Sets

Discrete Mathematics

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What is a Set

- A set is an unordered collection of objects

This is not a rigorous definition!!!

Every `conventional’ definition reduces the defined concept to a wider, more general, concept.

For example, `A cow is a big animal with horns and four legs at the corners’

There is no more general concept than sets. Therefore a rigorous definition is impossible.

If we use the `definition’ above, we get `naïve set theory’

Otherwise we need axiomatic set theory, Zermelo-Frenkel axioms (introduced by Skolem) ZF or ZFC.

Elements, Describing a Set

- The objects in a set are called **elements** or **members** of the set

  $a \in A$ – a is an element of A, a belongs to A
  $a \notin A$ – a is not an element of A, a does not belong to A

- One way to describe a set is to list its elements

  $\{0,1,2,3,4,5,6,7,8,9\}$ – the set of digits

  $\{a,b,c,\ldots,x,y,z\}$ – set of Latin letters, alphabet

- A set can be an element of another set
  Set of alphabets: $\{\{a,b,c,\ldots\}, \{\alpha,\beta,\gamma,\ldots\},\ldots\}$

Set Builder

- Big sets can be described using **set builder**:

  $\{x \mid P(x)\}$, the set of all x such that $P(x)$

  $\{x \mid$ there is y such that $x = 2y\}$, the set of even numbers

  $\parallel$

  $\{x \mid \exists y\ (x = 2y)\}$

  $\{x \mid x$ is a black cow$\}$

  $N = \{0,1,2,3,\ldots\}$, the set of natural numbers

  $Z = \{\ldots,-2,-1,0,1,2,3,\ldots\}$, the set of integers

  $Q = \{p/q \mid p,q$ are integers and $q \neq 0\}$, the set of rationals

  Z^+, Q^+, the sets of positive integers and positive rationals

  R, the set of real numbers

Russel's Paradox

- We should use the set builder construction very carefully
- Our understanding of sets is very broad. For example, a set can be its own element.
- Let U be the set of all sets that do not contain itself as an element:

  $U = \{x \mid x \notin x\}$

- Does U belong to itself?

  If yes, then $U \in U$, hence U does not satisfy the condition, and therefore $U \notin U$.

  If not, then $U \notin U$, hence U does satisfy the condition, and therefore $U \in U$.

Universe

- Although, in theory, elements of sets can be anything, in practice, it is not very convenient to allow such diversity. Say, if we are talking about numbers, all sets we can encounter have numbers as elements. If we work in propositional logic, then we are dealing with sets of statements, etc.
- This is why we usually have some sort of a universal set or a universe in mind, that contains all objects we may need.
- Example

  $\{ x \mid 1 \le x \le 10 \}$ What is this set?

  $\{ x \in Z \mid 1 \le x \le 10 \}$ the set of all integers from 1 to 10

  $\{ x \in Q \mid 1 \le x \le 10 \}$ the set of all rationals from 1 to 10

Equality of Sets, Subsets

- Two sets are equal if they have the same elements

  That is $A = B$ if and only if $\forall x\,(x \in A \leftrightarrow x \in B)$

  $\{1,3,5\} = \{5,3,1\}$

  $\{1,3,5\} = \{5,5,5,5,3,3,1,1,3,1\}$

  $\{1,3,5\} \neq \{1,3,\{5\}\}$

- Set B is a subset of a set A if every element of B is also an element of A.

  That is $B \subseteq A$ if and only if $\forall x\,(x \in B \rightarrow x \in A)$

Equality of Sets, Subsets (cntd)

$\{1,3\} \subseteq \{1,3,5\}$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & Z \arrow[dr, phantom, "\subseteq"] & & \\
Z^+ \arrow[ur, phantom, "\subseteq"] \arrow[dr, phantom, "\subseteq"] & & Q \arrow[r, phantom, "\subseteq"] & R \\
 & Q^+ \arrow[ur, phantom, "\subseteq"] & &
\end{tikzcd}$$

- Set B is not a subset of a set A if

  $\exists x\,(x \in B \wedge x \notin A)$

  There is an element in B that is not an element of A

- Another way to say that two sets are equal: each of them is a subset of the other

Equality of Sets, Subsets (cntd)

- A set B is said to be a proper subset of a set A, if it is a subset of A and is not equal to A.
- Symbolically: $B \subset A \quad \Leftrightarrow \quad (B \subseteq A) \wedge \exists x\, (x \in A \wedge x \notin B)$
- **Theorem**. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

  Proof.

  We have to prove that, for every x, if $x \in A$, then $x \in C$.

  Take any x such that $x \in A$. Since $A \subseteq B$, this implies that $x \in B$.

  Next, as $B \subseteq C$ and $x \in B$, we have $x \in C$.

  Q.E.D.

  Homework: give a formal proof, using rules of inference

Empty Set

- Empty set is a set that has no elements. $\varnothing$

  How many elements does this set contain? $\{\varnothing\}$

- **Theorem**. For any set A, (i) $\varnothing \subseteq A$, and (ii) $A \subseteq A$.

  Proof.

  (i) We have to prove that $\forall x\, (x \in \varnothing \rightarrow x \in A)$

  Since the premise $x \in \varnothing$ is always false, we conclude that the implication $x \in \varnothing \rightarrow x \in A$ is always true.

  Then we use the rule of universal generalization.

  (ii) Homework.

Cardinality

- Let A be a set. If there are exactly n distinct elements in A where n is a natural number, then we say that A is a finite set and that n is the cardinality of A.

  $$|A| = n$$

- Examples

  $|\varnothing| = 0$

  $|\{1,2,3,3\}| = 3$

  $|\{0,1,2,3,4,5,6,7,8,9\}| = 10$

  $|\{a,b,c,\ldots,x,y,z\}| = 26$

- Sets that are not finite are called infinite

  N, Z, Q, R are infinite

Power Set

- Given a set A, the power set of A is the set of all subsets of the set A. P(A)
- Example. Find P({1,2,{1,2}})
- **Theorem**. If A is a finite set, then $|P(A)| = 2^{|A|}$

  Proof.

  First, we define a systematic way of describing subsets of A.

  Let $A = \{ a_1, a_2, a_3, \ldots, a_n \}$.

  Then every subset of A can be defined by a string of n bits 0/1 such that a_i belongs to the subset if and only if i-th bit equals 1

  Say, $\varnothing$ is encoded with 000…0, A itself – with 111…1

  Such a representation is called Grey code (in Grimaldi), or characteristic function (everywhere else)

Power Set (cntd)

Finally, we observe that for each element/bit there are two possibilities, and its value does not depend on the value of other bits.

Thus, we have $2 \times 2 \times 2 \times \ldots \times 2 = 2^{|A|}$ possible subsets.

Q.E.D.

Homework

Exercises from the Book:
No. 1, 2, 4, 6, 8 (page 134)

- Give a formal proof of the theorem on slide 9-9.