MACM 101 (Surrey) Midterm 1, Fall 2018

Please write your answers in the exam booklet. **Show your work**: answers without explanations won't get full marks! This exam has **6** questions and is out of **40** marks (there are no bonus marks).

1. In a regular deck of 52 cards, 26 are red and 26 are black. Also, there are 4 suits of 13 cards each: spades, hearts, diamonds, and clubs.

   (a) (1 point) How many different ways can all 52 different cards be arranged in a line?

   Solution: 52!

   (b) (2 points) How many different ways can all 52 cards be arranged in a line, assuming all that matters is the suit of the card?

   Solution: $\frac{52!}{13!13!13!13!}$

   (c) (2 points) How many different ways can all 52 cards be arranged in a line if all the red cards come first, followed by all the black cards?

   Solution: $26! \cdot 26!$

   (d) (3 points) How many different ways can all 52 cards be distributed among 4 players so that each gets 5 or more cards? They don't all need to get the same number of cards, and the order of the cards in their hand doesn't matter.

   Solution: This is meant to be a combination with repetitions question. First give 5 cards to each player. Then there are $r = 32$ cards remaining to be distributed among $n = 4$ players. So the final answer is:

$$\begin{aligned}\binom{n+r-1}{r} &= \binom{4+32-1}{32} \\ &= \binom{35}{32}\end{aligned}$$

2. (a) (3 points) Define $\binom{n}{k}$ for all values of n and k that make sense.

Solution: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, for integers n and k such that $0 \le k \le n$.

(b) (1 point) In the expansion of $(a+b)^{100}$, what is the coefficient of the term $a^{60}b^{40}$?

Solution: $\binom{100}{40} = \binom{100}{60} = \frac{100!}{40!60!}$

(c) (5 points) Prove that this equation holds for all non-negative integers n:

$$n! + (n+1)! = \frac{(n+2)!}{n+1}$$

Solution: The technique to use here is to start with one side of the equation, and then by a series of small and obviously correct steps, transform it into the other side. In this solution, we start with the left-hand side and transform it into the right hand side:

$$\begin{aligned} n! + (n+1)! &= n!(1 + (n+1)) \\ &= n!(n+2) \\ &= n!(n+2) \cdot \frac{n+1}{n+1} \\ &= \frac{(n+2)(n+1)n!}{n+1} \\ &= \frac{(n+2)!}{n+1} \end{aligned}$$

3. (5 points) Give a short logical expression that is *logically equivalent* to $p \lor q$ that does *not* use $\lor$. Prove your expression is logically equivalent to $p \lor q$.

Solution: $\neg(\neg p \land \neg q)$ works. You can prove this with a truth table, or apply De Morgan's law.

$\neg p \to q$ also works. You can prove this with a truth table, or using the fact that $X \to Y = \neg X \lor Y$.

There are (many!) other solutions.

4. (5 points) Show that this argument is *invalid*:

$$\begin{array}{l} \neg p \lor q \\ \underline{\neg p \lor r} \\ \therefore q \lor r \end{array}$$

Solution: To prove an argument is invalid, you need to show that the associated conditional $[(\neg p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ is *not* a tautology. In other words, you need to find an assignment of truth values to all the variables that make the premises true, and the conclusion false.

One way to do this is to use a truth table for $[(\neg p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$. You can stop as soon as you find one row that makes the expression false.

Another way is to guess, or to use informal reasoning. For example, since both premises contain $\neg p$, if $p = 0$ then the premises are both true no matter the values of q and r. So if we set $q = 0$ and $r = 0$, then the conclusion is false. That means setting $p = q = r = 0$ makes the premises true and the conclusion false, which proves the argument is invalid.

5. Suppose $E(n)$ and $O(n)$ are defined as follows:

$$E(n) : n \text{ is exciting}$$
$$O(n) : n \text{ is obvious}$$

Assuming the universe of discourse is all integers, re-write each of the following English statements as logically equivalent statements:

(a) (2 points) 4 is neither exciting or obvious.

Solution: $\neg(E(4) \vee O(4))$

(b) (2 points) A number is exciting if, and only if, it's not obvious.

Solution: $\forall x. E(x) \leftrightarrow \neg O(x)$

(c) (2 points) No number is both obvious and exciting.

Solution: $\neg \exists x. O(x) \wedge E(x)$, or $\forall x. \neg(O(x) \wedge E(x))$

(d) (2 points) Numbers are obvious, unless they're exciting.

Solution: $\forall x. \neg E(x) \rightarrow O(x)$
"p unless q" is the same as "p if not q", which is logically equivalent to "not q implies p".

6. Assuming only logical statements with a single variable, state the rule of

(a) (2 points) *universal specification.*

Solution: If $\forall x.p(x)$ is true, then for any element a in the universe of discourse $p(a)$ is true.

(b) (3 points) *universal generalization.*

Solution: If an open statement $p(x)$ is proved to be true when x is replaced by an *arbitrary* element c chosen from the universe of discourse, then the universally quantified statement $\forall x.p(x)$ is true.