Discrete Mathematics

Spring 2017

Previous Lecture

- Equivalence Relations
- Equivalence Classes
- Partitions

Partial Orderings

Definition: A relation R on a set S is called a partial ordering, or partial order, if it is reflexive, antisymmetric, and transitive. The set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset

Example: Show that the ‘greater than or equal’ relation () is a partial ordering on the set of integers

Reflexivity: $a \geq a$ for every integer a
Antisymmetry: If $a \geq b$ and $b \geq a$, then $a = b$
Transitivity: If $a \geq b$ and $b \geq c$, then $a \geq c$

Partial Orderings

Example: Show that the divisibility relation ($|$) is a partial ordering on the set of integers

Reflexivity: $a|a$ for all integers a
Antisymmetry: If a and b are positive integers with $a|b$ and $b|a$, then $a = b$
Transitivity: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive

$(\mathbb{Z}^+, |)$ is a poset

Partial Orderings

Example: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S

Reflexivity: $A \subseteq A$ whenever A is a subset of S
Antisymmetry: If A and B are subsets of S with $A \subseteq B$ and $B \subseteq A$, then $A = B$
Transitivity: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Partial Orderings: Comparability

Definition: The elements a and b of a poset $(S, \preceq)$ are comparable if either $a \preceq b$ or $b \preceq a$. When a and b are elements of S so that neither $a \preceq b$ nor $b \preceq a$, then a and b are called incomparable

Definition: If $(S, \preceq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preceq$ is called a total order or a linear order. A totally ordered set is also called a chain

Definition: $(S, \preceq)$ is a well-ordered set if it is a poset such that $\preceq$ is a total ordering and every nonempty subset of S has a least element

Lexicographic Order

Definition: Given two posets $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$, the lexicographic ordering on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), that is, $(a_1, a_2) \prec (b1, b2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_2 b_2$

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries

discreet $\prec$ discrete, because these strings differ in the seventh position and $e \prec t$

discreet $\prec$ discreetness, because the first eight letters agree, but the second string is longer

Hasse Diagrams

Definition: A Hasse diagram is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties

Procedure for Constructing a Hasse Diagram
To represent a finite poset $(S, \preceq)$ using a Hasse diagram, start with the directed graph of the relation:

- Remove the loops (a, a) present at every vertex due to the reflexive property
- Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property
- Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex

Hasse Diagrams

Construction: Example for $(\{1,2,3,4\},\preceq)$

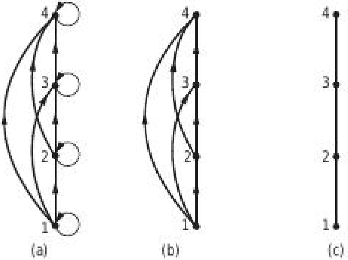
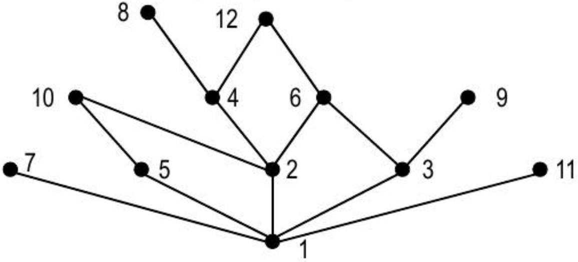


FIGURE 2 **Constructing the Hasse Diagram for $(\{1, 2, 3, 4\}, \leq)$.**

Hasse Diagrams

Construction: Example for $(\{1, 2, 3, \ldots, 12\}, |)$



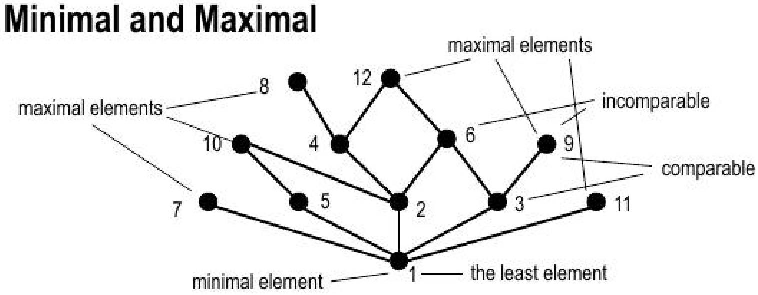
Hasse Diagrams

Minimal and Maximal

Elements a, b are said to be comparable if $(a, b) \in R$ or $(b, a) \in R$
Otherwise they are called incomparable
Element a is minimal if for any b if $(b, a) \in R$ then $a = b$
Element a is maximal if for any b if $(a, b) \in R$ then $a = b$
Element a is called the least element if for any b, $(a, b) \in R$
Element a is called the greatest element if for any b, $(b, a) \in R$

Hasse Diagrams

Construction: Example for $(\{1, 2, 3, \ldots, 12\}, |)$



Homework for practice (not graded)

- Exercises from the Book:
  - Section 9.6: Exercises 2, 4, 6, 8, 10, 16, 20, 24