Mathematical Induction

Discrete Mathematics
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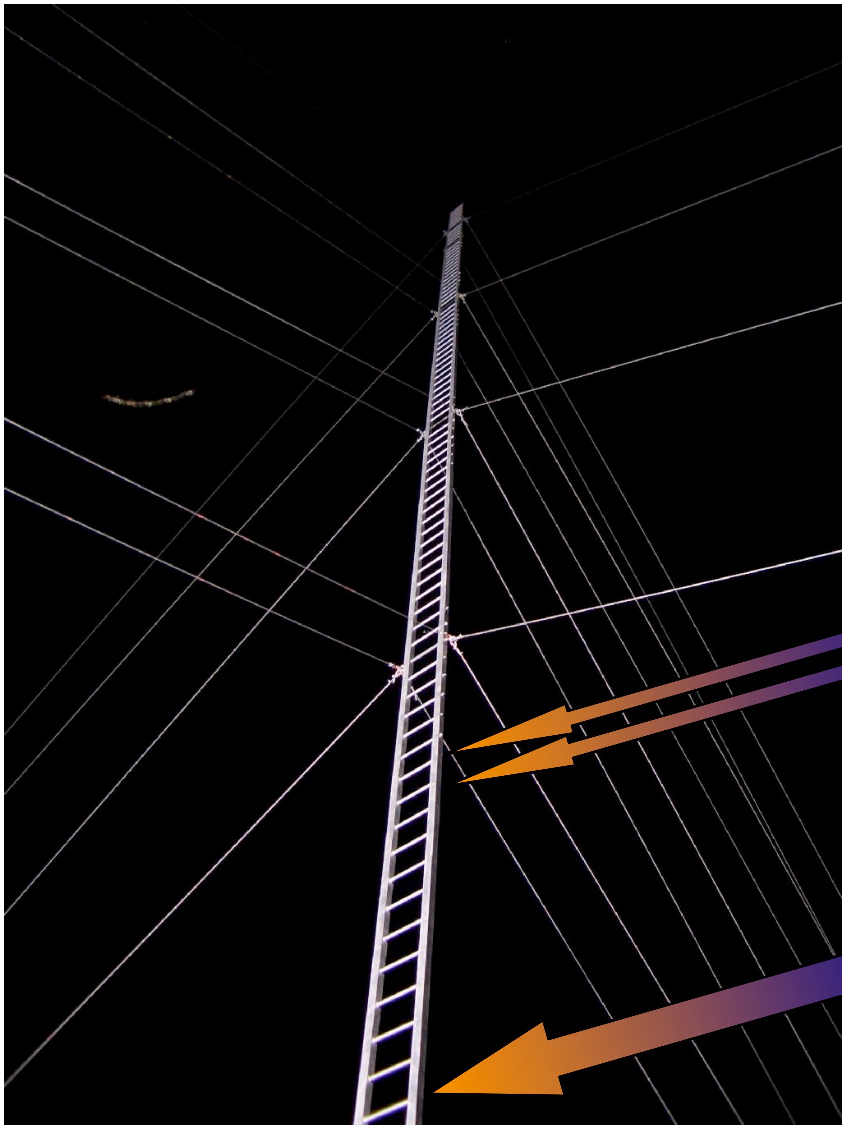
Principle of Mathematical Induction



Climbing an infinite ladder

Can we reach every step of it, if

For all k, standing on the rung k we can step on the rung $k + 1$

We can reach the first rung

Principle of Mathematical Induction

- Principle of mathematical induction:

  To prove that a statement that assert that some property $P(n)$ is true for all positive integers n, we complete two steps

  Basis step: We verify that $P(1)$ is true.

  Inductive step: We show that the conditional statement

  $$P(k) \to P(k+1) \text{ is true for all positive integers } k$$

- Symbolically, the statement

  $$(P(1) \land \forall k\,(P(k) \to P(k+1))) \to \forall n\, P(n)$$

- How do we do this?

  $P(1)$ is usually an easy property

  To prove the conditional statement, we assume that $P(k)$ is true (it is called inductive hypothesis) and show that under this assumption $P(k+1)$ is also true

Summation

- Prove that the sum of the first n natural numbers equals $\frac{n(n+1)}{2}$ that is
$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

- P(n): `the sum of the first n natural numbers …
- Basis step: P(1) means $1=\frac{1(1+1)}{2}$

- Inductive step: Make the inductive hypothesis, P(k) is true, i.e.
$$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$
Prove P(k + 1):
$$1+2+3+\cdots+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}$$

$$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$
$$=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

More Summation

- Prove that $1+2+2^2+\cdots+2^n = 2^{n+1}-1$
- Let P(n) be the statement `$1+2+2^2+\cdots+2^n = 2^{n+1}-1$' for the integer n

- Basis step: P(0) is true, as $2^0 = 1 = 2^{0+1}-1$
- Inductive step: We assume the inductive hypothesis

$$1+2+2^2+\cdots+2^k = 2^{k+1}-1$$

and prove P(k + 1), that is

$$1+2+2^2+\cdots+2^k+2^{k+1} = 2^{(k+1)+1}-1 = 2^{k+2}-1$$

We have

$$\begin{aligned}1+2+2^2+\cdots+2^k+2^{k+1} &= (2^{k+1}-1)+2^{k+1}\\ &= 2\times 2^{k+1}-1\\ &= 2^{k+2}-1\end{aligned}$$

The Cardinality of the Power Set

- Let's use induction to prove that $|P(A)| = 2^{|A|}$
- Let Q(n) denote the statement `an n-element set has 2^n subsets'
- Basis step: Q(0), and empty set has only one subset, empty
- Inductive step. We make the inductive hypothesis, a k-element set A has 2^k subsets

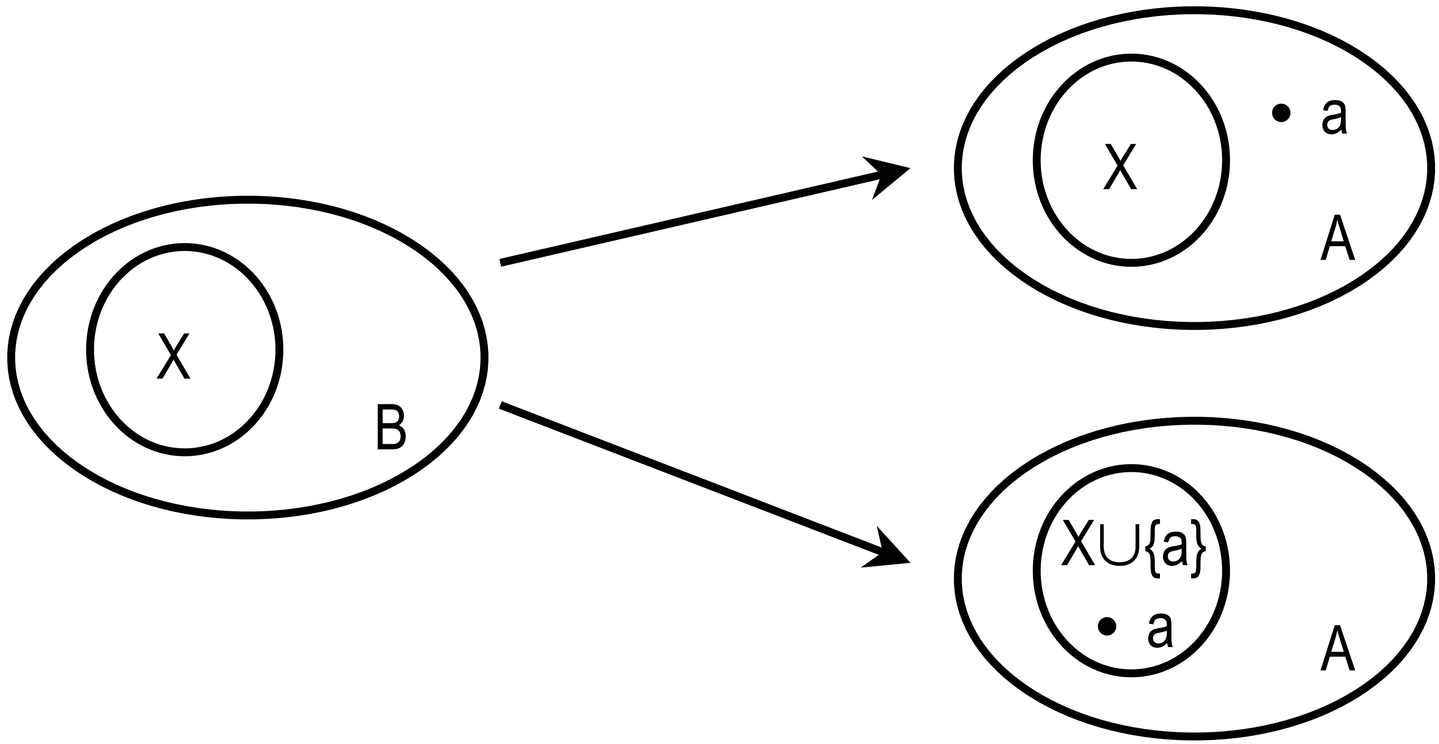
  We have to prove Q(k + 1), that is if a set A contains k + 1 elements, then $|P(A)| = 2^{k+1}$

  Fix an element $a \in A$, and set $B = A - \{a\}$.

  The set B contains k elements, hence $|P(B)| = 2^k$

  Every subset X of B corresponds to two subsets of A

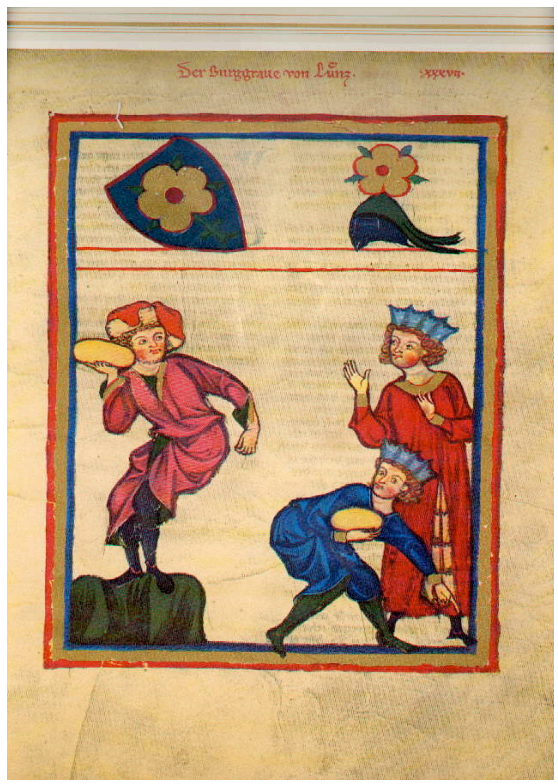
The Cardinality of the Power Set (cntd)



Therefore, $|P(A)| = 2 \cdot |P(B)| = 2 \times 2^k = 2^{k+1}$

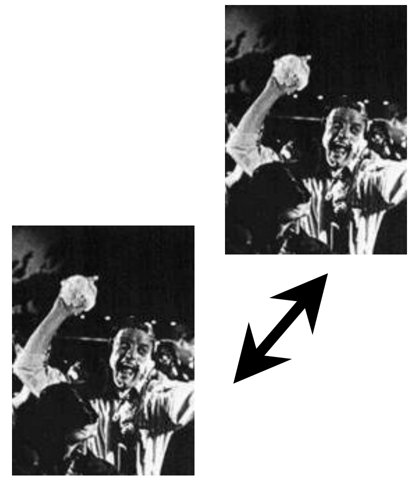
Odd Pie Fights

- An odd number of people stand in a yard at mutually distinct distances. At the same time each person throws a pie at their nearest neighbor, hitting this person. Show that there is at least one survivor, that is, at least one person who is not hit by a pie.



Odd Pie Fights (cntd)

- Let P(n) denote the statement `there is a survivor in the odd pie fight with 2n + 1 people'
- Basis step: P(1), there are 3 people



Of the three people, suppose that the closest pair is A and B, and C is the third person. Since distances between people are different, the distances between A and C, and B and C are greater than that between A and B.

Therefore, A and B throws pies at each other, and C survives.

Odd Pie Fights (cntd)

- Inductive step: Suppose that $P(k)$ is true, that is, in the pie fight with $2k + 1$ people there is a survivor.
- Consider the fight with $2(k + 1) + 1$ people.

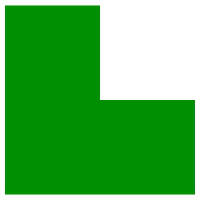
  Let A and B be the closest pair of people in this group of $2k + 3$ people. Then they throw pies at each other.

  If someone else throws a pie at one of them, then for the remaining $2k + 1$ people there are only $2k$ pies, and one of them survives.

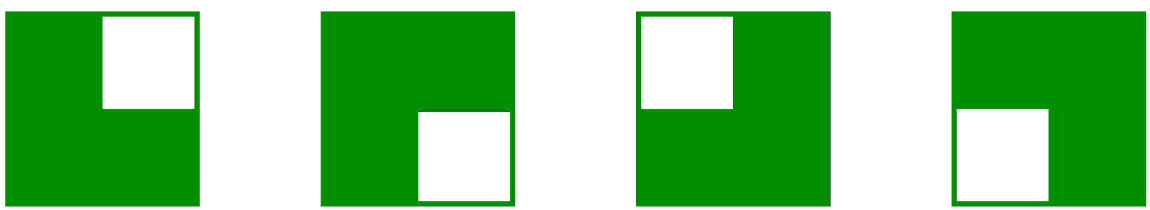
  Otherwise the remaining $2k + 1$ people throw pies at each other, playing the pie fight with $2k + 1$ people. By the inductive hypothesis, there is a survivor in such a fight.

Triomino

- Let n be a positive integer. Show that every $2^n \times 2^n$ checkerboard with one square removed can be tiled using triominos



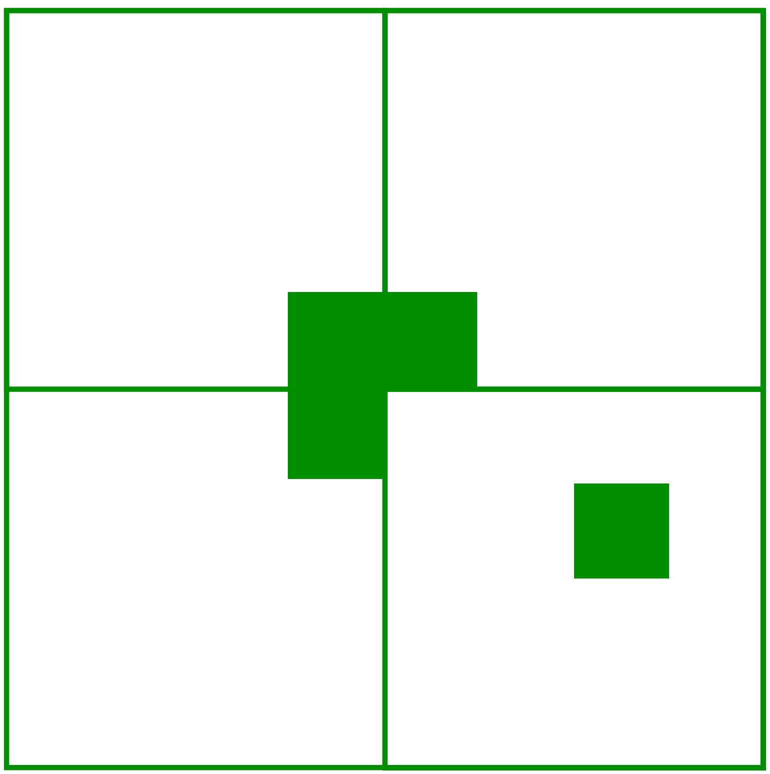
- P(n) denotes the statement above
- Basis step: P(1) is true, as 2×2 checkerboards with one square removed have one of the following shapes



Triomino (cntd)

- Inductive step: Suppose that P(k) is true that is every $2^k \times 2^k$ checkerboard with one square removed can be tiled with triominos.

  We have to prove P(k + 1), that is, every $2^{k+1} \times 2^{k+1}$ checkerboard without one square can be tiled.



Split the big checkerboard into 4 half-size checkerboards

Put one triomino as shown in the picture.

We have 4 $2^k \times 2^k$ checkerboards, each without one square. By the induction hypothesis, they can be tiled.

Analysis of Algorithms

- Consider the following problem

  There is a group of proposed talks to be given. We want to schedule as many talks as possible in the main lecture room. Let $t_1, t_2, \ldots, t_m$ be the talks, talk t_j begins at time b_j and ends at time e_j. (No two lectures can proceed at the same time, but a lecture can begin at the same time another one ends.) We assume that $e_1 \le e_2 \le \ldots \le e_m$.
- Greedy algorithm:

  At every step choose a talk with the earliest ending time among all those talks that begin after all talks already scheduled end.
- We prove that the greedy algorithm is optimal in the sense that it always schedules the most talks possible in the main lecture hall.

Greedy Algorithm

- Let P(n) be the proposition that if the greedy algorithm schedules n talks, then it is not possible to schedule more than n talks.
- Basis step. Suppose that the greedy algorithm has scheduled only one talk, t_1. This means that every other talk starts before e_1, and ends after e_1. Hence, at time e_1 each of the remaining talks needs to use the lecture hall. No two talks can be scheduled because of that. This proves P(1).
- Inductive step. Suppose that P(k) is true, that is, if the greedy algorithm schedules k talks, it is not possible to schedule more than k talks.

  We prove P(k + 1), that is, if the algorithm schedules k + 1 talks then this is the optimal number.

Greedy Algorithm (cntd)

- Suppose that the algorithm has selected k + 1 talks.

  First, we show that there is an optimal scheduling that contains t_1

  Indeed, if we have a schedule that begins with the talk t_i, i > 1, then this first talk can be replaced with t_1.

  To see this, note that, since $e_1 \leq e_i$, all talks scheduled after t_1 still can be scheduled.

  Once we included t_1, scheduling the talks so that as many as possible talks are scheduled is reduced to scheduling as many talks as possible that begin at or after time e_1.

  The greedy algorithm always schedules t_1, and then schedules k talks choosing them from those that start at or after e_1.

  By the induction hypothesis, it is not possible to schedule more than k such talks. Therefore, the optimal number of talks is k + 1.

Principle of Strong Induction

- Sometimes mathematical induction is not enough. We can use the principle of strong induction.
- To prove that $P(n)$ is true for all positive integers n, we complete two steps:
- Basis step: Verify that $P(1)$ is true.
- Inductive step: Show that the statement

  $$[\, P(1) \wedge P(2) \wedge \ldots \wedge P(k) \,] \rightarrow P(k+1)$$

  is true for all positive integers k.

Game with Matches

- Two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game.

- Show that if the two piles contain the same number of matches initially, then the second player can always guarantee a win.

Strategy for the Second Player

- Let $P(n)$ denote the statement `the second player wins when there are initially n matches in each pile'.
- Basis step: $P(1)$ is true, because in this case there is only one match in each pile, and the first player has only one choice, removing one match from one pile. Then the second player removes the match from the other pile and wins.
- Inductive step: Suppose that $P(j)$ is true for all j with $1 \le j \le k$.

  We prove that $P(k + 1)$ is true, that is, that the second player wins when each pile contain $k + 1$ matches.

  Suppose that the first player removes r matches from one pile leaving $k + 1 – r$ matches there.

  By removing the same number of matches from the other pile the second player creates the situation of two piles with $k + 1 – r$ matches in each. Apply the inductive hypothesis.

Why Induction Works? Well Ordering

- One of the axioms of positive integers is the principle of well-ordering:

  Every non-empty subset of $\mathbf{N}$ contains the least element.

- Note that the sets of all integers, rational numbers, and real numbers do not have this property.

- Suppose that mathematical induction is not valid.

  Then there is a predicate $P(n)$ such that $P(1)$ is true,

  $\forall k\,(P(k) \to P(k+1))$ is true, but there is n such that $P(n)$ is false

  Let $T \subseteq \mathbf{N}$ be the set of all n such that $P(n)$ is false.

  By the principle of well-ordering T contains the least element a

  As $P(1)$ is true, $a \neq 1$.

  We have $P(a-1)$ is true. However, since $P(a-1) \to P(a)$, we get a contradiction

Fallacies

- What is wrong in the following proof that every set of lines in the plane, no two of which are parallel, meet in a common point?
- Basis step: P(2) is true by the definition of parallel lines.
- Inductive step: Suppose P(k) is true, that is, every set of k lines meet in a common point.

  We prove P(k + 1).

  Consider a set of k + 1 lines in the plane, no two of which are parallel.

  By the induction hypothesis the first k of them meet in a point p.

  By the induction hypothesis the last k of them meet in a point q.

  If p and q were different points, then all the lines that contain both of them would be equal. A contradiction.

  Therefore, p = q and all the lines meet in this point

Recursively Defined Functions

- Induction mechanism can be used to define things.
- To define a function $f: N \to R$ we complete two steps:

  Basis step: define $f(1)$

  Inductive step: For all k define $f(k + 1)$ as a function of $f(k)$,

  or, more general, as a function of $f(1), f(2), \ldots, f(k)$.
- Give a recursive definition of $f(n) = 2^n$

  Basis step: $f(0) = 1$

  Inductive step: $f(k + 1) = 2 \cdot f(k)$.

Factorial

- Another useful recursively defined function is factorial
- $f(n) = n!$

  Basis step: $0! = 1$

  Inductive step: $(k + 1)! = k! \cdot (k + 1)$

n	n!
0	1
1	1
2	2
3	6
4	24

n	n!
5	120
6	720
7	5040
8	40320
9	362880

Fibonacci Numbers

- Usually, Fibonacci numbers are thought of as a sequence of natural numbers, but as we know such a sequence can also be viewed as a function from N.
- F(n)
- Basis step: F(1) = F(2) = 1
- Inductive step: F(k + 1) = F(k) + F(k – 1)

n	1	2	3	4	5	6	7	8	9	10	11	12	13
F(n)	1	1	2	3	5	8	13	21	34	55	89	144	233

Binet's formula

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$

where φ is the golden ratio

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618033988749$$

Recursively Defined Sets and Structures

- Induction can be used to define structures
- We need to complete the same two steps:

  Basis step: Define the simplest structure possible

  Inductive step: A rule, how to build a bigger structure from smaller ones.

Well Formed Propositional Statements

- What is a well formed statement?

  $(p \rightarrow q) \wedge \neg r$ is well formed

  $(p \rightarrow q) \neg \wedge r$ is not

- Recursive definition of well formed formulas
- Basis step: A primitive statement is a well formed statement
- Inductive step: If Φ and Ψ are well formed statements, then

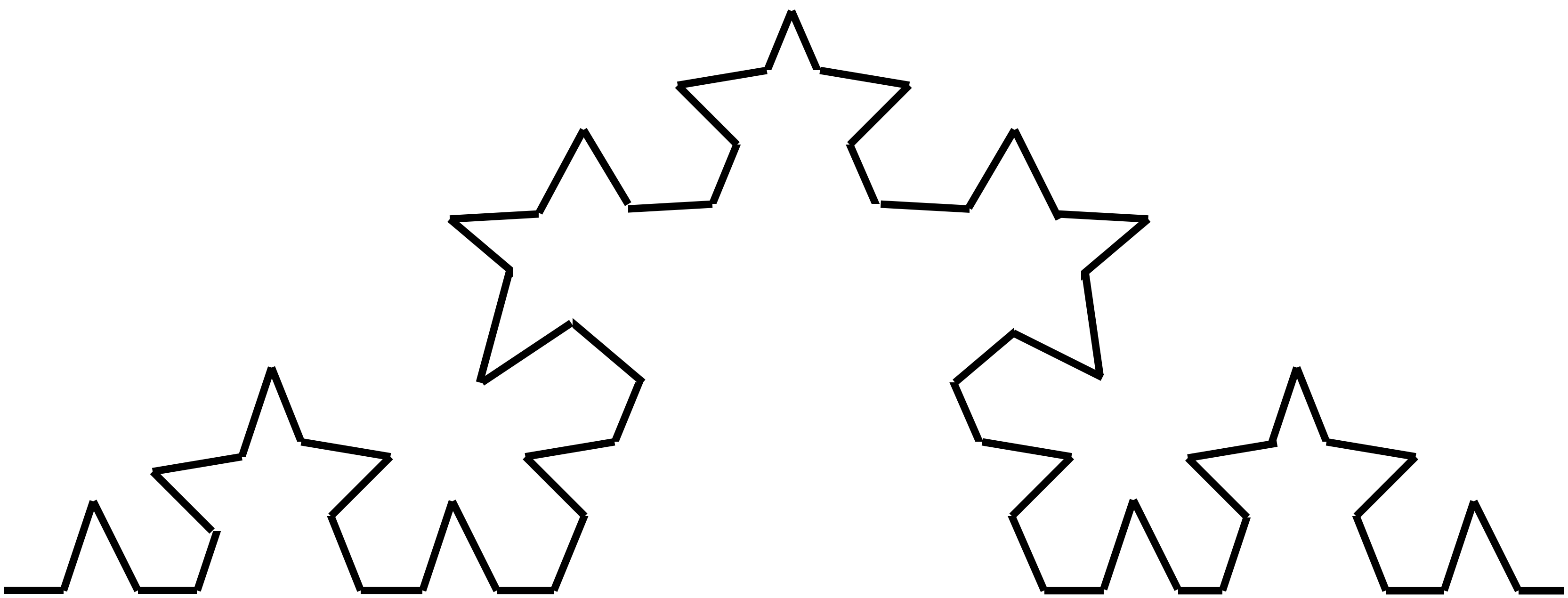
  $\neg \Phi$, $(\Phi \wedge \Psi)$, $(\Phi \vee \Psi)$, $(\Phi \rightarrow \Psi)$, $(\Phi \leftrightarrow \Psi)$, $(\Phi \oplus \Psi)$

  are well formed statements

- Such a definition can be used by various algorithms, for example, parsing

Fractals

- Fractals are curves defined recursively
- Basis step: Fractal of level 0 is just a segment
- Inductive step: Divide every segment of the fractal of level k into 3 equal parts and remove the middle one. Insert in this place two sides of a equilateral triangle



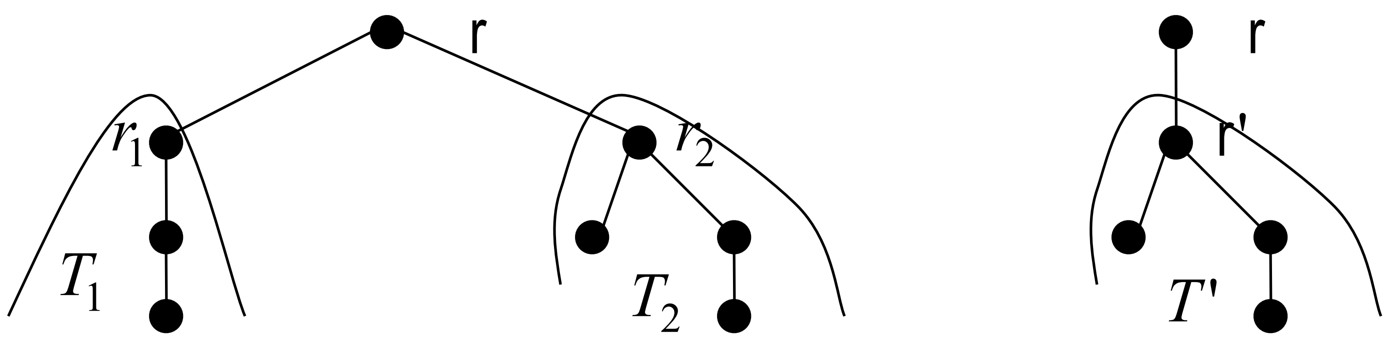
Rooted Trees

- A binary tree is a graph formed by the following recursive definition

- Basis case: A single vertex is a binary tree

- Inductive step: Suppose that T_1, T_2 are disjoint binary trees with roots r_1, r_2, respectively. Then the graph formed by starting with a root r, and adding an edge from r to each of the vertices r_1, r_2, is also a binary tree.

  Or T' is a binary tree with the root r'. Then the graph formed by starting with a root r, and adding an edge from r to r' is also a binary tree



Structural Induction

- To prove properties or design algorithms working with recursively defined structures we need structural induction
- To prove a statement using structural induction we complete two steps

  Basis step: Prove that the property is true for the simplest structure

  Inductive step: Assuming that the property is true for all simpler structures, prove it for a more complex structure

Structural Induction (cntd)

- Height of a binary tree, h(T). Recursive definition:
- Basis step: The height of a single vertex r is 0. h(r) = 0
- Inductive step: If a tree T is built from trees T_1, T_2 as shown in the inductive step, then h(T) = 1 + max(h(T_1), h(T_2))

- We prove that the number of vertices in a binary tree, n(T), satisfies the inequality $n(T) \le 2^{h(T)+1} - 1$
- Basis step: For a single vertex $1 = n(r) \le 2^{0+1} - 1 = 1$
- Inductive step: Let T be formed from T_1, T_2

  We have n(T) = 1 + n(T_1) + n(T_2)

$$\le 1 + (2^{h(T_1)+1} - 1) + (2^{h(T_2)+1} - 1)$$

$$\le 1 + 2(2^{\max(h(T_1),h(T_2))+1} - 1) = 1 + 2^{h(T)+1} - 2$$

$$= 2^{h(T)+1} - 1$$

Homework

Exercises from the Book:
No. 3, 4a, 7b (page 244)