Basic Counting Principles

Counting problems are of the following kind:

- “How many different 8-letter passwords are there?”
- “How many possible ways are there to pick 11 soccer players out of a 20-player team?”

Most importantly, counting is the basis for computing probabilities of discrete events.

- “What is the probability of winning the lottery?”

The Sum Rule

If a first task can be performed in \( n_1 \) ways, while a second task can be performed in \( n_2 \) ways, and the two tasks cannot be performed simultaneously, then performing either task can be done in any of \( n_1 + n_2 \) ways.

Example: The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors?

Example: A student can choose one computer project from one of three lists. The three lists contain 23, 15 and 19 possible choices. How many possible projects are there to choose from?

The Product Rule

Suppose that a procedure can be broken down into two successive tasks. If there are \( n_1 \) ways to do the first task and \( n_2 \) ways to do the second task after the first task has been done, then there are \( n_1 \cdot n_2 \) ways to do the procedure.

Generalized product rule:

If we have a procedure consisting of sequential tasks \( T_1, T_2, \ldots, T_n \) that can be done in \( n_1, n_2, \ldots, n_n \) ways, respectively, then there are \( n_1 \cdot n_2 \cdot \ldots \cdot n_n \) ways to carry out the procedure.

Example

How many different license plates are there that contain exactly three English letters?
Inclusion-Exclusion Example

How many bit strings of length 8 either start with a 1 or end with 00?

**Task 1:** Construct a string of length 8 that starts with a 1.

Since there are 128 ways to do Task 1 and 64 ways to do Task 2, does this mean that there are 192 bit strings either starting with 1 or ending with 00?

- If we want to use the sum rule in such a case, we have to subtract the cases when Tasks 1 and 2 are done at the same time.

**Task 2:** Construct a string of length 8 that ends with 00.

How many cases are there, that is, how many strings start with 1 **and** end with 00?
Another Approach – Tree Diagrams

How many bit strings of length four do not have two consecutive 1s?

<table>
<thead>
<tr>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
<th>Task 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1st bit)</td>
<td>(2nd bit)</td>
<td>(3rd bit)</td>
<td>(4th bit)</td>
</tr>
</tbody>
</table>

Permutations - Example

In a class of 10 students, 5 are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?

Consider the individual seating positions:

<table>
<thead>
<tr>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>5th</th>
</tr>
</thead>
<tbody>
<tr>
<td>pos</td>
<td>pos</td>
<td>pos</td>
<td>pos</td>
<td>pos</td>
</tr>
</tbody>
</table>

Permutations - Definitions

A permutation of a set of n distinct objects is an ordered arrangement of these objects.

For an integer \( n \geq 0 \), n factorial (denoted \( n! \)) is defined by

\[
0! = 1, \\
\quad n! = (n)(n-1)(n-2)\cdots(3)(2)(1), \text{ for } n \geq 1
\]

Counting Formula: Permutations with no repetition

The number of ways to arrange \( r \) (with \( 0 \leq r \leq n \)) objects from a set of \( n \) objects, in order, but with no repetition allowed is:

\[
P(n,r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) = \frac{n!}{(n-r)!}
\]

Permutations - Definitions

Counting Formula: Permutations with repetition

The number of ways to arrange \( r \) objects from a set of \( n \) objects, in order, with repetition allowed is: \( n^r \)

Example:

(a) What are the permutations of the letters in the word COMPUTER?

(b) If only 5 letters are used from the above, what is the number of permutations?

(c) If repetition of letters are allowed, how many 12-letter sequences are possible using the letters above?
Example

How many different strings can be made reordering the word “BALL”?

Permutations - Definitions

Counting Formula: Permutations w/ indistinguishable objects

The number of different (linear) permutations of \( n \) objects, where there are \( n_1 \) indistinguishable objects of type 1, \( n_2 \) indistinguishable objects of type 2, ... \( n_k \) indistinguishable objects of type \( k \), is:

\[
\frac{n!}{n_1! \cdot n_2! \cdots n_k!}
\]

where \( n = n_1 + n_2 + \ldots + n_k \)

Example

How many different strings can be made reordering the letters in the word “DATABASES”?