# Counting: Permutations and Combinations

Discrete Mathematics
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## What is Combinatorics

- Combinatorics, the study of arrangements of objects, is an important part of discrete mathematics. This subject was studied as long ago as the seventeenth century, when combinatorial questions arose in the study of gambling games.

  Enumeration is the counting of objects with certain properties

- Combinatorics is used in

  - Discrete probability: What is the probability to guess a 6-symbols password in the first attempt?

  - Analysis of algorithms: Why a comparison sort algorithm cannot be more efficient than O(n log n)?

  - Probabilistic proofs: Show that the local search algorithm with high probability does not find a good solution to a problem.

## Rules of Combinatorics

- Rule of Sum: if $A \cap B = \varnothing$, then $|A \cup B| = |A| + |B|$
- Rule of Product: $|A \times B| = |A| \times |B|$

# The Rule of Sum

- If a first task can be performed in $m$ ways, while a second task can be performed in $n$ ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.
- Example: Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors.
- Solution: There are 37 ways to choose a faculty member, and there are 83 ways to choose a student. Choosing a faculty member is never the same as choosing a student.

  By the rule of sum there are $37 + 83 = 120$ possible choices.

# The Rule of Product

- If a procedure can be broken down into two stages, and if there are $m$ possible outcomes of the first stage and if, for each of these outcomes, there are $n$ possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in $m \cdot n$ ways.
- Example: A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?
- Solution: The procedure of assigning offices consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways.

  By the rule of product, there are $12 \cdot 11 = 132$ ways to assign offices

## The Rule of Product (cntd)

- Example: The chairs of an auditorium are to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
- Solution: The procedure of labeling a chair consists of two tasks, namely, assigning one of the 26 letters and then assigning one of the 100 possible integers to the seat. By the rule of product, there are $26 \cdot 100 = 2600$ different labels.

- Example: How many functions are there from a set with $m$ elements to a set with $n$ elements?
- Solution: A function corresponds to a choice of one of the $n$ elements in the codomain for each of the $m$ elements of the domain. By the rule of product, there are $n \cdot n \cdot \ldots \cdot n = n^m$ functions

# Permutations

- Example: In how many ways can we select 3 students from a group of 5 student to stand in a line for a picture?
- Solution: First, note that the order in which we select students matters. There are 5 ways to select the first student. Once the first one is selected we are left with 4 ways to select the second student. After selecting the first 2 students there are 3 ways to select the third one.

  By the rule of product, there are $5 \cdot 4 \cdot 3 = 60$ ways to select students.
- Given a collection of $n$ distinct objects, any (linear) arrangement of these objects is called a permutation of the collection.

  A permutation of size $r$ $(0 \le r \le n)$ is any (linear) arrangement of $r$ distinct objects from the collection

# The Number of Permutations

- Similar to the example on the previous slide, the number $P(n,r)$ of permutations of size $r$ from a collection of $n$ objects can be found as follows:

  We choose $r$ elements out of $n$ and the order matters.

  There are $n$ ways to choose the first element,

  there are $n – 1$ ways to choose the second element

  $\vdots$

  there are $n – r + 1$ ways to choose element number $r$

  By the rule of product, $P(n,r) = n \cdot (n - 1) \cdot (n – 2) \cdot \ldots \cdot (n – r + 1)$
- Recall that $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n – 1) \cdot n$
- Therefore

$$P(n,r) = \frac{n!}{(n-r)!}$$

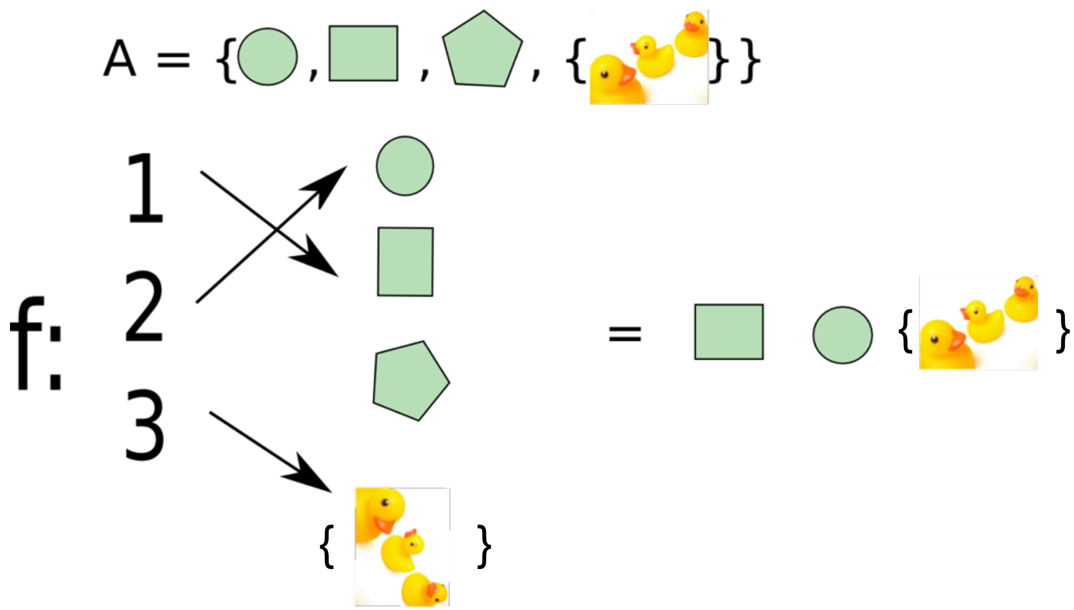
And the number of permutations

$$P(n,n) = n!$$

# The Number of Permutations (formally)

- Permutation of size r of elements of A is an injective function $f: \{1, \ldots, r\} \to A$

- Example: Permutation of size 3



# The Number of Permutations (formally) cntd

- We use method of mathematical induction to prove that
- $P(n, r) = n! / (n-r)!$
- Parameter of induction is r, that is our predicate is
- $Z(r) = [\ P(n,r) = n! / (n-r)!\ ]$
- Base step: $P(n,1) = n$ (obvious)

# The Number of Permutations (formally) cntd

- Inductive step: P(n, k) = n! / (n-k)! → P(n, k+1) = n! / (n - k - 1)!
- Proof:

To get the permutation we
1) Choose an element to put on the first place ==> n ways
2) Permute remaining n-1 elements on the places 2..k+1 ==> P(n-1, k)

By the product rule we have the total number of permutations
equal: n P(n-1, k).
So P(n, r) = P(n-1,k) n =
n (n-1)! / (n - 1 - k)! = n! / (n - k - 1)!

Exercise:
Product rule?
Cartesian product of what sets is considered here?

# The Number of Permutations (cntd)

- Example: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

- Solution: Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements.

  Thus, this number equals the number of permutations of size 3 from a set with 100 elements

  $$P(100,3) = 100 \cdot 99 \cdot 98 = 970200$$

# The Number of Permutations (cntd)

- Example: How many permutations of the letters ABCDEFGH contain the string ABC?

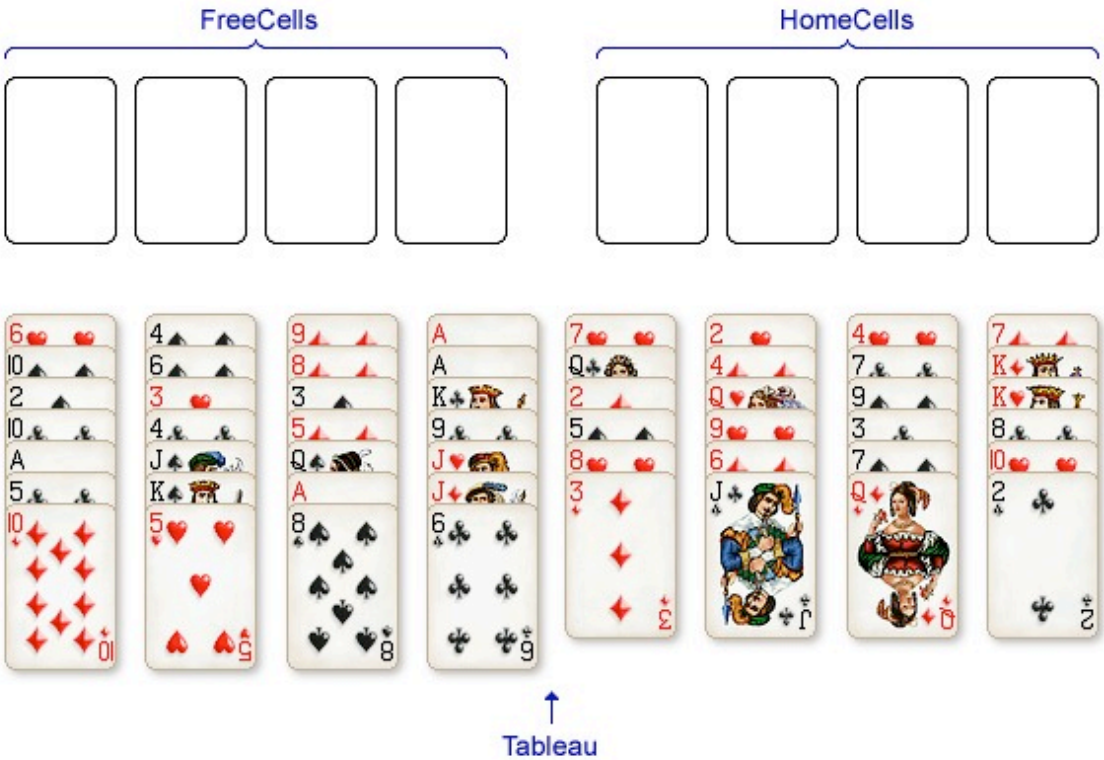
- Solution: Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G, and H. Since these six objects can occur in any order, there are

  $P(6,6) = 6! = 720$

  permutations of the letters ABCDEFGH in which ABC occurs as a block.

# More Examples

- In the FreeCell game a standard deck of 52 cards is arranged in 8 piles such that in the first 4 piles there are 7 cards and the last 4 piles contain 6 cards each. How many different FreeCell games are there?



## Permutations with Repetitions

- How many different 4-letter words (not necessarily meaningful) can be built permuting the letters of the word COOL?
- If all letters were distinct then the answer would be the number of all permutations of a 4-element set. However, in words we build we do not distinguish two O.
- So, words $O_1CLO_2$ and $O_2CLO_1$ are equal. For each of the words we are interested in, there are two words in which the two O's are distinguished.
- Therefore the answer is $\frac{4!}{2} = 12$

# Permutations with Repetitions (cntd)

- **Theorem.**

  If there are n objects with $n_1$ indistinguishable objects of a first type, $n_2$ indistinguishable objects of a second type, … , and $n_r$ indistinguishable objects of a type r, where $n_1 + n_2 + \ldots + n_r = n$, then there are

$$\frac{n!}{n_1! n_2! \ldots n_r!}$$

  (linear) arrangements of the given n objects.

- Each arrangement of this type is called a permutation with repetitions

# Proof

- Let us denote objects by capital letters and add subscripts to indistinguishable objects so that they become distinct.
- Then

$$A_1, A_2, \ldots, A_{n_1} \text{ - objects of the first type}$$
$$B_1, B_2, \ldots, B_{n_2} \text{ - objects of the second type}$$
$$\vdots$$

- How many permutations of these new distinct objects are there that correspond to the same permutation with repetitions?

$$A_1 \ldots B_1 \ldots B_2 \ldots A_2 \ldots A_{n_1} \ldots B_{n_2}$$

- If we take any permutation of A’s such that they on the same places as before, any permutation of B’s, …, we obtain a permutation of n objects that corresponds to the same permutation with repetitions

# Proof (cntd)

- Therefore, by the rule of product, there are

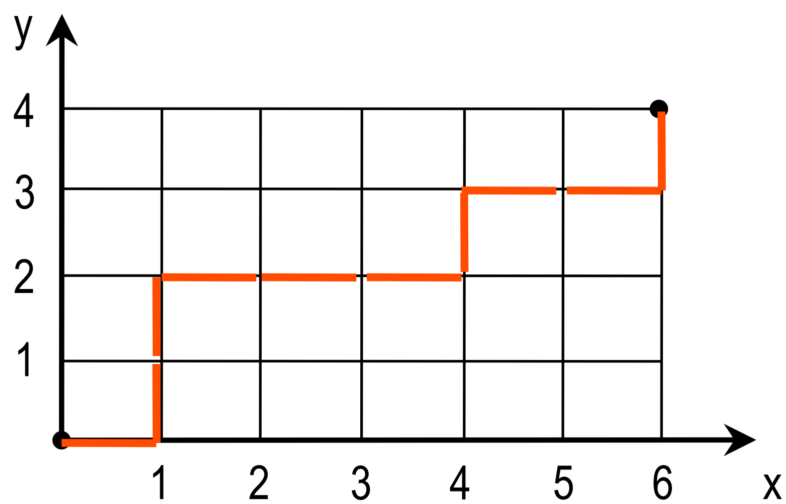
  $$n_1!n_2!\ldots n_r!$$

  such permutations.
- Since there are n! permutations of labeled objects, we obtain the required result.

Q.E.D.

# Example

- Determine the number of (staircase) paths in the xy-plane from (0,0) to (6,4), where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U).




- Every path like this can be encoded as a sequence of R's and U's
- For example, the path on the picture is encoded as RUURRRURRU

- Therefore, the number of paths equals to the number of permutations with repetitions: 6 R's and 4 U's:

$$\frac{10!}{6!4!} = 210$$

# Combinations

- How many different committees of three students can be formed from a group of four students?
- Solution: To answer this question, we need only to find the number of subsets with three elements from the set containing the four students. As is easily seen, there are four such subsets. Note that the order in which these students are chosen does not matter.
- An r-combination of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.
- The number of r-combinations of a set with n distinct elements is denoted by $C(n,r)$. Note that $C(n,r)$ is often denoted by $\binom{n}{r}$ and is called a binomial coefficient.

# Computing Binomial Coefficients

- **Theorem.**

  The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

- Proof.

  The permutations of size r can be obtained by forming C(n,r) r-combinations of the set, and then ordering the elements in each r-combination, which can be done in P(r,r) = r! ways. Therefore,

  $$P(n,r) = C(n,r) \cdot P(r,r)$$

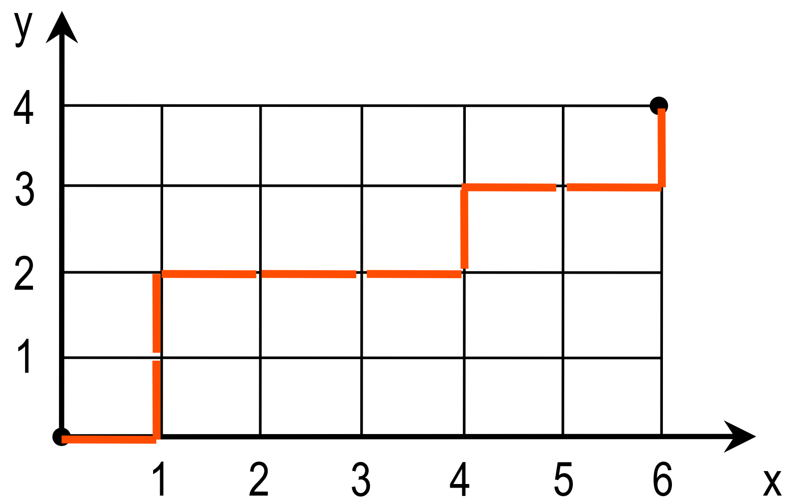
  This implies

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

Q.E.D.

# Example

- Reconsider the example with paths in the plain




- To get from (0,0) to (6,4) we need to make 10 steps. Among them 4 steps are upward and the rest to the right.
- Therefore every path corresponds to a selection from steps 1,2,…,10 four steps upward.
- Thus, the number of steps equals C(10,4) $= \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!} = 210$

## More Examples

- How many poker hands of five cards can be dealt from a standard deck of 52 cards?

# Homework

Exercises from the Book:

No. 1, 4, 11, 14, 24 (page 12)

No. 2, 9, 12, 15 (page 24)