ANSWER KEY

MACM 101 (Surrey) Sample Midterm, Fall 2019

Please write your answers in the space provided. **Show your work**: answers without explanations won't get full marks! This exam has **5** questions and is out of **35** marks (there are no bonus marks).

Question	Points	Score
1	10	
2	10	
3	2	
4	5	
5	8	
Total:	35	

1. Ms. Lomilrini's kindergarten class has 24 kids, 14 girls and 10 boys.

   (a) (1 point) How many different ways can all the kids be arranged in a line?

   Solution: 24!

   (b) (2 points) How many different ways can all the kids be arranged in a line if all the girls are before all the boys? In other words, the line consists of 14 girls followed by 10 boys.

   Solution: $14! \cdot 10!$

   (c) (2 points) How many different ways can all the kids stand in a line if all that is considered is if they are a boy or a girl?

   Solution: $\frac{24!}{10!14}$

   (d) (2 points) How many different ways can the kids be divided into 4 teams of 6 members each? The order of the kids on the teams doesn't matter.

   Solution:

$$
\begin{aligned}
&= \binom{24}{6}\binom{18}{6}\binom{12}{6}\binom{6}{6} \\
&= \frac{24!}{6!6!6!6!}
\end{aligned}
$$

(e) (3 points) Suppose the teacher has 30 indistinguishable pieces of candy. How many ways can she distribute the candies among the students with the requirement that each student gets at least 1 candy? If a student gets more than one piece, the order of the candy doesn't matter.

Solution: This is meant to be a combination with repetitions question. First give 1 candy to each kid, leaving 6 candies. Those 6 can then be distributed among the 24 kids in this many ways, where $r = 6$ and $n = 24$:

$$\begin{aligned}\binom{n+r-1}{r} &= \binom{24+6-1}{6} \\ &= \binom{29}{6}\end{aligned}$$

2. (a) (2 points) Simplify $\frac{100!-99!}{98!}$ to a single integer.

Solution:

$$\begin{aligned}\frac{100!-99!}{98!} &= \frac{99!(100-1)}{98!} \\ &= \frac{99!\cdot 99}{98!} \\ &= 99\cdot 99 \\ &= 99^2 \\ &= 9801\end{aligned}$$

(b) (2 points) Re-write the following expression using Σ-notation:

$$3+6+9+12+\ldots+99+102$$

Solution: Since $3\cdot 34 = 101$:

$$\sum_{i=1}^{34} 3i$$

It could also be written like this:

$$3\sum_{i=1}^{34} i$$

(c) (2 points) Calculate $\binom{3}{1}+\binom{4}{2}+\binom{5}{3}$ as a single integer.

Solution:

$$\begin{aligned}&= \binom{3}{1}+\binom{4}{2}+\binom{5}{3} \\ &= \frac{3!}{1!2!}+\frac{4!}{2!2!}+\frac{5!}{3!2!} \\ &= 3+6+10 \\ &= 19\end{aligned}$$

(d) (1 point) In the expansion of $(a+b)^{200}$, what is the coefficient of the term $a^{10}b^{190}$?

Solution: $\binom{200}{10} = \binom{200}{190} = \frac{200!}{10!190!}$

(e) (3 points) Find a much simpler expression that is equal to the following sum. Clearly show how you got your answer.

$$\binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \ldots + \binom{n}{n}2^n$$

Solution: Using the binomial theorem, this simplifies to 3^n:

$$\begin{aligned}
&= \binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \ldots + \binom{n}{n}2^n \\
&= \sum_{k=0}^{n} \binom{n}{k} 2^k \\
&= \sum_{k=0}^{n} \binom{n}{k} 2^k \cdot 1^{n-k} \\
&= (2+1)^n \quad \text{(apply binomial theorem to previous expression)} \\
&= 3^n
\end{aligned}$$

3. Define each of the following:

   (a) (2 points) $p \to q$ (p implies q), where p and q are logical propositions.

   > **Solution:** This can be defined using a truth table, or by clearly stating the truth value for $p \to q$ for all possibles values of p and q, e.g. $p \to q$ is false just when $p = 1$ and $q = 0$, and true in every other case.

4. (5 points) State the *Modus Ponens* rule of inference, and prove that it is valid.

Solution: *Modus ponens* says that if both premises p and $p \to q$ are true, then the conclusion q is also true:

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore q \end{array}$$

To prove it's valid, show that the corresponding expression $(p \land (p \to q)) \to q$ is a tautology.

5. Consider the following definitions:

$$E(x) : x \text{ is an even number}$$
$$P(x) : x \text{ is a prime number}$$
$$G(x, y) : x > y$$

Assuming the universe of discourse is all *positive integers* (i.e. 1,2,3,...), translate each of the following English statements into logically equivalent statements using just quantified logic and the above three open statements.

(a) (2 points) 11 is prime, but not even.

Solution: $P(11) \land \neg E(11)$

(b) (2 points) Every prime number is greater than 1.

Solution: $\forall x.P(x) \to G(x, 1)$

(c) (2 points) Some prime numbers are not even.

Solution: $\exists x.P(x) \land \neg E(x)$

(d) (2 points) There are an infinite number of primes.

Solution: $\forall x.P(x) \to \exists y.[P(y) \land G(y, x)]$