MACM 101 (Surrey) Midterm Review Questions, Spring 2018

1. In 10-pin bowling, the goal is to knock down 10 pins, numbered 1 to 10. You get to roll two balls, one after the other. If you knock down all 10 on your first ball, that's called a *strike*. If you don't get a strike, how many different patterns of pins are possible after the first ball?

   > **Solution:** After the first ball, each pin is either standing or knocked over. We can represent each pin with a *bit*, i.e. a 0 means knocked over, and 1 means standing. Since there are 10 pins, there are 10 bits, and so there are $2^{10} = 1024$ bit patterns. But the bit pattern of all 0s will never occur, because that means the bowler got a strike and so would not roll their second ball. Thus, there are **1023** possible patterns of pins for the second ball.

2. Recall that a *bit* is a 0 or a 1.

   (a) An *IPv4* Internet address has the form $W.X.Y.Z$, where W, X, Y, and Z are each integers from 0 to 255. For example, 142.58.102.68 is the IPv4 address for `www.sfu.ca`. How many different IPv4 addresses are possible?

   > **Solution:** Each of W, X, Y, and Z can take on 256 possible values, and so the total number of IPv4 addresses is $256 \cdot 256 \cdot 256 = 256^4$. Note that $256^4 = (2^8)^4 = 2^{32}$

   (b) How many bits are needed to represent one *IPv4* address?

   > **Solution:** Numbers from 0 to 255 can be represented with 8 bits, i.e. $2^8 = 256$. So four numbers from 0 to 255 need $4 \cdot 8 = 32$ bits. This also tells us that the total number of IPv4 addresses is 2^{32}.

   (c) An *IPv6* Internet address consists of 16 *octets*, where one octet is 8 bits. How many different IPv6 addresses are possible?

   > **Solution:** 16 octets is $16 \cdot 8 = 128$ bits, and so there are 2^{128} possible IPv6 addresses.

3. Prove that the following equality is true for all valid integers n and k:

$$\binom{n}{k} = \binom{n}{n-k}$$

Solution: By definition:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for } 0 \leq k \leq n$$

Then:

$$\begin{aligned}\binom{n}{n-k} &= \frac{n!}{(n-(n-k))!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}\end{aligned}$$

4. State the *binomial theorem*.

Solution: If n is a positive integer, and x and y are variables, then:

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \ldots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 \\ &= \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k} \\ &= \sum_{k=0}^{n}\binom{n}{n-k}x^ky^{n-k}\end{aligned}$$

5. An ice cream parlor sells 31 different flavors of ice cream. How many different ways can you select 3 scoops of ice cream where i) the order matters, and ii) the order doesn't matter?

Solution: i) If order doesn't matter, then the 3 scoops of 31 flavors can be chosen in $31 \cdot 31 \cdot 31 = 31^3 = 29791$ different ways.

ii) This is a combination with repetitions question where $n = 31$ and $r = 3$, and so the total number of possibilities is:

$$\binom{n+r-1}{r} = \binom{31+3-1}{3} = \binom{33}{3} = 5456$$

This is the same as asking how many ways 3 balls can be distributed among 31 containers.

6. State the *Modus Tollens* inference rule, and prove that it's valid.

Solution: *Modus Tollens* is this rule:

$$\begin{array}{c} \neg q \\ \underline{p \to q} \\ \therefore \neg p \end{array}$$

To prove that it's valid, you must show that $[\neg q \wedge (p \to q)] \to \neg p$ is a tautology. This can be done with a truth table.

7. Show that the following rule of inference is *not* valid.

$$\begin{array}{c} \neg p \\ \underline{p \to q} \\ \therefore \neg q \end{array}$$

Solution: To show that this rule is *not* valid, we can show that the corresponding conditional $(\neg p \wedge (p \to q)) \to \neg q$ is *not* a tautology. To show that a compound statement is not a tautology, we need to find and assignment of values to its variables that makes it false.

For $(\neg p \wedge (p \to q)) \to \neg q$ to be false, $(\neg p \wedge (p \to q))$ must be true and $\neg q$ must be false. If you construct a truth table, you will see that when $p = 0$ and $q = 1$ the expression false, and so it is not a tautology.

8. Suppose $p(x, y)$ means "$xy = 0$", and the universe of discourse is all real numbers. You may also use statements of the form "$a = b$", where a and b are numbers are variables. Translate each of the following statements in quantified logic, and also say whether the statement is true, false, or possibly true or false.

   (a) 3 times 5 is 0

   Solution: $p(3, 5)$ is false.

   (b) Either 3 times 5 is 0, or 6 times 0 is not 0.

   Solution: $p(3, 5) \vee \neg p(6, 0)$ is false.

   (c) x times y is 0 if, and only if, y times x is 0.

   Solution: $\forall x \forall y . p(x, y) \leftrightarrow p(y, x)$ is true.

   (d) If $xy = 0$, then either x or y is 0.

   Solution: $\forall x \forall y . p(x, y) \to (x = 0 \vee y = 0)$ is true.

   (e) If x or y is 0, then so is xy.

   Solution: $\forall x \forall y . (x = 0 \vee y = 0) \to p(x, y)$ is true.

   (f) A negative number times a positive number is never 0.

   Solution: $[(x < 0) \wedge (y > 0)] \to \neg p(x, y)$ is true. Note that 0 is neither negative nor positive.

   (g) 0 times any number is 0.

   Solution: $\forall x . p(0, x) \wedge p(x, 0)$ is true.

   (h) There's a number x such that no number times x is 0.

   Solution: $\exists x \forall y . \neg p(x, y) \wedge \neg p(y, x)$ is false.

   (i) For every number x, there's *exactly one* number that you can multiply x by to get 0.

   Solution: $\forall x \exists y . (p(x, y) \wedge (\forall z . p(x, z) \to y = z))$ is true.

9. Suppose $E(x, y)$ means "$x = y$", $G(x, y)$ means "$x > y$", and the universe of discourse is all real numbers. Translate each of the following statements into quantified logic, and also say whether the statement is true, false, or possibly true or false.

   (a) 5 is bigger than 2

   Solution: $G(5, 2)$ is true.

   (b) All numbers are equal to themselves.

   Solution: $\forall x.E(x, y)$ is true.

   (c) No number is greater than itself.

   Solution: $\forall x.\neg G(x, x)$ is true.

   (d) For any two different numbers, one is bigger than the other.

   Solution: $\forall x.\forall y.\neg E(x, y) \rightarrow [G(x, y) \vee G(y, x)]$ is true.

   (e) x equals y if, and only if, y equals x.

   Solution: $\forall x.\forall y.E(x, y) \leftrightarrow E(y, x)$ is true.

   (f) If x equals y and y equals z, then x equals z.

   Solution: $\forall x.\forall y.\forall z.[E(x, y) \wedge E(y, z) \rightarrow E(x, z)]$ is true.

   (g) There's no biggest number.

   Solution: $\forall x.\exists y.G(y, x)$ is true.

   (h) There's a biggest number.

   Solution: $\exists x.\forall y.\neg G(y, x)$ is false.