Functions

Discrete Mathematics
Evgeny Skvortsov

Previous Lecture

- Properties of binary relations
  - reflexivity
  - symmetricity
  - transitivity
  - anti-symmetricity
- Equivalence relations and partitions
- Partial and total orders
- Diagrams of orders

Orders

- A relation R on a set A is called a (partial) order if it is reflexive, transitive and anti-symmetric.

- Examples:
  - $a \leq b$ on the set of real numbers
  - $(a,b) \in Div$ if and only if a divides b

- Diagram of a partial order.

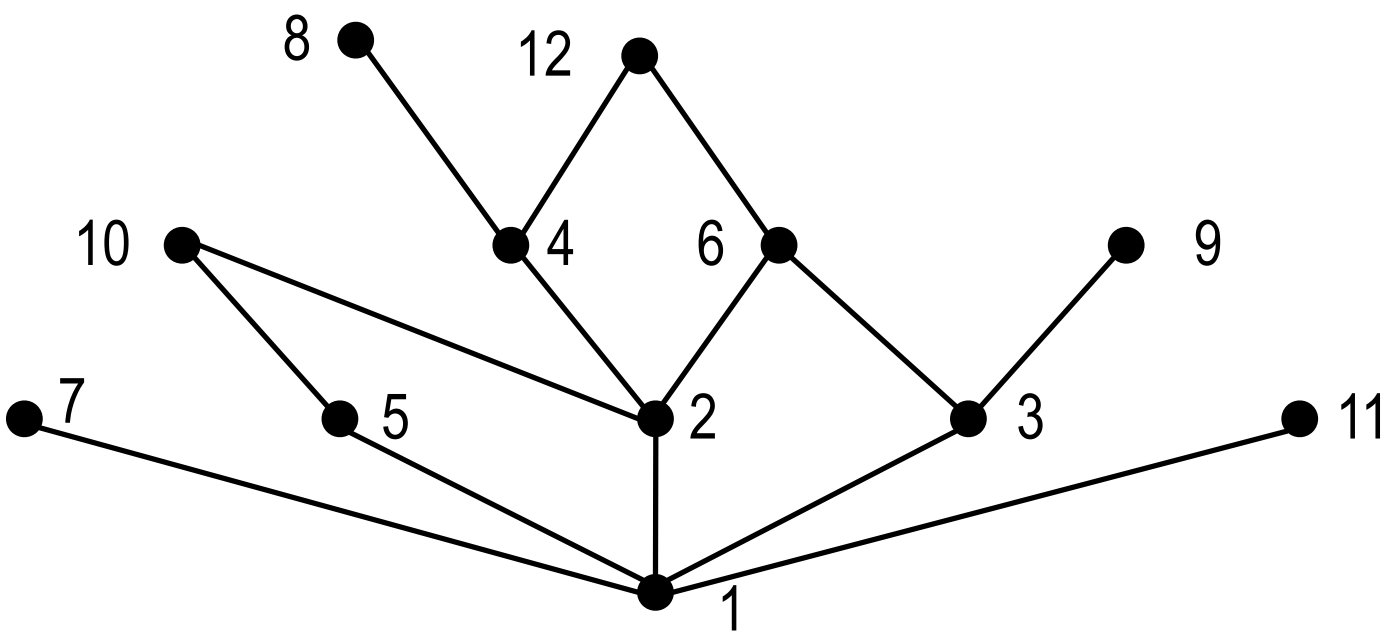
  Due to anti-symmetricity, all the elements of A are ranked with respect to the order R, that is b is ranked higher than a if $(a,b) \in R$.

  Due to transitivity, we do not need to know all pairs (a,b) from the relation, but only those, in which b is just higher than a.

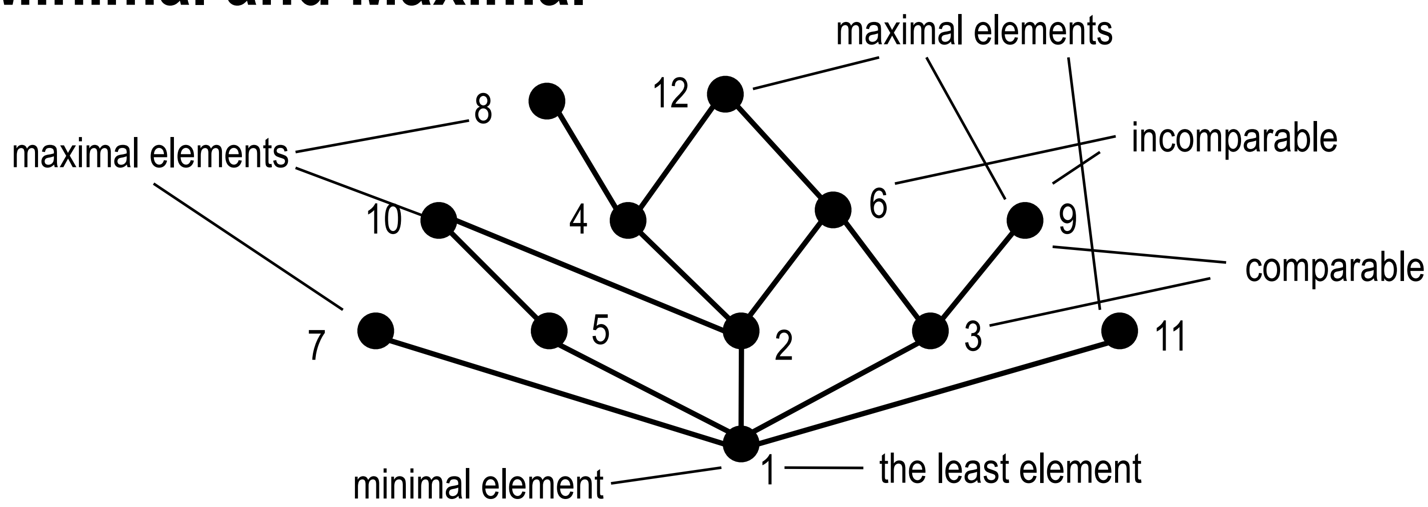
Diagram of a Partial Order

- Rules of drawing a diagram:
  if a is higher than b, put it higher
  connect every element only with elements that are just higher, so avoid triangles.

- Relation of divisibility on {1,2,…,12}



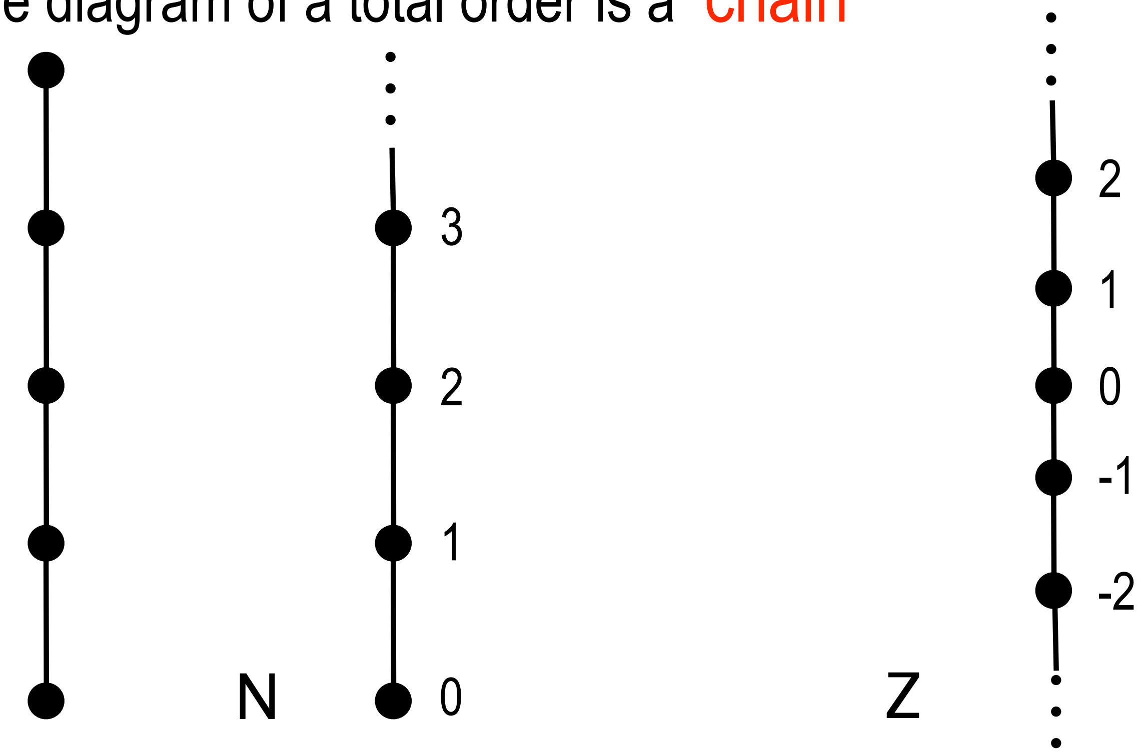
Minimal and Maximal



- Elements a,b are said to be comparable if $(a,b) \in R$ or $(b,a) \in R$
- Otherwise they are called incomparable
- Element a is minimal if for any b if $(b,a) \in R$ then $a = b$
- Element a is maximal if for any b if $(a,b) \in R$ then $a = b$
- Element a is called the least element if for any b, $(a,b) \in R$
- Element a is called the greatest element if for any b, $(b,a) \in R$

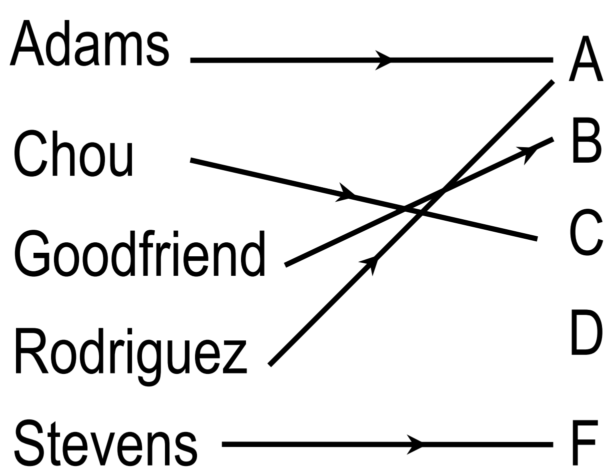
Total Order

- A partial order is said to be total if every two elements are comparable
- Sets N, Z, Q, R are totally ordered with respect to $\leq$
- The diagram of a total order is a chain



Functions

- In many instances we assign to each element of a set a particular element of a second set.
- For example, we may assign a grade to each student from a class
- What we get is a set of pairs (Student, Grade), that is a relation, but a very particular one
- A relation R from A to B is called a function from A to B, if for every $a \in A$ there is exactly one $b \in B$ such that $(a,b) \in R$. (Also mappings, transformations)



We use f,g,h to denote functions

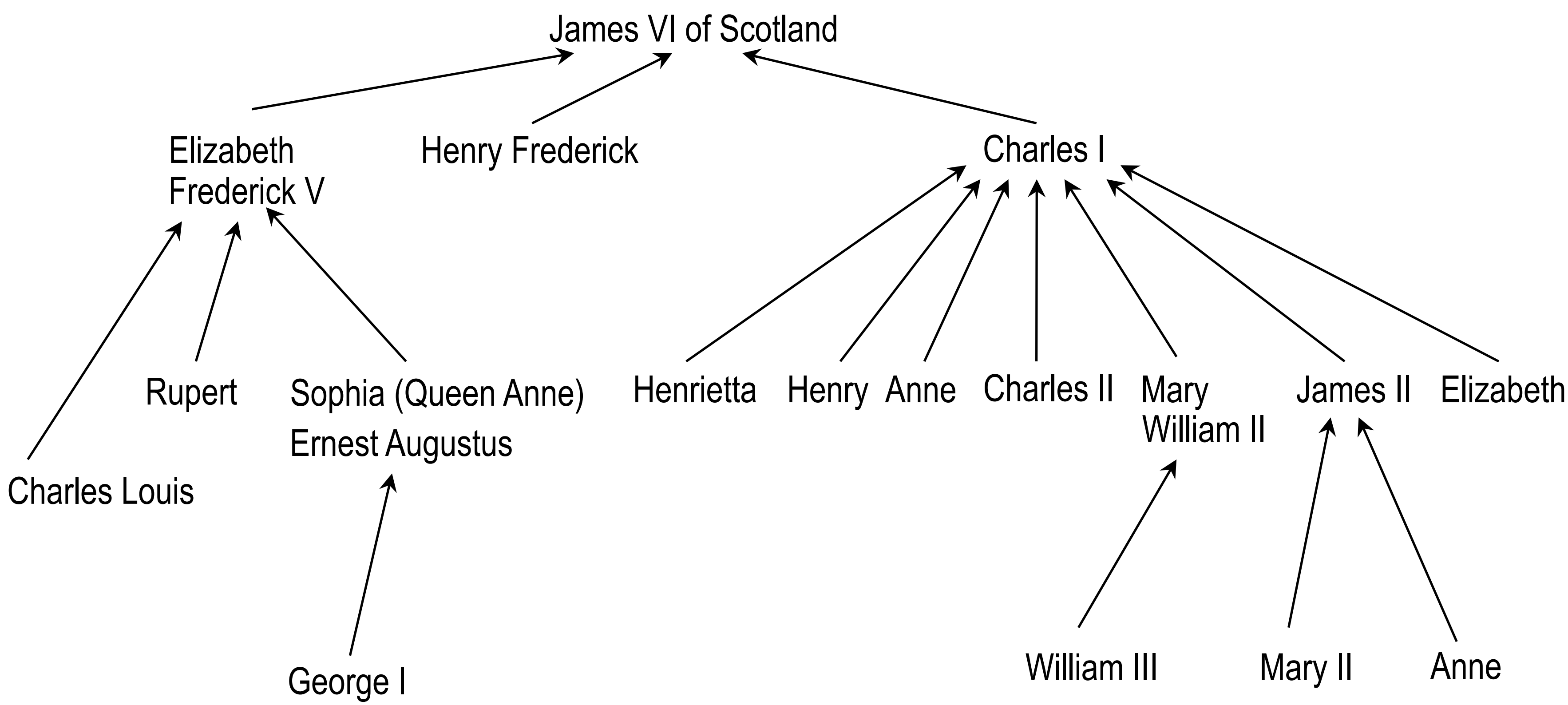
$f: A \rightarrow B$ $f(a) = b$

$f(\text{Rodriguez}) = A$

Example

- Consider the function from the set People to People:

  f(a) = b if b is the father of a.



(http://www.royal.gov.uk)

Domain and Codomain

- Let $f: A \rightarrow B$ be a function from A to B. Then A is called the **domain** of f, and B is called the **codomain** of f.

  If $f(a) = b$, then b is called the **image** of a, and a is called the **preimage** of b.

  The **range** of f is the set of all images of elements of A

  $$\text{range}(f) = \{ b \in B \mid \exists a \in A \;\; f(a) = b \}$$

  Also we say that f **maps** A to B

- In our example:

  domain = { Adams, Chou, Goodfriend, Rodriguez, Stevens }
  codomain = { A, B, C, D, F }
  range = { A, B, C, F }

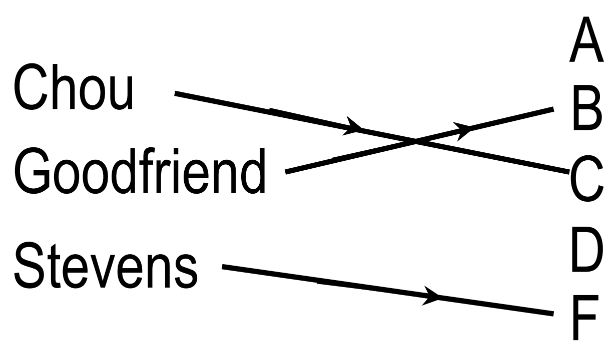
Restrictions and Extensions

- Let $f: A \rightarrow B$ be a function and $C \subseteq A$. The set

  $$f(C) = \{\, b \in B \mid b = f(a) \text{ for some } a \in A \,\}$$

  is called the **image** of C.
- Example: $f(\{\text{Adams, Rodriguez}\}) = \{A\}$

  $f(\{\text{Chou, Goodfriend, Stevens}\}) = \{B,C,F\}$
- Let $f: A \rightarrow B$ be a function and $C \subseteq A$. A function $f|_C : C \rightarrow B$ is called a **restriction** of f to C if $f|_C(a) = f(a)$ for all $a \in C$
- Example: Let $C = \{\text{Chou, Goodfriend, Stevens}\}$. Then the function



is a restriction of f

Restrictions and Extensions (cntd)

- Let $C \subseteq A$ and $f: C \to B$. Any function $g: A \to B$ such that $g(a) = f(a)$ for all $a \in C$ is called an **extension** of f.
- Let $A = R$, $B = Z$, $C = Z$, and $f: Z \to Z$ is defined as follows:

  $$f(a) = a$$

  Let g be the **floor function**:

  $g(x) = \lfloor x \rfloor$ = the greatest integer less than or equal to x

  Clearly, $g: R \to Z$, and $g(a) = a = f(a)$ for any integer a.

  Thus, g is an extension of f

Describing Functions

- Function is a relation, therefore we can use all methods of describing relations. Although the graph and the matrix are not very economical.

  { (Adams,A), (Chou,C), (Goodfriend,B), (Rodriguez,A), (Stevens,F)}
- Function table

Student	Grade
Adams	A
Chou	C
Goodfriend	B
Rodriguez	A
Stevens	F

Describing Functions (cntd)

- Numerical functions can be computed using a formula

  $f(x) = x^2$

  range(f) = { 0, 4, 9, …} – non-negative integers that are perfect squares

- The most general way is to use some algorithm to compute a function

  `The letter grade is A, if the numerical mark is in between 100 and 85; the letter grade is B, if …

  Functions in programming languages:

  int **floor**(float real) {…} in Java

  function floor(x: **real**): **integer** in Pascal

  def floor(x) in Python

One-to-One Functions

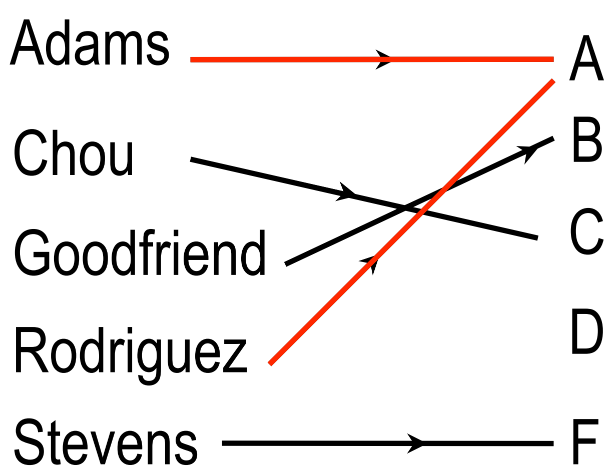
- A function f is said to be one-to-one, or injective, if and only if $f(a) = f(b)$ implies $a = b$.

  In other words no two elements are mapped into the same image.

  Contrapositive: if $a \neq b$ then $f(a) \neq f(b)$.

  Symbolically: $\forall a\, \forall b\, (f(a) = f(b) \rightarrow a = b)$

- Is this function injective?

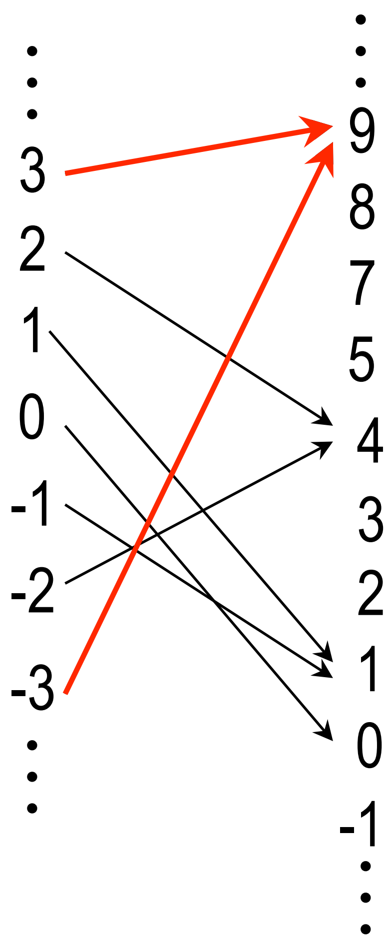


No!

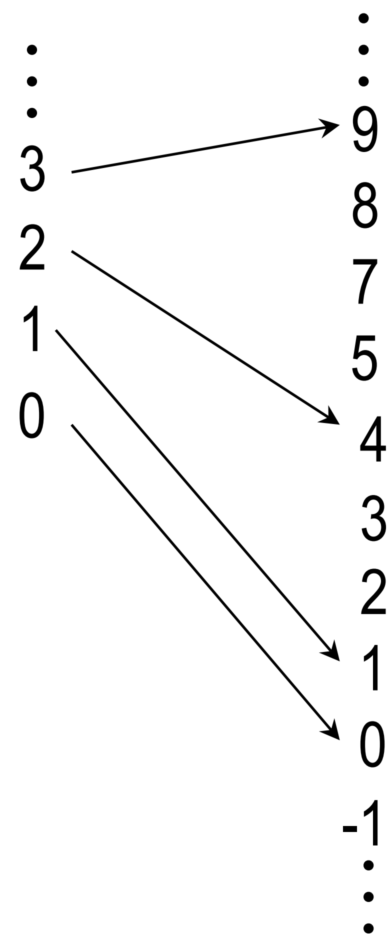
One-to-One Functions (cntd)

- Let’s consider the function $f(x) = x^2$ on Z

  Is it injective?



No!



Yes!
on N

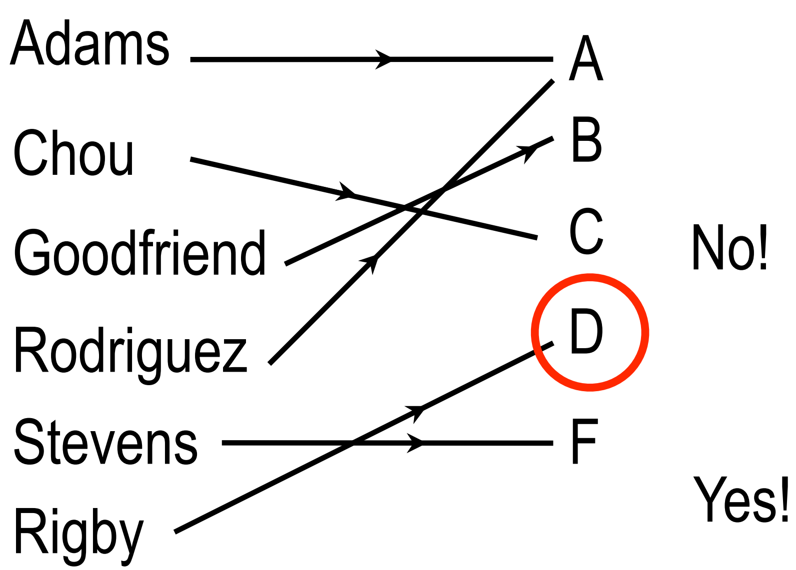
Onto Functions

- A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function is called a surjection if it is onto.
  Symbolically: $\forall b \exists a (f(a) = b)$

- Examples:
  - $f(x) = x + 1$
    Yes, because for any $b \in Z$ there is $a \in Z$ such that $a + 1 = b$
  - $f(x) = x^2$ on Z
    No, because $\sqrt{2}$ is not an integer
  - What about the same function on R^+
    Yes, the square root of a real number is a real number

Onto Functions (cntd)

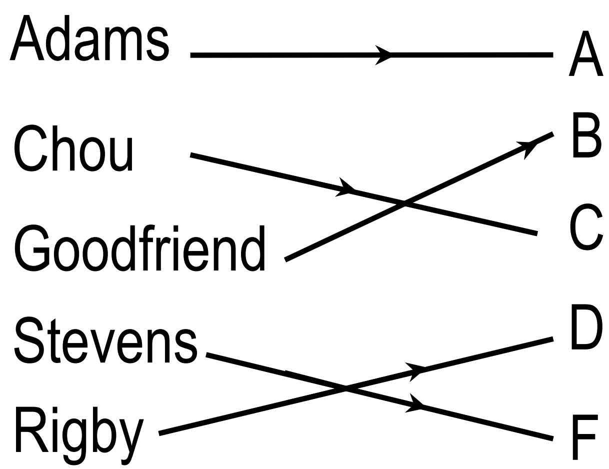
- More examples



- f(a) = b if b is the father of a

Bijections

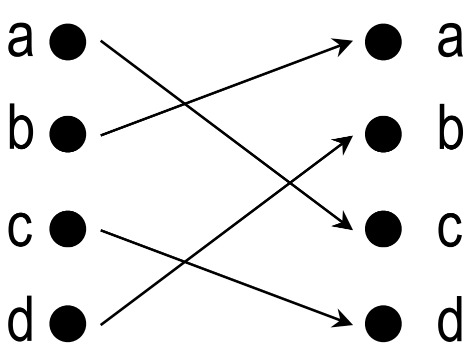
- A function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.

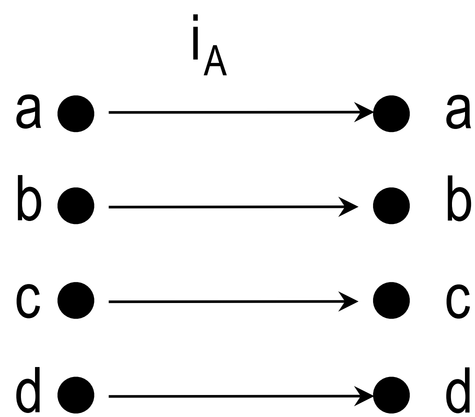


- If there is a bijection from a set A to a set B, then these sets in a certain sense are equal or identical.

Bijections (cntd)

- Numerical functions:
  - $f(x) = x + 1$ is a bijection on Z, Q, R, but not on N
  - $f(x) = x^2$ is a bijection on $\mathbf{R}^+$ but is not on any other numerical set
- A bijection from a set A to the same set A is called a permutation of A



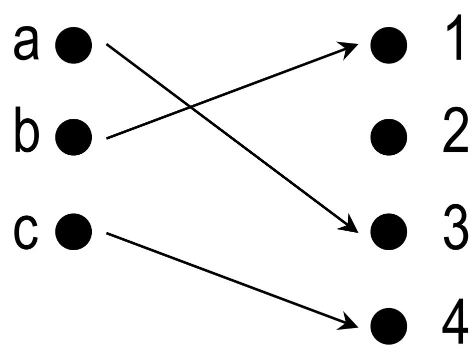


- The identity function on a set A is the function $i_A: A \to A$, where

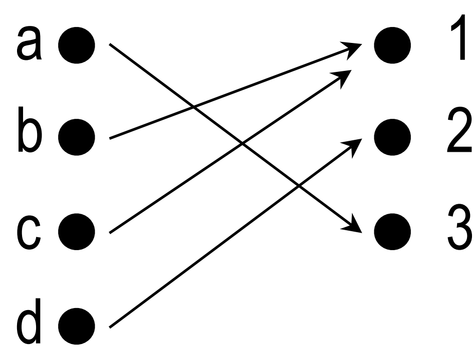
$$i_A(x) = x$$

Functions and Properties

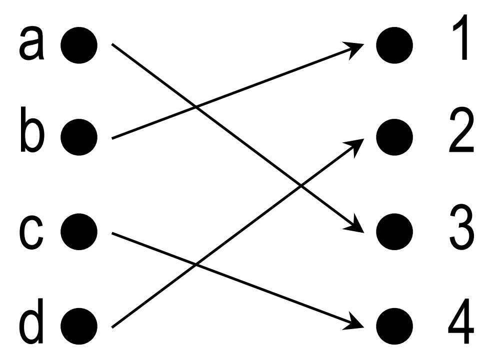
- Examples of different types of correspondences



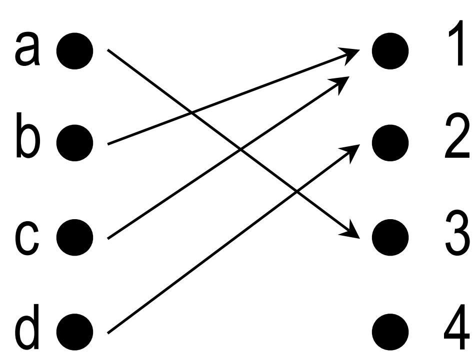
one-to-one, not onto



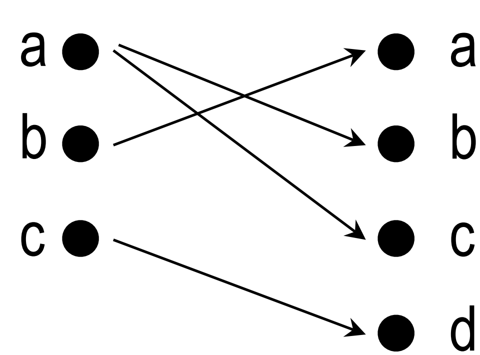
onto, not one-to-one



one-to-one and onto



neither one-to-one nor onto

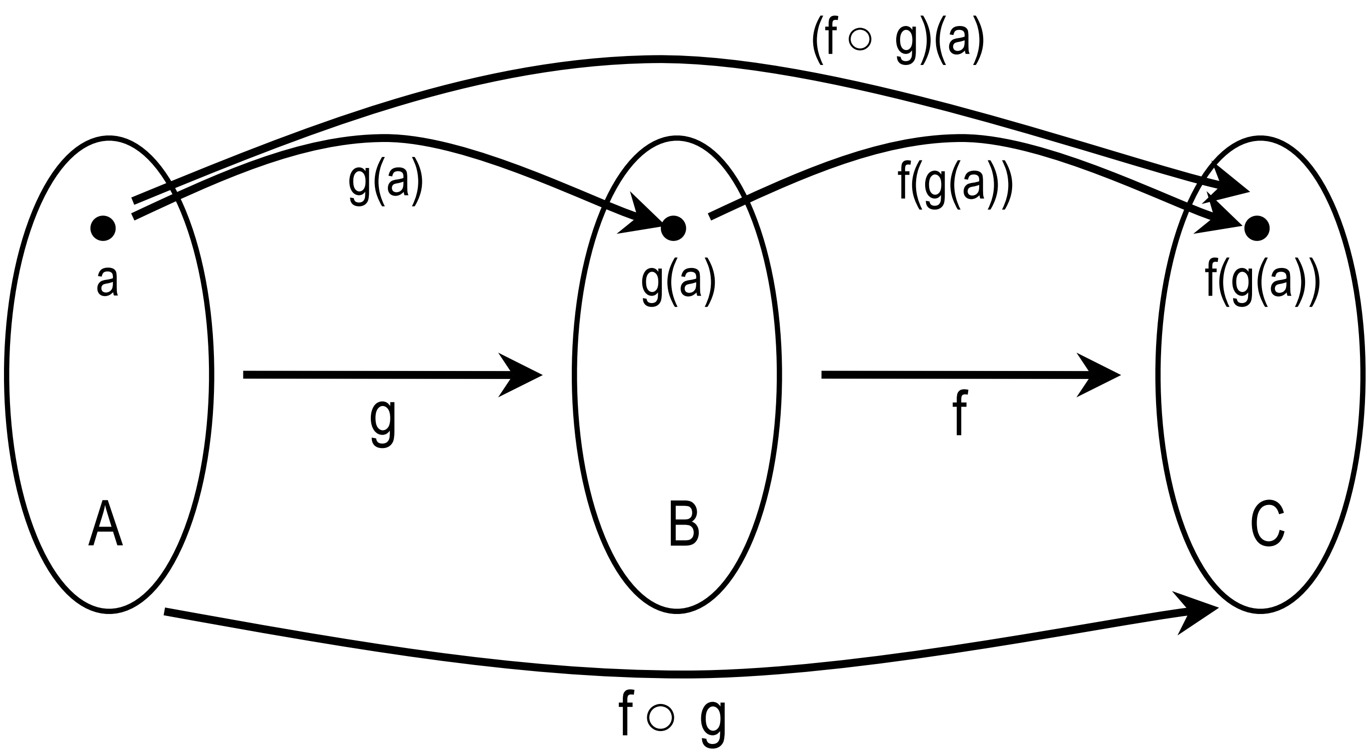


not a function

Composition of Functions

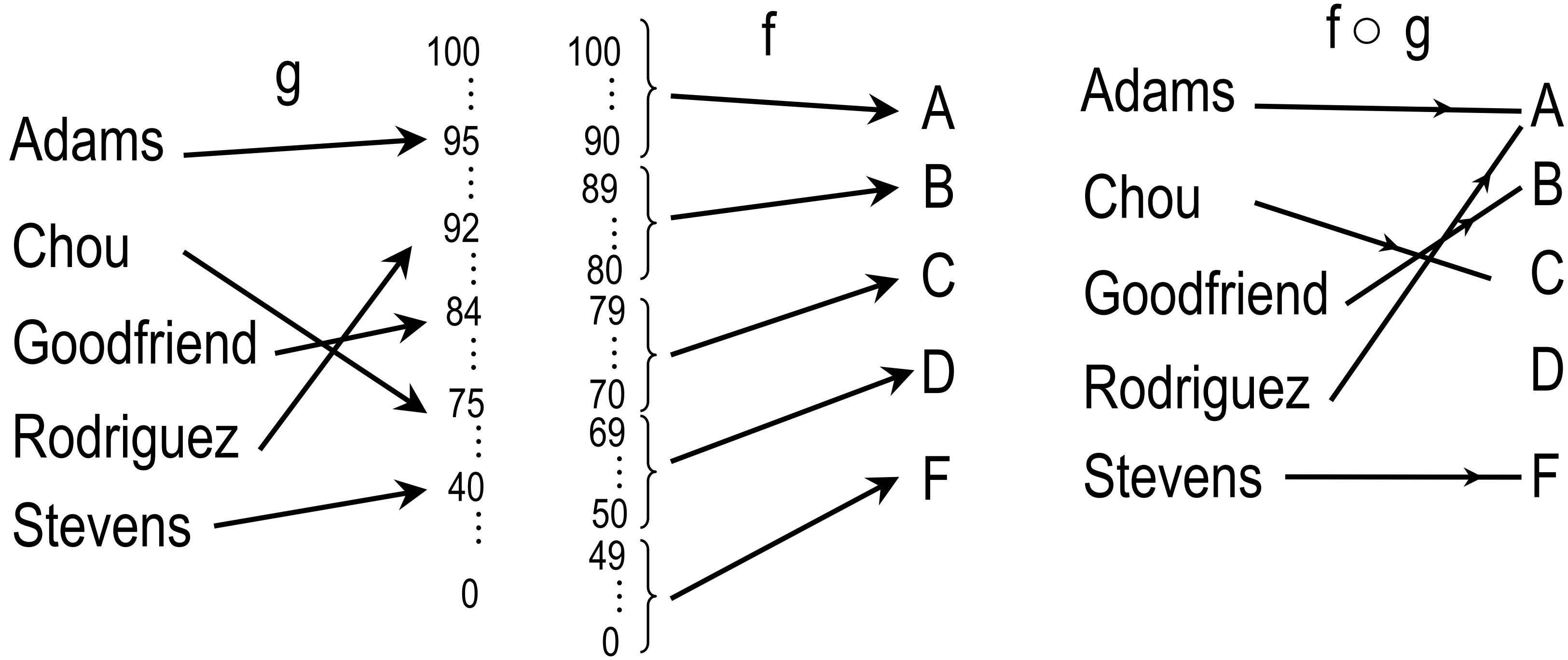
- Let g be a function from A to B and let f be a function from B to C. The composition of the functions f and g, denoted by $f \circ g$, is the function from A to C defined by

$$(f \circ g)(a) = f(g(a))$$



Composition of Functions (cntd)

- Suppose that the students first get numerical grades from 0 to 100 that are later converted into letter grade.



- Let f(a) = b mean `b is the father of a'.
  What is $f \circ f$?

Composition of Numerical Functions

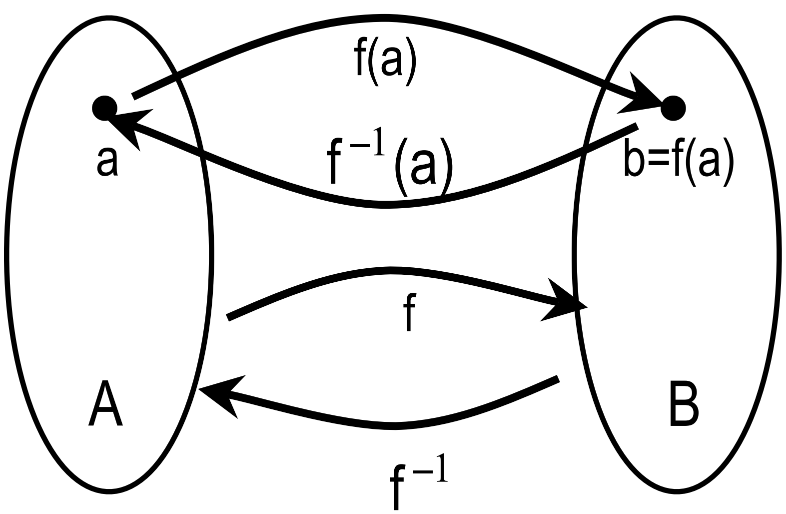
- Let $g(x) = x^2$ and $f(x) = x + 1$. Then $(f \circ g)(x) = f(g(x)) = g(x) + 1 = x^2 + 1$
- Thus, to find the composition of numerical functions f and g given by formulas we have to substitute g(x) instead of x in f(x).

Inverse Functions

- Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$.

  The inverse function is denoted by f^{-1}.

  Thus, $f^{-1}(b) = a$ if and only if $f(a) = b$



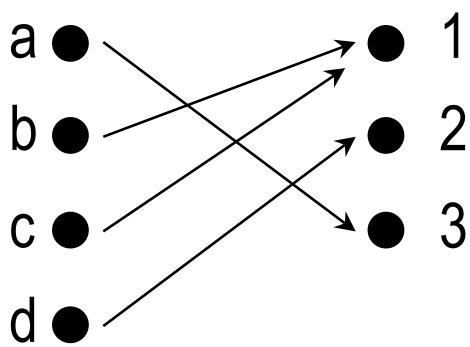
Note!
f^{-1} does not mean $\dfrac{1}{f(x)}$

$f \circ f^{-1} = i_B$

$f^{-1} \circ f = i_A$

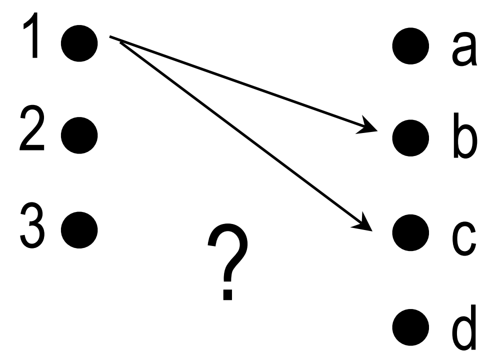
Inverse Functions (cntd)

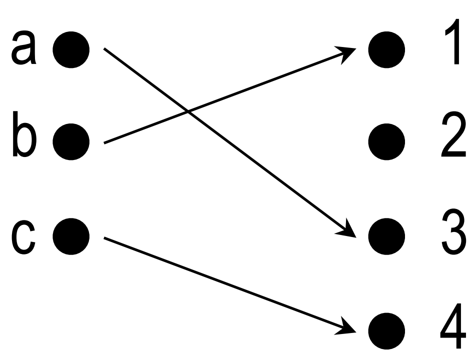
- If a function f is not a bijection, the inverse function does not exist. Why?
- If f is not a bijection, it is either not one-to-one, or not onto



onto, not one-to-one

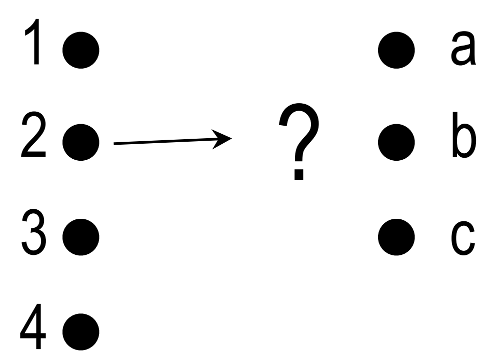
$f^{-1}(1) = ?$





one-to-one, not onto

$f^{-1}(2) = ?$



Homework

Exercises from the Book:

No. 1, 2, 6a,15, 16ace, 18 (page 258)