Growth of Functions

Discrete Mathematics
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Previous Lecture

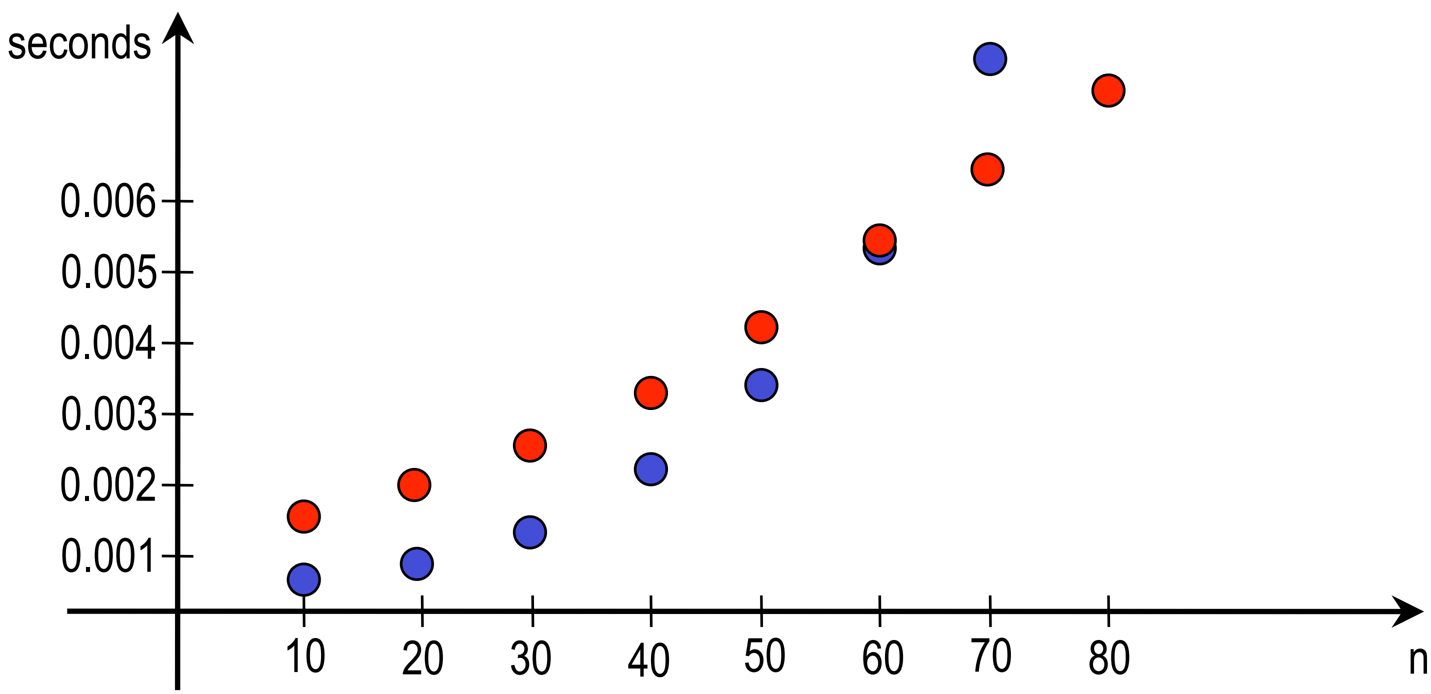
- Cardinality through bijections
- Comparing cardinalities
- Countable sets
- Integers and rationals

Complexity of Algorithms

- How to measure what the efficiency of an algorithm is?
- Sorting algorithms: given a sequence of numbers, arrange it in the increasing order.
- Longer sequences require more time.
- The (time) complexity of a sorting algorithm is a function f such that processing a sequence of length n requires $f(n)$ seconds.
- Not good:
  - computers are different, so, $f(n)$ is ill-defined
  - different sequences of the same length may require different time
- The (worst case) (time) complexity of a sorting algorithm is a function f such that processing a sequence of length n requires at most $f(n)$ steps.

Comparing Algorithms

- There are more than 20 different sorting algorithms. Which one is the best?
- Consider two of them: bubble sort and merge sort. We use the same computer, so we can measure in seconds, rather than in steps.



Comparing Functions

- Problems

  - the function f can be smaller than the function g on small values of n, but then it grows rapidly;

  - we may be comparing functions that count `steps' of different length, for example, every step counted by function f may be equal to two steps counted by function g.

- Solutions

  - start comparing functions not from $n = 1$, but from some sufficiently large n;

  - allow arbitrary constant factor for a function, for instance, do not distinguish $f(n) = n$ and $g(n) = 5n$.

Big-Oh

- Let f and g be functions from N to R. The function g dominates f (or f is dominated by g) if the exist numbers $m \in R$ and $k \in N$ such that for all $n > k$ we have $|f(n)| < m\,|g(n)|$.
- f is of order at most g
- $f \in O(g)$
- Thus, $O(g)$ is the class of all functions with domain N and codomain R that are of order at most g.
- Example.

  Let $f(n) = 5n$ and $g(n) = n^2$. Then $f \in O(g)$.

  Take $k = 5$ and $m = 1$.

  For any $n > k = 5$, we have $n^2 > 5n$.

Big-Oh (cntd)

- Example (cntd)

  Show that $g \notin O(f)$.

- First, we write the definition of Big-Oh in symbolic form:

  $$\exists k\ \exists m\ \forall n\ ((n > k) \to (|f(n)| < m|g(n)|))$$

- The negation of this is

  $$\forall k\ \forall m\ \exists n\ \neg((n > k) \to (|f(n)| < m\,|g(n)|))$$

  $$\Leftrightarrow\ \forall k\ \forall m\ \exists n\ ((n > k) \wedge (|f(n)| \geq m\,|g(n)|))$$

- In our case g plays the role of f, and f plays the role of g:

  $$\forall k\ \forall m\ \exists n\ ((n > k) \wedge (|g(n)| \geq m\,|f(n)|))$$

- Proving: take $n = \max\{ k, 5m \}$. Then

  $$n^2 = n \cdot n \geq \max\{ k, 5m\} \cdot n \geq 5m \cdot n \geq m \cdot g(n)$$

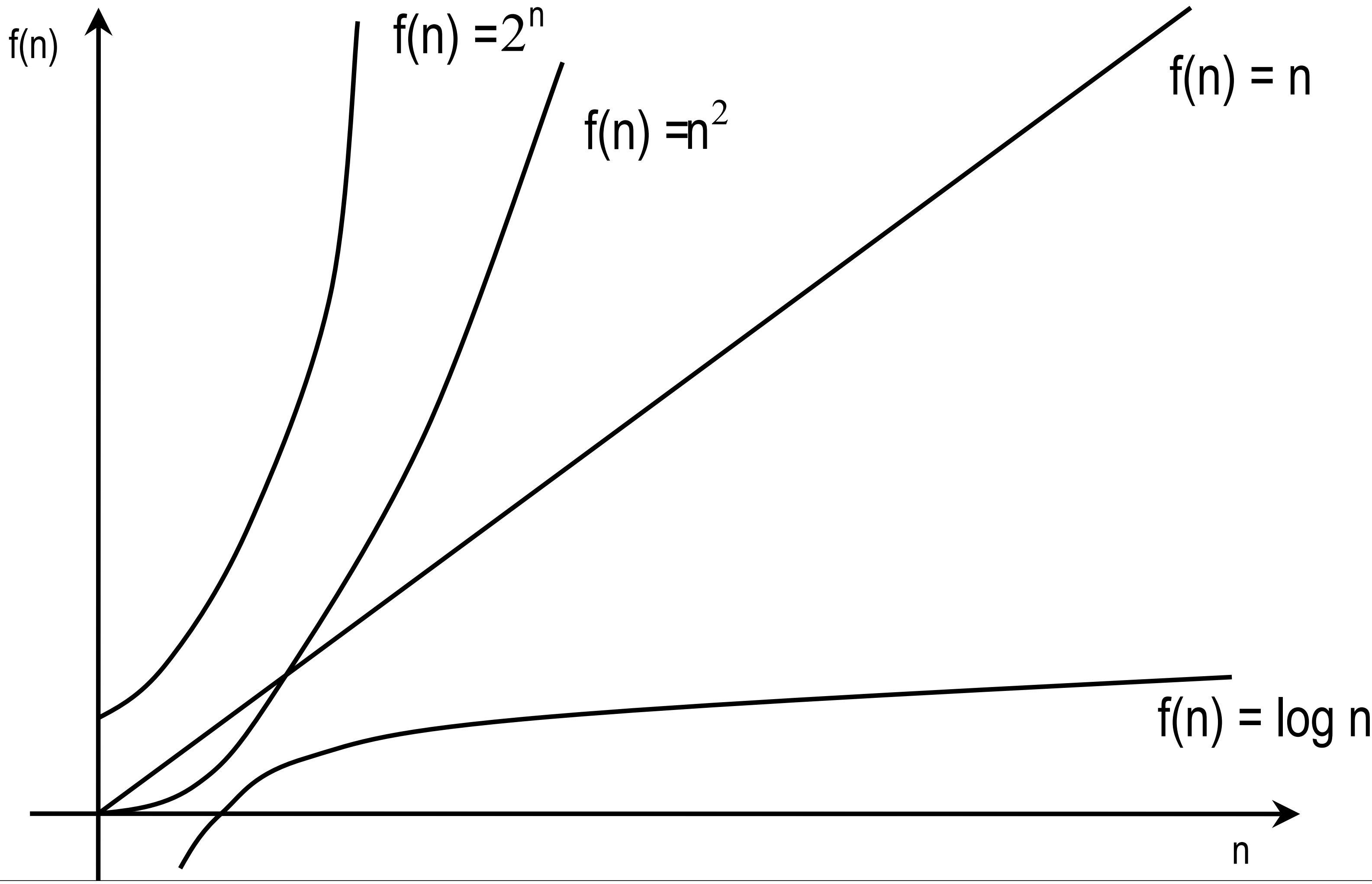
Big-Oh (cntd)

- More Big-Ohs
  - $5n^3 - 4n^2 + 17n - 110 \in O(n^3)$
  - $\log(n^k) \in O(\log(n))$
  - $\log_a n \in O(\log_b n)$

- If functions f and g are such that $f \in O(g)$ and $g \in O(f)$, then we say that f and g are of the same order, and denote it by $f \in \Theta(g)$
- All the pairs above are actually functions of the same order

Little-oh

- Some functions grow much faster than others



Little-oh (cntd)

- Let f and g be functions from **N** to **R**. The function f is said to be negligible for g if for any numbers $m \in \mathbf{R}$ there is a number $k \in \mathbf{N}$ such that for all $n > k$ we have $|f(n)| < m\,|g(n)|$.
- We write $f \in o(g)$
- In the case of domination m is usually big: we can find sufficiently big m so that the required inequality holds

  In the case of little-oh m is meant to be small: we cannot find sufficiently small m so that g becomes smaller than f.
- Example $5n \in o(n^2)$

  Take any m. Choose any $k > \frac{5}{m}$. For example, $k = \lfloor \frac{5}{m} \rfloor + 1$

  Then for any $n > k$ we have

$$5n = m \times \left(\frac{5}{m}\right) \times n < m \times n^2$$

More Little-ohs

- Little-oh pairs of some well known functions:
  - $n^{k-1} \in o(n^k)$
  - $\log^{k-1}(n) \in o(\log^k(n))$
  - $\log^k(n) \in o(n^m), \qquad m > 0$
  - $n^k \in o(a^n), \qquad a > 1$

Homework

Exercises from the Book:

No. 1ade, 3ac, 5, 10ab (page 293)