The Substitution method

$T(n) = 2T(n/2) + cn$

- **Guess**: $T(n) = O(n \log n)$
- **Proof** by Mathematical Induction:

Prove that $T(n) \le d\, n \log n$ for $d>0$

$T(n) \le 2(d \cdot n/2 \cdot \log n/2) + cn$

(where $T(n/2) \le d \cdot n/2 (\log n/2)$ *by induction hypothesis*)

$\le dn \log n/2 + cn$

$= dn \log n - dn + cn$

$= dn \log n + (c-d)n$

$\le dn \log n$ if $d \ge c$

- Therefore, $T(n) = O(n \log n)$

Quick Sort – Partitioning – algorithm

```
public int partition(Comparable[] arr, int low, int high) {
    Comparable pivot = arr[high]; // choose pivot
    int l = low;
    int r = high-1;
    while (l<=r) {
    // find bigger item on the left
      while (l<=r && arr[l].compareTo(pivot) <= 0)
      l++;

      // find smaller item on the right
      while (l<=r && arr[r].compareTo(pivot) >= 0)
      r--;

      if (l<r) {
        swap(arr[l], arr[r]);
        l++;
        r--;
      }
    }
    // put pivot to the correct location
    swap(arr[l], arr[high]);
    return r;
  }
```

Quick Sort – Partitioning – algorithm – Proof of correctness

- Loop invariant:

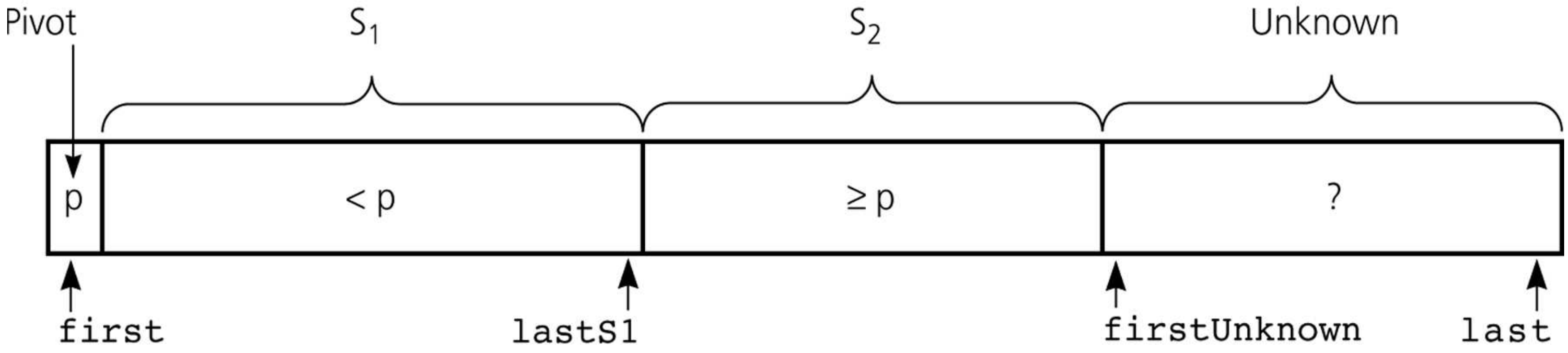
  at the beginning/end of each loop:
  1. arr[low]..arr[l-1] contains elements <= pivot
  2. arr[r+1]..arr[high-1] contains elements >= pivot

- When the loop is finished we have l=r+1, i.e.,
  1. arr[low]..arr[r] are <= pivot
  2. arr[r+1]..arr[high-1] are >= pivot
  3. arr[high]=pivot
- By swapping arr[high] with arr[l] (or arr[r]) we get a proper partitioning.

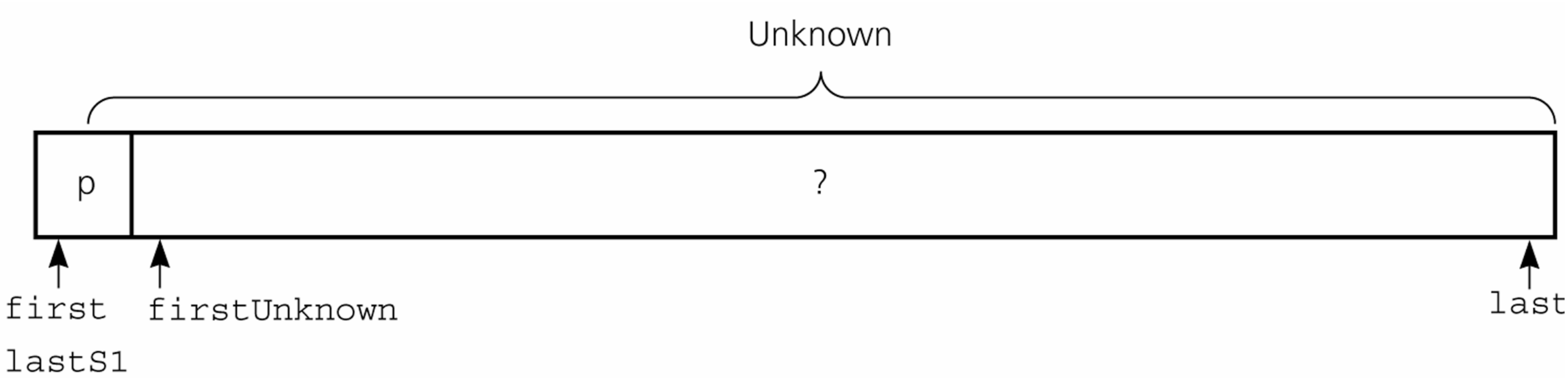
Quick Sort – Partitioning – another algorithm (textbook)

- Pivot is chosen to be the first element of the array (does not really matter)
- The array is divided to 4 parts (see bellow), initially "<p" and "≥p" parts are empty
- Invariant for the partition algorithm:

  The items in region S_1 are all less than the pivot, and those in S_2 are all greater than or equal to the pivot
- In each step the first element in "?" part is added either to "<p" or "≥p" part.



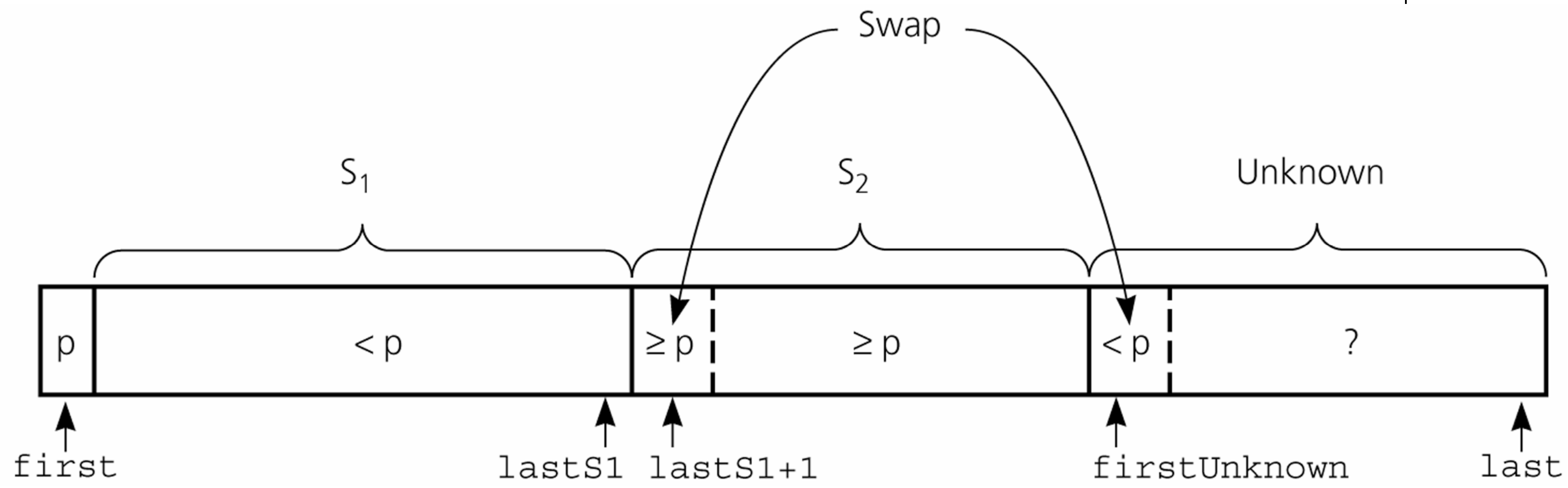
Quick Sort – Partitioning – another algorithm (textbook)



S_1: arr[first+1]..arr[lastS1] ⇒ empty
S_2: arr[lastS1+1]..arr[firstUnknown-1] ⇒ empty
?: arr[firstUnknown]..arr[last] ⇒ all elements but pivot

Initial state of the array

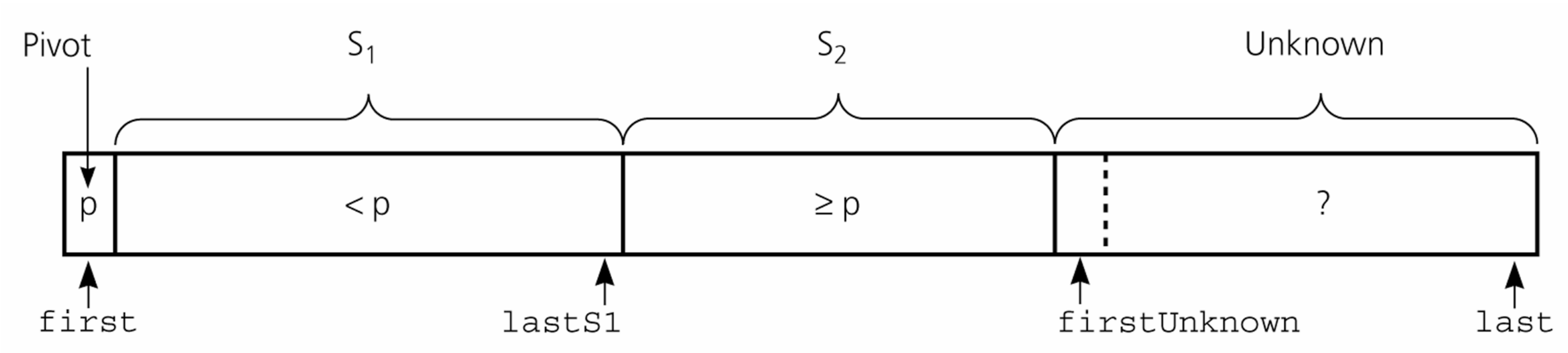
Quick Sort – Partitioning – another algorithm (textbook)



Processing arr[firstUnknown]: case “< pivot”

Move *arr[firstUnknown]* into S_1 by swapping it with *theArray[lastS1+1]* and by incrementing both *lastS1* and *firstUnknown.*

Quick Sort – Partitioning – another algorithm (textbook)



Processing arr[firstUnknown]: case "≥ pivot"

Moving *theArray[firstUnknown]* into S_2 by incrementing *firstUnknown.*

Quick Sort – Partitioning – another algorithm (textbook)

```
public int partition(Comparable[] arr, int first, int last) {
   Comparable pivot = arr[first]; // choose pivot

   // initially everything but pivot is unknown
   int lastS1 = first;
   for (int firstUnknown = first+1; firstUnknown <= last;
        firstUnknown++) {
     if (arr[firstUnknown].compareTo(pivot) < 0) {
       // item should be moved to S1
       lastS1++;
       swap(arr[lastS1],arr[firstUnknown]);
     }
     // else item should be moved to S2,
     // which will be increamenting firstUnknown in the loop
   }
   // put pivot to the correct location
   swap(arr[first], arr[lastS1]);
   return lastS1;
 }
```

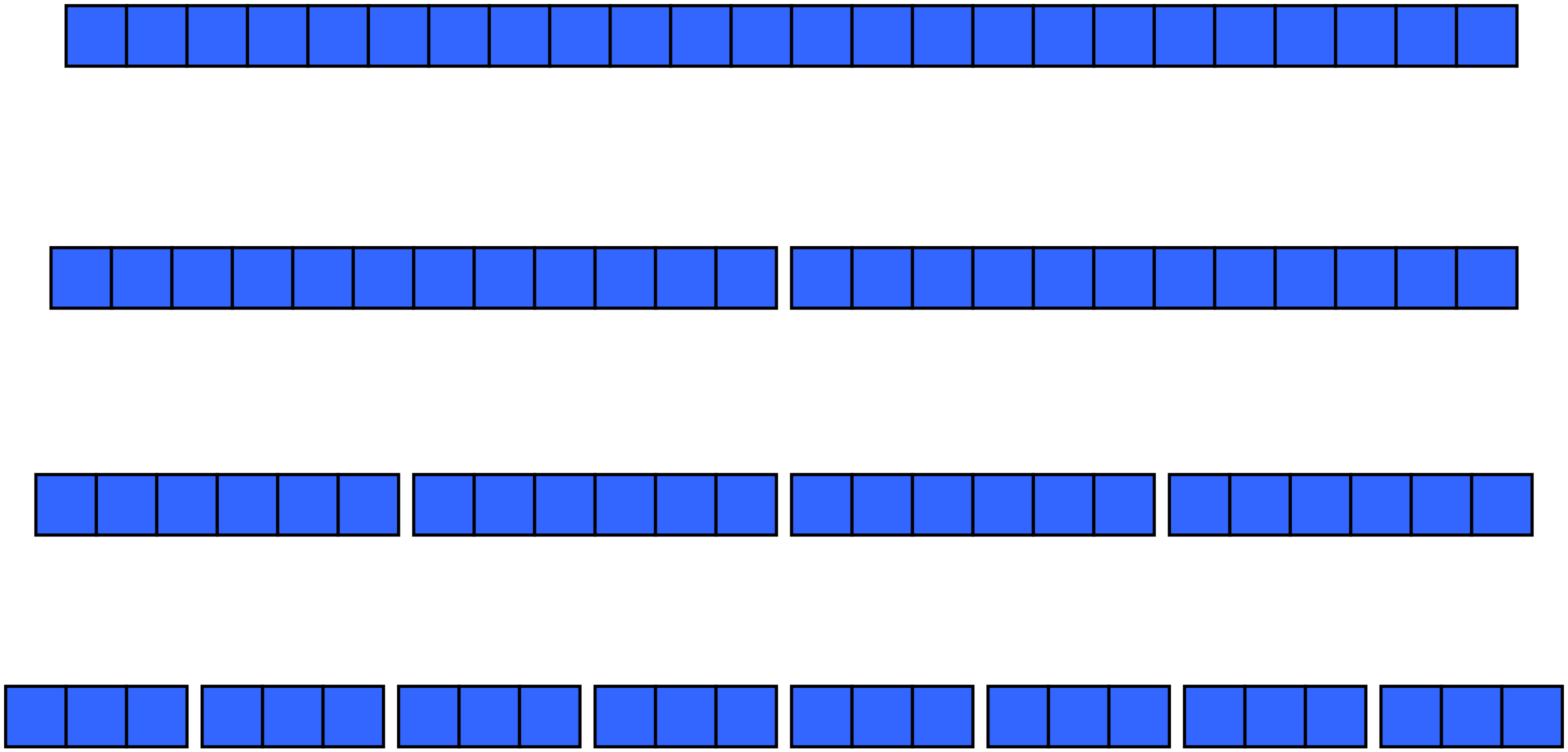
Quick Sort – Selection of pivot

- In the above algorithm we selected the pivot to be the last or the first element of subarray which we want to partition
- It turns out that the selection of pivot is crucial for performance of Quick Sort – see best and worst cases
- Other strategies used:
  - select 3 (or more elements) and pick the median
  - randomly select (especially used when the arrays might be originally sorted)
  - select an element “close to the median” in the subarray (there is a recursive linear time algorithm for that, see http://en.wikipedia.org/wiki/Selection_algorithm for details).

Analysis of Quick Sort: Best Case

- How much time do we need to partition an array of size *n*?
- O(n) using any of two algorithms
- **Best case:** Suppose each partition operation divides the array almost exactly in half

Best case Partitioning at various levels



Analysis of Quick Sort: Best Case

- How much time do we need to partition an array of size *n*?
- O(n) using any of two algorithms
- **Best case:** Suppose each partition operation divides the array almost exactly in half
- When could the best case happen?
- For example, array was sorted and the pivot is selected to be the middle element of the subarray.

Analysis of Quick Sort: Best Case

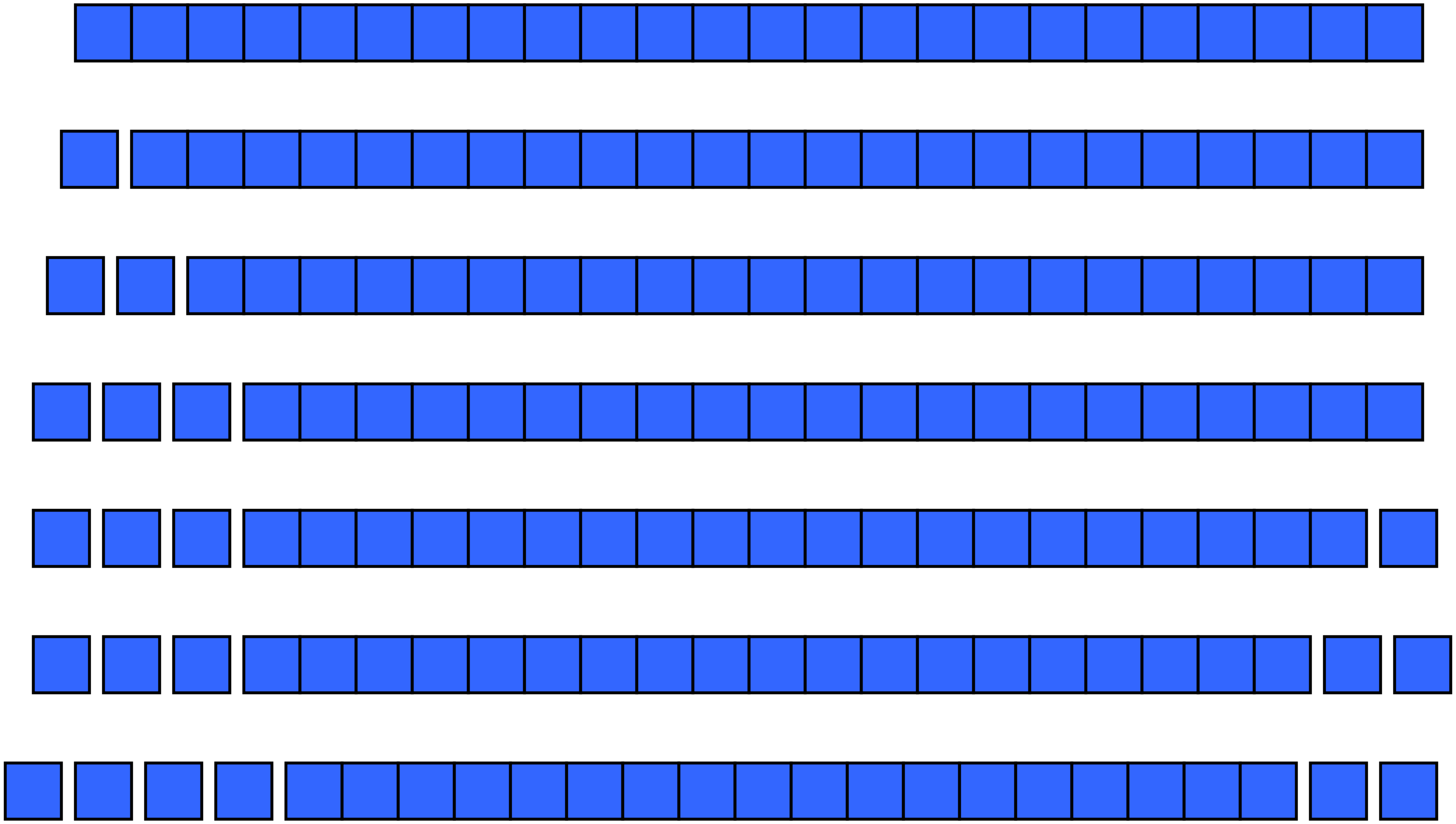
- **Best case:** Suppose each partition operation divides the array almost exactly in half
- The running time (time cost) can be expressed with the following recurrence:

  T(n) = 2.T(n/2) +
  T(partitioning array of size n)
  = 2.T(n/2) + O(n)
- The same recurrence as for merge sort, i.e., T(n) is of order O(n.log n).

Analysis of Quick Sort: Worst Case

- In the worst case, partitioning always divides the size `n` array into these three parts:
  - A length one part, containing the pivot itself
  - A length zero part, and
  - A length `n-1` part, containing everything else

Worst case partitioning



Analysis of Quick Sort: Worst Case

- In the worst case, partitioning always divides the size `n` array into these three parts:
  - A length one part, containing the pivot itself
  - A length zero part, and
  - A length `n-1` part, containing everything else
- When could this happen?
- Example: the array is sorted and the pivot is selected to be the first or the last element.

Analysis of Quick Sort: Worst Case

- The recurrent formula for the time cost of Quick Sort in the worst case:
  $T(n) = T(0) + T(n-1) + O(n)$
  $= T(n-1) + O(n)$

- By repeated substitution (or Master's theorem) we get the running time of Quick Sort in the worst case is $O(n^2)$
- Similar, situation as for Insertion Sort. Does it mean that the performance of Quick Sort is bad on average?

Quick Sort: Average Case

- If the array is sorted to begin with, Quick sort running time is terrible: $O(n^2)$
  (Remark: could be improved by random selection of pivot.)
- It is possible to construct other bad cases
- However, Quick sort runs *usually* (on average) in time $O(n.\log_2 n)$
  -> CMPT307 for detailed analysis
- The constant in front of $n.\log_2 n$ is so good that Quick sort is generally the fastest algorithm known.
- Most real-world sorting is done by Quick sort.

Exercise Problem on Quick Sort.

What is the running time of QUICKSORT when

a) All elements of array A have the same value ?

b) The array A contains distinct elements and in sorted decreasing order ?

Answer – 1^{st} algorithm

- Pivot is chosen to be the last element in the subarray.

  a) Whatever **pivot** you choose in each subarray it would result in WORST CASE PARTITIONING (*l=high*) and hence the running time is $O(n^2)$.

  b) Same is the case. Since you always pick the minimum element in the subarray as the **pivot** each partition you do would be a worst case partition and hence the running time is $O(n^2)$ again !

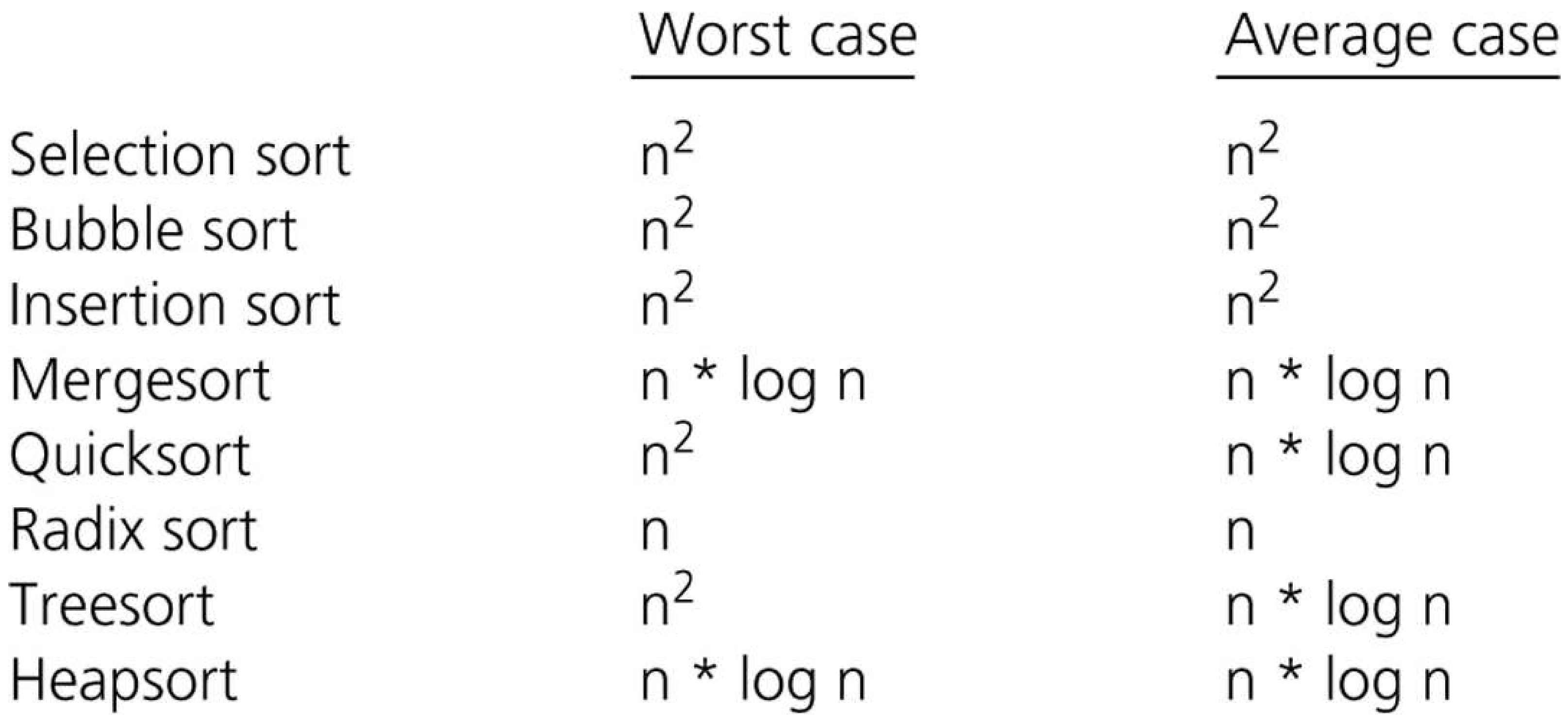
Answer – 2^{nd} algorithm

- Pivot is chosen to be the first element in the subarray

  a) Whatever **pivot** you choose in each subarray it would result in WORST CASE PARTITIONING (everything will be put to S_2 part) and hence the running time is $O(n^2)$.

  b) Same is the case. Since you always pick the maximum element in the sub array as the **pivot** each partition you do would be a worst case partition and hence the running time is $O(n^2)$ again !

A Comparison of Sorting Algorithms



	Worst case	Average case
Selection sort	n^2	n^2
Bubble sort	n^2	n^2
Insertion sort	n^2	n^2
Mergesort	n * log n	n * log n
Quicksort	n^2	n * log n
Radix sort	n	n
Treesort	n^2	n * log n
Heapsort	n * log n	n * log n

Approximate growth rates of time required for eight sorting algorithms

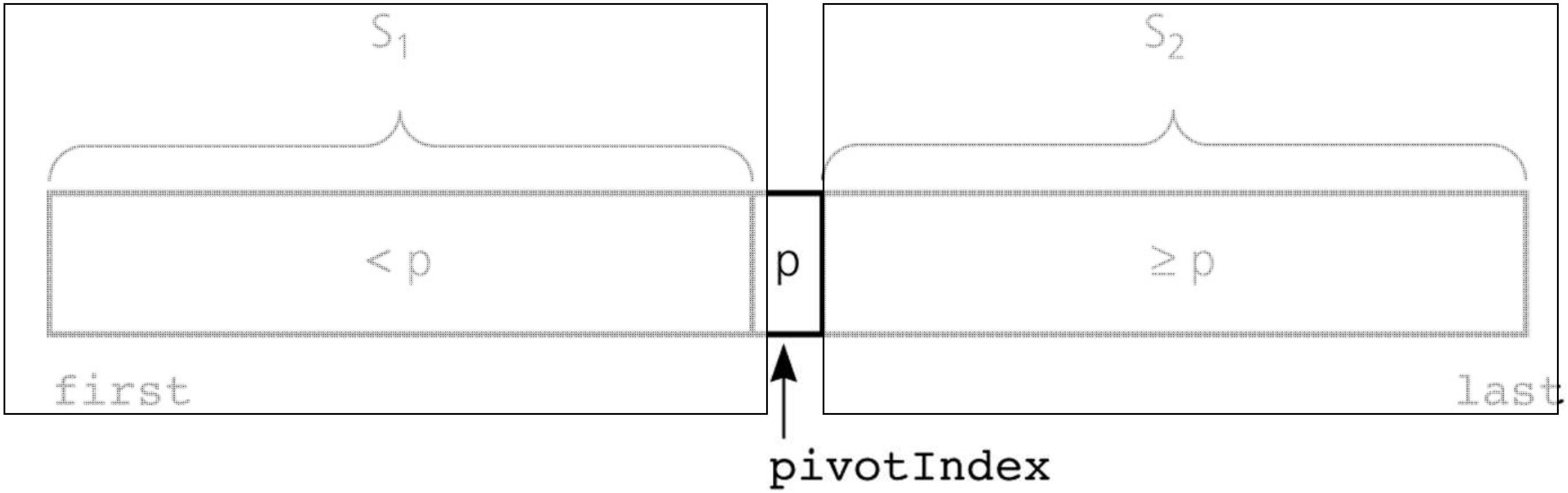
Finding the k-th Smallest Element in an Array (Selection Problem)

- One possible strategy: sort an array and just take the k-th element in the array
- This would require O(n.log n) time if use some efficient sorting algorithm

- Question: could we use partitioning idea (from Quicksort)?

Finding the k-th Smallest Element in an Array

- Assume we have partition the subarray as before.



If S1 contains k or more items -> S1 contains kth smallest item
If S1 contains k-1 items -> k-th smalles item is pivot *p*
If S1 contains fewer then k-1 items -> S2 contains kth smallest item

Finding the k-th Smallest Element in an Array

```
public Comparable select(int k, Comparable[] arr, int low, int high)
// pre: low <= high  and
//      k <= high-low+1 (number of elements in the subarray)
// return the k-th smallest element
// of the subarray arr[low..high]
{
    int pivotIndex = partition(arr, low, high);
    // Note: pivotIndex - low is the local index
    //   of pivot in the subarray
    if (k == pivotIndex - low + 1) {
          // the pivot is the k-th element of the subarray
          return arr[pivotIndex];
    } else if (k < pivotIndex - low + 1) {
          // the k-th element must be in S1 partition
          return select(k, arr, low, pivotIndex-1);
    } else { // k > pivotIndex - low +1
          // the k-th element must be in S2 partition
          // Note: there are pivotIndex-first elements in S1
          //       and one pivot, i.e., all smaller than
          //       elements in S2, so we have to recalculate
          //       index k
          return select(k - (pivotIndex-first+1), arr, pivotIndex+1, high);
    }
} // end kSmall
```

Finding the k-th Smallest Item in an Array

- The running time in the best case:
  $T(n) = T(n/2) + O(n)$
- It can be shown with repeated substitution that $T(n)$ is of order $O(n)$

- The running time in the worst case:
  $T(n) = T(n-1) + O(n)$
- This gives the time $O(n^2)$

- average case: $O(n)$
- By selecting the pivot close to median (using a recursive linear time algorithm), we can achieve $O(n)$ time in the worst case as well.