Relations

Discrete Mathematics
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Previous Lecture

- Venn diagrams
- Operations of
  - Intersection
  - Union
  - Symmetric difference
  - Complement
  - Difference
- Connection to logic
- Laws of set theory

Relations

- `**Relation**', the connection between things or people
- Between people, family relations

  `to be brothers' x is a brother of y

  `to be older' x is older than y

  `to be parents' x and y are parents of z

- Between things, numerical relations

  `to be greater than' $x < y$ on the set of real numbers

  `to be divisible by' x is divisible by y on the set of integers

- Between things and people, legal relations

  `to be an owner' x is an owner of y

Cartesian Product

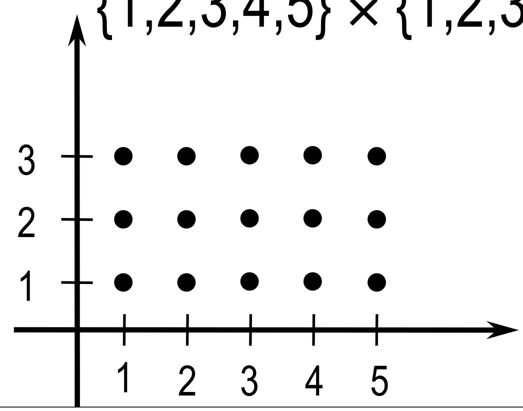
- The Cartesian product of sets A and B, denoted by $A \times B$, is the set of all ordered pairs of elements from A and B.

$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

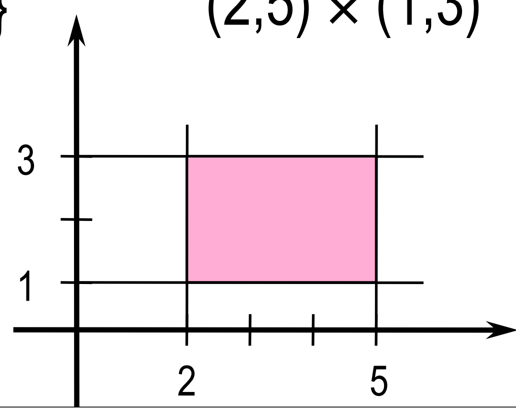
- The elements of the Cartesian product are ordered pairs. In particular, $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$.

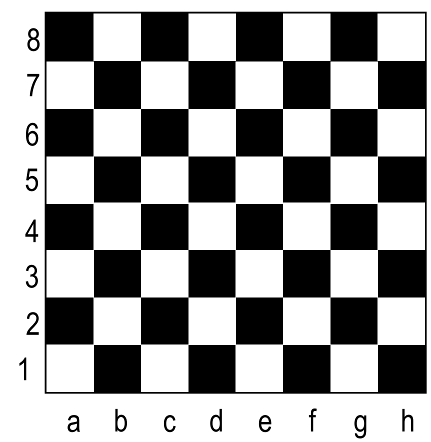
- If sets are thought of as `1-dimensional' objects, then Cartesian products are 2-dimensional

$\{1,2,3,4,5\} \times \{1,2,3\}$



$(2,5) \times (1,3)$





Cartesian Product of More Than Two Sets

- Instead of ordered pairs we may consider ordered triples, or, more general, k-tuples.

  (a,b,c), an ordered triple

  (a,b,c,d), an ordered quadruple

  $(a_1, a_2, \ldots, a_k)$ a k-tuple

- Triples, quadruples, and k-tuples are elements of Cartesian products of 3, 4, and k sets, respectively

  $$A \times B \times C = \{ (a,b,c) \mid a \in A,\ b \in B,\ c \in C\}$$

  $$A \times B \times C \times D = \{ (a,b,c,d) \mid a \in A,\ b \in B,\ c \in C,\ d \in D\}$$

  $$A_1 \times A_2 \times \ldots \times A_k = \{(a_1, a_2, \ldots, a_k) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_k \in A_k\}$$

Cardinality of Cartesian Product

- Theorem.

If A and B are finite then:

$$|A \times B| = |A| \cdot |B|$$

If $A_1, \ldots, A_k$ are finite then

$$|A_1 \times A_2 \times \ldots \times A_k| = |A_1| \times |A_2| \times \ldots \times |A_k|$$

- Proof

When creating an ordered pair (a,b), to every of the $|A|$ elements of A we can add any of the $|B|$ elements of B. Totally, we have $|A| \cdot |B|$ ordered pairs.

Q.E.D.

Cartesian Product, Intersection and Union

- Theorem. For any sets A,B,C

  (1) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

  (2) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

  (3) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

  (4) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

- Proof (of (2))

$$
\begin{aligned}
A \times (B \cup C) &= \{ (a,b) \mid a \in A \ \wedge b \in B \cup C \} \\
&= \{ (a,b) \mid a \in A \ \wedge \ (b \in B \ \vee \ b \in C) \} \\
&= \{ (a,b) \mid (a \in A \ \wedge \ b \in B) \ \vee (a \in A \ \wedge \ b \in C) \} \\
&= \{ (a,b) \mid a \in A \ \wedge \ b \in B\} \cup \{ (a,b) \mid a \in A \ \wedge \ b \in C \} \\
&= (A \times B) \cap (A \times C)
\end{aligned}
$$

Q.E.D.

Binary Relations

- A binary relation from set A to set B is any subset of $A \times B$.
  If $A = B$ then we say that the relation is on the set A

`x is a brother of y'	$\subseteq$	People $\times$ People
`x is older than y'	$\subseteq$	People $\times$ People
`x is an owner of y'	$\subseteq$	People $\times$ Properties
`x < y'	$\subseteq$	$R \times R$
`x divides y'	$\subseteq$	$Z \times Z$

More Relations

- Binary relations can be generalized to subsets of Cartesian products of more than two sets.
- Any subset of the Cartesian product of 3 sets is called a ternary relation

  `x and y are parents of z' is a subset of

  People × People × People
- Any subset of the Cartesian product of k sets is called a k-ary relation

$$\{ (a_1, a_2, \ldots, a_k) \mid a_1 + a_2 + \ldots + a_k = 3 \}$$ is a subset of R × R × … × R

Sets, Relations, and Predicates

- Observe that sets, relations and predicates are essentially the same object.

Unary predicate $P(x)$	Set $A = \{ x \mid P(x) \}$
Binary predicate $P(x,y)$	Binary relation $R = \{ (x,y) \mid P(x,y) \}$
Ternary predicate $P(x,y,z)$	Ternary relation $R = \{ (x,y,z) \mid P(x,y,z) \}$

Relational Databases

- A relational database is a collection of tables like

No.	Name	Student ID	Supervisor	Thesis title
1	Bradley Coleman	30101234	Petra Berenbrink	Algebraic graph theory
…	…	…	…	…

A table consists of a schema and an instance …

The instance of this table is a 5-ary relation, a subset of the Cartesian product

$Z^+ \times$ Names $\times$ 8-strings_of_digits $\times$ Names $\times$ Meaningful_Sentences

Describing Binary Relations

- A list of pairs.

  Among 6 people, Mark, Jerry, John, Randy, Aaron, and Ralph, Mark and Randy are brothers, and also John, Aaron and Ralph are brothers

  A = {Mark, Jerry, John, Randy, Aaron, Ralph }

  Brotherhood = { (x,y) | x is a brother of y}

  = { (Mark,Randy), (Randy,Mark), (John,Aaron), (Aaron,John), (John,Ralph), (Ralph,John), (Aaron,Ralph), (Ralph,Aaron) }

Describing Binary Relations (cntd)

- Matrix of a relation.

  Matrix of a relation $R \subseteq A \times B$ is a rectangle table, rows of which are labeled with elements of A (in any but fixed order), and columns are labeled with elements of B. We write 1 in the intersection of row a and column b if and only if $(a,b) \in R$; otherwise we write 0.

Brotherhood

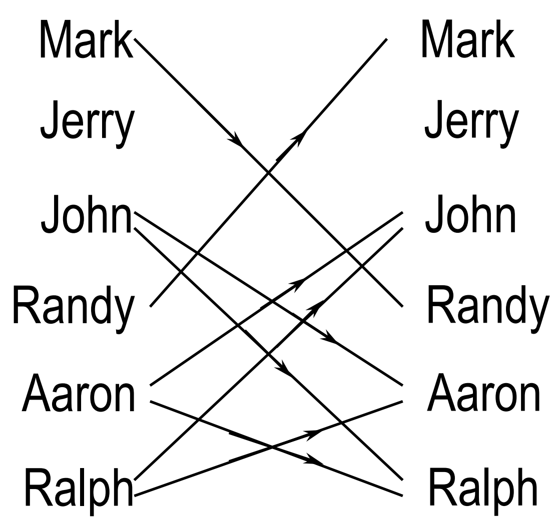
$$\begin{array}{c c}
 & \begin{matrix} \text{Mark} & \text{Jerry} & \text{John} & \text{Randy} & \text{Aaron} & \text{Ralph} \end{matrix} \\
\begin{matrix} \text{Mark} \\ \text{Jerry} \\ \text{John} \\ \text{Randy} \\ \text{Aaron} \\ \text{Ralph} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}
\end{array}$$

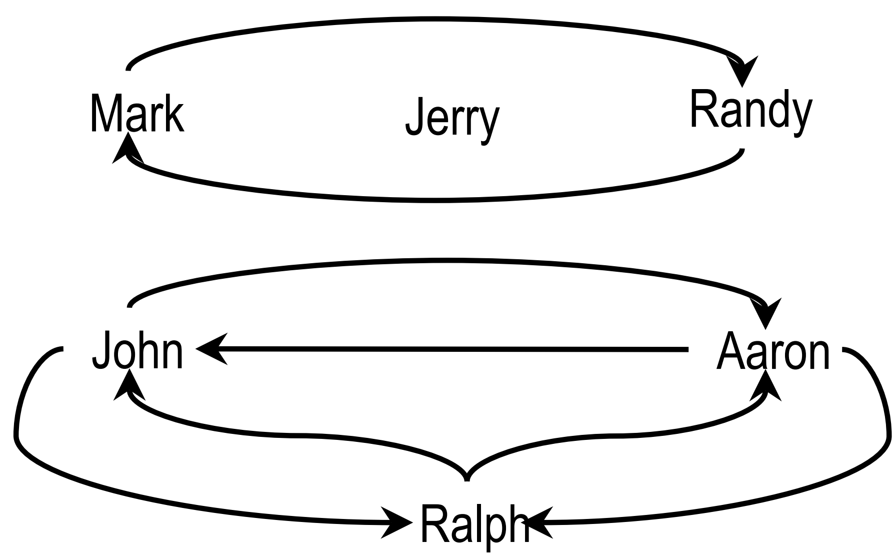
Describing Binary Relations (cntd)

- Graph of a relation

Graph of a relation $R \subseteq A \times B$ consists of two sets of vertices labeled by elements of A and B. A vertex a is connected to a vertex b with an edge (arc) if and only if $(a,b) \in R$.

If $A = B$ then we may use only one set of vertices





Properties of Binary Relations – Reflexivity

- From now on we consider only binary relations from a set A to the same set A. That is such relations are subsets of $A \times A$.
- A binary relation $R \subseteq A \times A$ is said to be reflexive if $(a,a) \in R$ for all $a \in A$.

$(a,b) \in R \subseteq Z \times Z$ if and only if $a \le b$

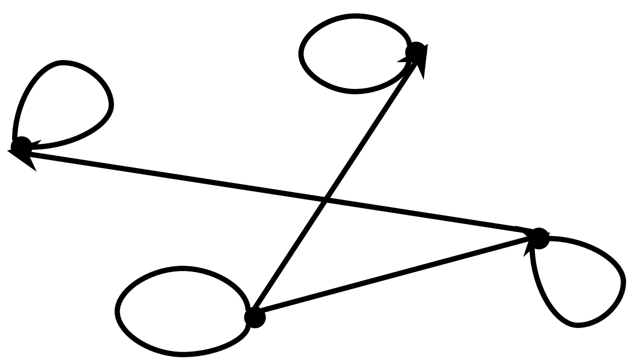
This relation is reflexive, because $a \le a$ for all $a \in Z$

Matrix:

$$\begin{pmatrix} 1 & * & * & * \\ * & 1 & * & * \\ * & * & 1 & * \\ * & * & * & 1 \end{pmatrix}$$

1's on the diagonal

Graph:



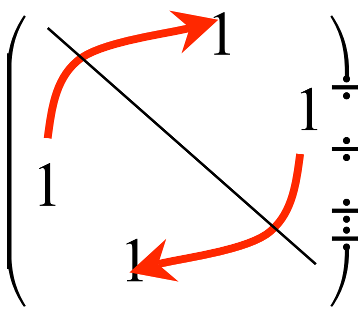
Loops at every vertex

Properties of Binary Relations – Symmetricity

- A binary relation $R \subseteq A \times A$ is said to be symmetric if, for any $a,b \in A$, if $(a,b) \in R$ then $(b,a) \in R$.

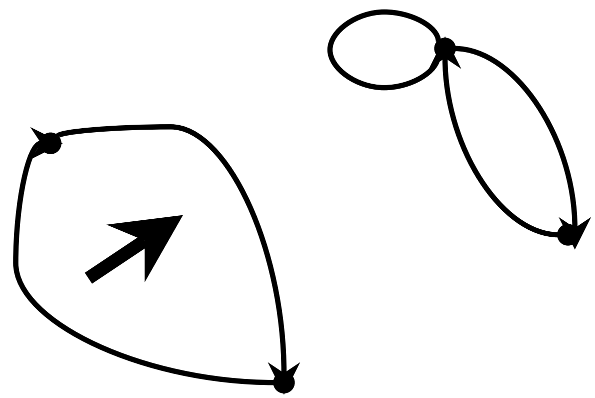
The relation Siblings (`x is a sibling of y') is symmetric, because if a is a sibling of y then y is a sibling of x

Matrix:



Matrix is symmetric w.r.t. the diagonal

Graph:



Graph is symmetric

Properties of Binary Relations – Transitivity

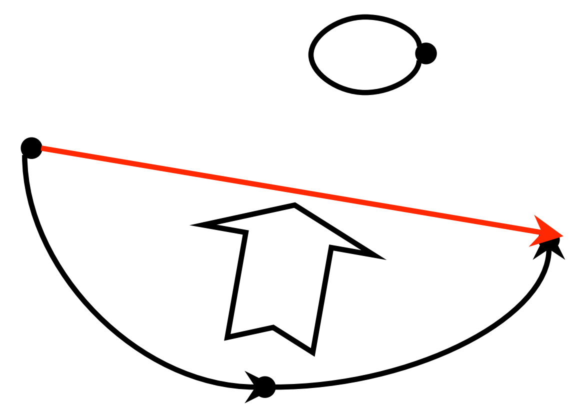
- A binary relation $R \subseteq A \times A$ is said to be **transitive** if, for any $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

The relation Div (`integer x divides y') is transitive, because if a divides b and b divides c, then a divides c

Matrix:

?

Graph:

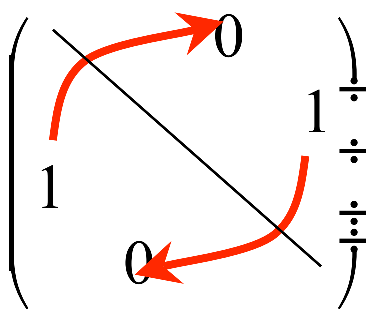


Properties of Binary Relations – Anti-Symmetricity

- A binary relation $R \subseteq A \times A$ is said to be anti-symmetric if, for any $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.

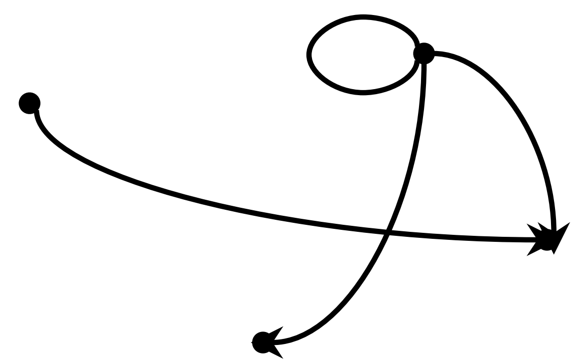
The relation Motherhood (\`x is the mother of y') is anti-symmetric, because if x is the mother of y then y is not the mother of x

Matrix:



Matrix is anti-symmetric w.r.t. the diagonal

Graph:



There are no edges going towards each other

Examples

	reflexive	symmetric	transitive	anti-symmetric
Brotherhood x is a brother of y				
Neighborhood x is a neighbor of y				
$x \le y$				
x,y are intergers and x divides y				

Homework

Exercises from the Book:

No. 1, 4, 5a (page 252)

- Prove part (3) of the theorem on slide 11-7
- Are the following relations reflexive? symmetric? transitive? anti-symmetric?
    - Motherhood: `x is the mother of y'
    - Intersect: `straight lines x and y intersect'