Pigeonhole Principle

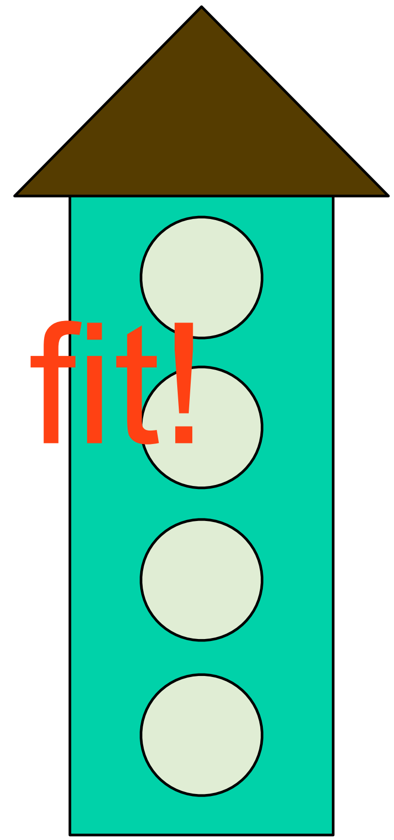
Discrete Mathematics
Evgeny Skvortsov

Previous Lecture

- Rules of sum and product
- Permutations, permutations with repetitions
- Combinations, combinations with repetitions
- Binomial theorem, properties of binomial coefficients

Pigeonhole Principle

- If m pigeons occupy n pigeonholes and $m > n$, then at least one pigeonhole has two or more pigeons roosting in it.



He knows it and chooses a roommate.

Pigeon principle proof

- Obvious?
- Allmost.

Let A be the set of pigeons, B be the set of holes, $|A| > |B|$.

Let f be a function
f(x) = <the hole, where x roosts>

f maps the set pigeons to the set holes.
Can f be injection?
No way, because $|A|>|B|$!

QED

Examples

- Among any group of 367 (or more) people, there must be at least two with the same birthday, because there are only 366 possible birthdays
- In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letter in the Latin alphabet

More Examples

- Among any group of $n + 1$ integers, there must be at least two with the same remainder when divided by n
- For every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion

Solution: Let n be a positive integer.

Consider the $n + 1$ integers $1, 11, 111, \dots, 11\dots1$ (where the last integer in this list is the integer with $n + 1$ 1s in its decimal expansion).

At least two of them have the same remainder when divided by n. Let they have the form $a \cdot n + c$ and $b \cdot n + c$.

The larger of these integers less the smaller one, that is $(a - b) \cdot n$ or $(b - a) \cdot n$, is a multiple of n, which has a decimal expansion consisting entirely of 0s and 1s.

Yet Another Example

- While on a four-week vacation, Herbert will play at least one set of tennis each day, but he will not play more than 40 sets total during this time. Prove that no matter how he distributes his sets during the four weeks, there is a span of consecutive days during which he will play exactly 15 sets.
- Solution: For $1 \le i \le 28$, let x_i be the total number of sets Herbert will play from the start of the vacation to the end of ith day.

  Then $1 \le x_1 < x_2 < \cdots < x_{28} \le 40$ and

  $x_1 + 15 < x_2 + 15 < \cdots < x_{28} + 15 \le 55$

  We have the 28 distinct numbers $x_1, x_2, \ldots, x_{28}$ and the 28 distinct numbers $x_1 + 15, x_2 + 15, \ldots, x_{28} + 15$

Yet Another Example (cntd)

- (solution continued)

  These 56 numbers can take only 55 different values, so at least two of them are equal.

  If for $1 \le j < i \le 28$ we have $x_j + 15 = x_i$, then from the start of day $j + 1$ to the end of day x_i, Herbert will play exactly 15 sets of tennis

Generalized Pigeonhole Principle

- The ceiling, $\lceil x \rceil$, of a real number x is the least integer n such that $x \le n$. For example, $\lceil 3.14 \rceil = 4$.

- If n objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{n}{k} \rceil$ objects.
- Proof: By contradiction.

  Let us suppose that none of the boxes contains more than $\lceil \frac{n}{k} \rceil - 1$ objects.

  Then the total number of objects is at most

$$k\left(\left\lceil \frac{n}{k} \right\rceil - 1\right) < k\left(\left(\frac{n}{k} + 1\right) - 1\right) = n$$

  A contradiction.

Examples Again

- Among 100 people there are at least $\lceil \frac{100}{12} \rceil = 9$ who were born in the same month.

- Suppose that a computer science laboratory has 15 workstations and 10 servers. A cable can be used to directly connect a workstation to a server. For each server, only one direct connection to that server can be active at any time. We want to guarantee that at any time any set of 10 or fewer workstations can simultaneously access different servers via direct connections.

  Although we could do this by connecting every workstation directly to every server (using 150 connections), what is the minimum number of direct connections needed to achieve the goal?

Examples Again (cntd)

- Let us label workstations $W_1, W_2, \ldots, W_{15}$ and the servers $S_1, S_2, \ldots, S_{10}$

  Let us connect W_k to S_k for $k = 1,2,\ldots,10$ and each of $W_{11}, W_{12}, W_{13}, W_{14}$, and W_{15} to all 10 servers. We have a total of 60 direct connections.

  Clearly any set of 10 or fewer workstations can simultaneously access different servers.

  Now suppose that there are fewer than 60 direct connections.

  Then some server is connected to at most 5 workstations. (If all servers were connected to at least 6 workstations, there would be at least 60 connections.)

  This means that the remaining 9 servers are not enough to allow other 10 workstations to simultaneously access different servers.

Ramsey Numbers

- Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.
- Solution:

  Let A be one of the six people.

  Of the five other people in the group, there are either three or more who are friends of A, or three or more who are enemies of A. Indeed, by the generalized pigeonhole principle, when 5 objects are divided into 2 sets, one of the sets contains at least $\lceil 5/2 \rceil = 3$ objects.

  In the former case, suppose that B, C, and D are friends of A.

  If any of these 3 individuals are friends, then these two and A form a group of 3 friends.

  Otherwise, B, C, and D form a group of 3 enemies.

Ramsey Numbers (cntd)

- The Ramsey number $R(m,n)$, where m and n are positive integers greater than or equal to 2, denotes the minimum number of people at a party such that there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies.
- In the example above we showed that $R(3,3) \le 6$.
- Show that $R(3,3) = 6$

Frank Ramsey

More Examples

- An auditorium has a seating capacity of 800. How many seats must be occupied to guarantee that at least two people seated in the auditorium have the same first and last initials.
- Let ABCD be a square with AB = 1. Show that if we select 5 points in the interior of this square, there are at least two whose distance apart is less than $1/\sqrt{2}$

Pigeonhole works for infinite sets

- Example:
- A number x is algebraic if it is a root of some polynomial with rational coefficients.
- For instance $\sqrt{2}$ is algebraic because it's a root of $x^2 - 1 = 0$
- **Theorem**: There exist real numbers that are not algebraic.
- **Proof:**
- There are $\aleph_0$ algebraic numbers and there are $\aleph_1$ real numbers.
- Let algebraic numbers are pigeons, real numbers are holes and "x roosts in y" is "x = y". If all reals are rationals then by pigeonhole principle two different real numbers are equal, which can not be true. QED

Homework

Exercises from the Book:
No. 4, 10 (see Example 5.45, p.275), 14, 18 (page 277)

- Determine Ramsey number R(4,4)