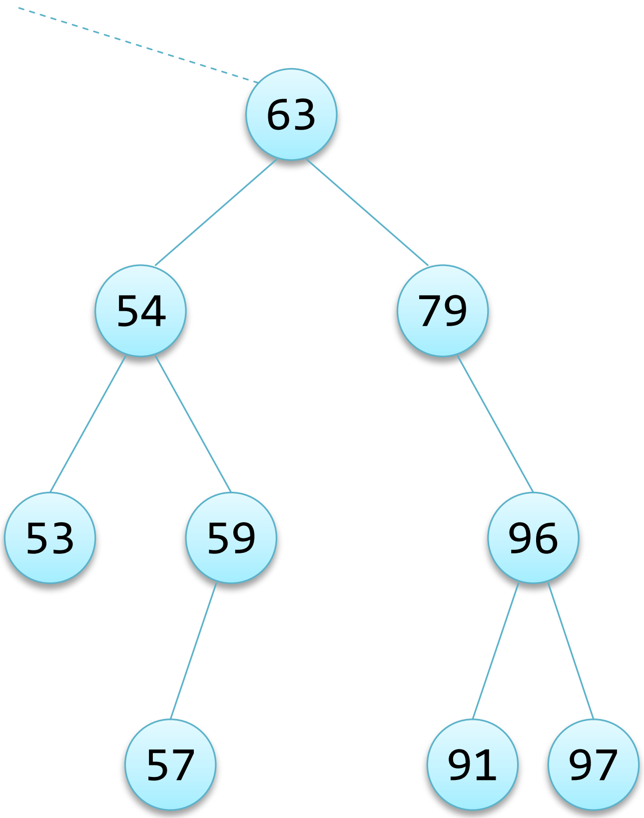
Looking Ahead

remove 59

Step 1 - we need to find the node to remove and its parent

it's useful to know if nd is a left or right child

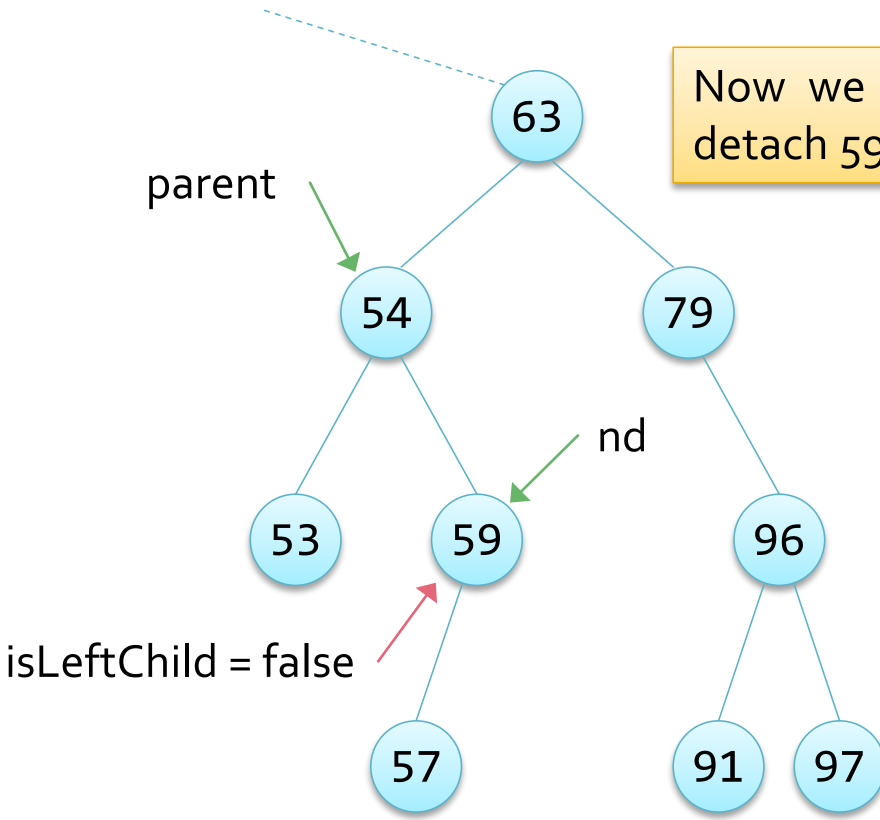
```
while (nd != target)
        if (nd == NULL)
                return
        if (target < nd->data)
                parent = nd
                nd = nd->left
                isLeftChild = true
        else
                parent = nd
                nd = nd->right
                isLeftChild = false
```



Left or Right?

remove 59

Now we have enough information to detach 59 and attach its child to 54



Removing a Node With 2 Children

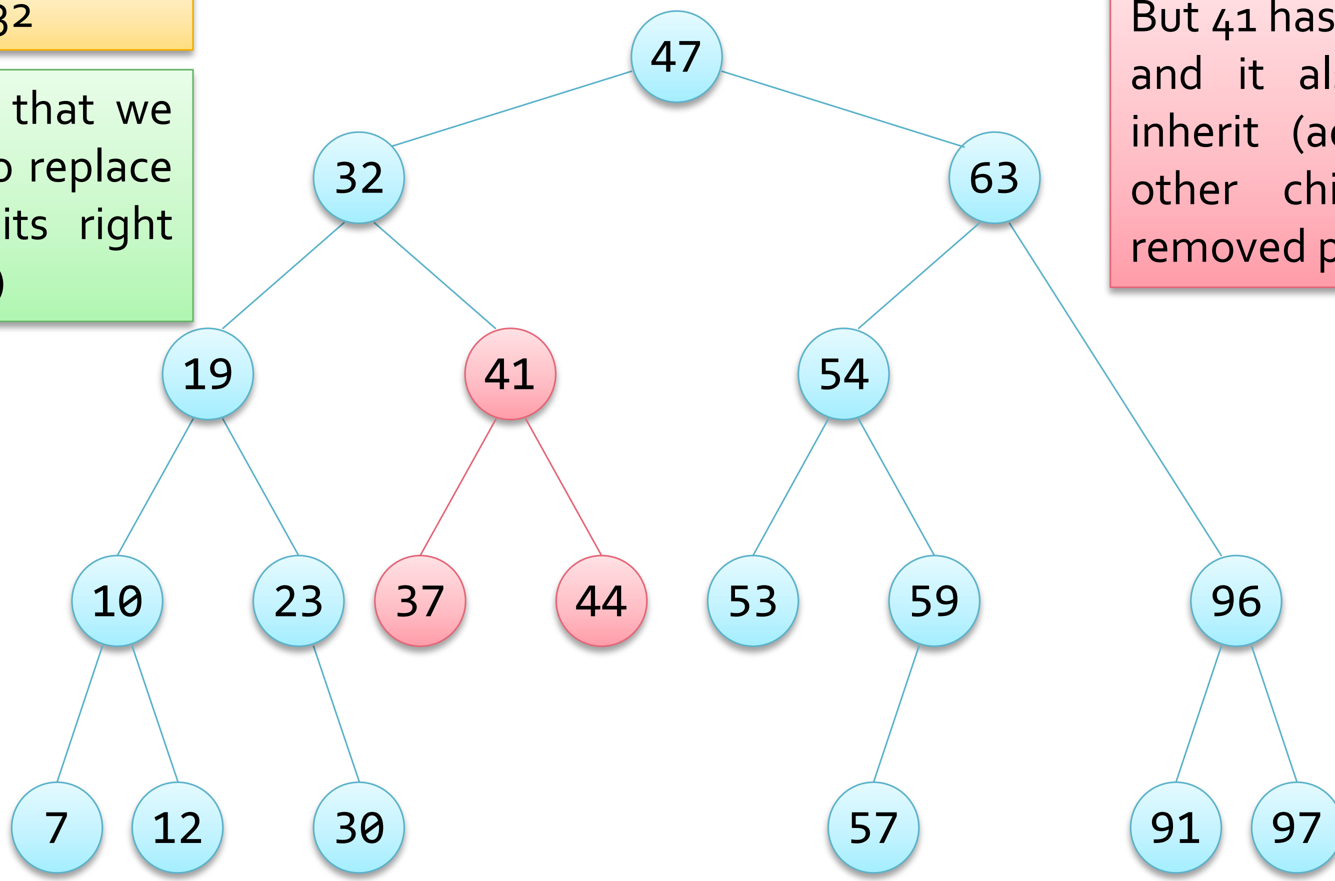
- The most difficult case is when the node to be removed has two children
  - The strategy when the removed node had one child was to replace it with its child
  - But when the node has two children problems arise
- Which child should we replace the node with?
  - We could solve this by just picking one …
- But what if the node we replace it with also has two children?
  - This will cause a problem

Removed Node Has 2 Children

remove 32

let's say that we decide to replace it with its right child (41)

But 41 has 2 children, and it also has to inherit (adopt?) the other child of its removed parent



Find the Predecessor

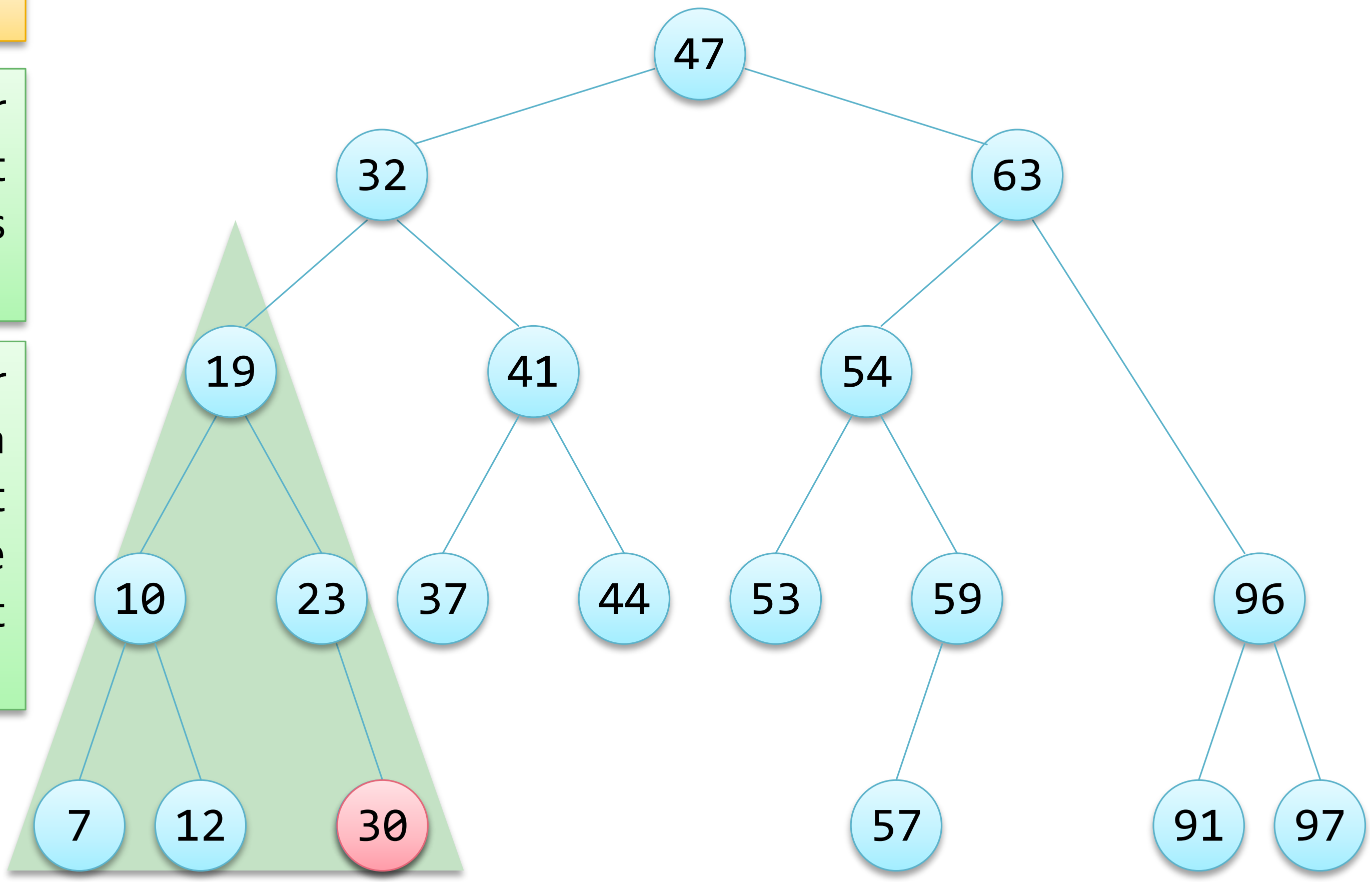
- When a node has two children, instead of replacing it with one of its children find its *predecesor*
  - A node's predecessor is the *right most* node of its *left subtree*
  - The predecessor is the node in the tree with the *largest* value *less* than the node's value
- The predecesor cannot have a right child and can therefore have at most one child
  - Why?

Predecessor Node

32's predecessor

the predecessor of 32 is the right most node in its left subtree

The predecessor *cannot* have a right child as it wouldn't then be the right most node



Why Use the Predecessor?

- The predecssor has some useful properties
  - Because of the BST property it must be the largest value less than its ancestor's value
    - It is to the right of all of the nodes in its ancestor's *left* subtree so must be greater than them
    - It is less than the nodes in its ancestor's *right* subtree
  - It can have only one child
- These properties make it a good candidate to replace its ancestor

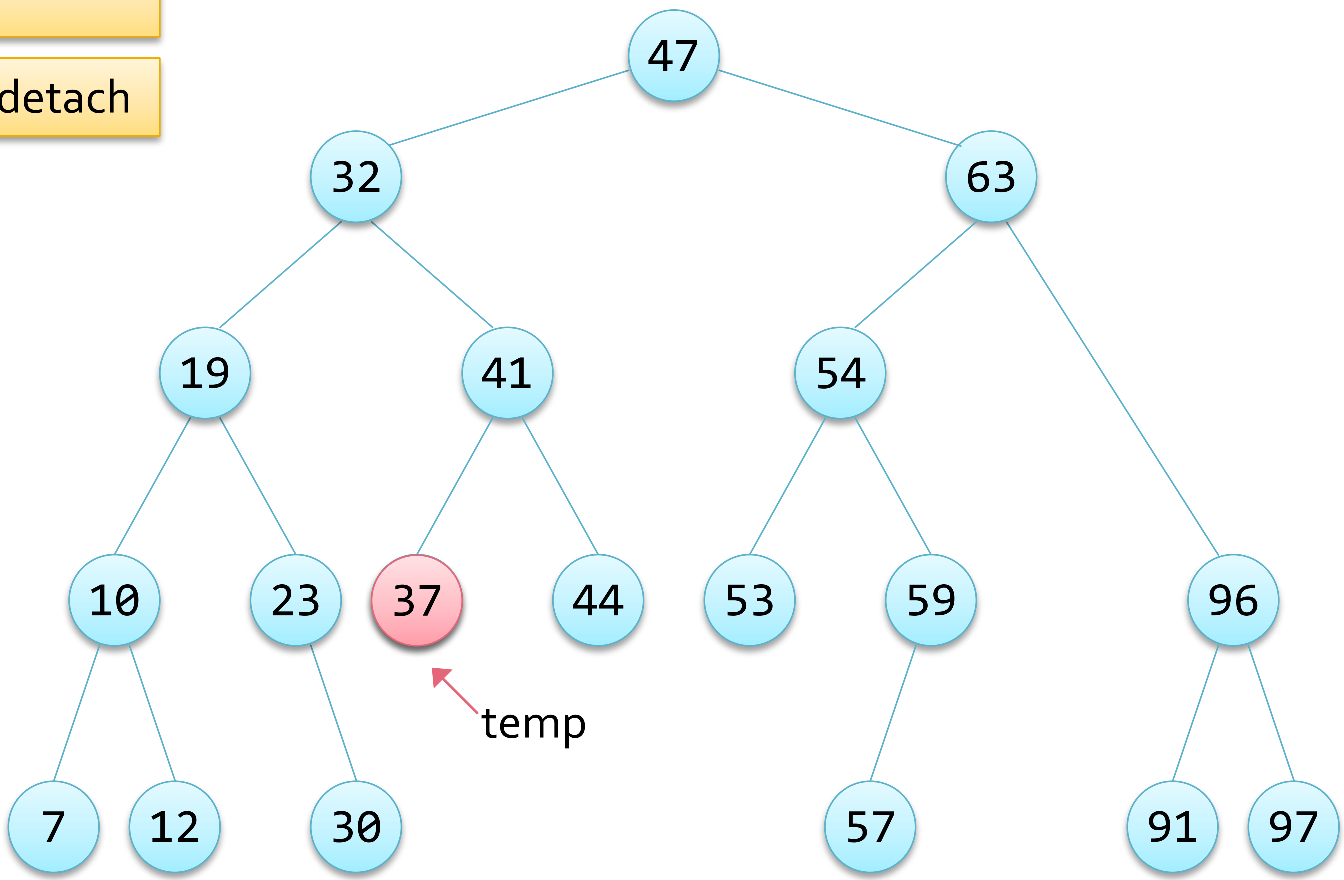
What About the Successor?

- The successor to a node is the *left* most child of its *right* subtree
  - It has the *smallest* value *greater* than its ancestor's value
  - And cannot have a left child
- The successor can also be used to replace a removed node
  - Pick either the precedessor or successor!

Removed Node Has 2 Children

remove 32

find successor and detach

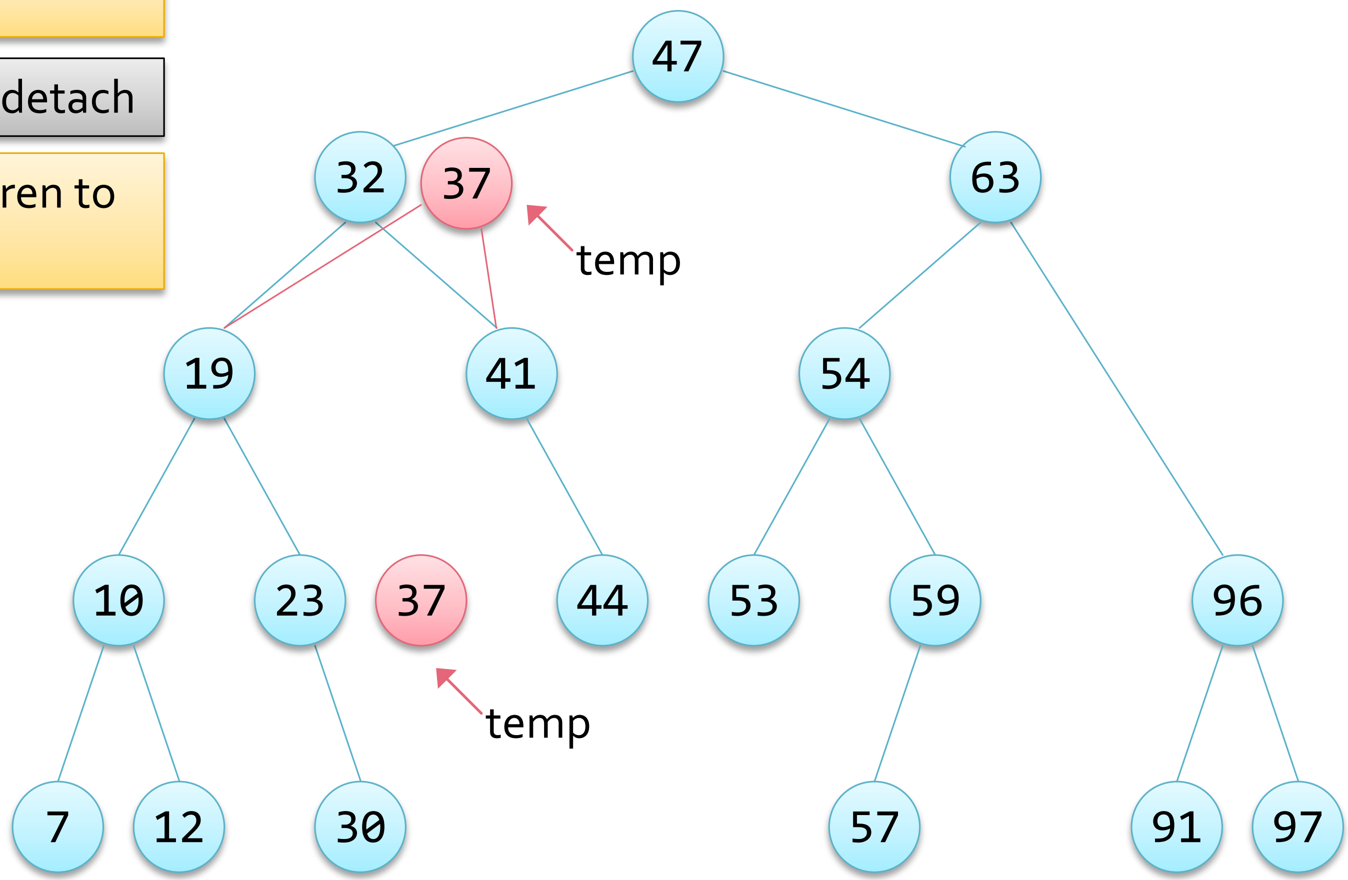


Removed Node Has 2 Children

remove 32

find successor and detach

attach node's children to its successor



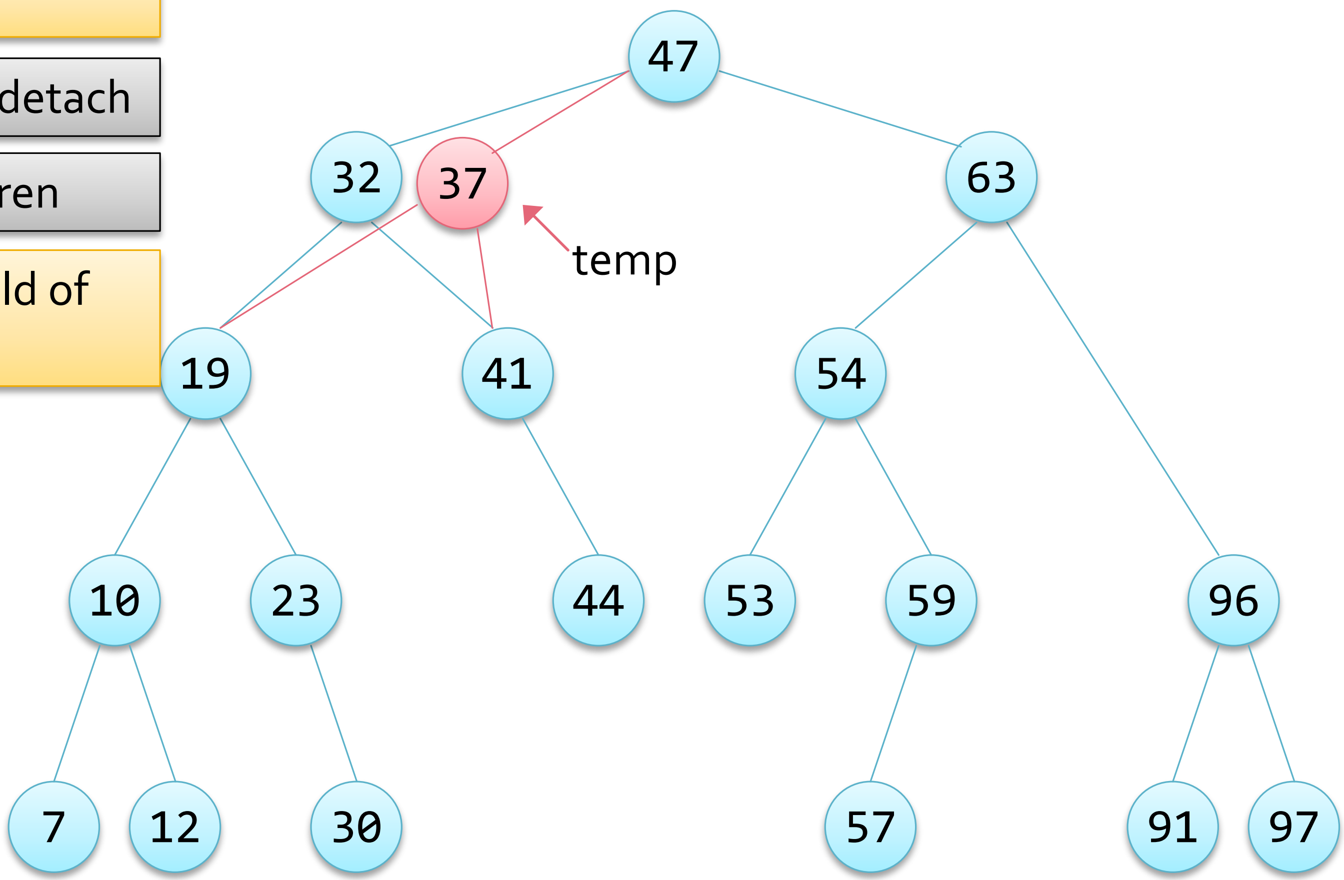
Removed Node Has 2 Children

remove 32

find successor and detach

attach node's children

make successor child of node's parent



Removed Node Has 2 Children

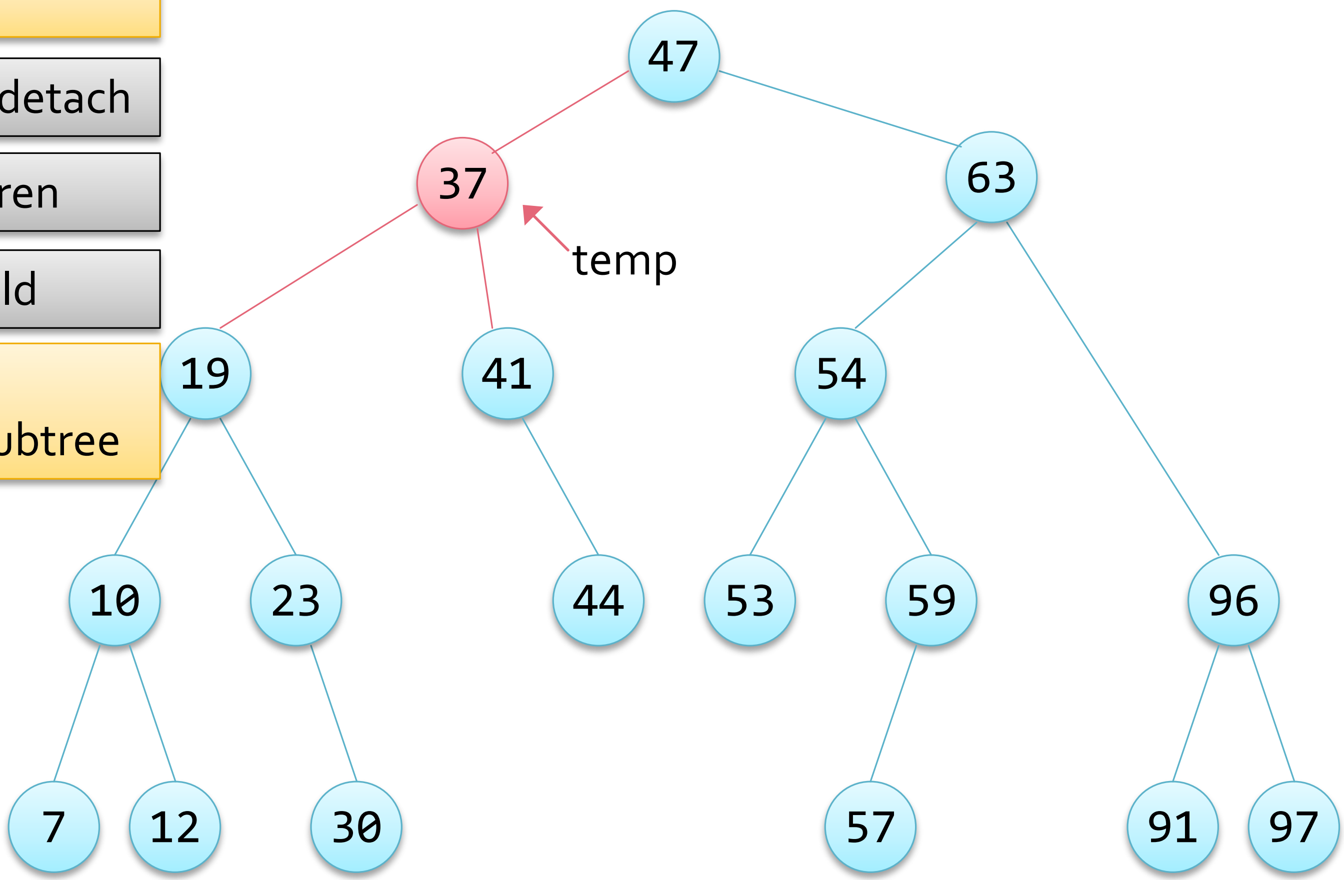
remove 32

find successor and detach

attach node's children

make successor child

in this example the successor had no subtree

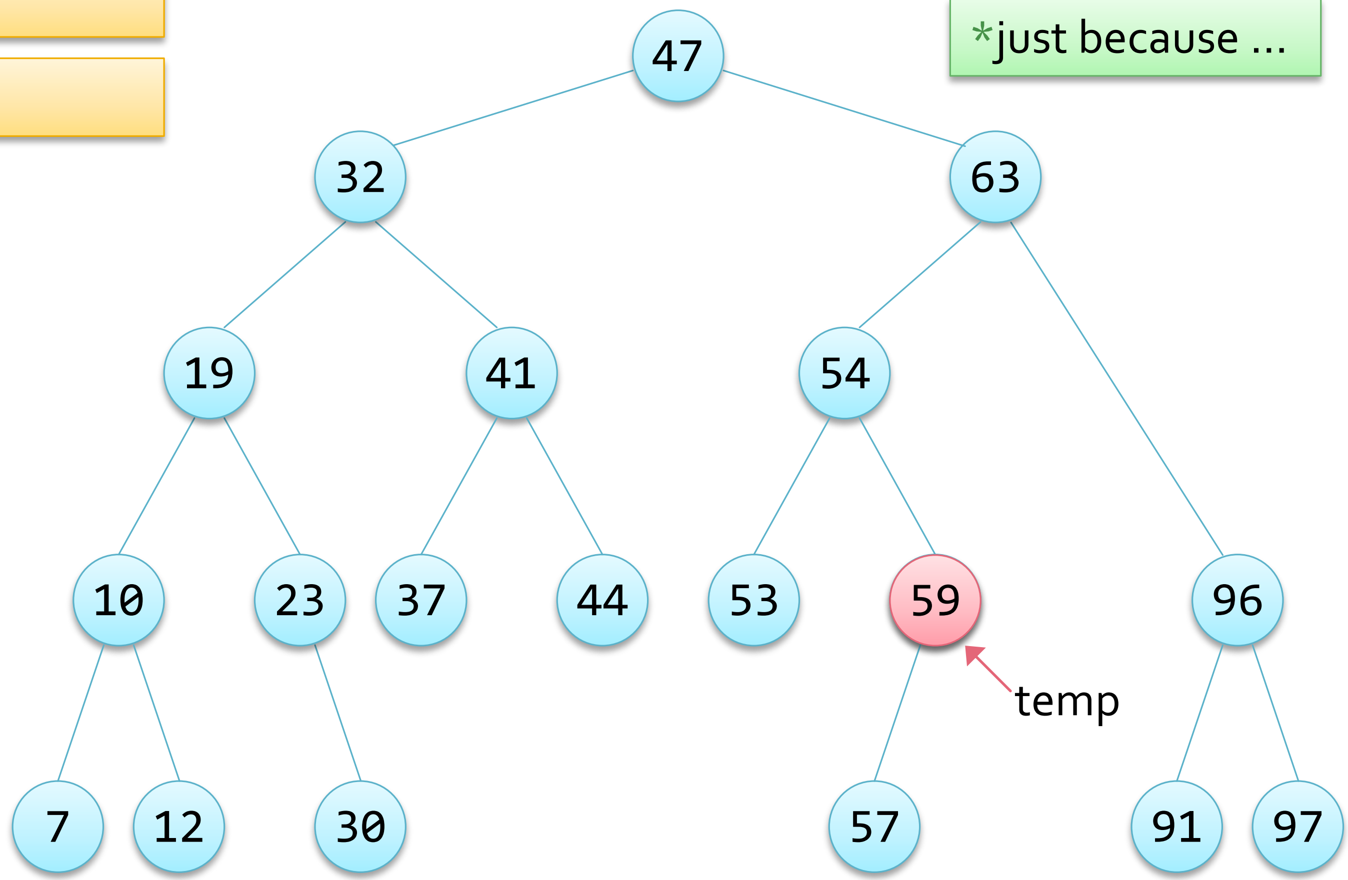


Removed Node Has 2 Children

remove 63

find predecessor*

*just because ...

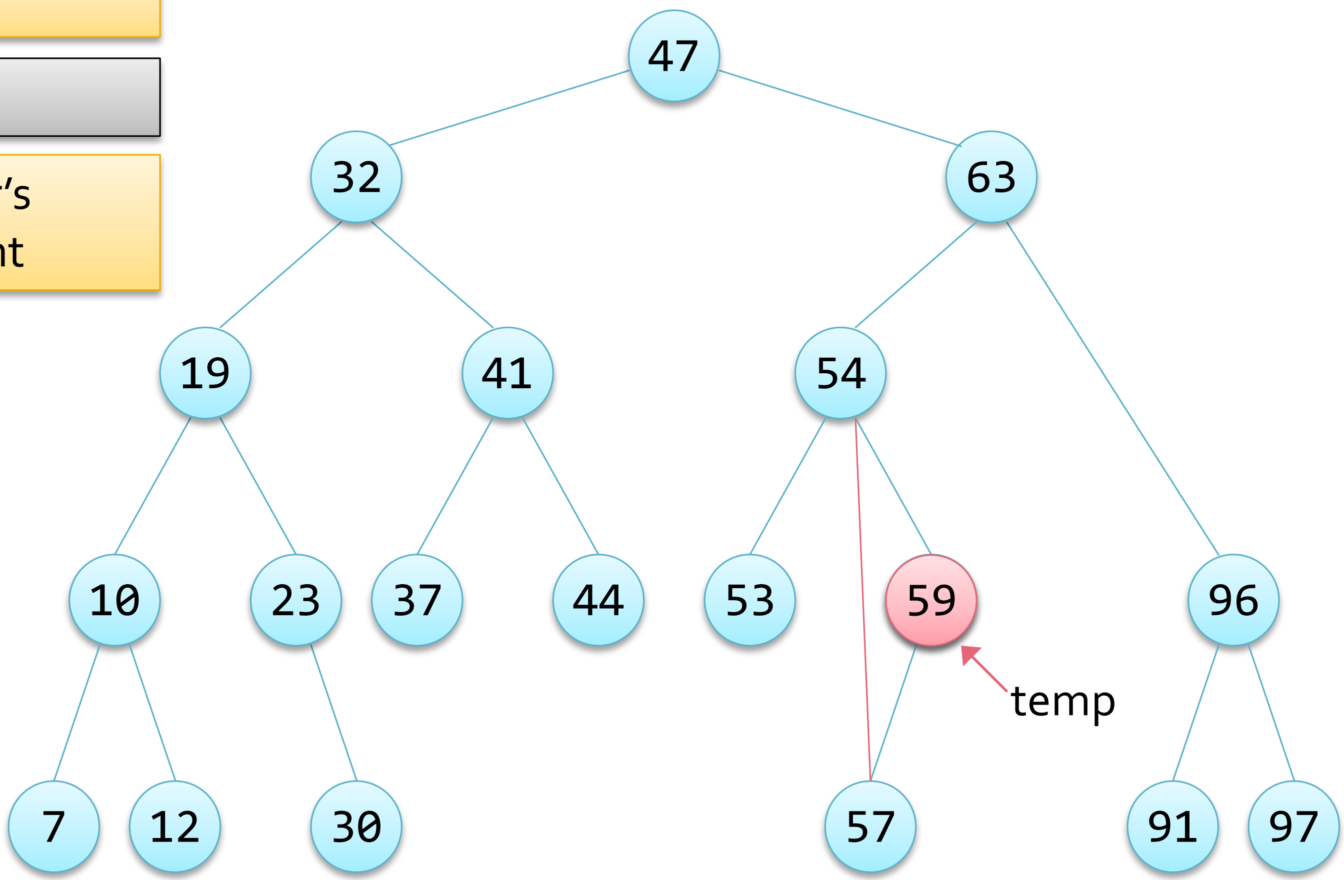


Removed Node Has 2 Children

remove 63

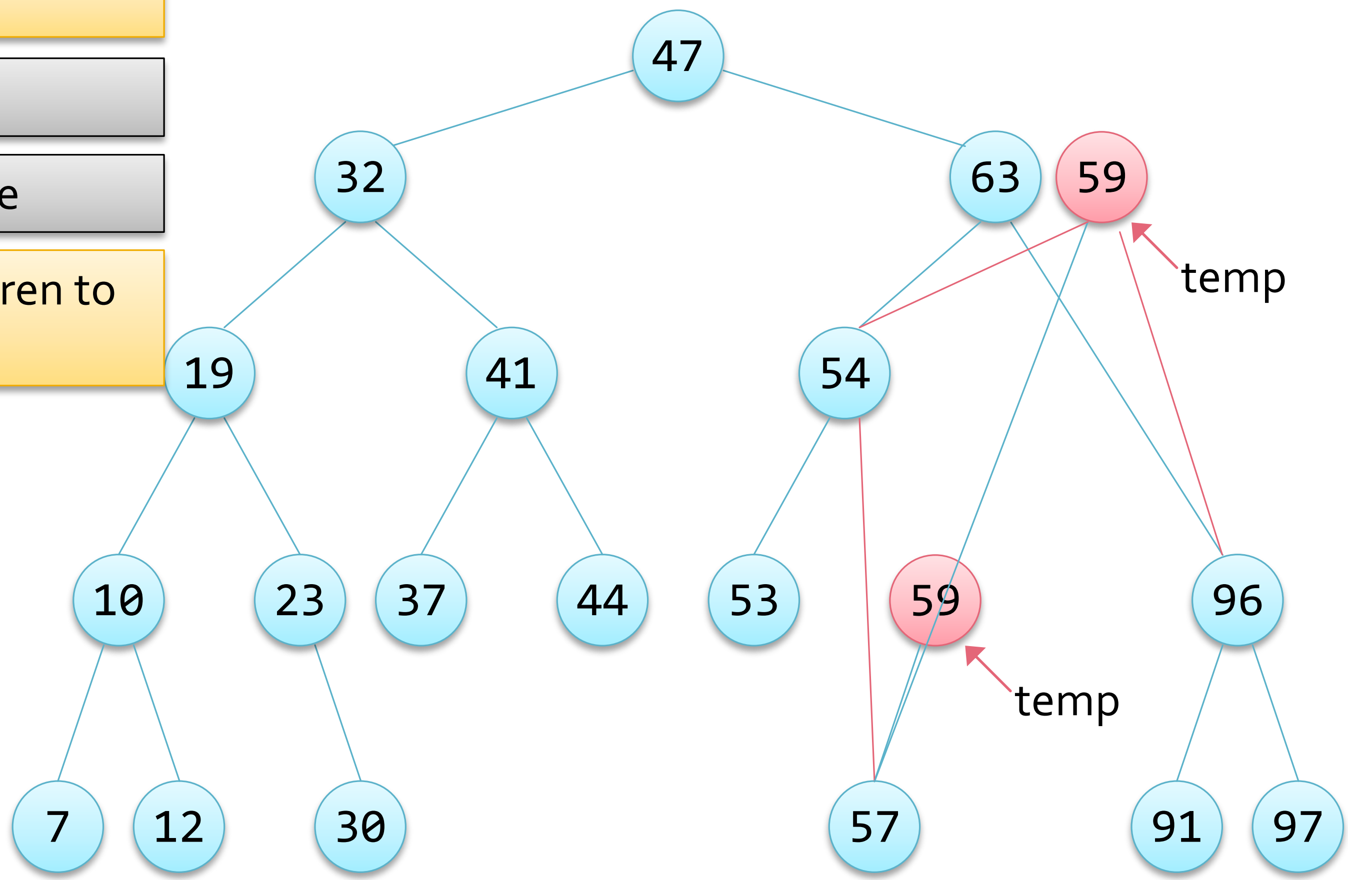
find predecessor

attach predecessor's subtree to its parent



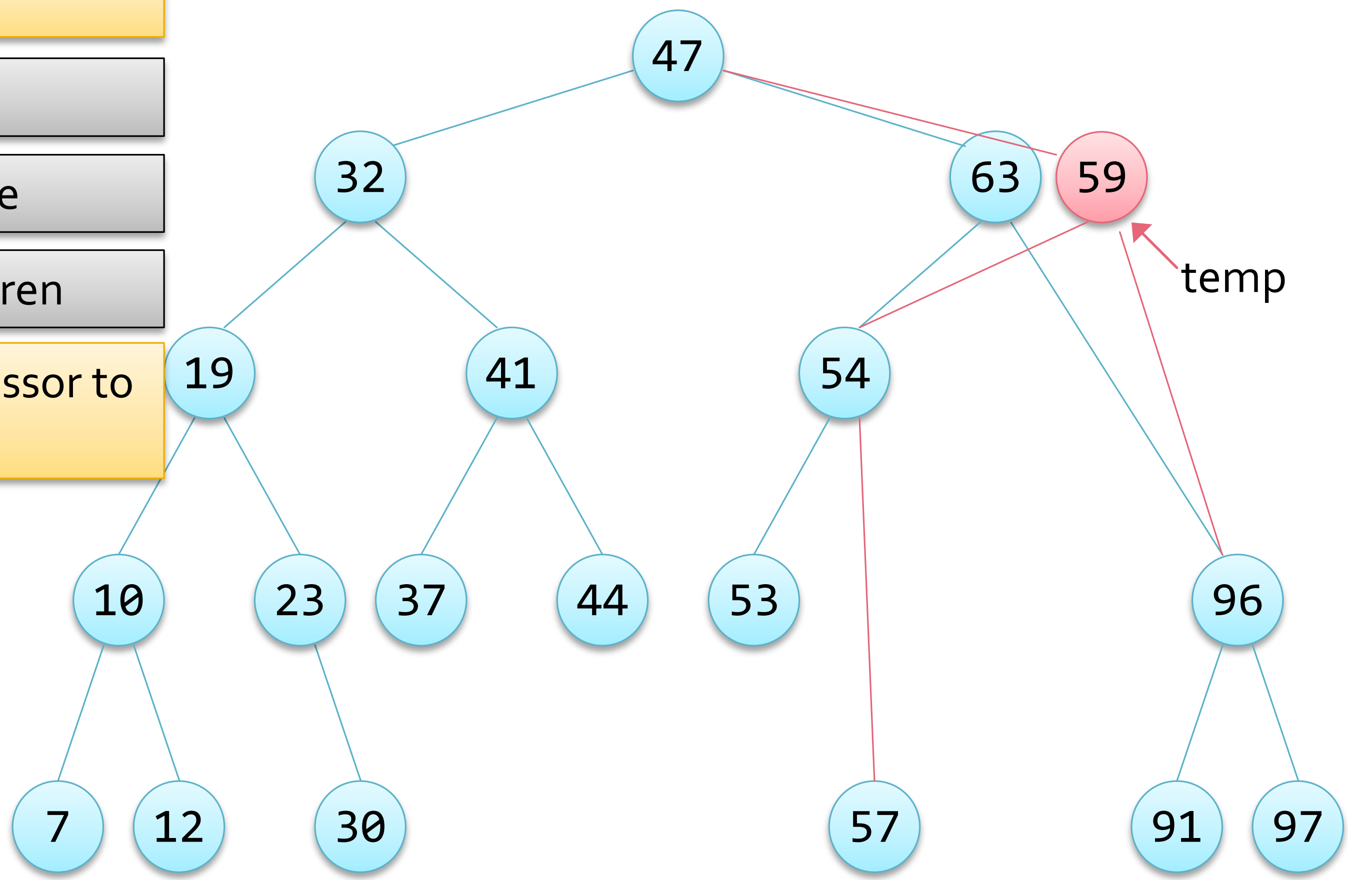
Removed Node Has 2 Children

- remove 63
- find predecessor
- attach pre's subtree
- attach node's children to predecessor



Removed Node Has 2 Children

- remove 63
- find predecessor
- attach pre's subtree
- attach node's children
- attach the predecessor to the node's parent



Removed Node Has 2 Children

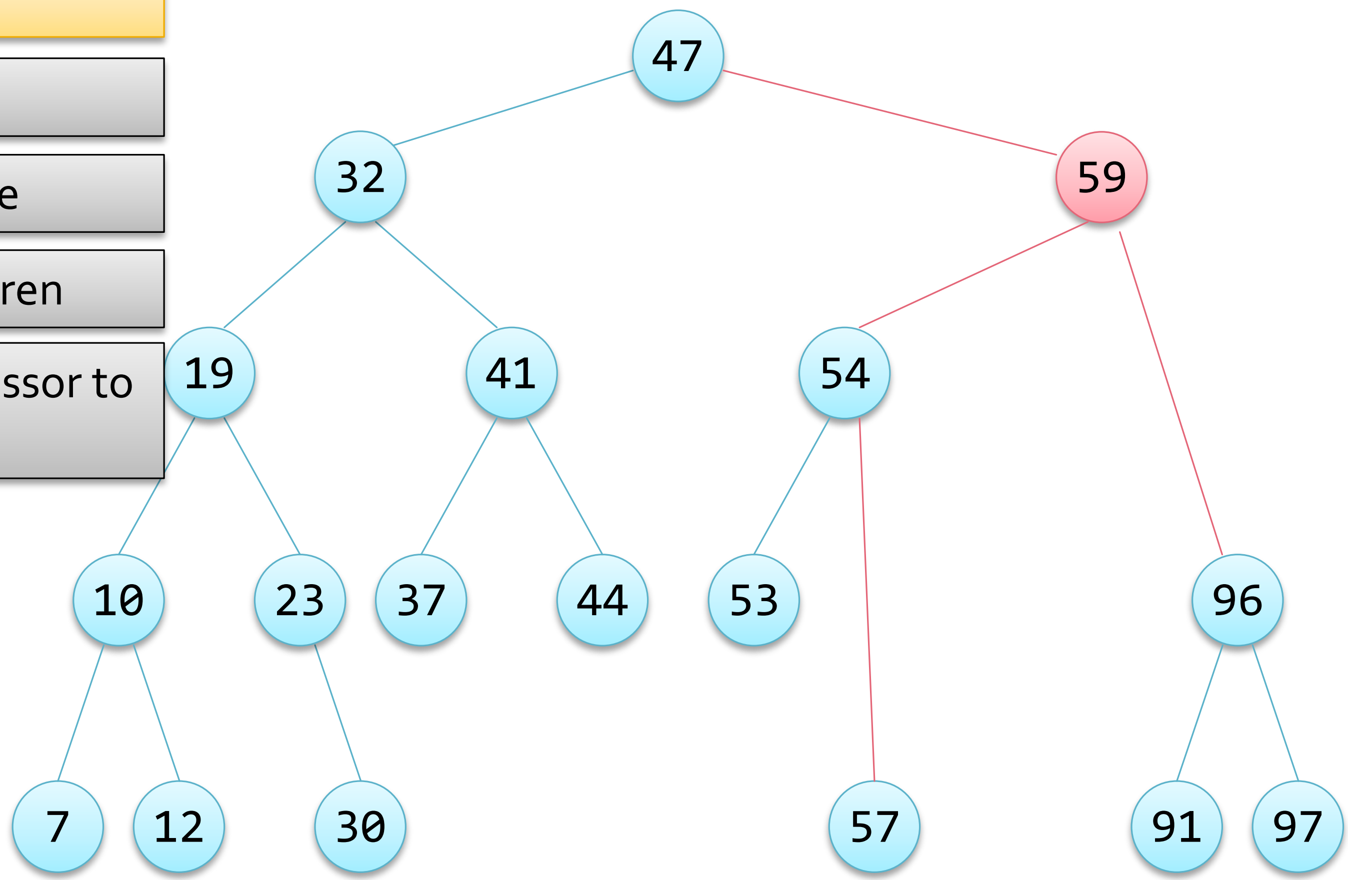
remove 63

find predecessor

attach pre's subtree

attach node's children

attach the predecessor to the node's parent



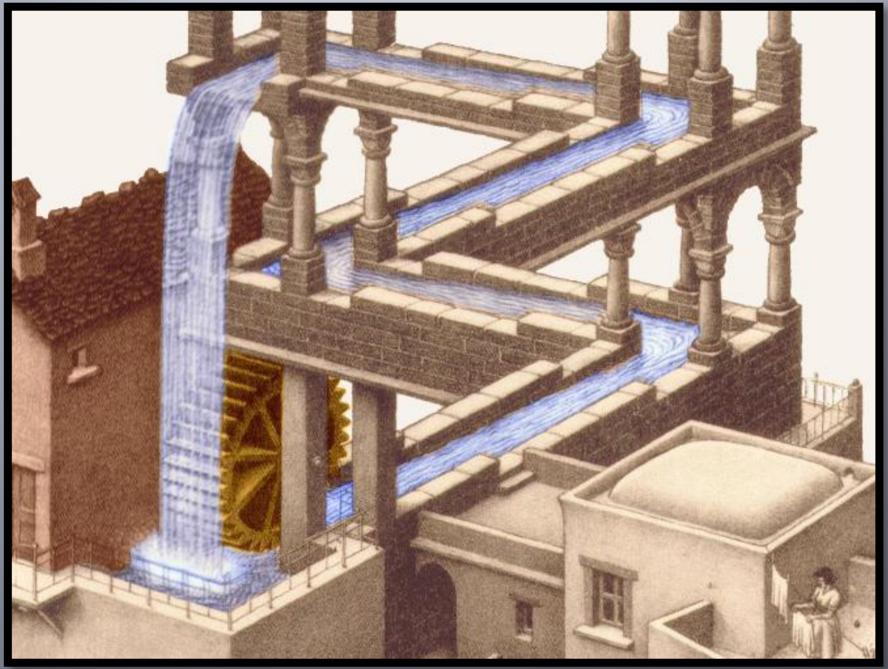
Removal Alternatives - 1

- Instead of removing a BST node mark it as removed in some way
  - Set the data object to *null*, for example
- And change the insertion algorithm to look for empty nodes
  - And insert the new item in an empty node that is found on the way down the tree

Removal Alternatives - 2

- An alternative to the removal approach for nodes with 2 children is to replace the data
  - The data from the predecessor node is copied into the node to be removed
  - And the predecessor node is then removed
    - Using the approach described for removing nodes with one or no children
- This avoids some of the complicated pointer assignments

BST Efficiency

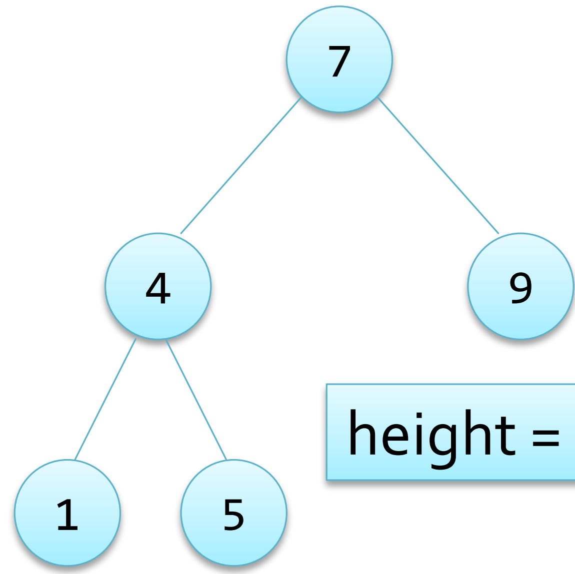


BST Efficiency

- The efficiency of BST operations depends on the *height* of the tree
  - All three operations (search, insert and delete) are *O(height*)
- If the tree is complete the height is $\lfloor \log(n) \rfloor$
  - What if it isn't complete?

Height of a BST

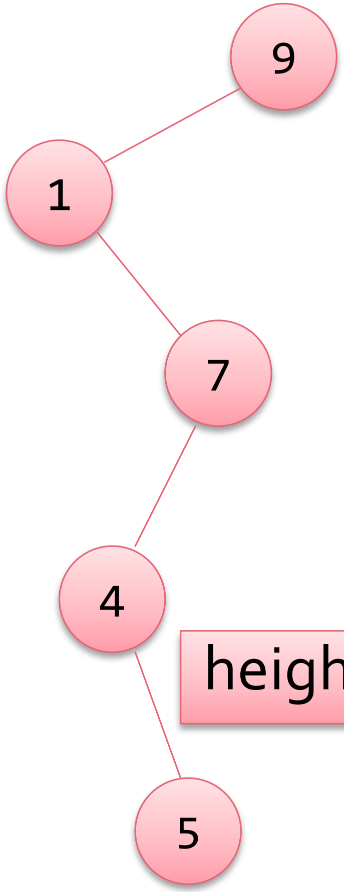
- Insert 7
- Insert 4
- Insert 1
- Insert 9
- Insert 5
- It's a complete tree!



height = $\lfloor \log(5) \rfloor = 2$

Height of a BST

- Insert 9
- Insert 1
- Insert 7
- Insert 4
- Insert 5
- It's a linked list with a lot of extra pointers!



height = n-1 = 4 = O(n)

Balanced BSTs

- It would be ideal if a BST was always close to complete
  - i.e. balanced
- How do we guarantee a balanced BST?
  - We have to make the structure and / or the insertion and deletion algorithms more complex
    - e.g. **red – black** trees.

Sorting and Binary Search Trees

- It is possible to sort an array using a binary search tree
  - Insert the array items into an empty tree
  - Write the data from the tree back into the array using an *InOrder* traversal
- Running time = n*(insertion cost) + traversal
  - Insertion cost is O(h)
  - Traversal is O(n)
  - Total = O(n) * O(h) + O(n), i.e. O(n * h)
  - If the tree is balanced = O(n * log(n))