Automatic Reasoning

Discrete Mathematics
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Previous Lecture

- Valid and invalid arguments
- Arguments and tautologies
- Rules of inference
  - Modus ponens
  - Rule of syllogism
  - Modus tollens
  - Rule of disjunctive syllogism
  - Rule of Proof by Cases
  - Rule of Contradiction
  - Rule of Simplification
  - Rule of Amplification

Rules of Inference

Modus ponens

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad\quad} \\ \therefore\ q \end{array}$$

Rule of syllogism

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore\ p \rightarrow r \end{array}$$

Modus Tollens

$$\begin{array}{l} p \rightarrow q \\ \underline{\neg q \qquad} \\ \therefore\ \neg p \end{array}$$

Rule of Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \underline{\neg p \qquad} \\ \therefore\ q \end{array}$$

Rule of Proof by Cases

$$\begin{array}{l} p \rightarrow r \\ \underline{q \rightarrow r} \\ \therefore\ (p \vee q) \rightarrow r \end{array}$$

Rule of Contradiction

$$\begin{array}{l} \underline{\neg p \rightarrow F} \\ \therefore\ p \end{array}$$

Rule of Simplification

$$\begin{array}{l} \underline{p \wedge q} \\ \therefore\ p \end{array}$$

Rule of Amplification

$$\begin{array}{l} \underline{p \qquad\quad} \\ \therefore\ p \vee q \end{array}$$

Logic Puzzles

- A prisoner must choose between two rooms each of which contains either a lady, or a tiger. If he chooses a room with a lady, he marries her, if he chooses a room with a tiger, he gets eaten by the tiger. The rooms have signs on them:

I	II
at least one of these rooms contains a lady	a tiger is in the other room

It is known that either both signs are true or both are false

Logic Puzzles (cntd)

- Notation:

  p - the first room contains a lady

  q - the second room contains a lady

- Premises:

  $$(p \vee q) \to \neg p,$$

  $$\neg p \to (p \vee q)$$

Logic Puzzles (cntd)

- Argument

$(p \vee q) \rightarrow \neg p$,
$\neg p \rightarrow (p \vee q)$

Step	Reason
1. $\neg p \rightarrow (p \vee q)$	premise
2. $p \vee p \vee q$	expression for implication
3. $p \vee q$	idempotent law
4. $(p \vee q) \rightarrow \neg p$	premise
5. $\neg p$	modus ponens
6. q	rule of disjunctive syllogism to 3 and 5

Conjunctive Normal Form

- A literal is a primitive statement (propositional variable) or its negation

  $p, \ \neg p, \ q, \ \neg q$

- A clause is a disjunction of one or more literals

  $p \vee q, \ p \vee \neg q \vee r, \ \neg q, \ \neg s \vee s \vee \neg r \vee \neg q$

- A statement is said to be a Conjunctive Normal Form (CNF) if it is a conjunction of clauses

$$p \wedge (p \vee \neg q) \wedge (\neg r \vee \neg p)$$
$$p \wedge q \wedge (\neg r \vee \neg p)$$
$$(\neg r \vee q) \wedge (p \vee \neg q \vee \neg s \vee r) \wedge (\neg r \vee \neg p)$$
$$\neg r$$

CNF Theorem

- **Theorem**

  Every statement is logically equivalent to a certain CNF.

  Proof (sketch)

  Let Φ be a (compound) statement.

  Step 1. Express all logic connectives in Φ through negation, conjunction, and disjunction. Let Ψ be the obtained statement.

  Step 2. Using DeMorgan's laws move all the negations in Ψ to individual primitive statements. Let Θ denote the updated statement

  Step 3. Using distributive laws transform Θ into a CNF.

Example

Find a CNF logically equivalent to $(p \rightarrow q) \rightarrow r$

Step 1. $\neg(\neg p \vee q) \vee r$

Step 2. $(p \wedge \neg q) \vee r$

Step 3. $(p \vee r) \wedge (\neg q \vee r)$

Rule of Resolution

- $$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore\ q \vee r \end{array}$$

  $q \vee r$ is called resolvent

- The corresponding tautology $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

  ``Jasmine is skiing or it is not snowing.
  It is snowing or Bart is playing hockey.''

  p - `it is snowing'

  q - `Jasmine is skiing'

  r - `Bart is playing hockey'

  ``Therefore, Jasmine is skiing or Bart is playing hockey''

Computerized Logic Inference

- Convert the premises into CNF
- Convert the *negation* of the conclusion into CNF
- Consider the collection consisting of all the clauses that occur in the obtained CNFs
- Use the rule of resolution to obtain the empty clause ($\varnothing$). If it is possible, then the argument is valid. Otherwise, it is not.

Why empty clause?

The only way to produce the empty clause is to apply the resolution rule to a pair of clauses of the form p and $\neg p$. Therefore, the collection of clauses is contradictory. In other words, for any choice of truth values for the primitive statements, if premises are true, the conclusion cannot be false.

Example

- A lady and a tiger
- Premises: $\neg p \rightarrow (p \vee q)$, $(p \vee q) \rightarrow \neg p$
- Negation of the conclusion: $\neg q$
- Clauses:

  $\neg p$, $\neg p \vee \neg q$, $p \vee q$, $\neg q$
- Argument:

$p \vee q$	premise	$\varnothing$	resolvent
$\neg q$	premise		
p	resolvent		
$\neg p$	premise		

Homework

Exercises from the Book:
No. 5, 9a (page 84-85)

- Prove that resolution is a valid rule of inference
- Same arrangements as in the `A lady or a tiger' problem. This time if a lady is in Room I, then the sign on it is true, but if a tiger is in it, then the sign is false. If a lady is in Room II, then the sign on it is false, and if a tiger is in it, then the sign is true. Signs are

I	II
both rooms contain ladies	both rooms contain ladies