CMPT 225

Algorithm Analysis

Algorithm Analysis

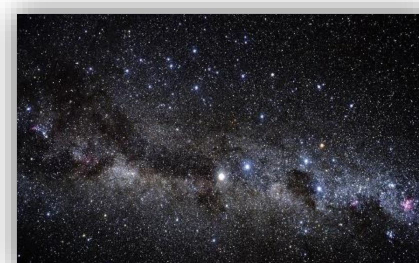
- Measuring algorithm efficiency
  - Timing
  - Counting
- Cost functions
  - Cases
    - Best case
    - Average case
    - Worst case
  - Searching
  - Sorting
- O Notation
  - O notation's mathematical basis
  - O notation classes
  - Θ and Ω notations

Counting

Algorithm Analysis

- Algorithms can be described in terms of time and space efficiency

- Time
  - How long, in *ms*, does my algorithm take to solve a particular problem?
  - How much does the time increase as the problem size increases?
  - How does my algorithm compare to other algorithms?
- Space
  - How much memory space does my algorithm require to solve the problem?



Usability

- Choosing an appropriate algorithm can make a significant difference in the usability of a system
  - Government and corporate databases with many millions of records, which are accessed frequently
  - Online search engines and media platforms
  - Big data
  - Real time systems where near instantaneous response is required
    - From air traffic control systems to computer games

Comparing Algorithms

- There are often many ways to solve a problem
  - Different algorithms that produce the same results
    - e.g. there are numerous *sorting* algorithms
- We are usually interested in how an algorithm performs when its input is large
  - In practice, with today's hardware, *most* algorithms will perform well with small input
  - There are exceptions to this, such as the *Traveling Salesman Problem*
    - Or the recursive Fibonacci algorithm presented previously …

Measuring Algorithms

- It is possible to *count* the number of operations that an algorithm performs
  - By a careful visual walkthrough of the algorithm or by
  - Inserting code in the algorithm to count and print the number of times that each line executes

profiling

- It is also possible to *time* algorithms
  - Compare system time before and after running an algorithm
    - More sophisticated timer classes exist
  - Simply timing an algorithm may ignore a variety of issues

Timing Algorithms

- It may be useful to time how long an algorithm takes to rum
  - In some cases it may be *essential* to know how long an algorithm takes on a particular system
    - e.g. air traffic control systems
    - Running time may be a strict requirement for an application
- But is this a good *general* comparison method?
  - Running time is affected by a number of factors other than algorithm efficiency

Running Time is Affected By

- CPU speed
- Amount of main memory
- Specialized hardware (e.g. graphics card)
- Operating system
- System configuration (e.g. virtual memory)
- Programming language
- Algorithm implementation
- Other programs
- System tasks (e.g. memory management)
- …

Counting

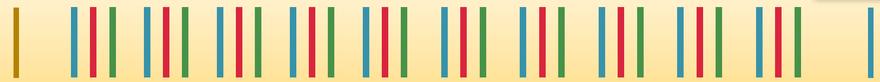
- Instead of *timing* an algorithm, *count* the number of instructions that it performs
- The number of instructions performed may vary based on
  - The size of the input
  - The organization of the input
- The number of instructions can be written as a cost function on the input size

A Simple Example

```
void printArray(int arr[], int n){
  for (int i = 0; i < n; ++i){
    cout << arr[i] << endl;
  }
}
```

32 operations

Operations performed on an array of length 10



declare and initialize *i*

perform comparison, print array element, and increment *i*:10 times

make comparison when *i* = 10

Cost Functions

- Instead of choosing a particular input size we will express a cost function for input of size *n*
  - We assume that the running time, *t*, of an algorithm is proportional to the number of operations
- Express *t* as a function of *n*
  - Where *t* is the time required to process the data using some algorithm *A*
  - Denote a cost function as $t_A(n)$
    - i.e. the running time of algorithm *A*, with input size *n*

A Simple Example

```
void printArray(int arr[], int n){
  for (int i = 0; i < n; ++i){
    cout << arr[i] << endl;
  }
}
```

t = 3n + 2

Operations performed on an array of length **n**	1	3n	1
	declare and initialize *i*	perform comparison, print array element, and increment *i*: *n* times	make comparison when *i* = *n*

What's an Operation?

- In the example we assumed two things
  - Neither of which are strictly true ...
- Any C++ statement counts as a single operation
  - Unless it is a function call
- That all operations take the same amount of time
  - Some fundamental operations are faster than others
  - What is a fundamental operation in a high level language is multiple operations in assembly
- These are both simplifying assumptions

Input Varies

- The number of operations often varies based on the size of the input
  - Though not always – consider array lookup
- In addition algorithm performance may vary based on the *organization* of the input
  - For example consider searching a large array
  - If the target is the first item in the array the search will be very fast

Best, Average and Worst Case

- Algorithm efficiency is often calculated for three broad cases of input
  - Best case
  - Average (or "usual") case
  - Worst case
- This analysis considers how performance varies for *different* inputs of the *same* size

Analyzing Algorithms

- It can be difficult to determine the exact number of operations performed by an algorithm
  - Though it is often still useful to do so
- An alternative to counting all instructions is to focus on an algorithm's *barometer instruction*
  - The barometer instruction is the instruction that is executed the most number of times in an algorithm
  - The number of times that the barometer instruction is executed is usually proportional to its running time

Cost Functions for Searching

Searching

- It is often useful to find out whether or not a list contains a particular item
  - Such a search can either return true or false
  - Or the position of the item in the list
- If the array isn't sorted use *linear search*
  - Start with the first item, and go through the array comparing each item to the target
  - If the target item is found return true (or the index of the target element)

Linear Search

```
int linearSearch(int arr[], int n, int x){
   for (int i=0; i < n; i++){
      if(arr[i] == x){
         return i;
      }
   } //for
   return -1; //target not found
}
```

The function returns as soon as the target item is found

return -1 to indicate that the item has not been found

Worst case cost function: $t_{\text{linear search}} = 3n+2$

Linear Search Barometer Instruction

- Search an array of *n* items
- The barometer instruction is equality checking (or *comparisons* for short)
  - `arr[i] == x;`
  - There are actually two other barometer instructions
    - What are they?
- How many comparisons does linear search perform?

the *barometer operation* is the most frequently executed operation

look for barometer operations in loops, or functions, or recursive calls

```
int linearSearch(int arr[], int n, int x){
	for (int i=0; i < n; i++){
		if(arr[i] == x){
			return i;
		}
	} //for
	return -1; //target not found
}
```

Linear Search Comparisons

- Best case
  - The target is the first element of the array
  - Makes 1 comparison
- Worst case
  - The target is not in the array or
  - The target is at the last position in the array
  - Makes *n* comparisons in either case
- Average case
  - Is it (*best case* + *worst case*) / 2, i.e. (n + 1) / 2?

Linear Search: Average Case

- There are two situations when the worst case occurs
  - When the target is the last item in the array
  - When the target is not there at all
- To calculate the average cost we need to know how often these two situations arise
  - We can make assumptions about this
  - Though these assumptions may not hold for a particular use of linear search

Assumptions

- A1: The target is not in the array half the time
  - Therefore half the time the entire array has to be checked to determine this
- A2: There is an equal probability of the target being at any array location
  - If it is in the array
  - That is, there is a probability of $1/n$ that the target is at some location i

Cost When Target Not Found

- Work done if the target is *not* in the array
  - *n* comparisons
  - This occurs with probability of 0.5 (A1)

Cost When Target Is Found

- Work done if target is in the array:
  - 1 comparison if target is at the 1st location
    - Occurs with probability $1/n$ (A2)
  - 2 comparisons if target is at the 2nd location
    - Also occurs with probability $1/n$
  - i comparisons if target is at the i^{th} location
- Take the weighted average of the values to find the total expected number of comparisons (E)
  - $E = 1*1/n + 2*1/n + 3*1/n + \ldots + n * 1/n$ *or*
  - $E = (n + 1) / 2$

Average Case Cost

- Target is *not* in the array: n comparisons
- Target *is* in the array $(n + 1) / 2$ comparisons
- Take a weighted average of the two amounts:
  - $= (n * ½) + ((n + 1) / 2 * ½)$
  - $= (n / 2) + ((n + 1) / 4)$
  - $= (2n / 4) + ((n + 1) / 4)$
  - $= (3n + 1) / 4$
- Therefore, on average, we expect linear search to perform $(3n + 1) / 4$ comparisons

Linear Search and Sorted Arrays

- If we sort the target array first we can change the linear search average cost to approximately *n* / 2
  - Once a value equal to or greater than the target is found the search can end
    - So, if a sequence contains 8 items, on average, linear search compares 4 of them,
    - If a sequence contains 1,000,000 items, linear search compares 500,000 of them, etc.
- However, if the array is sorted, it is possible to do *much better* than this by using binary search

Binary Search Algorithm

```
int binarySearch(int arr[], int n, int x){
    int low = 0;
    int high = n - 1;
    int mid = 0;
    while (low <= high){
        mid = (low + high) / 2;
        if(x == arr[mid]){
            return mid;
        } else if(x > arr[mid]){
            low = mid + 1;
        } else { //x < arr[mid]
            high = mid - 1;
        }
    } //while
    return -1; //target not found
}
```

Index of the last element in the array

Note: if, else if, else

Analyzing Binary Search

- The algorithm consists of three parts
  - Initialization (setting lower and upper)
  - While loop including a return statement on success
  - Return statement which executes on failure
- Initialization and return on failure require the same amount of work regardless of input size
- The number of times that the while loop iterates depends on the size of the input

Binary Search Iteration

- The while loop contains an *if*, *else if*, *else* statement
- The first if condition is met when the target is found
  - And is therefore performed at most once each time the algorithm is run
- The algorithm usually performs 5 operations for each iteration of the while loop
  - Checking the while condition
  - Assignment to mid
  - Equality comparison with target
  - Inequality comparison
  - One other operation (setting either lower or upper)

Barometer instructions

Best Case

- In the best case the target is the midpoint element of the array
  - Requiring just one iteration of the while loop

binary search (arr, 11)

index	0	1	2	3	4	5	6	7
arr	1	3	7	11	13	17	19	23

mid = (0 + 7) / 2 = 3

Worst Case

- What is the worst case for binary search?
  - Either the target is not in the array, or
  - It is found when the search space consists of one element
- How many times does the while loop iterate in the worst case?

binary search (arr, 20)

index	0	1	2	3	4	5	6	7
arr	1	3	7	11	13	17	19	23

mid = (0 + 7) / 2 = 3 (4 + 7) / 2 = 5 (6 + 7) / 2 = 6 (7 + 7) / 2 = 7

Analyzing the Worst Case

- Each iteration of the while loop halves the search space
  - For simplicity assume that n is a power of 2
    - So $n = 2^k$ (e.g. if $n = 128$, $k = 7$ or if $n = 8$, $k = 3$)
- How large is the search space?
  - After the first iteration the search space is $n/2$
  - After the second iteration the search space is $n/4$
  - After the k^{th} iteration the search space consists of just one element
    - Note that as $n = 2^k$, $k = \log_2 n$
  - The search space of size 1 still needs to be checked
  - Therefore at most $\log_2 n + 1$ iterations of the while loop are made in the worst case

or $n/2^1$

or $n/2^2$

$n/2^k = n/n = 1$

Cost function: $t_{\text{binary search}} = 5(\log_2(n)+1)+4$

Average Case

- Is the average case more like the best case or the worst case?
  - What is the chance that an array element is the target
    - 1/*n* the first time through the loop
    - 1/(*n*/2) the second time through the loop
    - … and so on …
- It is more likely that the target will be found as the search space becomes small
  - That is, when the while loop nears its final iteration
  - We can conclude that the average case is more like the worst case than the best case

Binary Search vs Linear Search

n	Linear Search $(3n+1)/4$	Binary Search $\log_2(n)+1$
10	8	4
100	76	8
1,000	751	11
10,000	7,501	14
100,000	75,001	18
1,000,000	750,001	21
10,000,000	7,500,001	25

Simple Sorting

Simple Sorting

- As an example of algorithm analysis let's look at two simple sorting algorithms
  - Selection Sort and
  - Insertion Sort
- Calculate an approximate cost function for these two sorting algorithms
  - By analyzing how many operations are performed by each algorithm
  - This will include an analysis of how many times the algorithms' loops iterate

Selection Sort

- The array is divided into sorted part and unsorted parts
- Expand the sorted part by swapping the first unsorted element with the smallest unsorted element
  - Starting with the element with index 0, and
  - Ending with the last but one element (index $n - 1$)
- Requires two processes
  - Finding the smallest element of a sub-array
  - Swapping two elements of the array

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
arr	1	3	5	7	21	27	29	23	19	13	31	15	17	9	25	11

smallest: 9

The algorithm is on its fifth iteration

Find the smallest element in arr[4:15]

Selection Sort

- The array is divided into sorted part and unsorted parts
- Expand the sorted part by swapping the first unsorted element with the smallest unsorted element
  - Starting with the element with index 0, and
  - Ending with the last but one element (index $n-1$)
- Requires two processes
  - Finding the smallest element of a sub-array
  - Swapping two elements of the array

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
arr	1	3	5	7	9	27	29	23	19	13	31	15	17	21	25	11

smallest: 9

The algorithm is on its fifth iteration

Find the smallest element in arr[4:15]

Swap smallest and first unsorted elements

Selection Sort Algorithm

```
void selectionSort(int arr[], int n){
    for(int i = 0; i < n-1; ++i){
        int smallest = getSmallest(arr, i, n);
        swap(arr, i, smallest);
    }
}
```

```
int getSmallest(int arr[], int start, int end){
    int smallest = start;
    for(int i = start + 1; i < end; ++i){
        if(arr[i] < arr[smallest]){
            smallest = i;
        }
    }
    return smallest;
}
```

note: end is 1 past the last legal index

```
void swap(int arr[], int i, int j){
    int temp = arr[i];
    arr[i] = arr[j];
    arr[j] = temp;
}
```

Selection Sort Analysis – Swap

```
void selectionSort(int arr[], int n){
    for(int i = 0; i < n-1; ++i){
        int smallest = getSmallest(arr, i, n);
        swap(arr, i, smallest);
    }
}
```

n-1 swaps

3(*n*-1)

```
int getSmallest(int arr[], int start, int end){
    int smallest = start;
    for(int i = start + 1; i < end; ++i){
        if(arr[i] < arr[smallest]){
            smallest = i;
        }
    }
    return smallest;
}
```

Swap always performs three operations

```
void swap(int arr[], int i, int j){
    int temp = arr[i];
    arr[i] = arr[j];
    arr[j] = temp;
}
```

Selection Sort Analysis – Smallest

```
void selectionSort(int arr[], int n){
    for(int i = 0; i < n-1; ++i){
        int smallest = getSmallest(arr, i, n);
        swap(arr, i, smallest);
    }
}
```

called *n*-1 times

n-1 swaps

3(*n*-1)

```
int getSmallest(int arr[], int start, int end){
    int smallest = start;
    for(int i = start + 1; i < end; ++i){
        if(arr[i] < arr[smallest]){
            smallest = i;
        }
    }
    return smallest;
}
```

4(*start*-*end*-1)+4 operations

we can substitute *n* for *end*

the *start* index is the index of *i* in the calling function

the values of which are the sequence {0,1,2,…, n-2}

1:*n*-1, 2:*n*-1, …, *n*-1:*n*-1,

average = (*n*-1+1) / 2 = *n*/2

average cost = 4(*n*/2)+4

Selection Sort Analysis – Smallest

```
void selectionSort(int arr[], int n){
    for(int i = 0; i < n-1; ++i){
        int smallest = getSmallest(arr, i, n);
        swap(arr, i, smallest);
    }
}
```

called n-1 times

$(n-1)(4(n/2)+4)$

$= (n-1)(2n + 4)$

$= 2n^2-2n+4(n-1)$

n-1 swaps

$3(n-1)$

for loop: $3(n-1)+2$

Cost function: $t_{selection\ sort} = 2n^2 - 2n + 10(n-1) + 2$

Barometer Operation

- The barometer operation for selection sort is in the loop that finds the smallest item
  - Since operations in that loop are executed the greatest number of times
- The loop contains four operations
  - Compare *i* to *end*
  - Compare *arr*[*i*] to smallest
  - Change smallest
  - Increment *i*

The barometer instructions

```
int getSmallest(arr[], start, end)
      smallest = start
      for(i = start + 1; i < end; ++i)
            if(arr[i] < arr[smallest])
                  smallest = i
      return smallest
```

Barometer Operations

Unsorted elements	Barometer
n	n-1
n-1	n-2
…	…
3	2
2	1
1	0
	$n(n-1)/2$

Selection Sort Cases

- How is selection sort affected by the organization of the input?
  - The only work that varies based on the input organization is whether or not smallest is assigned the value of *arr*[*i*]
- What is the worst case organization?
- What is the best case organization?
- The difference between best case and worst case is quite small
  - $(n-1)(3(n/2)) + 10(n-1) + 2$ in the best case and
  - $(n-1)(4(n/2)) + 10(n-1) + 2$ in the worst case

Selection Sort Summary

- Ignoring leading constants, selection sort performs the following work
  - $n*(n-1)/2$ barometer operations, regardless of the original order of the input
  - $n-1$ swaps
- The number of comparisons dominates the number of swaps
- The organization of the input only affects the leading constant of the barometer operations

Insertion Sort

- The array is divided into sorted part and unsorted parts
- The sorted part is expanded one element at a time
  - By moving elements in the sorted part up one position until the correct position for the first unsorted element is found
    - Note that the first unsorted element is stored so that it is not lost when it is written over by this process
  - The first unsorted element is then copied to the insertion point

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
arr	5	11	27	31	21	1	29	23	19	13	7	15	17	9	25	3

temp: 21

The algorithm is on its fourth iteration

Find the correct position for arr[4]

Insertion Sort

- The array is divided into sorted part and unsorted parts
- The sorted part is expanded one element at a time
  - By moving elements in the sorted part up one position until the correct position for the first unsorted element is found
    - Note that the first unsorted element's value is stored so that it is not lost when it is written over by this process
  - The first unsorted element is then copied to the insertion point

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
arr	5	11	21	27	31	1	29	23	19	13	7	15	17	9	25	3

temp: 21

The algorithm is on its fourth iteration

Find the correct position for arr[4]

Move up elements in the sorted part until the position for 21 is found

Work Performed in Inserting Values

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
arr	5	11	21	27	31	1	29	23	19	13	7	15	17	9	25	3

- How much work was performed by expanding the sorted part of the array by one element?
  - The value 21 was stored in a variable, *temp*
  - The values 27 and 31 were compared to 21
    - And moved up one position in the array
  - The value 11 was compared to 21, but not moved
  - The value of *temp* was written to *arr*[2]
- How much work will be performed expanding the sorted part of the array to include the value 1?
- How much work will be performed expanding the sorted part of the array to include the value 29?

Insertion Sort Algorithm

```
void insertionSort(int arr[], int n){
    for(int i = 1; i < n; ++i){
        temp = arr[i];
        int pos = i;
        // Shuffle up all sorted items > arr[i]
        while(pos > 0 && arr[pos - 1] > temp){
            arr[pos] = arr[pos – 1];
            pos--;
        } //while
        // Insert the current item
        arr[pos] = temp;
    }
}
```

What are the barometer operations?

How often are they performed?

It depends on the values in the array

Insertion Sort Algorithm

```
void insertionSort(int arr[], int n){
    for(int i = 1; i < n; ++i){
        temp = arr[i];
        int pos = i;
        // Shuffle up all sorted items > arr[i]
        while(pos > 0 && arr[pos - 1] > temp){
            arr[pos] = arr[pos - 1];
            pos--;
        } //while
        // Insert the current item
        arr[pos] = temp;
    }
}
```

outer loop
n-1 times

inner loop body
how many times?

worst case: *pos* – 1 times for each iteration

pos ranges from 1 to *n*-1; *n*/2 on average

What is the worst case organization?

outer loop runs *n*-1 times: $n * (n - 1) / 2$

Insertion Sort Worst Case Cost

Sorted Elements	Worst-case Search	Worst-case Move
0	0	0
1	1	1
2	2	2
…	…	…
n-1	n-1	n-1
	$n(n-1)/2$	$n(n-1)/2$

Insertion Sort Worst Case

- In the worst case the array is in reverse order
- Every item has to be moved all the way to the front of the array
  - The outer loop runs n-1 times
    - In the first iteration, one comparison and move
    - In the last iteration, n-1 comparisons and moves
    - On average, n/2 comparisons and moves
  - For a total of $n * (n-1) / 2$ comparisons and moves

Insertion Sort Best Case

- The efficiency of insertion sort *is* affected by the state of the array to be sorted
- What is the best case?
  - In the best case the array is already completely sorted!
  - No movement of any array element is required
  - Requires *n* comparisons

Insertion Sort: Average Case

- What is the average case cost?
  - Is it closer to the best case?
  - Or the worst case?
- If *random* data is sorted, insertion sort is usually closer to the worst case
  - Around $n * (n\text{-}1) / 4$ comparisons
- And what do we mean by average input for a sorting algorithm in anyway?

Introduction to QuickSort

QuickSort Introduction

- Quicksort is a more efficient sorting algorithm than either selection or insertion sort
  - It sorts an array by repeatedly *partitioning* it
- Partitioning is the process of dividing an array into sections (partitions), based on some criteria
  - Big and small values
  - Negative and positive numbers
  - Names that begin with *a-m*, names that begin with *n-z*
  - Darker and lighter pixels

Partitioning an Array

Partition this array into *small* and *big* values using a partitioning algorithm

31	12	07	23	93	02	11	18

Partitioning an Array

Partition this array into *small* and *big* values using a partitioning algorithm

We will partition the array around the last value (18), we'll call this value the *pivot*

Use two indices, one at each end of the array, call them *low* and *high*

31	12	07	23	93	02	11	18

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Partition this array into *small* and *big* values using a partitioning algorithm

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31	12	07	23	93	02	11	18

arr[*low*] (31) is greater than the pivot and should be on the right, we need to swap it with something

Partitioning an Array

Partition this array into *small* and *big* values using a partitioning algorithm

31	12	07	23	93	02	11	18

We will partition the array around the last value (18), we'll call this value the *pivot*

Use two indices, one at each end of the array, call them *low* and *high*

arr[*low*] (31) is greater than the pivot and should be on the right, we need to swap it with something

arr[*high]* (11) is less than the pivot so swap with arr[*low*]

Partitioning an Array

Partition this array into *small* and *big* values using a partitioning algorithm

We will partition the array around the last value (18), we'll call this value the *pivot*

Use two indices, one at each end of the array, call them *low* and *high*

11	12	07	23	93	02	31	18

Partitioning an Array

Partition this array into *small* and *big* values using a partitioning algorithm

We will partition the array around the last value (18), we'll call this value the *pivot*

Use two indices, one at each end of the array, call them *low* and *high*

11	12	07	23	93	02	31	18

increment *low* until it needs to be swapped with something

then decrement *high* until it can be swapped with *low*

Partitioning an Array

Partition this array into *small* and *big* values using a partitioning algorithm

We will partition the array around the last value (18), we'll call this value the *pivot*

Use two indices, one at each end of the array, call them *low* and *high*

11	12	07	02	93	23	31	18

increment *low* until it needs to be swapped with something

then decrement *high* until it can be swapped with *low*

and then swap them

Partitioning Algorithm

Partition this array into *small* and *big* values using a partitioning algorithm

We will partition the array around the last value (18), we'll call this value the *pivot*

Use two indices, one at each end of the array, call them *low* and *high*

11	12	07	02	93	23	31	18

repeat this process until

high and *low* are the same

Partitioning an Array

Partition this array into *small* and *big* values using a partitioning algorithm

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Use two indices, one at each end of the array, call them *low* and *high*

11	12	07	02	18	23	31	93

repeat this process until

high and *low* are the same

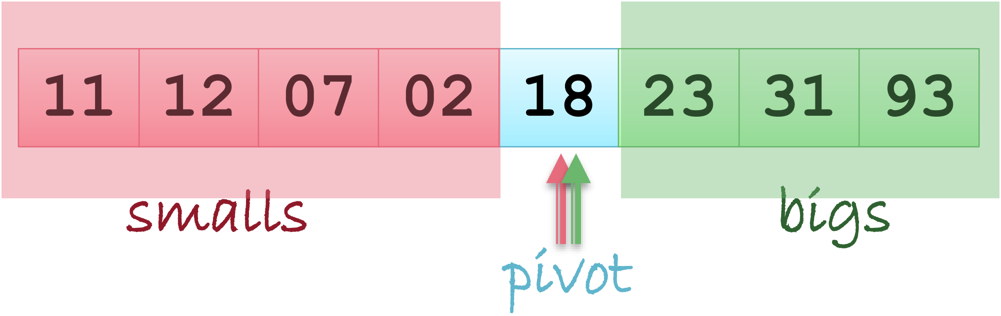
We'd like the pivot value to be in the centre of the array, so we will swap it with the first item greater than it

Partitioning an Array

Partition this array into *small* and *big* values using a partitioning algorithm

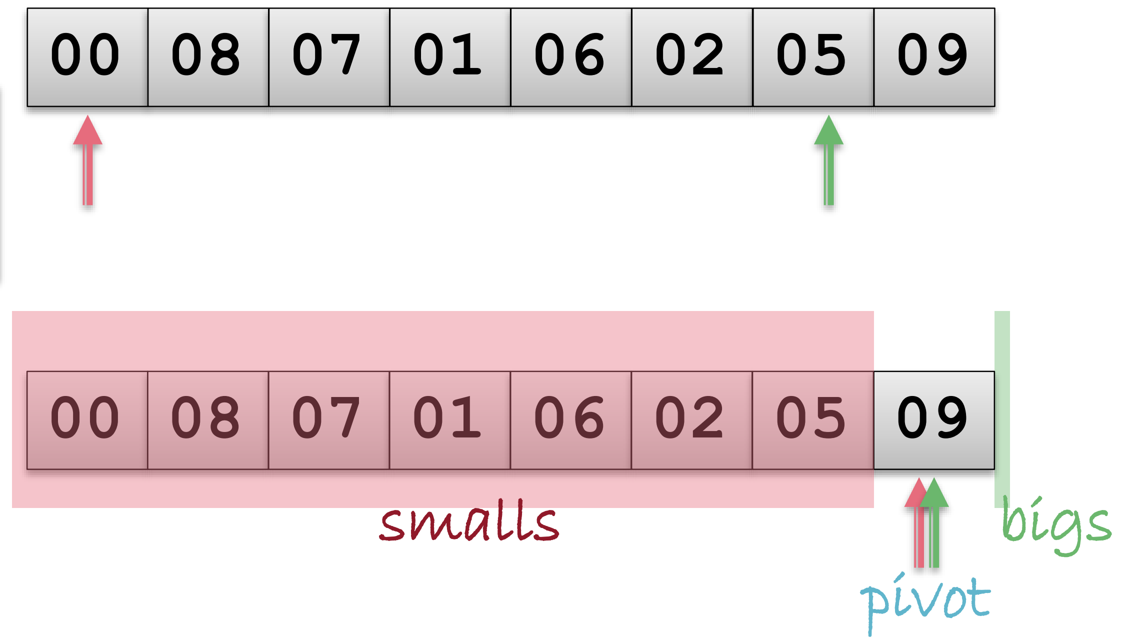
We will partition the array around the last value (18), we'll call this value the *pivot*

Use two indices, one at each end of the array, call them *low* and *high*



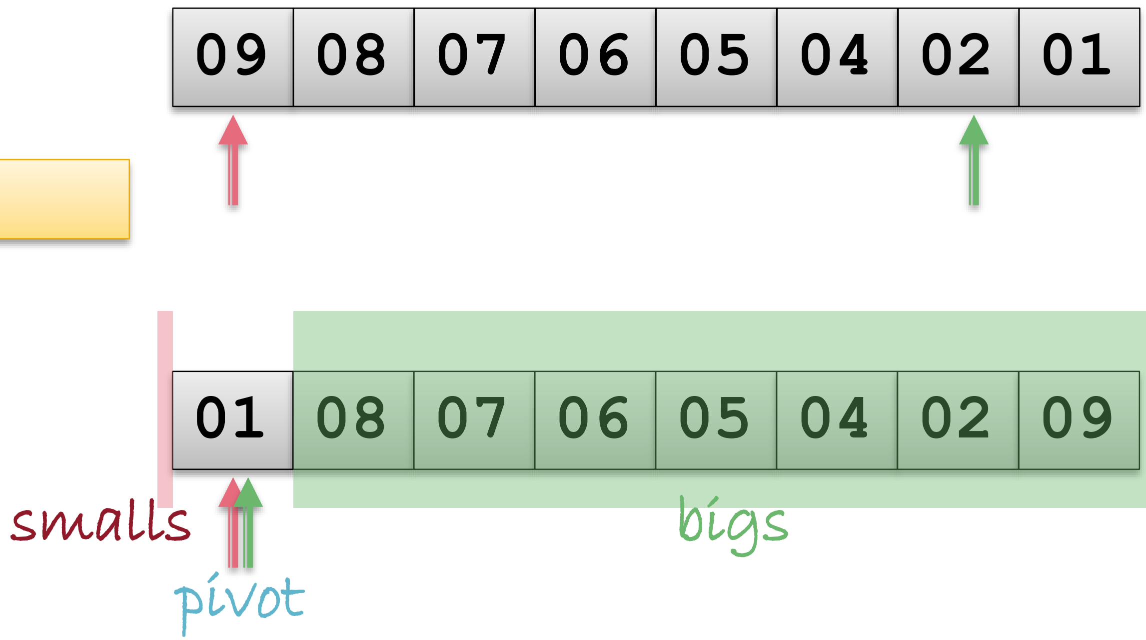
Partitioning Question

Use the same algorithm to partition this array into small and big values



Partitioning Question

Or this one:



Quicksort

- The quicksort algorithm works by *repeatedly partitioning* an array
- Each time a sub-array is partitioned there is
  - A sequence of *small* values,
  - A sequence of *big* values, and
  - A *pivot* value *which is in the correct position*
- Partition the small values, and the big values
  - Repeat the process until each sub-array being partitioned consists of just one element

Quicksort Algorithm

- The quicksort algorithm repeatedly partitions an array until it is sorted
  - Until all partitions consist of at most one element
- A simple iterative approach would halve each sub-array to get partitions
  - But partitions are not necessarily the same size
  - So the start and end indexes of each partition are not easily predictable

Uneven Partitions

47	70	36	97	03	67	29	11	48	09	53
36	09	29	48	03	11	47	53	97	61	70
36	09	03	11	29	47	48	53	61	70	97
08	01	11	29	36	47	48	53	61	70	97
09	03	11	29	36	47	48	53	61	70	97
03	09	11	29	36	47	48	53	61	70	97

Keeping Track of Indexes

- One way to implement quicksort might be to record the index of each new partition
- But this is difficult and requires a reasonable amount of space
  - The goal is to record the start and end index of each partition
  - This can be achieved by making them the parameters of a recursive function

Recursive Quicksort

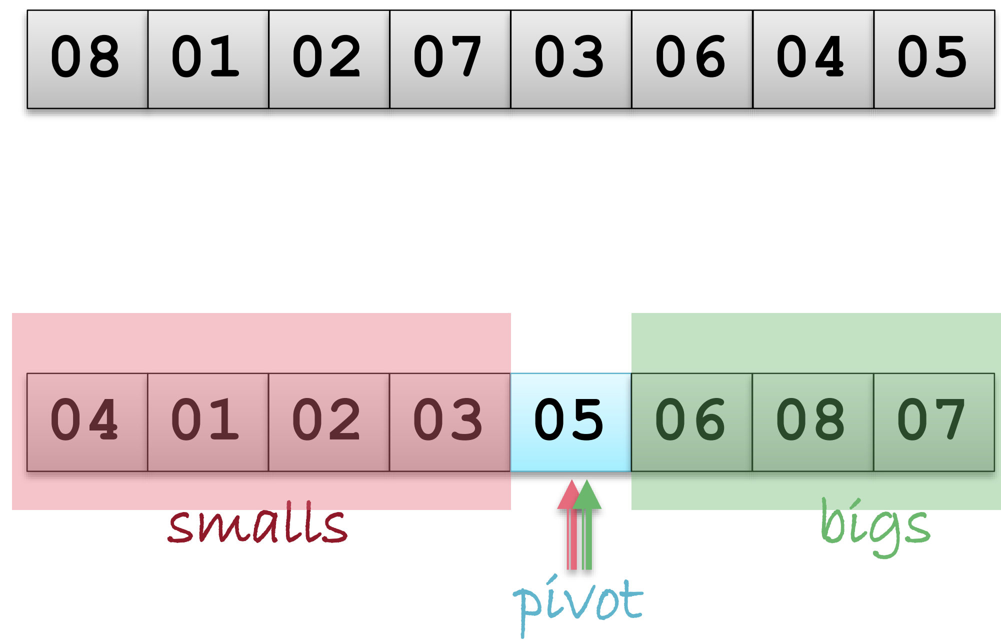
```
void quicksort(arr[], int low, int high){
  if (low < high){
    pivot = partition(arr, low, high);
    quicksort(arr, low, pivot – 1);
    quicksort(arr, pivot + 1, high);
  }
}
```

Quicksort Analysis

- How long does Quicksort take to run?
  - Let's consider the best and the worst case
  - These differ because the partitioning algorithm may not always do a good job
- Let's look at the best case first
  - Each time a sub-array is partitioned the pivot is the exact midpoint of the slice (or as close as it can get)
    - So it is divided in half
  - What is the running time?

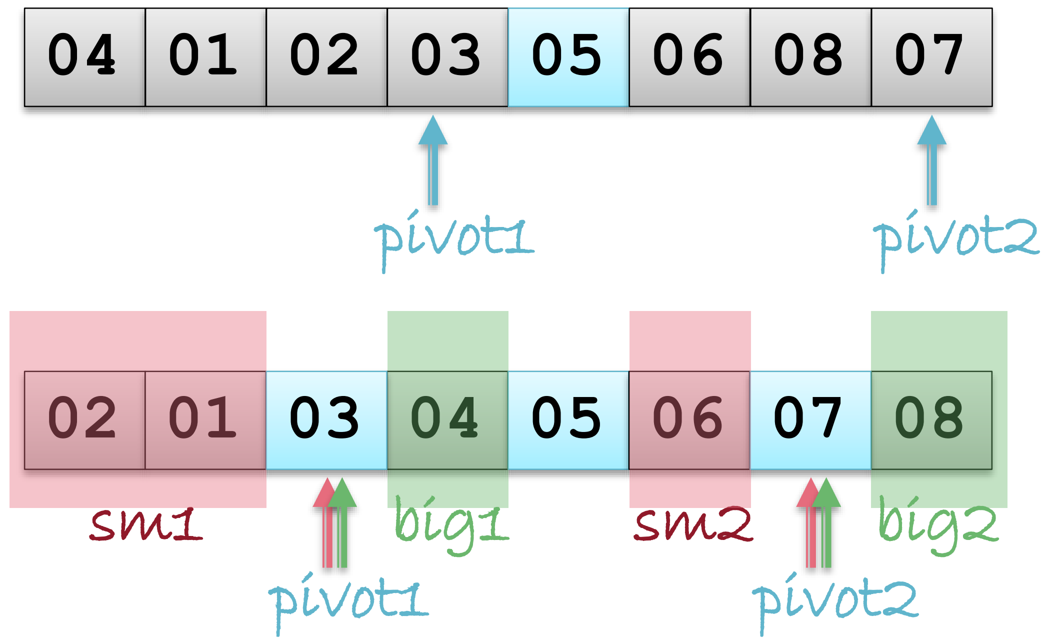
Quicksort Best Case

First partition



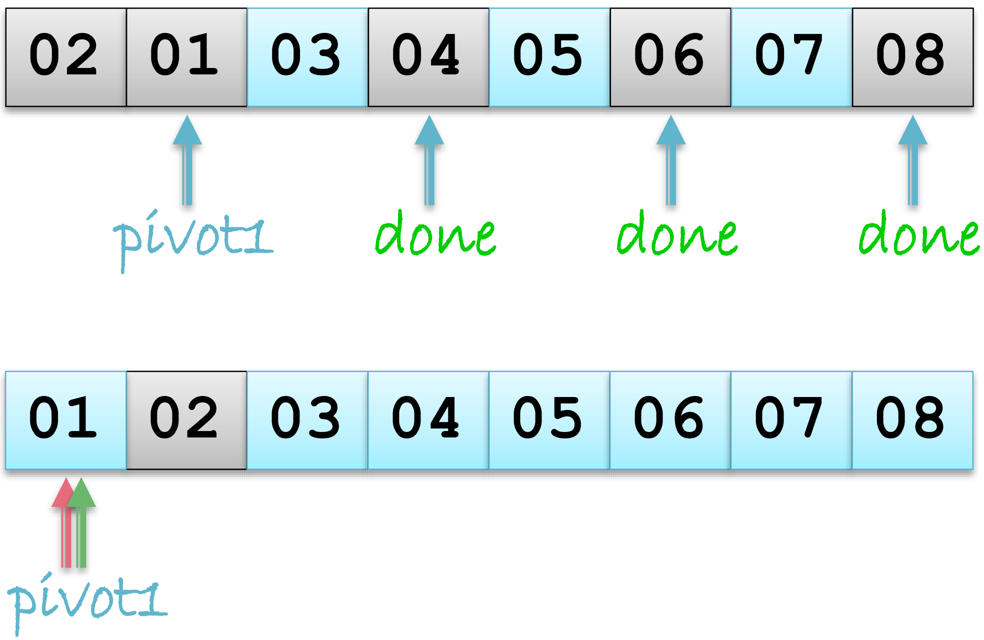
Quicksort Best Case

Second partition



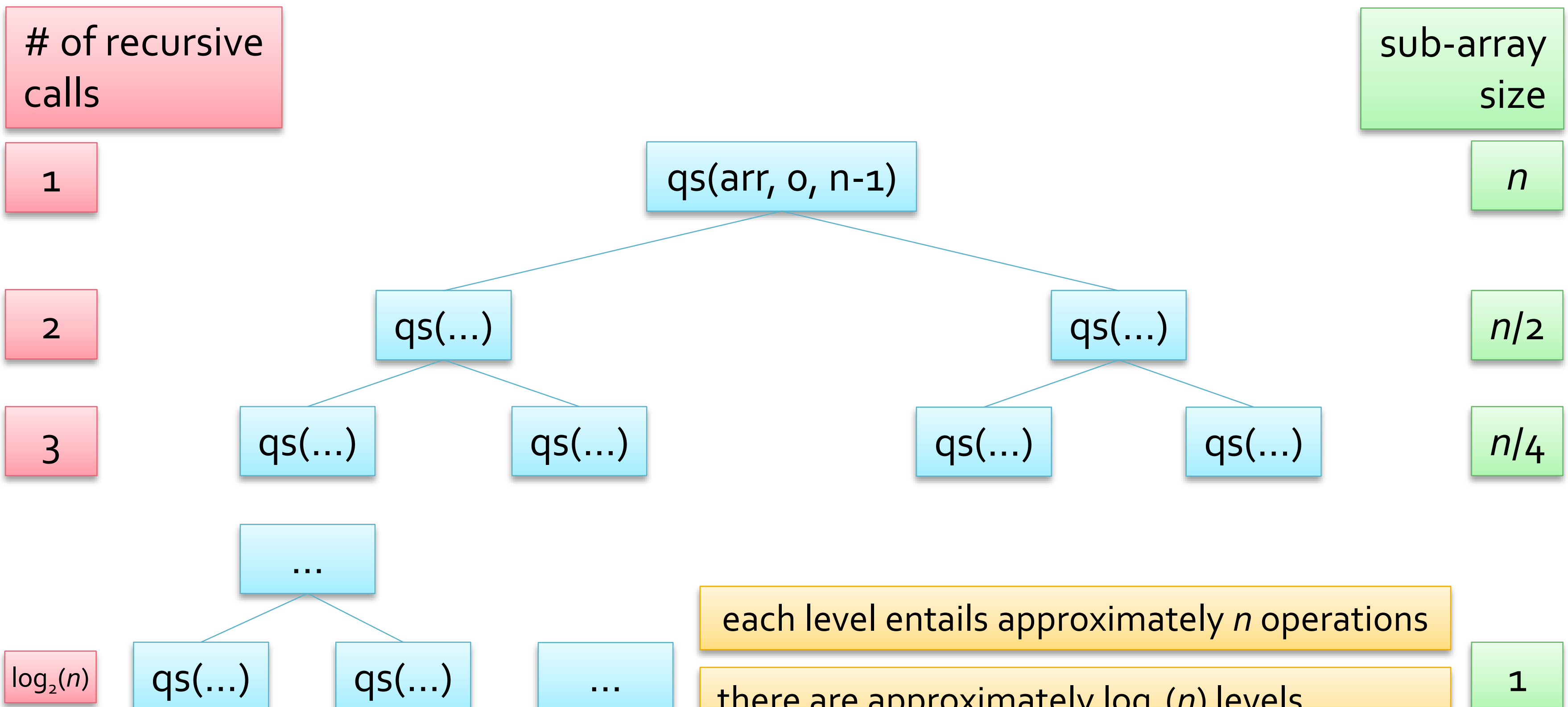
Quicksort Best Case

Third partition



Quicksort Recursion Tree

Assume the best case – each partition splits its sub-array in half



each level entails approximately n operations

there are approximately $\log_2(n)$ levels

approximately $\log_2(n) * n$ operations in total

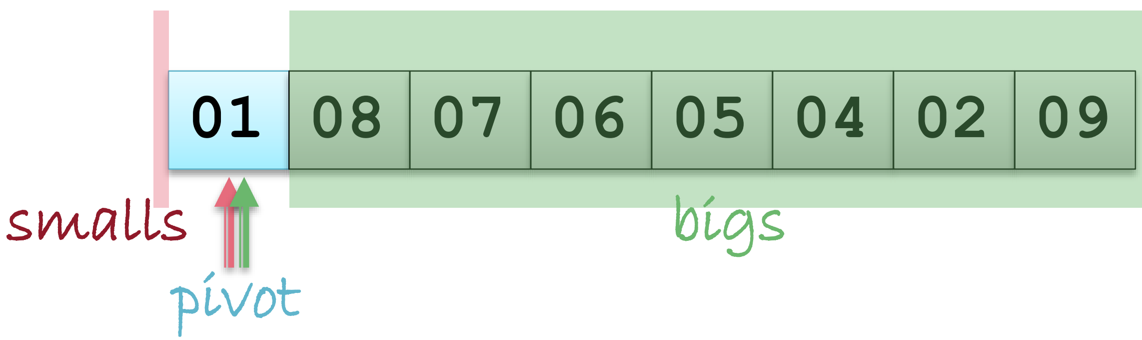
Quicksort Best Case

- Each sub-array is divided in half in each partition
  - Each time a series of sub-arrays are partitioned *n* (approximately) comparisons are made
  - The process ends once all the sub-arrays left to be partitioned are of size 1
- How many times does *n* have to be divided in half before the result is 1?
  - $\log_2(n)$ times
  - Quicksort performs $n * \log_2 n$ operations in the best case

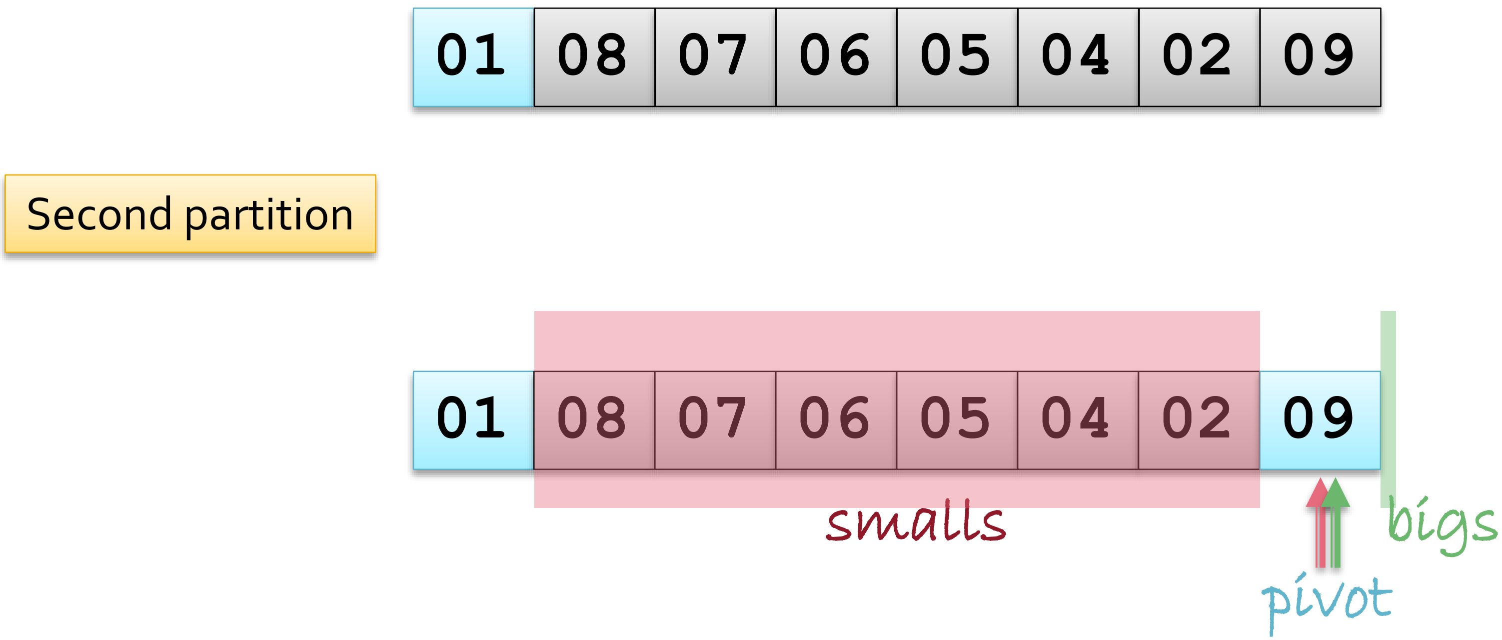
Quicksort Worst Case

09	08	07	06	05	04	02	01

First partition

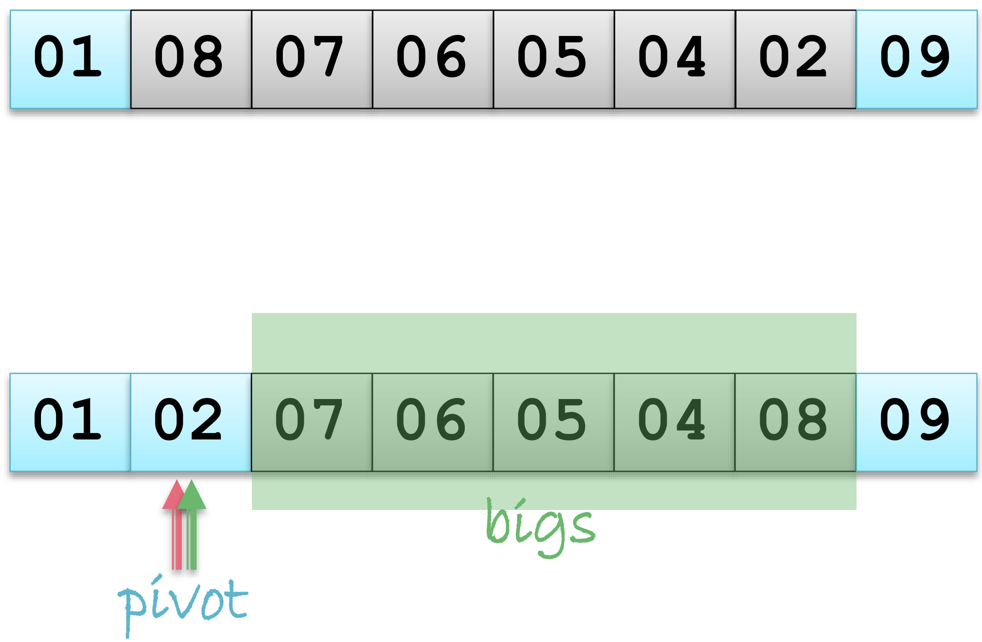


Quicksort Worst Case



Quicksort Worst Case

Third partition



Quicksort Worst Case

01	02	07	06	05	04	08	09

Fourth partition

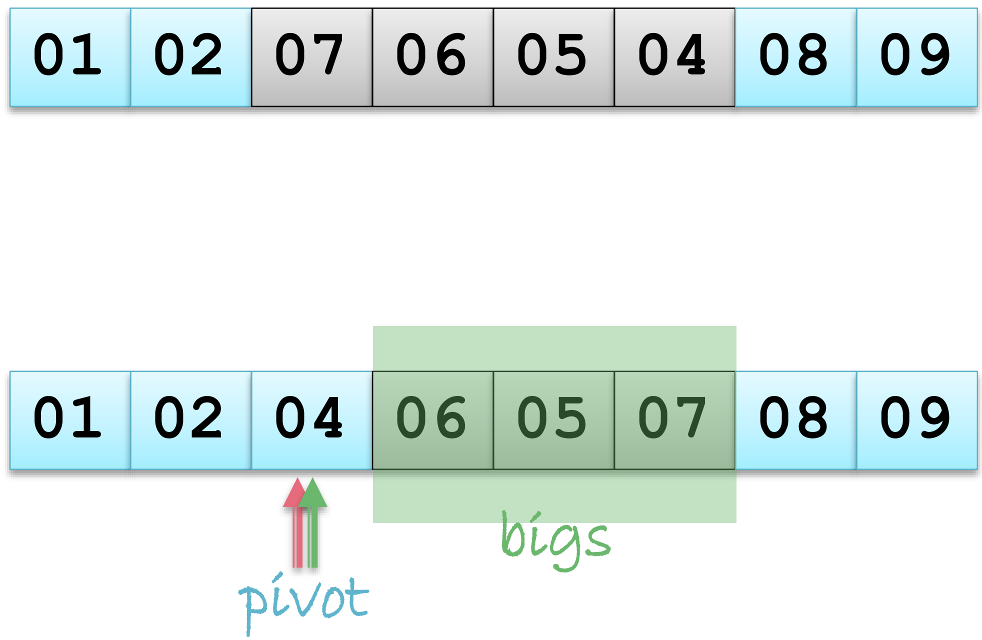
01	02	07	06	05	04	08	09

smalls

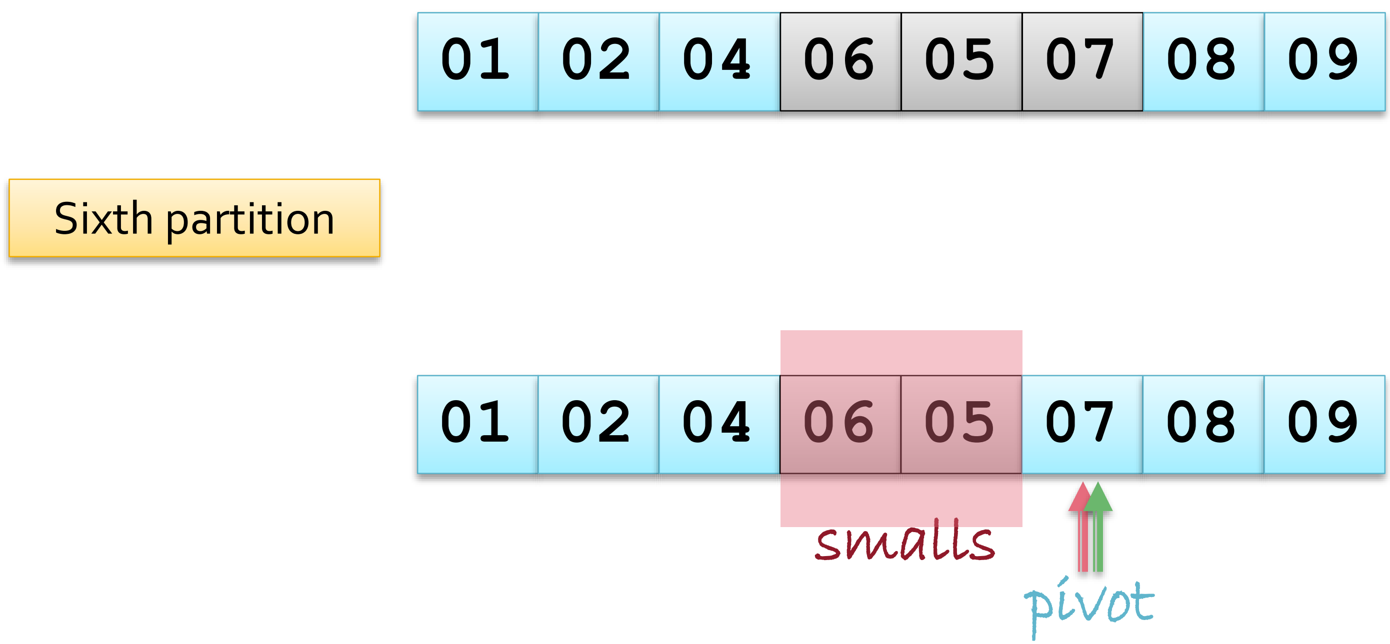
pivot

Quicksort Worst Case

Fifth partition



Quicksort Worst Case



Quicksort Worst Case

01	02	04	06	05	07	08	09

Seventh partition!

01	02	04	05	06	07	08	09

pivot

Quicksort Recursion Tree

Assume the worst case – each partition step results in a single sub-array

# of recursive calls		sub-array size
1	qs(arr, 0, n-1)	*n*
2	qs(…)	*n*-1
3	qs(…)	*n*-2
	…	
n	qs(…)	1

each level entails on average *n*/2 operations

there are approximately *n* levels

approximately $n^2/2$ operations in total

Quicksort Worst Case

- Every partition step ends with no values on one side of the pivot
  - The array has to be partitioned *n* times, not $\log_2(n)$ times
  - So in the worst case Quicksort performs around n^2 operations
- The worst case usually occurs when the array is nearly sorted (in either direction)

Quicksort Average Case

- With a large array we would have to be very, very unlucky to get the worst case
  - Unless there was some reason for the array to already be partially sorted
- The average case is much more like the best case than the worst case
- There is an easy way to fix a partially sorted arrays to that it is ready for quicksort
  - Randomize the positions of the array elements!

Comparisons

Ignoring Leading Constants

- Calculation of a detailed cost function can be onerous and dependent on
  - Exactly how the algorithm was implemented
    - Implementing selection sort as a single function would have resulted in a different cost function
  - The definition of a single discrete operation
    - How many operations is this: *mid* = (*low* + *high*) / 2?
- We are often interested in how algorithms behave as the problem size increases

Comparing Algorithm Performance

- There can be many ways to solve a problem
  - Different algorithms that produce the same result
    - There are numerous sorting algorithms
- Compare algorithms by their behaviour for large input sizes, i.e., as *n* gets large
  - On today's hardware, *most* algorithms perform quickly for small *n*
- Interested in growth rate as a function of *n*
  - Sum an array: *linear* growth
  - Sort with selection sort: *quadratic* growth

Comparing Algorithms

- Measuring the performance of an algorithm can be simplified by
  - Only considering the highest order term
    - i.e. only consider the number of times that the barometer instruction is executed
  - And ignoring leading constants
- Consider the selection sort algorithm
  - $t_{\text{selection sort}} = 2n^2 - 2n + 10(n-1) + 2$
  - Its cost function approximates to n^2

Algorithm Summary

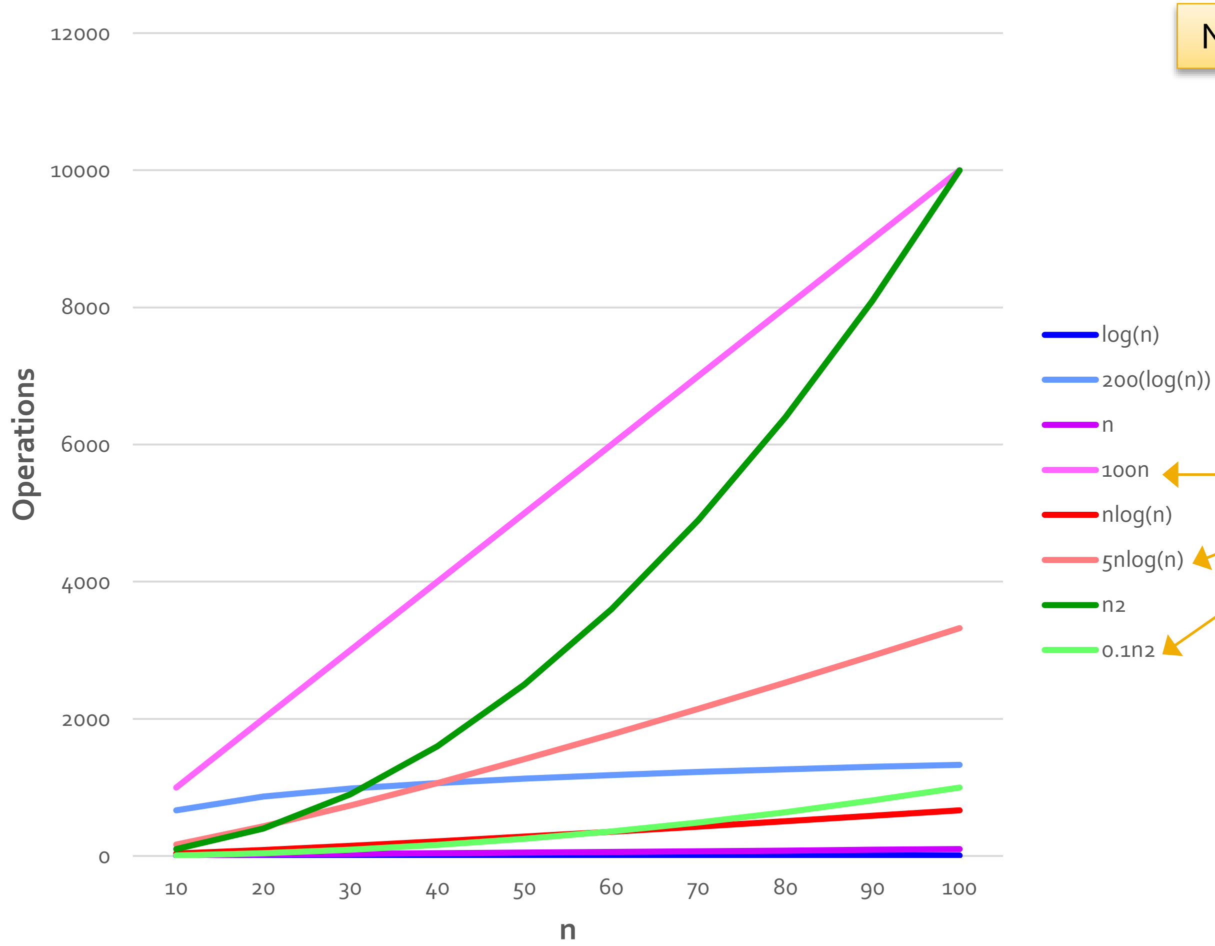
- What are the approximate number of barometer operations for the algorithms we looked at?
  - Ignoring leading constants
- Linear search: n — worst and average case
- Binary search: $\log_2 n$ — worst and average case
- Selection sort: n^2 — all cases
- Insertion sort: n^2 — worst and average case
- Quicksort: $n(\log_2(n))$ — best and average case

Algorithm Growth Rate

- What do we want to know when comparing two algorithms?
  - Often, the most important thing is how quickly the time requirements increase with input size
  - e.g. If we double the input size how much longer does an algorithm take?
- Here are some graphs …

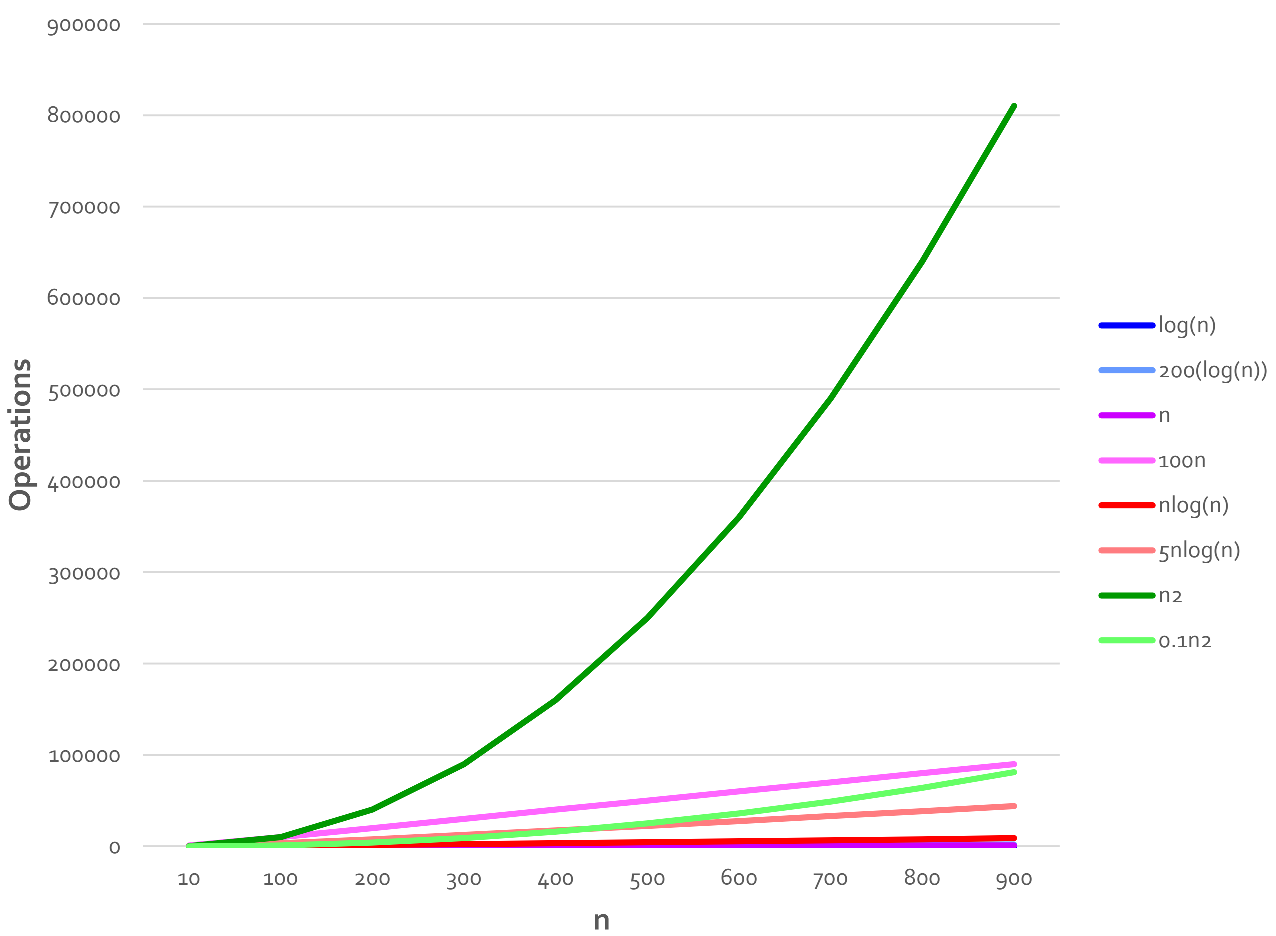
Small *n*

Note that *n* is very small ...



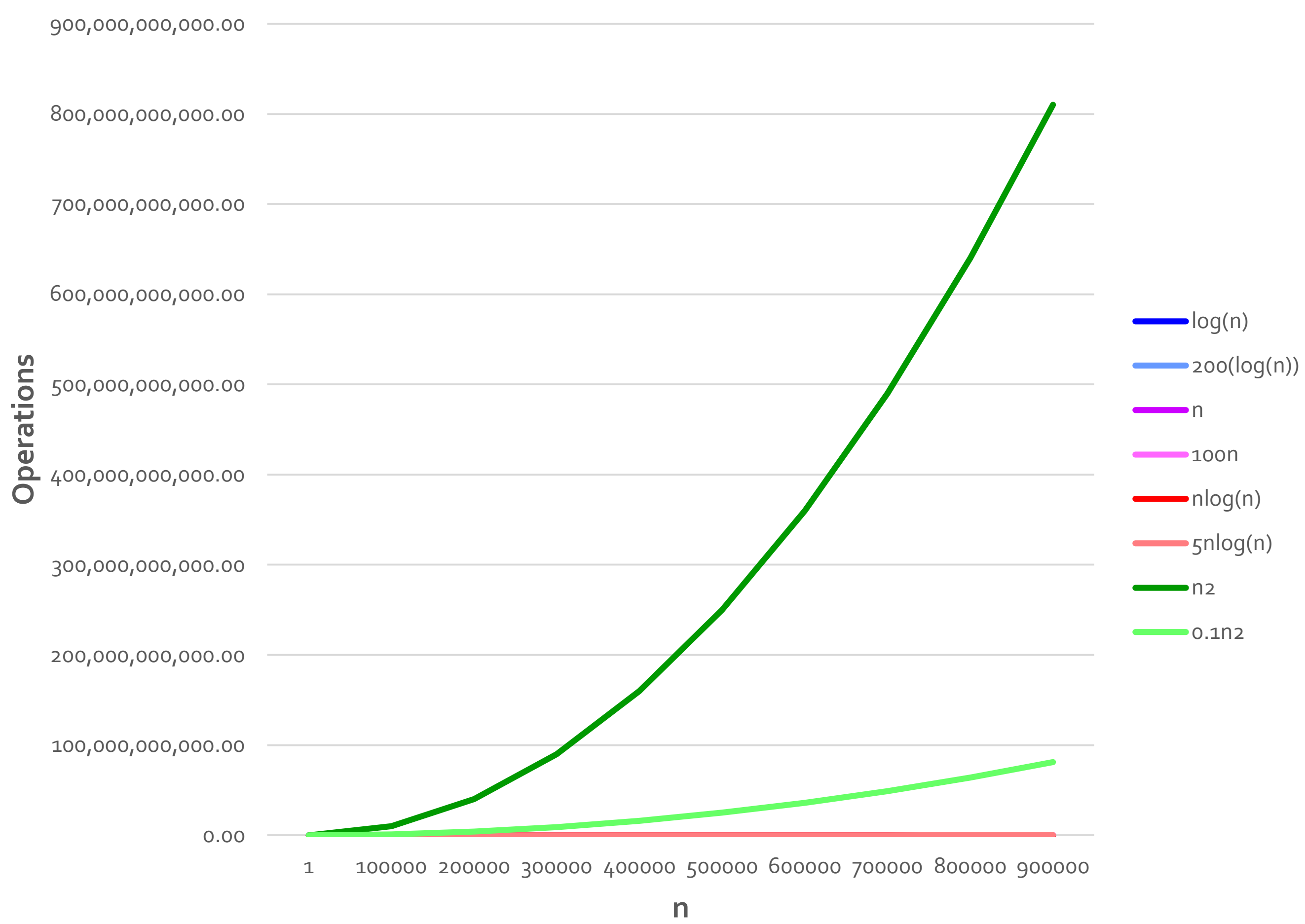
Arbitrary functions for the sake of illustration

Larger *n*

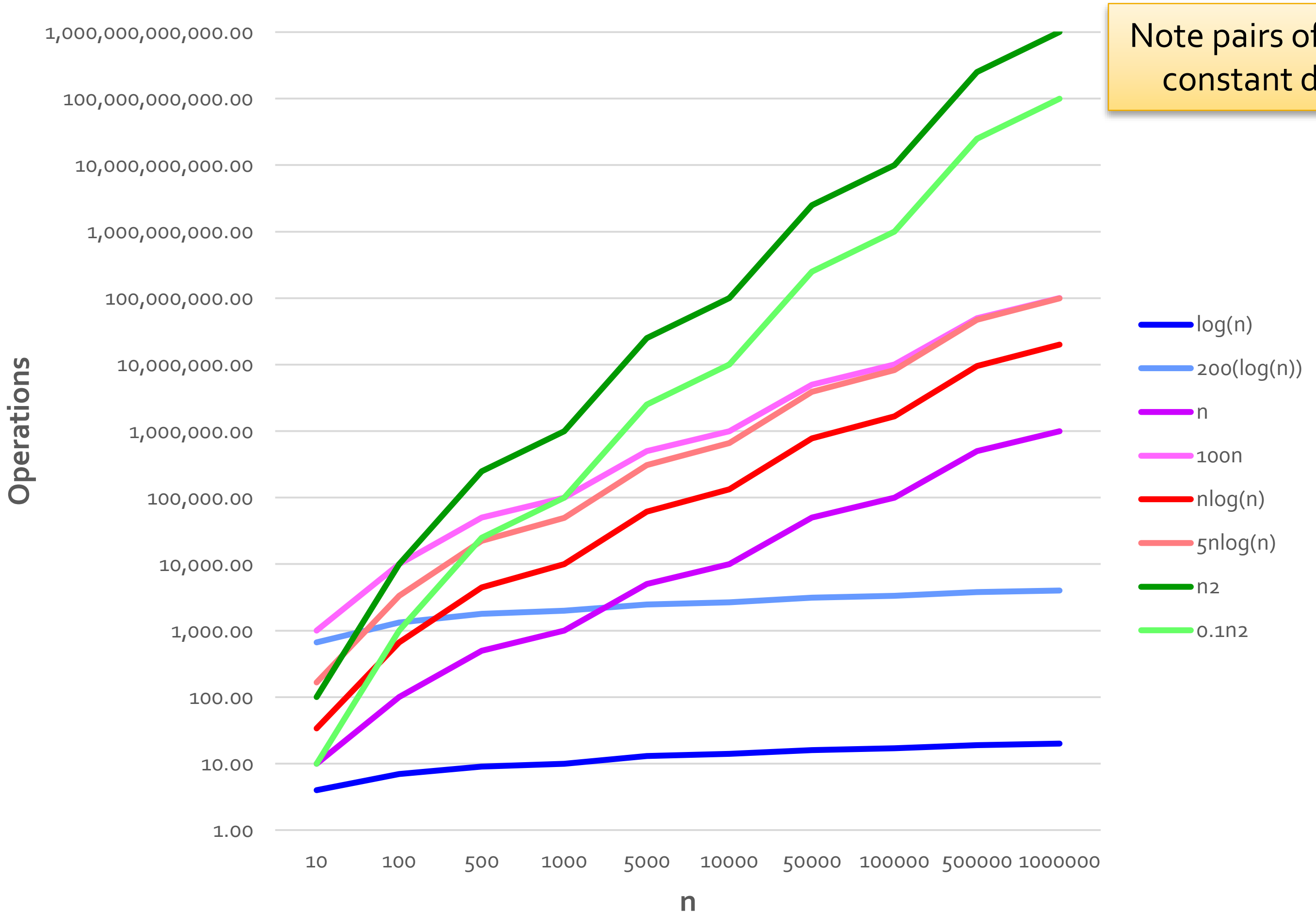


900000
800000
700000
600000
500000
400000
300000
200000
100000
0
Operations
10
100
200
300
400
500
600
700
800
900
n
log(n)
200(log(n))
n
100n
nlog(n)
5nlog(n)
n2
0.1n2

Much Larger *n*



Logarithmic Scale



Note pairs of growth rates a constant distance apart

O Notation

O Notation Introduction

- Exact counting of operations is often difficult (and tedious), even for simple algorithms
  - And is often not much more useful than estimates due to the relative importance of other factors
- *O Notation* is a mathematical language for evaluating the running-time of algorithms
  - O-notation evaluates the *growth rate* of an algorithm

Example of a Cost Function

- Cost Function: $t_A(n) = n^2 + 20n + 100$
  - Which term in the function is the most important?
- It depends on the size of *n*
  - $n = 2$, $t_A(n) = 4 + 40 + \underline{100}$
    - The constant, 100, is the dominating term
  - $n = 10$, $t_A(n) = 100 + \underline{200} + 100$
    - $20n$ is the dominating term
  - $n = 100$, $t_A(n) = \underline{10{,}000} + 2{,}000 + 100$
    - n^2 is the dominating term
  - $n = 1000$, $t_A(n) = \underline{1{,}000{,}000} + 20{,}000 + 100$
    - n^2 is still the dominating term

The general idea is …

- Big-*O* notation does not give a precise formulation of the cost function for a particular data size
- It expresses the general behaviour of the algorithm as the data size n grows very large so ignores
  - lower order terms and
  - constants
- A Big-*O* cost function is a simple function of n
  - n, n^2, $\log_2(n)$, etc.

Order Notation (Big O)

- Express the number of operations in an algorithm as a function of *n*, the problem size
- Briefly
  - Take the dominant term
  - Remove the leading constant
  - Put O(…) around it
- For example, $f(N) = 2n^2 - 2n + 10(n-1) + 2$
  - i.e. $O(n^2)$

But …

- Of course leading constants matter
  - Consider two algorithms
    - $f_1(n) = 20n^2$
    - $f_2(n) = 2n^2$
  - Algorithm 2 runs ten times faster
- Let's consider machine speed
  - If machine 1 is ten times faster than machine 2 it will run the same algorithm ten times faster
- Big O notation ignores leading constants
  - It is a hardware independent analysis

O Notation, More Formally

- Given a function $T(n)$
  - Say that $T(N) = O(f(n))$ if $T(n)$ is at most a constant times $f(n)$
    - Except perhaps for some small values of n
- Properties
  - Constant factors don't matter
  - Low-order terms don't matter
- Rules
  - For any k and any function $g(n)$, $k*g(n) = O(f(n))$
    - e.g., $5n = O(n)$

Why Big O?

- An algorithm is said to be *order* $f(n)$
  - Denoted as $O(f(n))$
- The function $f(n)$ is the algorithm's growth rate function
  - If a problem of size n requires time proportional to n then the problem is $O(n)$
    - e.g. If the input size is doubled so is the running time

Big O Notation Definition

- An algorithm is *order* $f(n)$ if there are positive constants k and m such that
  - $t_A(n) \leq k * f(n)$ for all $n \geq m$
    - i.e. find constants k and m such that the cost function is less than or equal to k * a simpler function for all n greater than or equal to m
- If so we would say that $t_A(n)$ is $O(f(n))$

Constants *k* and *m*

- Finding a constant $k \mid t_A(n) \leq k * f(n)$ shows that t is $O(f(n))$
  - e.g. If the cost function was $n^2 + 20n + 100$ and we believed this was $O(n)$
    - We claim to be able to find a constant $k \mid t_A(n) \leq k * f(n)$ for all values of n
    - Which is not possible
- For some small values of n lower order terms may dominate
  - The constant m addresses this issue

Or In English…

- The idea is that a cost function can be approximated by another, simpler, function
  - The simpler function has 1 variable, the data size n
  - This function is selected such that it represents an *upper bound* on the value of $t_A(n)$
- Saying that the time efficiency of algorithm A $t_A(n)$ is $O(f(n))$ means that
  - A cannot take more than $O(f(n))$ time to execute, and
  - The cost function $t_A(n)$ *grows at most as* fast as $f(n)$

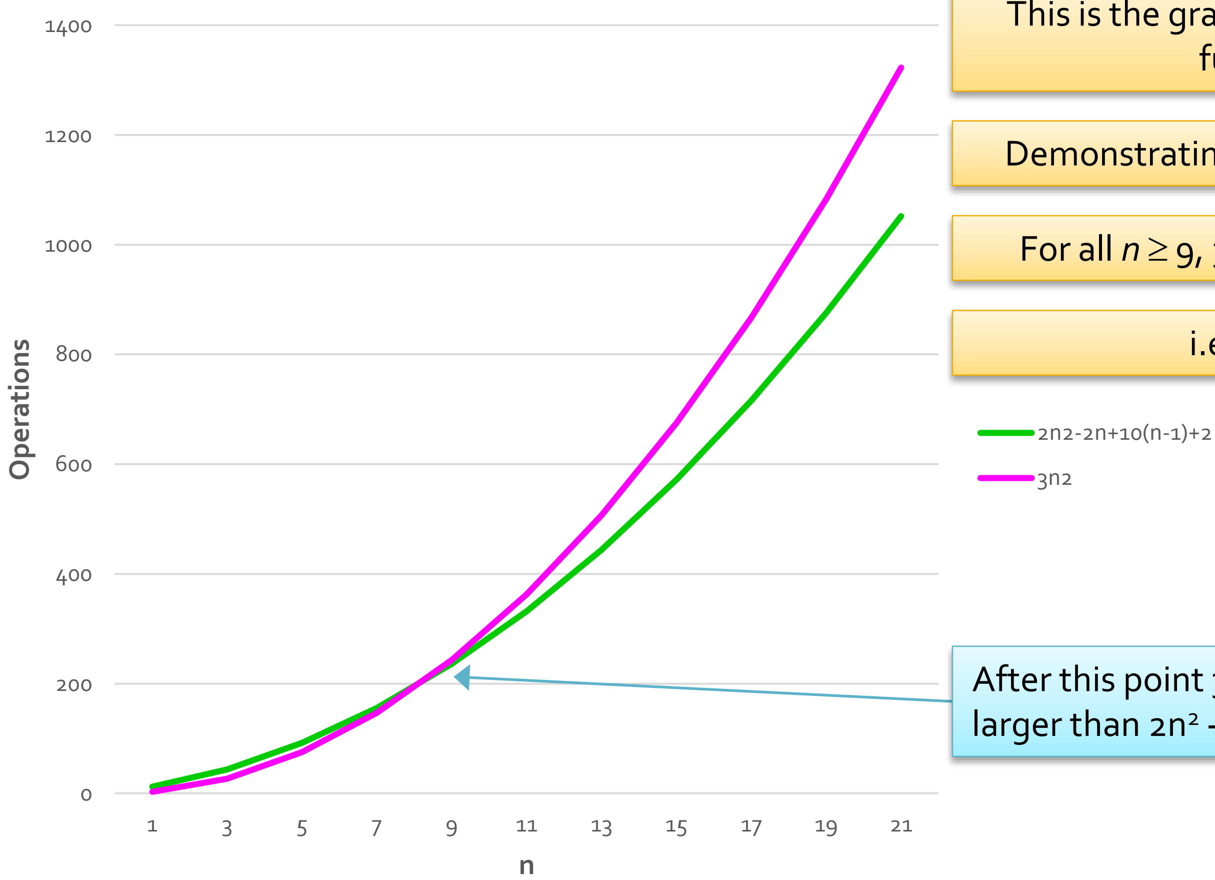
Big O Example

- An algorithm's cost function is $3n + 12$
  - If we can find constants m and k such that:
  - $k * n \geq 3n + 12$ for all $n \geq m$ then
  - The algorithm is $O(n)$
- Find values of k and m so that this is true
  - $k = 4$, and
  - $m = 12$ then
  - $4n \geq 3n + 12$ for all $n \geq 12$

Another Big O Example

- A cost function is $2n^2 - 2n + 10(n-1) + 2$
- If we can find constants m and k such that:
  - $k * n^2 \geq 2n^2 - 2n + 10(n-1) + 2$ for all $n \geq m$ then
  - The algorithm is $O(n^2)$
- Find values of k and m so that this is true
  - $k = 3$, and
  - $m = 9$ then
  - $3n^2 > 2n^2 - 2n + 10(n-1) + 2$ for all $n \geq 9$

And Another Graph



This is the graph of the selection sort cost function and $3n^2$

Demonstrating that selection sort is $O(n^2)$

For all $n \geq 9$, $3(n^2) \geq 2n^2 - 2n + 10(n-1) + 2$

i.e. $k = 3$ and $m = 9$

After this point $3n^2$ is always going to be larger than $2n^2 - 2n + 10(n-1) + 2$

O Notation Examples

- All these expressions are $O(n)$
  - n, $3n$
  - $61n + 5$
  - $22n - 5$
- All these expressions are $O(n^2)$
  - n^2
  - $9n^2$
  - $18n^2 + 4n - 53$
- All these expressions are $O(n \log n)$
  - $n(\log n)$
  - $5n(\log 99n)$
  - $18 + (4n - 2)(\log (5n + 3))$

Arithmetic and O Notation

- $O(k * f) = O(f)$ if k is a constant
  - e.g. $O(23 * O(\log n))$, simplifies to $O(\log n)$
- $O(f + g) = \max[O(f), O(g)]$
  - $O(n + n^2)$, simplifies to $O(n^2)$
- $O(f * g) = O(f) * O(g)$
  - $O(m * n)$, equals $O(m) * O(n)$
  - Unless there is some known relationship between m and n that allows us to simplify it, e.g. $m < n$

Typical Growth Rate Functions

- Growth rate functions are typically one of the following
  - $O(1)$
  - $O(\log n)$
  - $O(n)$
  - $O(n*\log n)$
  - $O(n^2)$
  - $O(n^k)$
  - $O(2^n)$

O(1) – Constant Time

- We write $O(1)$ to indicate something that takes a constant amount of time
  - Array look up
  - Swapping two values in an array
  - Finding the minimum element of an *ordered* array takes $O(1)$ time
    - The minimum value is either the first or the last element of the array
  - Binary or linear search best case
- *Important*: constants can be large
  - So in practice $O(1)$ is not *necessarily* efficient
  - It tells us is that the algorithm will run at the same speed no matter the size of the input we give it

O(log*n*) – Logarithmic Time

- $O(\log n)$ algorithms run in *logarithmic* time
  - Binary search average and worst case
  - The logarithm is assumed to be base 2 unless specified otherwise
- Doubling the size of *n* doubles increases the number of operations by one
- Algorithms might be $O(\log n)$
  - If they are divide and conquer algorithms that halve the search space for each loop iteration or recursive call

O(n) – Linear Time

- $O(n)$ algorithms run in *linear* time
  - Linear search
  - Summing the contents of an array
  - Traversing a linked list
  - Insertion sort or bogo sort best case
- Doubling the size of n doubles the number of operations
- Algorithms might be O(n)
  - If they contain a single loop
    - That iterates from 0 to n (or some variation)
  - Make sure that loops only contain constant time operations
    - And evaluate any function calls

O(*n*log*n*)

- $O(n\log n)$
  - Mergesort in all cases
  - Heap sort in all cases
  - Quicksort in the average and best case
  - $O(n\log n)$ is the best case for comparison sorts
    - We will not prove this in CMPT 225
- Growth rate is faster than linear but still slow compared to $O(n^2)$
- Algorithms are $O(n\log n)$
  - If they have one process that is linear that is repeated $O(\log n)$ times

$O(n^2)$ – Quadratic Time

- $O(n^2)$ algorithms run in *quadratic* time
  - Selection sort in all cases
  - Insertion sort in the average and worst case
  - Bubble sort in all cases
  - Quicksort worst case
- Doubling the size of n quadruples the number of operations
- Algorithms might be $O(n^2)$
  - If they contain nested loops
    - As usual make sure to check the number of iterations in such loops
    - And that the loops do not contain non-constant function calls

O(n^k) – Polynomial Time

- $O(n^k)$ algorithms are referred to as running in *polynomial* time
  - If k is large they can be very slow
- Doubling the size of n increases the number of operations by 2^k

$O(2^n)$ – Exponential Time

- $O(2^n)$ or $O(k^n)$ algorithms are referred to as running in *exponential* time
- Very slow, and if there is no better algorithm implies that the problem is intractable
  - That is, problems of any reasonable size cannot be solved in a reasonable amount of time
    - Such as over the lifetime of the human race …
- $O(n!)$ algorithms are even slower
  - Traveling Salesman Problems (and many others)
  - Bogo sort in the average case

Worst, Average and Best Case

- The *O* notation growth rate of some algorithms varies depending on the input
- Typically we consider three cases:
  - *Worst case*, usually (relatively) easy to calculate and therefore commonly used
  - *Average case*, often difficult to calculate
  - *Best case*, usually easy to calculate but less important than the other cases
    - Relying on the input having the best case organization is not generally a good idea

Other Related Notations

- Ω (omega) notation
  - Gives a lower bound
    - Whereas O notation is an upper bound
  - There exists some constant k, such that $k * f(n)$ is a lower bound on the cost function $g(n)$
- Θ (theta) notation
  - Gives an upper and lower bound
  - $k_1 * f(n)$ is an upper bound on $g(n)$ and $k_2 * f(n)$ is a lower bound on $g(n)$

Upper and Lower Bound