Combinatorics: Binomial Coefficients

Discrete Mathematics
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Previous Lecture

- Rules of sum and product
- Permutations, permutations of size r
- Permutations with repetitions

Combinations

- How many different committees of three students can be formed from a group of four students?
- Solution: To answer this question, we need only to find the number of subsets with three elements from the set containing the four students. As is easily seen, there are four such subsets. Note that the order in which these students are chosen does not matter.
- An r-combination of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.
- The number of r-combinations of a set with n distinct elements is denoted by C(n,r). Note that C(n,r) is often denoted by $\binom{n}{r}$ and is called a binomial coefficient.

Computing Binomial Coefficients

- **Theorem.**

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

- Proof.

The permutations of size r can be obtained by forming C(n,r) r-combinations of the set, and then ordering the elements in each r-combination, which can be done in P(r,r) = r! ways. Therefore,

$$P(n,r) = C(n,r) \cdot P(r,r)$$

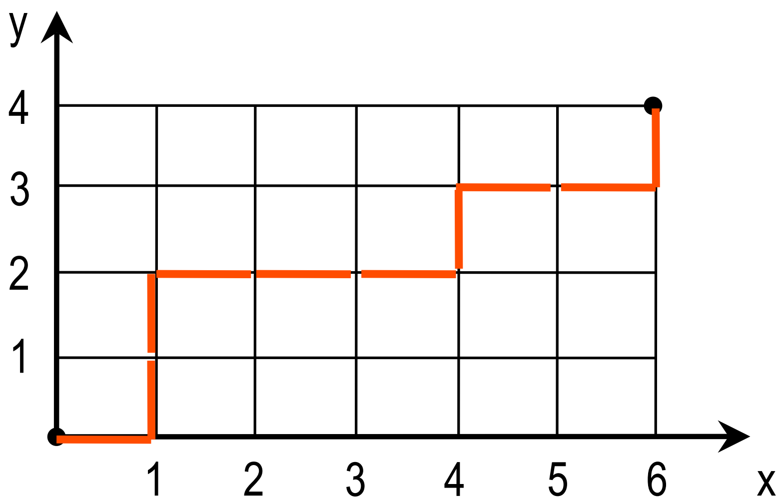
This implies

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

Q.E.D.

Example

- Reconsider the example with paths in the plain



- To get from (0,0) to (6,4) we need to make 10 steps. Among them 4 steps are upward and the rest to the right.
- Therefore every path corresponds to a selection from steps 1,2,…,10 four steps upward.
- Thus, the number of steps equals C(10,4) $= \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!} = 210$

More Examples

- How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Combinations with Repetitions

- How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matter, and there are at least four pieces of each type of fruit in the bowl?
- Solution (brute force): List all possibilities

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple,1 pear	2 pears, 1 apple, 1 orange

Combinations with Repetitions (cntd)

- A better way: four pieces of fruit

$$\overbrace{\underbrace{*}_{\text{apples}} \;\Big|\; \underbrace{*\quad *}_{\text{oranges}} \;\Big|\; \underbrace{*}_{\text{pear}}}$$

- Every choice of fruits corresponds to an arrangement of 4 stars and 2 bars.

  We have six positions to place a symbol, and two of them must be bars. Therefore the number we are looking for is

$$C(6,2) = \frac{6!}{2!(6-2)!} = \frac{6\times 5}{1\times 2} = 15$$

Combinations with Repetitions (cntd)

- **Theorem.**

  There are $C(n + r - 1,n - 1)$ r-combinations from a set with n elements when repetitions of elements are allowed.

- Proof.

  We use the same idea as in the example above.

  We represent the r members of our selection by stars, and separate from each other the n types of elements by $n – 1$ bars.

  There are $n + r – 1$ places for stars and bars, and bars must be on $n – 1$ positions.

Q.E.D.

Example

- How many solutions does the equation

  $$x + y + z = 11$$

  have, where x, y, and z are nonnegative integers?

  (In other words, in how many ways can we represent 11 as the sum of 3 nonnegative summands?)

- Solution:

  A solution corresponds to a way of selecting 11 items from a set with three elements so that x items of type one, y items of type two, and z items of type three are chosen.

  Hence, the number of solutions is equal to the number of 11-combinations with repetitions from a set with 3 elements

$$C(3+11-1, 3-1) = C(13,2) = \frac{13!}{2!(13-2)!} = \frac{13 \times 12}{1 \times 2} = 78$$

Examples

- A coin is flipped 8 times where each flip comes up either heads or tails. How many possible outcomes
  - are there in total?
  - contain exactly 3 heads?
  - contain at least 3 heads?
  - contain the same number of heads and tails?

Examples

- 100 tickets, numbered 1,2,3,…, 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award prizes if
  - there are no restrictions?
  - the person holding ticket 47 wins the grand prize?
  - the person holding ticket 47 wins one of the prizes?
  - the person holding ticket 47 does not win a prize?
  - the people holding tickets 19 and 47 both win prizes?
  - the people holding tickets 19, 47, and 73 all win prizes?
  - the people holding tickets 19, 47, 73, and 97 all win prizes?
  - none of the people holding tickets 19, 47, 73, and 97 wins a prize?
  - the grand prize winner is a person holding ticket 19, 47, 73, or 97?

A Binomial

- A binomial is simply the sum of two terms, such as x + y
- We are to determine the expansion of $(x+y)^n$
- Let us start with $(x+y)^3$

$$(x+y)^3 = (x+y)\times(x+y)\times(x+y)$$

Every term in the expansion is obtained as the product of a term from the first binomial, a term from the second binomial, and a term from the third binomial

$$= xxx + xxy + xyx + xyy + yxx + xxy + xyx + xyy$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Each of the terms xxy, xyx, and yxx is obtained by selecting y from one of the 3 binomials. Therefore, the coefficient 3 at x^2y is, actually, the number of 1-combinations from a set with 3 elements

The Binomial Theorem

- **Theorem.**

  Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{k}x^{n-k}y^k + \dots + \binom{n}{n-1}x^{n-1}y + \binom{n}{n}y^n$$

- Proof.

  The terms in the product when it is expanded are of the form $x^{n-k}y^k$ for k = 0,1,2,…, n.

  To count the number of terms of the form $x^{n-k}y^k$, note that to obtain such a term it is necessary to choose k y's from the n sums (so that the other n – k terms in the product are x's).

  Therefore, the coefficient of $x^{n-k}y^k$ is $\binom{n}{k}$

Q.E.D.

Examples

- Expand $(x+y)^4$

$$(x+4)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

- Expand $(x+2y)^5$

$$(x+2y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4(2y) + \binom{5}{2}x^3(2y)^2 + \binom{5}{3}x^2(2y)^3$$
$$+ \binom{5}{4}x(2y)^4 + \binom{5}{5}(2y)^5$$
$$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$$

Properties of Binomial Coefficients

- For any nonnegative integer n and any r with $0 \le r \le n$

$$\binom{n}{r} = \binom{n}{n-r}$$

- Indeed, $$\binom{n}{r} = \frac{n!}{r!(n-r)} \text{ and}$$

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$$

Properties of Binomial Coefficients (cntd)

- For any nonnegative integer n

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

- Proof 1: By the Binomial Theorem

$$2^n=(1+1)^n=\binom{n}{0}1^n+\binom{n}{1}1^{n-1}1+\binom{n}{2}1^{n-2}1^2+\dots$$

$$+\binom{n}{n-1}1^1 1^{n-1}+\binom{n}{n}1^n=$$

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-1}+\binom{n}{n}$$

- Proof 2: Recall that C(n, r) is the number of r-element subsets of a set with n elements. Therefore, the sum on the left side is the number of all subsets of an n-element set. We know this number equals 2^r

Pascal’s Identity

- For any nonnegative integer n and any r with $0 \le r \le n$

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Blaise Pascal

- Proof: As we know C(n+1, r) is the number of r-element subsets of an (n + 1)-element set. Take a set T with n + 1 elements.

  Pick an element $a \in T$ and set $S = T – \{a\}$.

  Every r- element subset of T either does not contain a and hence is an r-element subset of S (there are C(n, r) subsets of this type),

  or it contains a and the remaining elements form an (r – 1)-element subset of S (there are C(n, r-1) subsets of this type)

Pascal's Triangle

- Pascal's identity and the simple observation that $\binom{n}{0} = \binom{n}{n} = 1$ allow us to give an inductive definition of binomial coefficients. It is convenient to arrange them into a triangle

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Computing Binomial Coefficients

- Computing factorials and binomial coefficients can be very difficult. Fortunately, there are many ways to simplify the computation
- Stirling formula: $n! \approx \sqrt{2\pi n}\,\frac{n^n}{e^n}$, where $e = 2.718281828459\ldots$
- Gamma function: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$

  $\Gamma(n + 1) = n!$
- Computing binomial coefficients:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \approx \frac{\sqrt{2\pi n}\frac{n^n}{e^n}}{\sqrt{2\pi r}\frac{r^r}{e^r}\sqrt{2\pi(n-r)}\frac{n^{n-r}}{e^{n-r}}} =$$

$$= \sqrt{\frac{n}{r(n-r)}}\left(\frac{n}{r}\right)^r\left(\frac{n}{n-r}\right)^{n-r}$$

Homework

Exercises from the Book:

No. 22a, 29, 30 (page 25)

No. 5a, 7, 10 (page 34)

- See examples on the slides 12 and 13