

MIDTERMS AND ASSIGNMENTS

SEE THE WEBSITE

Operations on Sets

Discrete Mathematics
Evgeny Skvortsov

Previous Lecture

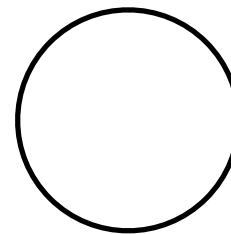
- Sets and elements
- Subsets, proper subsets, empty sets
- Universe
- Cardinality
- Power set

Venn Diagrams

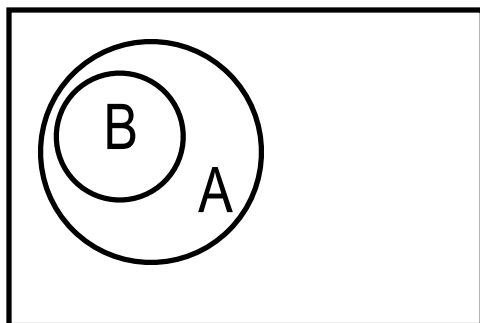
- Often it is convenient to visualize various relations between sets. We use **Venn diagrams** for that.



universe



set

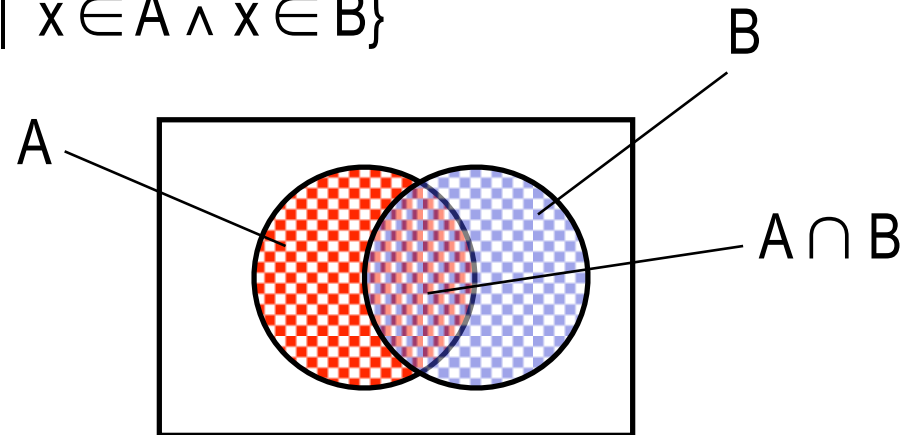


B is a subset of A

Intersection

- The **intersection** of sets A and B , denoted by $A \cap B$, is the set that contains those elements in both A and B .

- $A \cap B = \{x \mid x \in A \wedge x \in B\}$



- Examples

$$\{1,3,5,7\} \cap \{2,3,4,5,6\} = \{3,5\}$$

$$\{\text{Jan.}, \text{Feb.}, \text{Dec.}\} \cap \{\text{Jan.}, \text{Feb.}, \text{Mar.}\} = \{\text{Jan.}, \text{Feb.}\}$$

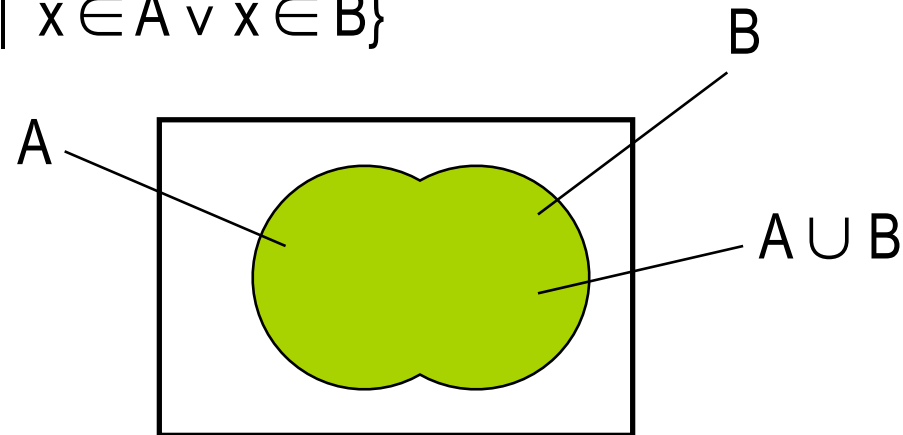
$$\{x \mid \exists y \ x=2y\} \cap \{x \mid \exists y \ x=3y\} = \{x \mid \exists y \ x=6y\}$$

$$\mathbb{Z} \cap \mathbb{Q}^+ = \mathbb{Z}^+$$

Union

- The **union** of sets A and B , denoted by $A \cup B$, is the set that contains those elements that are either in A or in B .

- $A \cup B = \{x \mid x \in A \vee x \in B\}$



- Examples

$$\{\text{Mon, Tue, Wed, Thu, Fri}\} \cup \{\text{Sat, Sun}\} = \{\text{Mon, Tue, Wed, Thu, Fri, Sat, Sun}\}$$

$$\{1, 3, 5, 7\} \cup \{2, 3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6, 7\}$$

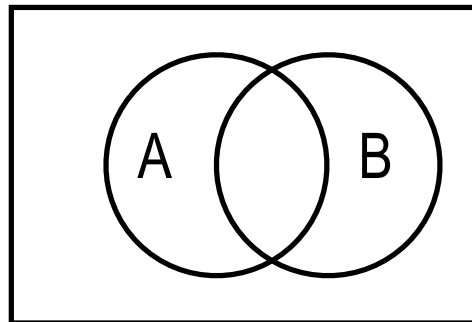
Disjoint Sets and Principle of Inclusion-Exclusion

- Sets A and B are said to be **disjoint** if $A \cap B = \emptyset$.

Sets $\{\text{Mon, Tue, Wed, Thu, Fri}\}$ and $\{\text{Sat, Sun}\}$ are disjoint.

- Principle of inclusion-exclusion. For any finite sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|$$



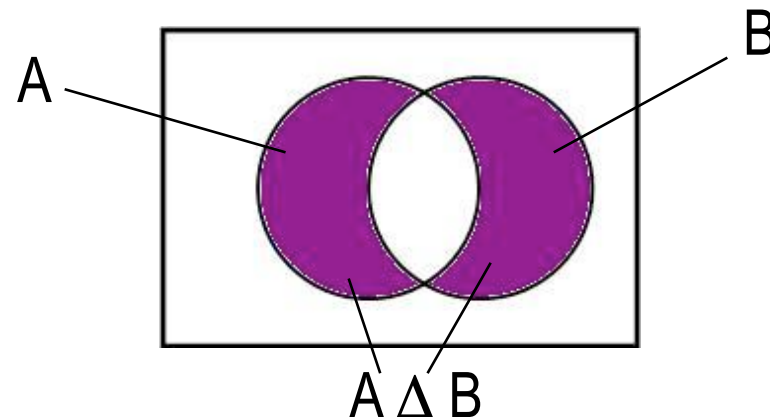
To count elements in $A \cup B$ we first count elements of A , then elements of B . Elements of $A \cap B$ are counted twice, so, we subtract the number of such elements

- If A and B are disjoint, then $|A \cup B| = |A| + |B|$

Symmetric Difference

- The symmetric difference of sets A and B , denoted by $A \Delta B$, is the set that contains those elements that are either in A or in B , but not in both.

- $A \Delta B = \{x \mid x \in A \oplus x \in B\}$



- Example
 $\{\text{Jan.}, \text{Feb.}, \text{Mar.}\} \Delta \{\text{Dec.}, \text{Jan.}, \text{Feb.}\} = \{\text{Dec.}, \text{Mar.}\}$

Disjoint Sets and Symmetric Difference

● **Theorem.** Sets A and B are disjoint if and only if
$$A \cup B = A \Delta B$$

● **Proof.**

Notice first that $A \Delta B \subseteq A \cup B$.

Suppose A and B are disjoint. To prove the equality, it suffices to show that $A \cup B \subseteq A \Delta B$.

Take $x \in A \cup B$. It belongs to A or B , but $x \notin A \cap B$, as the intersection is empty. Therefore, $x \in A \Delta B$.

We prove by contraposition. Assume $A \cap B \neq \emptyset$. Say $x \in A \cap B$

Then $x \in A \cup B$.

However, from $x \in A \cap B$ we conclude that $x \notin A \Delta B$.

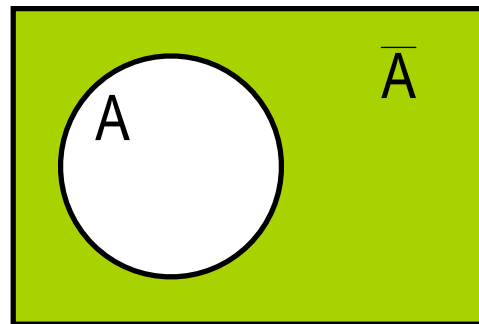
Therefore, $A \cup B \neq A \Delta B$.

Q.E.D.

Complement

- Let A be a set and U a universe, $A \subseteq U$. The **complement** of A , denoted by \bar{A} , is the set that comprises all elements of U that do not belong to A .

$$\bar{A} = \{x \mid x \in U \text{ and } x \notin A\} = \{x \mid x \notin A\}$$

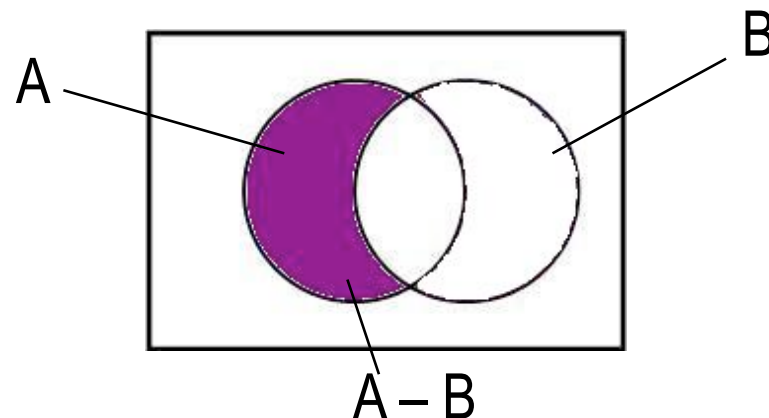


- Let the universe be the set of all integers, and $A = \{x \mid \exists y \ x=2y\}$. Then \bar{A} is the set of all odd numbers.
- The universe is the Latin alphabet, $A = \{a, e, i, o, u, y\}$. Then $\bar{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, z\}$.

Difference

- The difference of sets A and B (or relative complement of B in A), denoted by $A - B$, is the set containing those elements that are in A , but not in B .

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$



- $\{1,3,5\} - \{1,2,3\} = \{5\}$
- Clearly, $\bar{A} = U - A$.

Laws of Set Theory

- Similar to logic connectives and formulas, expressions built from set operations and sets also satisfy some laws.

Theorem. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof. We will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \supseteq \overline{A} \cup \overline{B}$

Prove that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Take $x \in \overline{A \cap B}$.

By the definition, $x \notin A \cap B$. Therefore, $x \notin A$ or $x \notin B$.

Hence $x \in \overline{A}$ or $x \in \overline{B}$. Thus, $x \in \overline{A} \cup \overline{B}$

Now we prove that $\overline{A \cap B} \supseteq \overline{A} \cup \overline{B}$. Take $x \in \overline{A} \cup \overline{B}$.

By definition, $x \in \overline{A}$ or $x \in \overline{B}$. Therefore, $x \notin A$ or $x \notin B$.

This implies $x \notin A \cap B$. And, finally, $x \in \overline{A \cap B}$.

Q.E.D.

Another Proof

- Another way to prove equalities for sets is to use the set builder construction and some logic.

$$\begin{aligned}\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\ &= \{x \mid \neg(x \in A \cap B)\} && \text{by definition of does not belong symbol} \\ &= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\ &= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by De Morgan's law} \\ &= \{x \mid (x \notin A) \vee (x \notin B)\} && \text{by def. of does not belong symbol} \\ &= \{x \mid (x \in \overline{A}) \vee (x \in \overline{B})\} && \text{by definition of complement} \\ &= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\ &= \overline{A} \cup \overline{B}\end{aligned}$$

Q.E.D.

Sets and Logic

- If we look closer at the second proof, we notice that there is a very tight connection between set operations and logic connectives

\neg corresponds to complement $—$

\vee corresponds to union \cup

\wedge corresponds to intersection \cap

\oplus corresponds to symmetric difference Δ

0 (false) corresponds to the empty set \emptyset

1 (truth) corresponds to the universe U

More Laws of Set Theory

$$A \cup \emptyset = A$$

$$A \cap U = A$$

Identity laws

$$A \cup U = U$$

$$A \cap \emptyset = \emptyset$$

Domination laws

$$A \cup A = A$$

$$A \cap A = A$$

Idempotent laws

$$\overline{\overline{A}} = A$$

Complementation law

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Commutative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Associative laws

More Laws of Set Theory (cntd)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Distributive laws

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

De Morgan's laws

$$A \cap (A \cup B) = A$$

$$A \cup (A \cap B) = A$$

Absorption laws

$$A \cup \bar{A} = U$$

$$A \cap \bar{A} = \emptyset$$

Complement laws

Homework

Exercises from the Book:

No. 1, 4, 6, 8bc, 16, 17bc (page 146 – 147)