MACM 101 Sample Questions
Fall 2019, Surrey

IMPORTANT! The following questions are just a collection of practice questions. This is *not* necessarily the same length, or mix of questions, as the actual final exam.

1. In the following questions, the letters are the regular 26 letters a, b, c, ... z.
   (a) How many different 4-letter strings are there?
   (b) How many different 4-letter strings are there such that not all the letters are the same?
   (c) How many different 4-letter strings are there such that no two letters are the same?
   (d) How many different 4-letter strings are there such that two, or more, of the letters are the same?

2. How many different ways can cars be stopped at a 4-way stop? Take into account the direction they are going: left, right, or straight. What if there are pedestrians who want to cross as well?

3. Prove that if $\binom{n}{k} = \binom{n}{k+1}$, then n must be odd.

4. Give a propositional logic expression having only the variables r, y, and g that is true just when exactly one of r, y, or g is true, and false otherwise.

5. Give a *logically equivalent* expression for $\neg p$ that does *not* use $\neg$, and prove that it is logically equivalent to $\neg p$. The only symbols you can use in your answer are p, $\wedge$, $\vee$, $\rightarrow$, T_0, and F_0 (you can use each of them 0 or more times).

6. Define the following propositional logic terms, and also give a short example of each:
   (a) *Tautology*
   (b) *Contradiction*

7. (a) Prove that if $\forall x.p(x) \vee \forall x.q(x)$ is true, then $\forall x.[p(x) \vee q(x)]$ is also true.
   (b) Prove that if $\forall x.[p(x) \vee q(x)]$ is true, then $\forall x.p(x) \vee \forall x.q(x)$ is *not* necessarily true.

8. Suppose that $p(x)$ and $q(x)$ are open statements in the same universe. Prove:
   (a) $\exists x.[p(x) \wedge q(x)] \Rightarrow [\exists x.p(x) \wedge \exists x.q(x)]$
   (b) $[\exists x.p(x) \wedge \exists x.q(x)] \not\Rightarrow \exists x.[p(x) \wedge q(x)]$

9. Assuming A and B are sets in the same universe, prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

10. (a) State the principle of (weak) mathematical induction.
    (b) Using mathematical induction, prove that this equation holds for all positive integers n:

$$\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$$

11. In the following paragraph, an *incorrect* proof by induction is given. Explain exactly what is wrong with the proof.

    Let $S(n)$ represent that statement "any group of n jelly beans are all the same color". Assume that n is a positive integer, and each jelly bean is one color.

    We're going to use induction prove that $S(n)$ true for all positive integers n.

    The base case is $n = 1$, and clearly $S(1)$ is true. That is, if you have a group consisting of just one jelly bean, then all the jelly beans in that group are the same color.

    For the inductive case, suppose $S(k)$ is true, i.e. for any group of k jelly beans all the jelly beans are the same color. Now consider any group of $k+1$ jelly beans. Pick one of the jelly beans and call it x. If you remove x, then the remaining k jelly beans are all the same color c (by our inductive hypothesis). Now pick some jelly bean, other than x, from the $k+1$ jelly beans, and call it y. If you remove y, then the remaining k jelly beans are, again, all the same color c. Therefore, we've proven that $S(k+1)$ is true, and so, by induction, it follows that $S(n)$ is true for all positive integers n. In other words, all jelly beans are the same color.

12. (a) Define a *divides* b, i.e. $a|b$.

    (b) Suppose $a|b$ and $b|a$. What are all possible values of a and b that make this true? Prove your answer is correct.

    (c) Give the definition of a *composite number*.

    (d) How many different divisors does 10! have?

    (e) Give a number with exactly 100 different divisors.

    (f) Suppose n is the product of exactly 100 different primes. How many different divisors does n have?

    (g) What is the greatest common divisor of 217 and 301? Show the steps of how to calculate it using the Euclidean algorithm.

13. Prove that if p is a prime greater than 3, then the square of p is one more than a multiple of 24.

14. (a) Define the *Cartesian product* (*cross product*) $A \times B$.

    (b) Suppose A and B are both finite, and $|A| = m$ and $|B| = n$. What is $|A \times B|$?

    (c) Suppose A and B are both finite sets that each contain 2 or more elements. Explain why it's impossible for $A \times B$ to have size 31.

    (d) Give the definition of a *function* from a set A to a set B.

    (e) Suppose A and B are both finite, and $|A| = m$ and $|B| = n$. How many binary relations are there from A to B?

    (f) Suppose A is a set with 3 elements, and B is a set with 2 elements. How many binary relations from A to B are *not* functions?

    (g) If A and B are finite sets, how many binary relations from A to B are *not* functions?

(h) Define a *1-to-1* (*injective*) function.

(i) Suppose $f : A \to B$ and $g : B \to C$. Define the *composition* $f \circ g$ of f and g. Be sure to clearly state the domain and codomain of $f \circ g$.

(j) Define what it means for a function f to be *invertible*.

15. For each of the following questions, choose *true* or *false*.

(a) *True* or *False*: Functions are relations.

(b) *True* or *False*: The ordered pair $(2, 3)$ equals the ordered pair $(3, 2)$.

(c) *True* or *False*: If f is an onto function, then the range of f equals the codomain of f.

(d) *True* or *False*: For the function $f : A \to B$, A is the domain of f and B is the range of f.

(e) *True* or *False*: If $f(n) = n^2$ and $f : \mathbb{Z} \to \mathbb{Z}$, then $\mathbb{Z}$ is the range of f.

(f) *True* or *False*: If a function is onto, then it is 1-to-1.

(g) *True* or *False*: If a function's range and codomain are the same, then its onto.

(h) *True* or *False*: If A and B are finite, and f is a function $f : A \to B$, then if $|A| < |B|$ it's impossible for f to be onto.

(i) *True* or *False*: If A and B are finite, and $f : A \to B$ is a bijection, then $|A| = |B|$.

(j) *True* or *False*: A function is invertible if, and only if, it is a bijection.

(k) *True* or *False*: The function 1_A is a bijection.

16. Is there a binary operator on $\mathbb{R}$ that has both 0 and 1 has an identity element? Prove your answer is correct.

17. (a) Define g *dominates* f, where $f : \mathbb{Z}^+ \to \mathbb{R}$ and $g : \mathbb{Z}^+ \to \mathbb{R}$.

(b) Prove that $3n^2 + 6$ is dominated by $n^2 - 10$.

(c) Prove that $n^2 - 10$ is dominated by $3n^2 + 6$.

(d) Prove that n^2 does *not* dominate n^3.