MACM 101 Sample Questions
Fall 2019, Surrey
ANSWER KEY

IMPORTANT! The following questions are just a collection of practice questions. This is *not* necessarily the same length, or mix of questions, as the actual final exam.

1. In the following questions, the letters are the regular 26 letters a, b, c, ... z.

   (a) How many different 4-letter strings are there?

   > **Solution:** $26^4 = 456976$

   (b) How many different 4-letter strings are there such that not all the letters are the same?

   > **Solution:** $26^4 - 26 = 456950$

   (c) How many different 4-letter strings are there such that no two letters are the same?

   > **Solution:** $26 \cdot 25 \cdot 24 \cdot 23 = 358800$

   (d) How many different 4-letter strings are there such that two, or more, of the letters are the same?

   > **Solution:** $26^4 - 26 \cdot 25 \cdot 24 \cdot 23 = 98176$

2. How many different ways can cars be stopped at a 4-way stop? Take into account the direction they are going: left, right, or straight. What if there are pedestrians who want to cross as well?

   > **Solution:** There are 4 roads into a 4-way stop, and each road could be in one of 4-states: no car, a car turning left, a car turning right, or a car going straight. Thus, there are $4^4 = 256$ different possible situations at a 4-way stop.
   >
   > Now lets add pedestrians. Each of the 4 roads could be crossed in two different directions, and so there are a total of 8 possible pedestrians crossing. Assuming at most one pedestrian per crossing, there are $2^8 = 256$ possible ways pedestrians could cross. Combined with the cars, that means there are $4^4 \cdot 2^8 = 256 \cdot 256 = 65536$ 4-way stop configurations of cars and pedestrians. That's a lot — 4-way stops can be tricky!

3. Prove that if $\binom{n}{k} = \binom{n}{k+1}$, then n must be odd.

Solution: You can show $n = 2k + 1$ by a step-by-step simplification of $\binom{n}{k} = \binom{k}{k+1}$.

4. Give a propositional logic expression having only the variables r, y, and g that is true just when exactly one of r, y, or g is true, and false otherwise.

Solution: One such expression is: $(r \wedge \neg y \wedge \neg g) \vee (\neg r \wedge y \wedge \neg g) \vee (\neg r \wedge \neg y \wedge g)$.

5. Give a *logically equivalent* expression for $\neg p$ that does *not* use $\neg$, and prove that it is logically equivalent to $\neg p$. The only symbols you can use in your answer are p, $\wedge$, $\vee$, $\rightarrow$, T_0, and F_0 (you can use each of them 0 or more times).

Solution: $p \rightarrow F_0$ works. To prove it's logically equivalent to $\neg p$, note that if p is 1, then $p \rightarrow F_0$ is the same as $1 \rightarrow 0$, which is 0. If p is 0, then $p \rightarrow F_0$ is the same as $0 \rightarrow 0$, which is 1. Thus $p \rightarrow F_0$ always has the same truth value as $\neg p$, and so they're logically equivalent.

6. Define the following propositional logic terms, and also give a short example of each:

   (a) *Tautology*

   Solution: A tautology is a compound statement that is *true* for all assignments of truth values to its variables. For example, $p \vee \neg p$ and $p \rightarrow p$ are tautologies.

   (b) *Contradiction*

   Solution: A contradiction is a compound statement that is *false* for all assignments of truth values to its variables. For example, $p \wedge \neg p$ and $\neg(p \rightarrow p)$ are contradictions.

7. (a) Prove that if $\forall x.p(x) \vee \forall x.q(x)$ is true, then $\forall x.[p(x) \vee q(x)]$ is also true.

   Solution: If $\forall x.p(x) \vee \forall x.q(x)$ is true, then either $\forall x.p(x)$ is true, or $\forall x.q(x)$ is true (or both are true). Suppose $\forall x.p(x)$ is true. Then for all elements a in the universe, $p(a)$ is true, and thus $p(a) \vee q(a)$ is true (it's a basic law of propositional logic that if p is true, then so is $p \vee q$ — it's called *disjunction amplification*). Since a is any element in the universe, $\forall x.[p(x) \vee q(x)]$ is true.

   A similar argument shows that if $\forall x.q(x)$ is true, then $\forall x.[p(x) \vee q(x)]$ is true. Thus, these two facts together prove that $\forall x.[p(x) \vee q(x)]$ is true whenever $\forall x.p(x) \vee \forall x.q(x)$ is true.

(b) Prove that if $\forall x.[p(x) \vee q(x)]$ is true, then $\forall x.p(x) \vee \forall x.q(x)$ is *not* necessarily true.

Solution: To prove this we just have to show an example universe where $\forall x.[p(x) \vee q(x)]$ is true, but $\forall x.p(x) \vee \forall x.q(x)$ is false.

Consider a universe with just two members, a and b. Suppose that $p(a)$ is true and $q(a)$ is false, and $p(b)$ is false and $q(b)$ is true. Then it is not hard to see that $\forall x.[p(x) \vee q(x)]$ is true. However, $\forall x.p(x) \vee \forall x.q(x)$ is false because it's not the case that p is true of all elements, and it is not the case that q is true of all elements.

It helps to think about this intuitively. Suppose in a group of 10 people, 5 people speak French (and no English), and 5 speak English (and no French). Then all of them speak either French or English, but it is clearly not the case that they either all speak French, or all speak English.

8. Suppose that $p(x)$ and $q(x)$ are open statements in the same universe. Prove:

(a) $\exists x.[p(x) \wedge q(x)] \Rightarrow [\exists x.p(x) \wedge \exists x.q(x)]$

Solution: Suppose $\exists x.[p(x) \wedge q(x)]$ is true. Then there is some element a such that $p(a) \wedge q(a)$. It follows from this that $p(a)$ is true, and thus $\exists x.p(x)$. Similarly, it also follows that $q(a)$ is true, and so $\exists x.q(x)$. Thefore, $\exists x.p(x) \wedge \exists x.q(x)$ is true.

(b) $[\exists x.p(x) \wedge \exists x.q(x)] \not\Rightarrow \exists x.[p(x) \wedge q(x)]$

Solution: To prove this, we need only show that there is at least one universe where $\exists x.p(x) \wedge \exists x.q(x)$ is true but $\exists x.[p(x) \wedge q(x)]$ is false. Consider the universe of integers, and suppose $p(x)$ means x is even, and $q(x)$ means x is odd. Then $\exists x.p(x) \wedge \exists x.q(x)$ is true becuase it states there are some numbers that are even, and there are some numbers that are odd. But $\exists x.[p(x) \wedge q(x)]$ states there is at least one number that is both odd and even, but this is false. Therefore, $[\exists x.p(x) \wedge \exists x.q(x)]$ does *not* logically imply $\exists x.[p(x) \wedge q(x)]$.

9. Assuming A and B are sets in the same universe, prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Solution: We need to show both $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

To prove $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$, suppose x is some element in $\overline{A \cup B}$. Then:

$$\begin{aligned}
& x \in \overline{A \cup B} \\
\Rightarrow\ & x \notin (A \cup B) \\
\Rightarrow\ & \neg(x \in A \cup B) \\
\Rightarrow\ & \neg(x \in A \vee x \in B) \\
\Rightarrow\ & \neg(x \in A) \wedge \neg(x \in B) && \text{(by De Morgan's law)} \\
\Rightarrow\ & x \notin A \wedge x \notin B \\
\Rightarrow\ & x \in \overline{A} \wedge x \in \overline{B} \\
\Rightarrow\ & x \in \overline{A} \cap \overline{B}
\end{aligned}$$

To prove $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$, suppose x is some element in $\overline{A} \cap \overline{B}$. Then:

$$\begin{aligned}
& x \in \overline{A} \cap \overline{B} \\
\Rightarrow\ & x \in \overline{A} \wedge x \in \overline{B} \\
\Rightarrow\ & x \notin A \wedge x \notin B \\
\Rightarrow\ & \neg(x \in A) \wedge \neg(x \in B) \\
\Rightarrow\ & \neg(x \in A \vee x \in B) && \text{(by De Morgan's law)} \\
\Rightarrow\ & \neg(x \in A \cup B) \\
\Rightarrow\ & x \notin A \cup B \\
\Rightarrow\ & x \in \overline{A \cup B}
\end{aligned}$$

Note that because the empty set is a subset of all sets, we can assume there is at least one element, x, in the set on the left side of $\subseteq$.

10. (a) State the principle of (weak) mathematical induction.

Solution: Let $S(n)$ be any open statement, where n is a positive integer. Suppose the following two things are true:

1. $S(1)$
2. $S(k) \Rightarrow S(k+1)$, for all positive integers k

Then $S(n)$ is true for all positive integers n.

(b) Using mathematical induction, prove that this equation holds for all positive integers n:

$$\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$$

Solution: Let $S(n)$ be the proposition $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$.
For the base case, $S(1)$, it is easy to verify that $2^{1-1} = 2^1 - 1$.
For the inductive case, we need to show that $S(k) \Rightarrow S(k+1)$ is true for all positive integers k. To prove a statement of the form $P \Rightarrow Q$, we assume P is true and then show that Q is also true. So, we assume $S(k)$ is true, and our goal is to prove $S(k+1)$.
$S(k+1)$ is:

$$\sum_{i=1}^{k+1} 2^{i-1} = 2^{k+1} - 1$$

The left-hand side of the equation can be transformed into the right-hand side like this:

$$\begin{aligned} L.H.S. &= \sum_{i=1}^{k+1} 2^{i-1} \\ &= (2^{1-1} + 2^{2-1} + 2^{3-1} + \ldots + 2^{k-1}) + 2^{(k+1)-1} \\ &= (2^k - 1) + 2^k && \text{(since } S(k) \text{ is true)} \\ &= 2 \cdot 2^k - 1 \\ &= 2^{k+1} - 1 \\ &= R.H.S. \end{aligned}$$

This proves the inductive step, and so, by the principle of mathematical induction, $S(n)$ is true for all positive integers n.

11. In the following paragraph, an *incorrect* proof by induction is given. Explain exactly what is wrong with the proof.

    Let $S(n)$ represent that statement "any group of n jelly beans are all the same color". Assume that n is a positive integer, and each jelly bean is one color.

    We're going to use induction prove that $S(n)$ true for all positive integers n.

    The base case is $n = 1$, and clearly $S(1)$ is true. That is, if you have a group consisting of just one jelly bean, then all the jelly beans in that group are the same color.

    For the inductive case, suppose $S(k)$ is true, i.e. for any group of k jelly beans all the jelly beans are the same color. Now consider any group of $k+1$ jelly beans. Pick one of the jelly beans and call it x. If you remove x, then the remaining k jelly beans are all the same color c (by our inductive hypothesis). Now pick some jelly bean, other than x, from the $k+1$ jelly beans, and call it y. If you remove y, then the remaining k jelly beans are, again, all the same color c. Therefore, we've proven that $S(k+1)$ is true, and so, by induction, it follows that $S(n)$ is true for all positive integers n. In other words, all jelly beans are the same color.

Solution: The general argument in the inductive case does not work for the case $n = 2$. Try it and see. Consider a group of 2 jelly beans. Suppose they are yellow and green. If you take out the yellow one, then the remaining jelly beans are green. If you take out the green one, then the remaining ones are yellow. But obviously the two "taken out" jelly beans are different colors, so the argument does not hold.

So the flaw in the proof is that it assumes x and y are the *same* color. But that's wrong: all that can be logically concluded is that if you take out x, then the remaining jelly beans are all the same color c_1, and if you take out y, then the remaining jelly beans are all the same color c_2. The proof incorrectly assumes $c_1 = c_2$! As we showed above, c_1 is *not* equal to c_2 when $n = 2$.

This is a classic example of an incorrect proof by induction. You can replace $S(n)$ with many other propositions, e.g. $S(n)$ could mean "any n people are the same age".

12. (a) Define a *divides* b, i.e. $a|b$.

Solution: If a and b are both integers, and $b \neq 0$, then b divides a if there is an integer n such that $a = bn$.

(b) Suppose $a|b$ and $b|a$. What are all possible values of a and b that make this true? Prove your answer is correct.

Solution: If $a|b$ and $b|a$, then either $a = b$ or $a = -b$.
To see why, we have from the definition of divides that both $b = ma$ and $a = nb$ for some integers m and n. Thus $b = m(nb)$, which simplifies to $mn = 1$ ($b \neq 0$, so it's safe to cancel the bs). The only integer solutions to $mn = 1$ are either $m = n = 1$, or $m = n = -1$. So if $m = n = 1$, then $a = nb = 1 \cdot b = b$, and if $m = n = -1$, $a = nb = -1 \cdot b = -b$.

(c) Give the definition of a *composite number*.

Solution: A composite is an integer greater than 1 that is not prime. Or, equivalently, an integer greater than 1 that has three, or more, different divisors.

(d) How many different divisors does 10! have?

Solution: The prime factorization of 10!:

$$\begin{aligned} 10 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \\ &= 1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \cdot 3^2 \cdot (2 \cdot 5) \\ &= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \end{aligned}$$

Thus, 10! has $(8+1)(4+1)(2+1)(1+1) = 9 \cdot 5 \cdot 3 \cdot 2 = 270$ different divisors.

(e) Give a number with exactly 100 different divisors.

> **Solution:** There are many numbers that could be given here. Some are: $2^{49} \cdot 3$, $7^{19} \cdot 23^4$, $11 \cdot 37^4 \cdot 101^9$.
>
> In general, a positive integer $n = p_1^{e_1} p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$, where $p_1, \ldots, p_k$ are all different primes, will have exactly 100 different divisors just when $(e_1 + 1)(e_2 + 1) \cdot \ldots \cdot (e_k + 1) = 100$.

(f) Suppose n is the product of exactly 100 different primes. How many different divisors does n have?

> **Solution:** Each subset of the primes making up n corresponds to a divisor, so n has 2^{100} divisors.

(g) What is the greatest common divisor of 217 and 301? Show the steps of how to calculate it using the Euclidean algorithm.

> **Solution:** $\mathrm{GCD}(217, 301) = 7$. It can be calculated as follows:
>
> $$\begin{aligned} \mathrm{GCD}(217, 301) &= \mathrm{GCD}(217, 301) \\ &= \mathrm{GCD}(217, 84) \\ &= \mathrm{GCD}(133, 84) \\ &= \mathrm{GCD}(49, 84) \\ &= \mathrm{GCD}(35, 84) \\ &= \mathrm{GCD}(35, 49) \\ &= \mathrm{GCD}(35, 14) \\ &= \mathrm{GCD}(21, 14) \\ &= \mathrm{GCD}(7, 14) \\ &= \mathrm{GCD}(7, 7) \end{aligned}$$

13. Prove that if p is a prime greater than 3, then the square of p is one more than a multiple of 24.

> **Solution:** Let p be a prime greater than 3. We need to show that $p^2 - 1$ is a multiple of 24. Since $p^2 - 1 = (p-1)(p+1)$, we know that both $p - 1$ and $p - 2$ are even (since p is odd), and also that one of $p - 1$ or $p - 2$ must be divisible by 4 (because one of any two consecutive even numbers must be a mutiple of 4). Thus $(p-1)(p+1)$ is a multiple of 8. Furthermore, at least one of $p-1$, p, and $p+1$ must be a multiple of 3, and it can't be p because $p > 3$. Thus $(p-1)(p+1)$ is both a multiple of 3 and 8, so it is also a multiple of 24.

14. (a) Define the *Cartesian product* (*cross product*) $A \times B$.

Solution: For any two sets A and B, $A \times B = \{(a,b) \mid a \in A, b \in B\}$.

(b) Suppose A and B are both finite, and $|A| = m$ and $|B| = n$. What is $|A \times B|$?

Solution: $|A \times B| = mn = |A||B|$

(c) Suppose A and B are both finite sets that each contain 2 or more elements. Explain why it's impossible for $A \times B$ to have size 31.

Solution: We know that $|A \times B| = |A| \cdot |B| = 31$. Since 31 is prime, it's only divisors are 1 and 31, and so either $|A| = 1$ or $|B| = 1$. But this is impossible because both A and B have at least two elements.

(d) Give the definition of a *function* from a set A to a set B.

Solution: A function from A to B is a binary relation where every element of A occurs *exactly once* as the first element of an ordered pair in the relation.

(e) Suppose A and B are both finite, and $|A| = m$ and $|B| = n$. How many binary relations are there from A to B?

Solution: $2^{mn} = 2^{|A||B|}$

(f) Suppose A is a set with 3 elements, and B is a set with 2 elements. How many binary relations from A to B are *not* functions?

Solution: $2^{2 \cdot 2} - 2^3 = 8$

(g) If A and B are finite sets, how many binary relations from A to B are *not* functions?

Solution: $2^{|A||B|} - |B|^{|A|}$

(h) Define a *1-to-1* (*injective*) function.

Solution: A function $f : A \to B$ is 1-to-1 if each element of B appears at most once as the image of an element of A.

(i) Suppose $f : A \to B$ and $g : B \to C$. Define the *composition* $f \circ g$ of f and g. Be sure to clearly state the domain and codomain of $f \circ g$.

Solution: The composition of f and g is defined as $(g \circ f)(a) = g(f(a))$ for each $a \in A$, and $f \circ g : A \to C$.

(j) Define what it means for a function f to be *invertible*.

Solution: A function f is invertible if there is function $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$, where 1_A is the identity function on A, and 1_B is the identity function on B.

15. For each of the following questions, choose *true* or *false*.

(a) *True* or *False*: Functions are relations.

Solution: True

(b) *True* or *False*: The ordered pair $(2, 3)$ equals the ordered pair $(3, 2)$.

Solution: False

(c) *True* or *False*: If f is an onto function, then the range of f equals the codomain of f.

Solution: True

(d) *True* or *False*: For the function $f : A \to B$, A is the domain of f and B is the range of f.

Solution: False

(e) *True* or *False*: If $f(n) = n^2$ and $f : \mathbb{Z} \to \mathbb{Z}$, then $\mathbb{Z}$ is the range of f.

Solution: False

(f) *True* or *False*: If a function is onto, then it is 1-to-1.

Solution: False

(g) *True* or *False*: If a function's range and codomain are the same, then its onto.

Solution: True

(h) *True* or *False*: If A and B are finite, and f is a function $f : A \to B$, then if $|A| < |B|$ it's impossible for f to be onto.

Solution: True

(i) *True* or *False*: If A and B are finite, and $f : A \to B$ is a bijection, then $|A| = |B|$.

Solution: True

(j) *True* or *False*: A function is invertible if, and only if, it is a bijection.

Solution: True

(k) *True* or *False*: The function 1_A is a bijection.

Solution: True

16. Is there a binary operator on $\mathbb{R}$ that has both 0 and 1 has an identity element? Prove your answer is correct.

Solution: No, there is no such operator.

We'll prove the more general fact that if a binary operator f has an identity element, then it must be unique. Suppose e_1 and e_2 are identity elements for f. Then $f(e_1, e_2) = e_2$ because e_1 is an identity element, and also, $f(e_1, e_2) = e_1$ since e_2 is an identity element. Since e_1 and e_2 are both equal to $f(e_1, e_2)$, it follows that $e_1 = e_2$.

Since $0 \neq 1$, no binary operator on $\mathbb{R}$ can have both 0 and 1 as an identity element.

17. (a) Define g *dominates* f, where $f : \mathbb{Z}^+ \to \mathbb{R}$ and $g : \mathbb{Z}^+ \to \mathbb{R}$.

Solution: g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$ such that $|f(n)| \leq m|g(n)|$ for all $n \in \mathbb{Z}^+$, where $n \geq k$.

(b) Prove that $3n^2 + 6$ is dominated by $n^2 - 10$.

Solution: To show that $n^2 - 10$ dominates $3n^2 + 6$, we must find $m \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$ such that $3n^2 + 6 \leq m(n^2 - 10)$ for all $n \in \mathbb{Z}^+$, where $n \geq k$.

By inspecting the inequality, it seems that $m = 5$ might work as a value for m (since that will give a $5n^2$ term on the right-hand side which is bigger than the $3n^2$ term on the left-hand side).

When $m = 5$, the inequality simplifies to $3n^2 + 6 \leq 5(n^2 - 10)$, or $3n^2 + 6 \leq 5n^2 - 50$. Subtracting $3n^2$ from both sides gives $6 \leq 2n^2 - 50$. This inequality is clearly true if n is bigger than, say, 100, and so we set $k = 100$.

Setting $m = 5$ and $k = 100$ satisfies the inequality, and so $n^2 - 10$ dominates $3n^2 + 6$.

(c) Prove that $n^2 - 10$ is dominated by $3n^2 + 6$.

Solution: To show that $3n^2 + 6$ dominates $n^2 - 10$, we must find $m \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$ such that $n^2 - 10 \leq m(3n^2 + 6)$ for all $n \in \mathbb{Z}^+$, where $n \geq k$.

By inspecting the inequality, it seems that $m = 1$ might work as a value for m (since that will give a $3n^2$ term on the right-hand side which is bigger than the n^2 term on the left-hand side).

When $m = 1$, the inequality simplifies to $n^2 - 10 \leq 3n^2 + 6$. Subtracting n^2 from both sides gives $-10 \leq 2n^2$. This inequality is clearly true if n is 1 or more, and so we set $k = 1$.

Setting $m = 1$ and $1 = 100$ satisfies the inequality, and so $3n^2 + 6$ dominates $n^2 - 10$.

(d) Prove that n^2 does *not* dominate n^3.

Solution: We'll use proof by contradiction here. So assume n^2 *did* dominate n^3. Then that means there must be $m \in \mathbb{R}^+$ and $k \in \mathbb{Z}^+$ such that $n^3 \leq mn^2$ for all $n \in \mathbb{Z}^+$ where $n \geq k$.

Dividing both sides of the inequality by n^2 gives $n < m$. But this is impossible: m is a fixed and unchanging constant value, while n is a variable that can be as big as we like. No matter what value m is set to, n can be made bigger.

This is a contradiction, and so n^2 does *not* dominate n^3