Bijection and Cardinality

Discrete Mathematics
Evgeny Skvortsov

Previous Lecture

- Functions
- Describing functions
- Injective functions
- Surjective functions

Injective and Surjective

- A function f is said to be one-to-one, or injective, if and only if $f(a) = f(b)$ implies $a = b$.

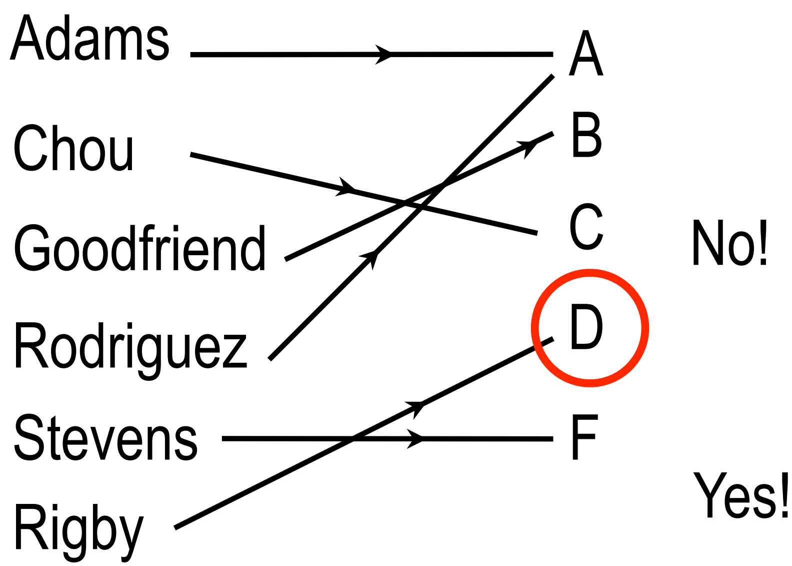
  In other words no two elements are mapped into the same image.

  Contrapositive: if $a \neq b$ then $f(a) \neq f(b)$.

- A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is called a surjection if it is onto.

Onto Functions (cntd)

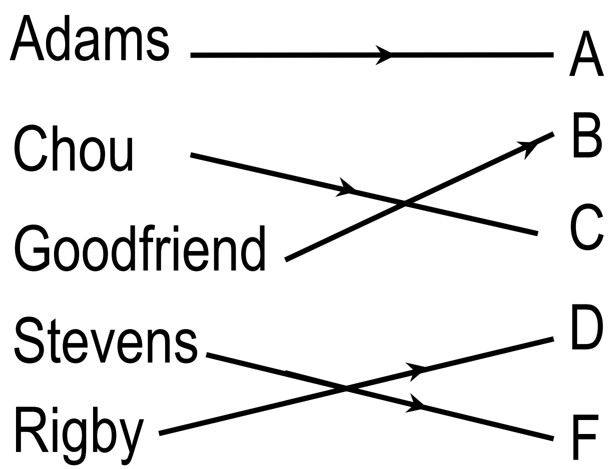
- More examples



- f(a) = b if b is the father of a

Bijections

- A function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.

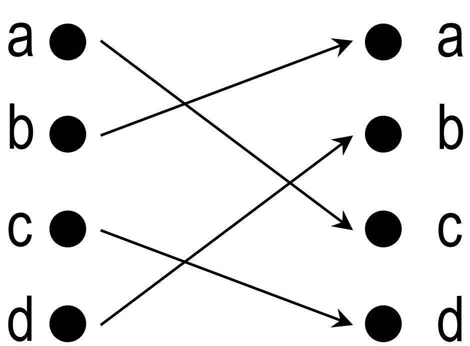


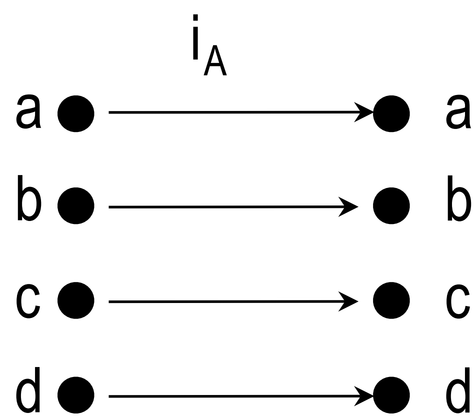
- If there is a bijection from a set A to a set B, then these sets in a certain sense are equal or identical.

Bijections (cntd)

- Numerical functions:
  - $f(x) = x + 1$ is a bijection on Z, Q, R, but not on N
  - $f(x) = x^2$ is a bijection on $\mathbf{R}^+$ but is not on any other numerical set

- A bijection from a set A to the same set A is called a permutation of A



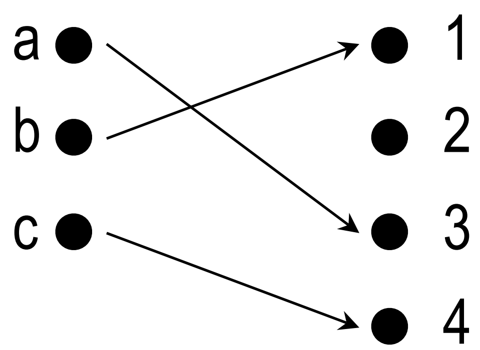


- The identity function on a set A is the function $i_A: A \to A$, where

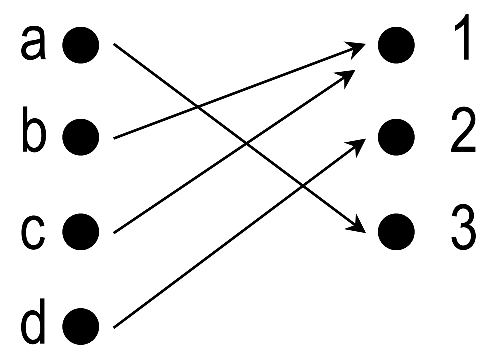
$$i_A(x) = x$$

Functions and Properties

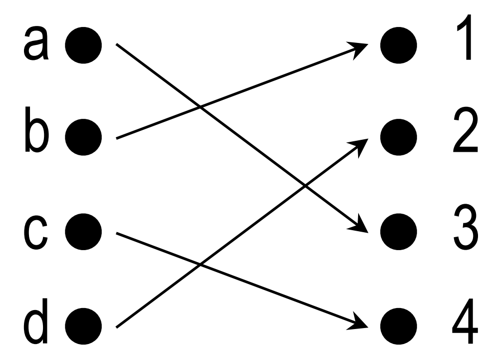
- Examples of different types of correspondences



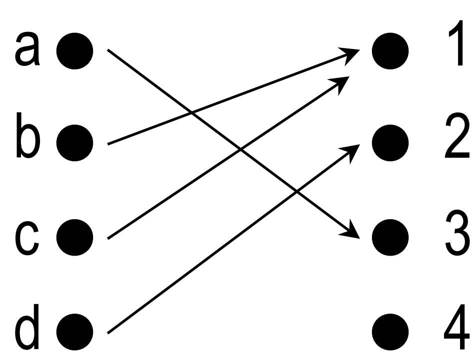
one-to-one, not onto



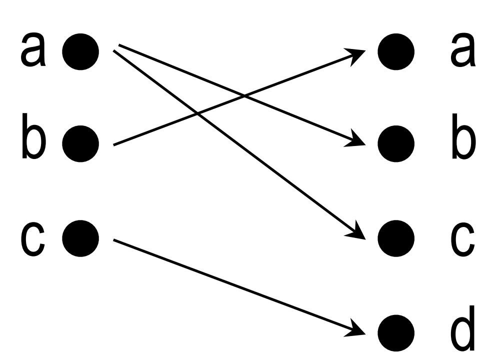
onto, not one-to-one



one-to-one and onto



neither one-to-one nor onto

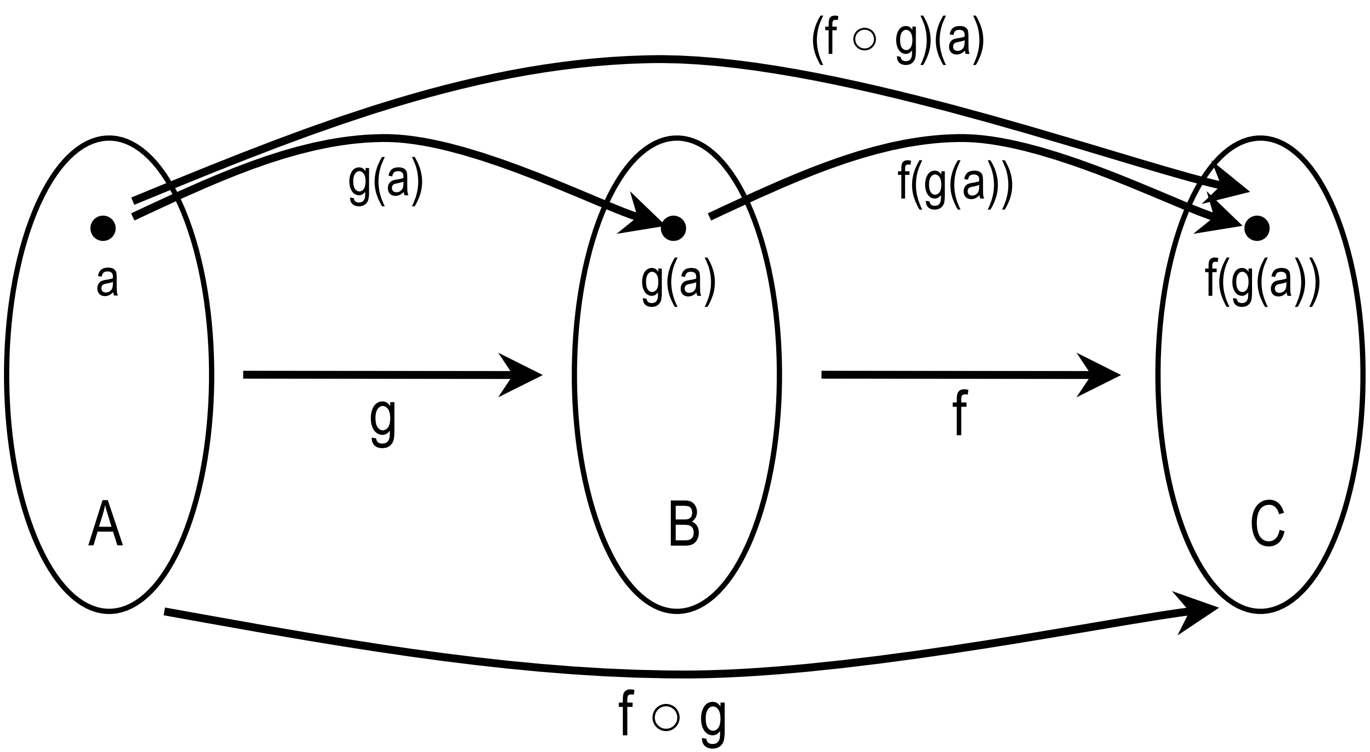


not a function

Composition of Functions

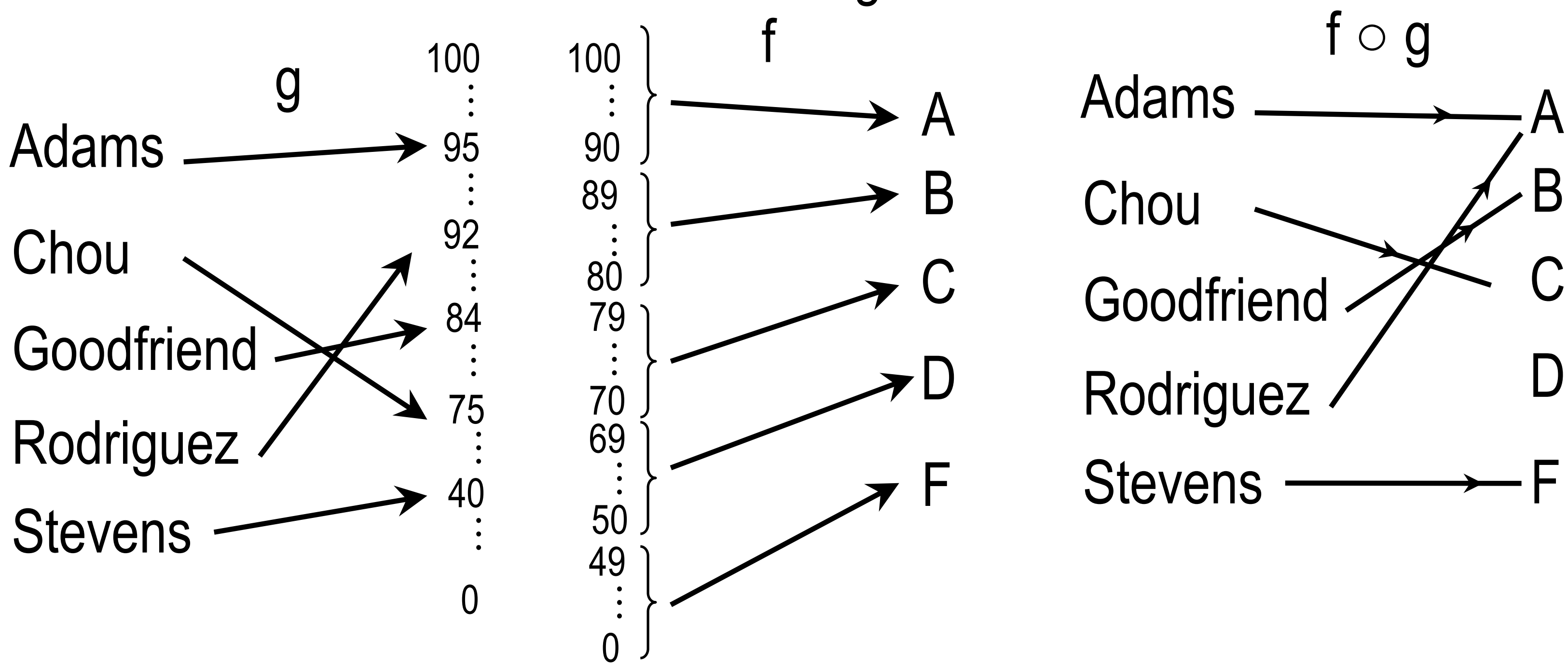
- Let g be a function from A to B and let f be a function from B to C. The composition of the functions f and g, denoted by $f \circ g$, is the function from A to C defined by

$$(f \circ g)(a) = f(\, g(\, a\,))$$



Composition of Functions (cntd)

- Suppose that the students first get numerical grades from 0 to 100 that are later converted into letter grade.



- Let $f(a) = b$ mean `b is the father of a'.
  What is $f \circ f$?

Composition of Numerical Functions

- Let $g(x) = x^2$ and $f(x) = x + 1$. Then
  $(f \circ g)(x) = f(g(x)) = g(x) + 1 = x^2 + 1$

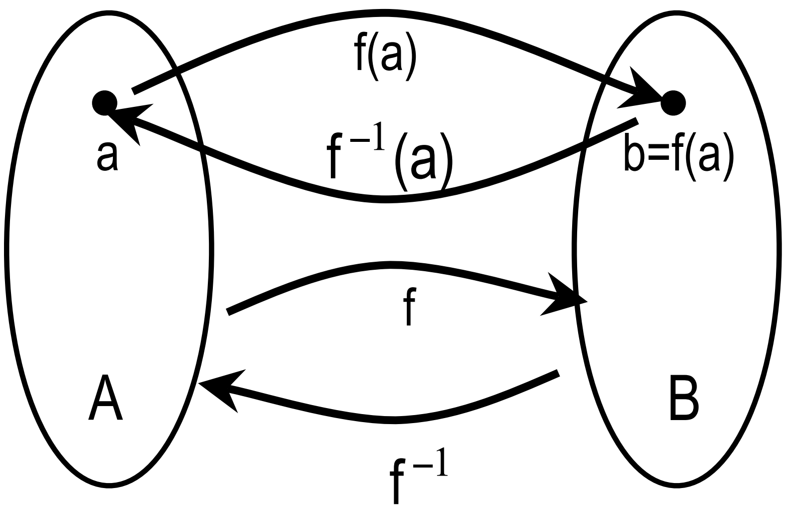
- Thus, to find the composition of numerical functions f and g given by formulas we have to substitute $g(x)$ instead of x in $f(x)$.

Inverse Functions

- Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.

  The inverse function is denoted by f^{-1}.

  Thus, $f^{-1}(b) = a$ if and only if $f(a) = b$



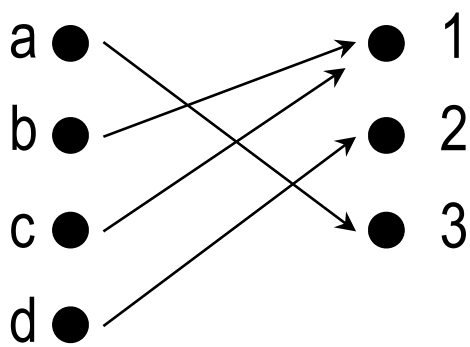
Note!
f^{-1} does not mean $\dfrac{1}{f(x)}$

$f \circ f^{-1} = i_B$

$f^{-1} \circ f = i_A$

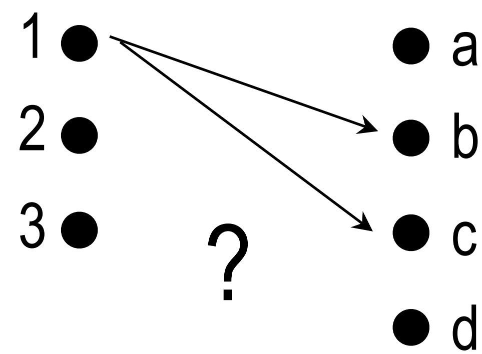
Inverse Functions (cntd)

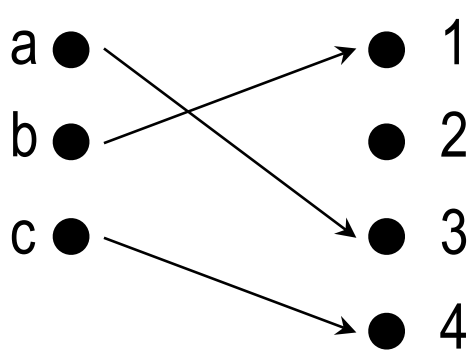
- If a function f is not a bijection, the inverse function does not exist. Why?
- If f is not a bijection, it is either not one-to-one, or not onto



onto, not one-to-one

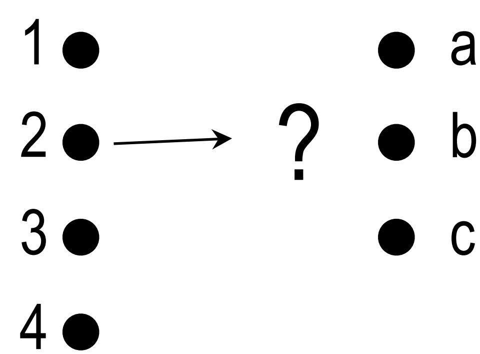
$f^{-1}(1) = ?$





one-to-one, not onto

$f^{-1}(2) = ?$



How to Count Elements in a Set

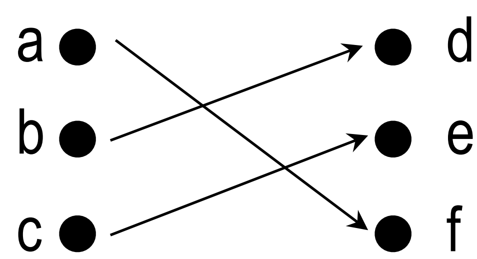
- How many elements are in a set?
- Easy for finite sets, just count the elements.
- What about infinite sets? Does it make sense at all to ask about the number of elements in an infinite set?
- Can we say that this infinite set is larger than that infinite set?
- Which set is larger: the set of all integers or the set of even integers?

  the set of all integers or the set of all rationals?

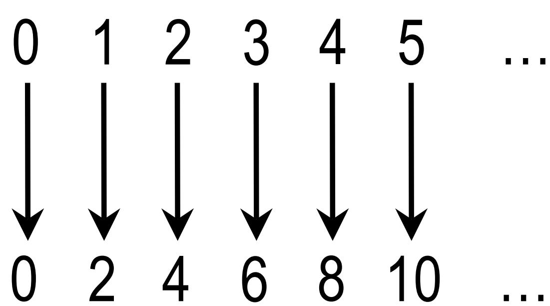
  the set of all integers or the set of all reals?

Cardinality and Bijections

- If A and B are finite sets, it is not hard to see that they have the same cardinality if and only if there is a bijection from A to B
- For example, $|\{a,b,c\}| = |\{d,e,f\}|$



- Sets A and B (finite or infinite) have the same cardinality if and only if there is a bijection from A to B.
- $|N| = |2N|$



The function $f: N \rightarrow 2N$, where
$f(x) = 2x$,
is a bijection

Comparing Cardinalities

- Let A and B be sets. We say that $|A| \leq |B|$ if there is an injective function from A to B.

0 2 4 6 8 10 …

0 1 2 3 4 5 6 7 8 9 10 11 …

The function $f(x) = x$ is an injective function from 2N to N. Therefore $|2N| \leq |N|$

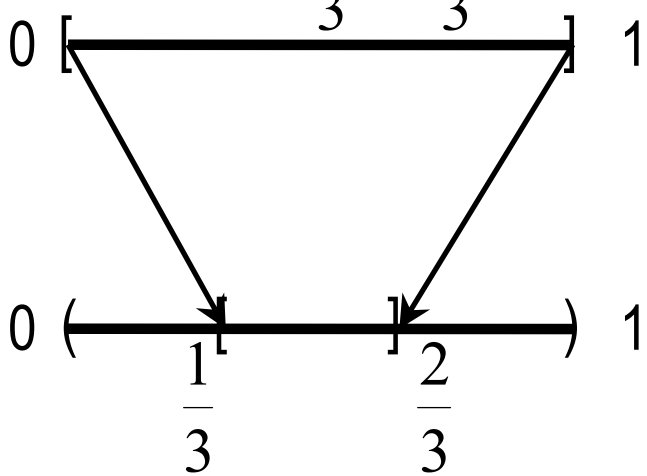
- If there is an injective function from A to B, but not from B to A, we say that $|A| < |B|$
- If there is an injective function from A to B and an injective function from B to A, then we say that A and B have the same cardinality
- Exercise: Prove that a bijection from A to B exists if and only if there are injective functions from A to B and from B to A.

Example

- Let A be the closed interval [0;1] (it includes the endpoints) and B – the open interval (0;1) (it does not include the endpoints)
- There are injective functions f and g from A to B and B to A, respectively.

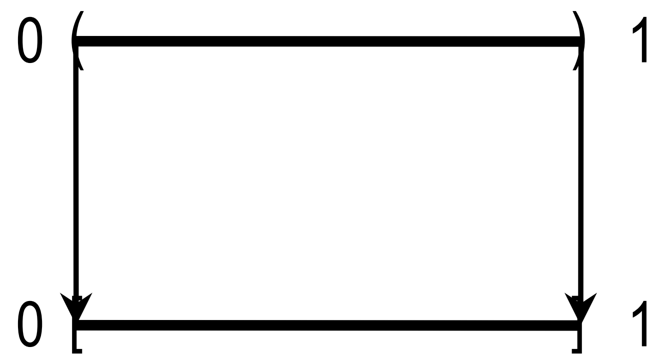
$f: A \to B$

$f(x) = \frac{1}{3}x + \frac{1}{3}$



$f: B \to A$

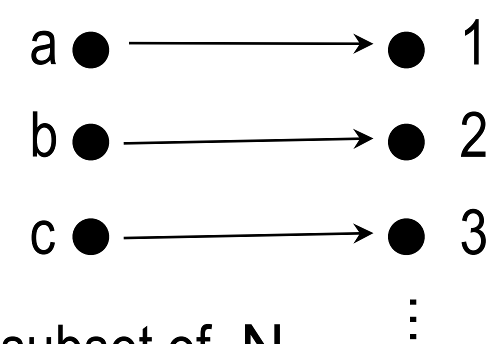
$f(x) = x$



Countable and Uncountable

- A set A is said to be countable if $|A| \le |\mathbf{N}|$
- This is because an injective function from A to $\mathbf{N}$ can be viewed as assigning numbers to the elements of A, thus counting them
- Sets that are not countable are called uncountable
- Countable sets:

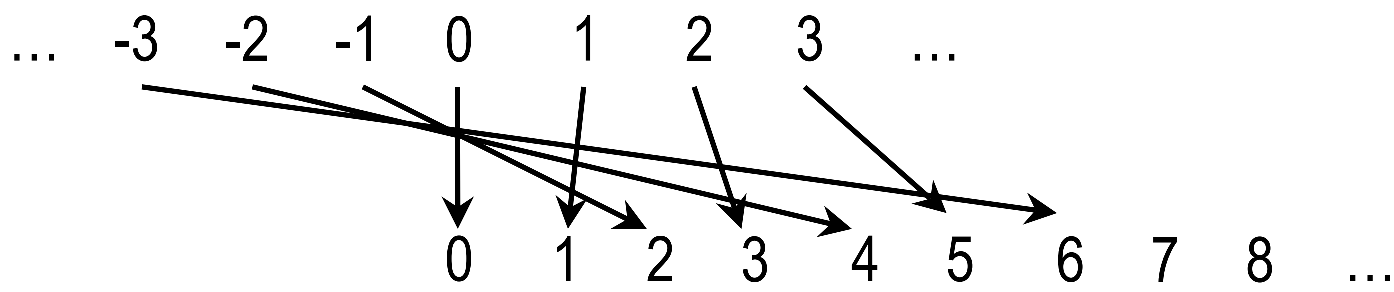
  finite sets



  any subset of $\mathbf{N}$

More Countable Sets

- The set of all integers is countable



- In other words we can make a list of all integers

  0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, …

- The cardinality of the set of all natural numbers is denoted by $\aleph_0$

More Countable Sets (cntd)

- The set of positive rational numbers is countable
- Every rational number can be represented as a fraction p/q
  We do not insist that p and q do not have a common divisor

$$\begin{array}{ccccc}
1/1 & 2/1 & 3/1 & 4/1 & \cdots \\
1/2 & 2/2 & 3/2 & 4/2 & \cdots \\
1/3 & 2/3 & 3/3 & 4/3 & \cdots \\
1/4 & 2/4 & 3/4 & 4/4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

- This gives an injection from $\mathbb{Q}^+$ to $\mathbf{N}$. The converse injection is
  f(x) = x + 1

The Smallest Infinite Set

- Theorem.
  If A is an infinite set, then $|A| \geq \aleph_0$

- Proof requires mathematical induction. Wait for a few days.

Uncountable Sets

- Can we make a list of all real numbers?
- Every real number can be represented as an infinite decimal fraction, like 3.1415926535897932384626433832795028841971…
- Suppose we have constructed a list of all real numbers

$$
\begin{array}{ll}
1. & a_{10}.a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}a_{17}\ldots \\
2. & a_{20}.a_{21}a_{22}a_{23}a_{24}a_{25}a_{26}a_{27}\ldots \\
3. & a_{30}.a_{31}a_{32}a_{33}a_{34}a_{35}a_{36}a_{37}\ldots \\
4. & a_{40}.a_{41}a_{42}a_{43}a_{44}a_{45}a_{46}a_{47}\ldots \\
5. & a_{50}.a_{51}a_{52}a_{53}a_{54}a_{55}a_{56}a_{57}\ldots \\
\vdots & \vdots
\end{array}
$$

Here the a_{ij} are digits 0,1,2,…,9

Let

$$b_i = \begin{cases} 4, & \text{if } a_{ii} \neq 4, \\ 5 & \text{otherwise} \end{cases}$$

- It is not hard to see that the number $0.b_1b_2b_3b_4b_5b_6b_7\ldots$ is not in this list

Cantor's Theorem

- Theorem (Cantor). For any set $|P(A)| > |A|$.

  Proof.

  Suppose that there is a bijection $f: A \to P(A)$.

  We find a set that does not belong to the range of f. A contradiction with the assumption that f is bijective.

  Consider the set $T = \{ a \in A \mid a \notin f(a) \}$

  If T is in the range of f, then there is $t \in A$ such that $f(t) = T$.

  Either $t \in T$ or $t \notin T$.

  If $t \in T$ then $t \in f(t)$, and we get $t \notin T$.

  If $t \notin T$ then $t \in T$.

  Q.E.D.

Paradox... again?

- **Corollary: Collection of all possible sets is not a set.**
- If it was a set X, then we would have that
- $|X| \geq |P(X)|$, because $X \supseteq P(X)$ and
- $|X| < |P(X)|$.
- This is another paradox of the naive set theory, like Russell's paradox. :-(

Cantor's Theorem (cntd)

- This method is called Cantor's diagonalization method
- The cardinality of P(A) is denoted by $2^{|A|}$
- Thus, we obtain an infinite series of infinite cardinals

$$|N| = \aleph_0$$

$$2^{\aleph_0} = \aleph_1 \quad (= |R|) \qquad \text{<-- Exercise}$$

$$2^{\aleph_1} = \aleph_2$$

$$\vdots$$

Continuum Hypothesis

- We just proved that $\aleph_0 < |R|$. Does there exist a set A such that $\aleph_0 < |A| < |R|$?

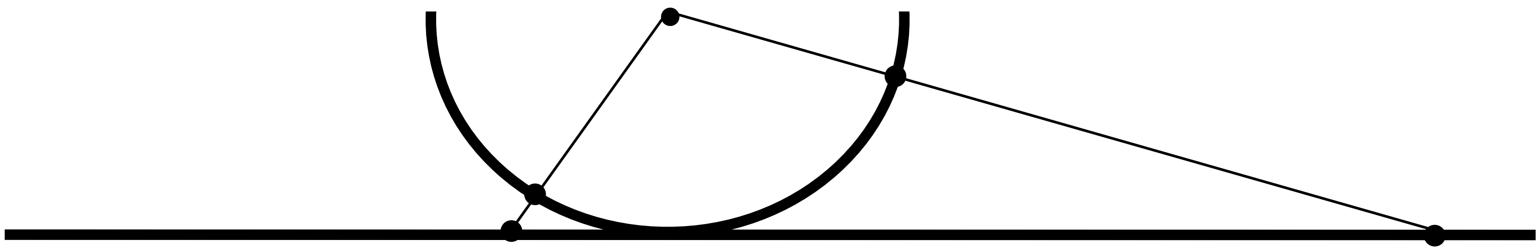
- The negative answer to this question is known as the continuum hypothesis.

- Continuum hypothesis is the first problem in the list of Hilbert's problems

- Paul Cohen resolved the question in 1963. The answer is shocking: You can think either way.

More Uncountable Sets

- For any real numbers a, b, the open interval (a;b) has the same cardinality as R



Homework

Exercises from the Book:
No. 1def, 2b, 4 (page A-32)

- Construct a bijective mapping between the closed interval [0;1] and the square $[0;1] \times [0;1]$