

# Predicates and Quantifiers

## What Propositional Logic Cannot Do

- We saw that some declarative sentences are not statements without specifying the value of 'indeterminates'

“ $x + 2$  is an even number”

“If  $x + 1 > 0$ , then  $x > 0$ ”

“A man has a brother”

- Some valid arguments cannot be expressed with all our machinery of tautologies, equivalences, and rules of inference

All dogs go to heaven.

Plutto is a dog.

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$\vdash$  Plutto goes to heaven

## Open Statements or Predicates

- Sentences like ' $x$  is greater than 3' or 'person  $x$  has a brother' are not true or false unless the variable is assigned some particular value.
- Sentence ' $x$  is greater than 3' consists of 2 parts.
  - The first part,  $x$ , is called the **variable** or the **subject** of the sentence.
  - The second part – the predicate, ' $\text{is greater than } 3$ ' – refers to a property the subject can have.
- Sentences that have such structure are called **open statements** or **predicates**
- We write  $P(x)$  to denote a predicate with variable  $x$

## Unary, Binary, and so on

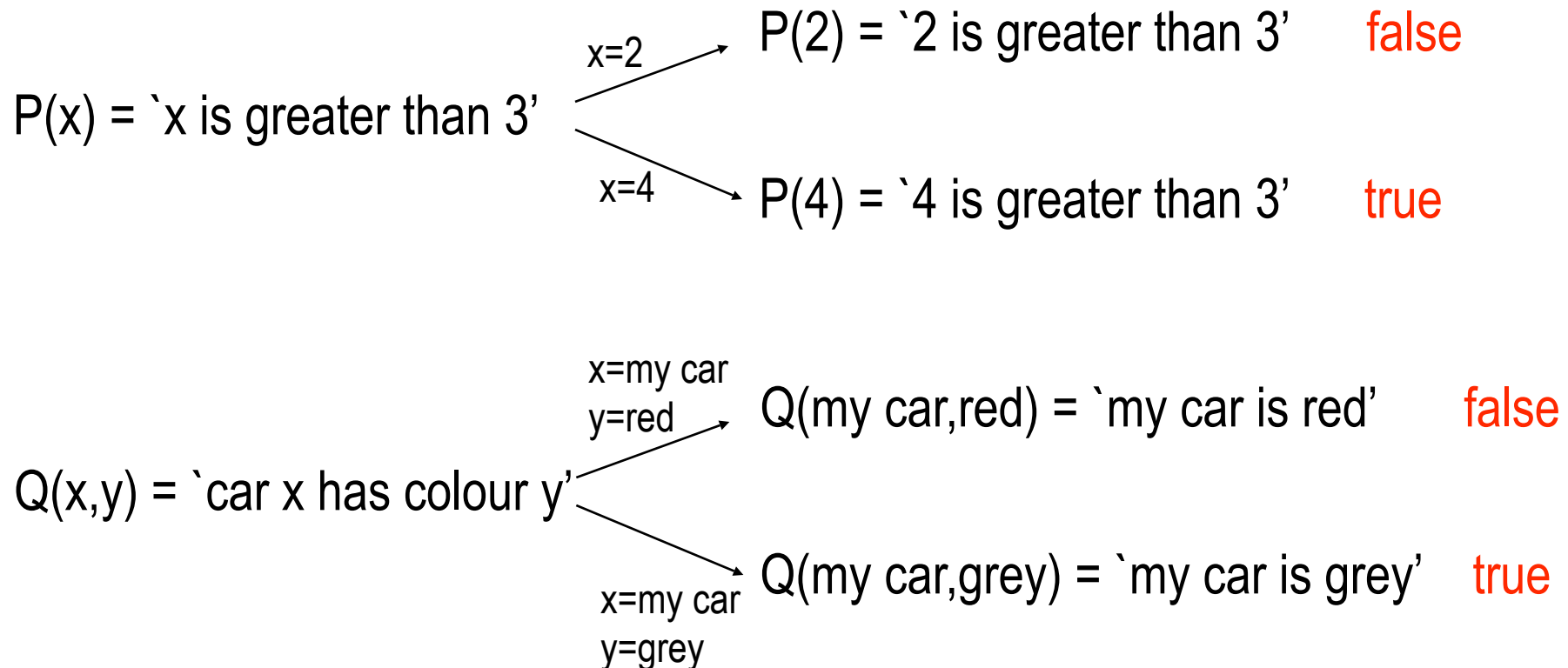
$\left. \begin{array}{l} \text{'x is greater than 3'} \\ \text{'x is my brother'} \\ \text{'x is a human being'} \end{array} \right\}$  contain only 1 variable, **unary** predicates  
 $P(x)$

$\left. \begin{array}{l} \text{'x is greater than y'} \\ \text{'x is the mother of y'} \\ \text{'car x has colour y'} \end{array} \right\}$  contain 2 variables, **binary** predicates  
 $Q(x,y)$

$\left. \begin{array}{l} \text{'x divides y + z'} \\ \text{'x sits between y and z'} \\ \text{'x is a son of y and z'} \end{array} \right\}$  contain 3 variables, **ternary** predicates  
 $R(x,y,z)$

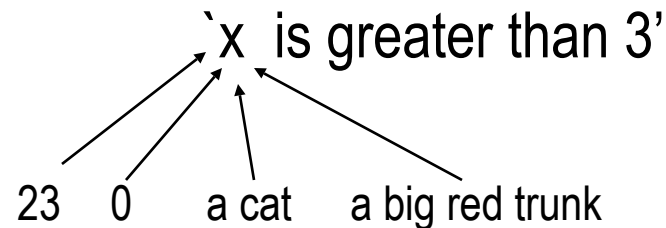
## Assigning a Value

- When a variable is assigned a value, the predicates turns into a statement, whose truth value can be evaluated.



# Universe

- We cannot assign a variable of a predicate ANY value. We need to obtain a meaningful statement!



- Every variable of a predicate is associated with a **universe** or **universe of discourse**, and its values are taken from this universe

'x is greater than 3'

x is a number

'x is my brother 3'

x is a human

'x is an animal'

x is a ???

'car x has colour y'

x is a car

y is a colour

# Relational Databases

- A **relational database** is a collection of **tables** like

No.	Name	Student ID	Supervisor	Thesis title
1	Bradley Coleman	30101234	Petra Berenbrink	Algebraic graph theory
...	...	...	...	...

A table consists of a schema and an instance. A schema is a collection of attributes, where each attribute has an associated universe of possible values. An instance is a collection of rows, where each row is a mapping that associates with each attribute of the schema a value in its universe.

- Every table is a predicate that is true on the rows of the instance and false otherwise.

# Quantifiers

● One way to obtain a statement from a predicate is to assign all its variables some values

● Another way to do that is to use expressions like

‘For every ...’

‘There is ... such that ...’

‘A ... can be found ...’

‘Any ... is ...’

‘Every man is mortal’

‘There is  $x$  such that  $x$  is greater than 3’

‘There is a person who is my father’

‘For any  $x$ ,  $x^2 \geq 0$ ’

quantification



# Universal Quantifiers

- Abbreviates constructions like
  - For all ...
  - For any ...
  - Every ...
  - Each ...
- Asserts that a predicate is true for all values from the universe
  - 'Every man is mortal'
  - 'All lions are fierce'
  - 'For any  $x$ ,  $x^2 \geq 0$ '
- Notation:  $\forall$
- $\forall x P(x)$  means that for every value  $a$  from the universe  $P(a)$  is true

## Universal Quantifiers (cntd)

`For any  $x$ ,  $x^2 \geq 0$  '                      true!

`Every car is red'                      false! my car is not red

●  $\forall x P(x)$  is false if and only if **there is** at least one value  $a$  from the universe such that  $P(a)$  is false

● Such a value  $a$  is called a **counterexample**

Thus to disprove that `Every man is mortal' it suffices to recall the movie `Highlander'



# Existential Quantifiers

- Abbreviates constructions like
  - For some ...
  - For at least one ...
  - There is ...
  - There exists ...
- Asserts that a predicate is true for at least one value from the universe
  - 'There is a living king'
  - 'Some people are fierce'
  - 'There are  $x$  such that  $x^2 \geq 10$ '
- Notation:  $\exists$
- $\exists x P(x)$  means that there is a value  $a$  from the universe such that  $P(a)$  is true

## Existential Quantifiers (cntd)

`There is a red car'

true! my friend's car is red

`For some  $x$ ,  $x^2 < 0$ '

false!

- $\exists x P(x)$  is false if and only if for all  $a$  from the universe  $P(a)$  is false
- Disproving an existential statement is difficult!

# Quantifiers and Negations

## Summarizing

	true	false
$\forall x P(x)$	For every value $a$ from the universe $P(a)$ is true	There is a counterexample – a value $a$ from the universe such that $P(a)$ is false
$\exists x P(x)$	There is a value $a$ from the universe such that $P(a)$ is true	For all values $a$ from the universe $P(a)$ is false

## Observe that

$\forall x P(x)$  is false if and only if  $\exists x \neg P(x)$  is true

$\exists x P(x)$  is false if and only if  $\forall x \neg P(x)$  is true

## Example

- What is the negation of each of the following statements?

Statement	Negation
All lions are fierce $\forall x P(x)$	There is a peaceful lion
Everyone has two legs $\forall x P(x)$	There is a person having more than two legs, one leg, or no legs at all
Some people like coffee $\exists x P(x)$	All people hate coffee
There is a lady in one of these rooms (Some rooms contain a lady)	There is no lady anywhere

## Multiple quantifiers

- Often open statements have more than one variable. In this case we need more than one quantifier.

$P(x,y)$  = “car  $x$  has colour  $y$ ”

$\forall x \forall y P(x,y)$  “every car is painted all colours

$\exists x \exists y P(x,y)$  “there is a car that is painted some colour

$\forall x \exists y P(x,y)$  “every car is painted some colour

$\exists x \forall y P(x,y)$  “there is a car that is painted all colours

# Open and Bound Variables

- In the statement

“car  $x$  has some colour”  $\exists y P(x,y)$

variables  $x$  and  $y$  play completely different roles.

- Variable  $y$  is **bound** by the existential quantifier. Effectively it disappeared from the statement.

- Variable  $x$  is not bound, it is **free**.

- Another example:

“ $x$  is the least number”  $Q(x,y) = \text{“}x \text{ is less than } y\text{”}$

$\forall y Q(x,y)$  or  $\forall y (x < y)$

“ $x$  is the greatest number”  $\forall y Q(y,x)$  or  $\forall y (y < x)$



- Quantifying some variables helps creating new predicates

Let  $P(x,y)$  mean “lion  $x$  likes  $y$ ”

$R(x) = \forall y \ P(x,y)$       "lion x likes everything"       $R(x)$  is always false  
(say it using quantifiers:  $\forall x \neg R(x)$     or     $\forall x \neg (\forall y \ P(x,y))$  )

$S(x) = \exists y \ P(x,y)$       “lion  $x$  has favorite food”

## Quantifiers and Compound Statements

- Quantifiers can be use together with logic connectives

“Every car is either red or blue”

$P(x)$  - “car  $x$  is red”

$Q(x)$  - “car  $x$  is blue”

$$\forall x (P(x) \vee Q(x))$$

“Everyone who knows a current password can logon onto the network”

$P(x)$  - “ $x$  knows a current password”

$Q(x)$  - “ $x$  can logon onto the network”

$$\forall x (P(x) \rightarrow Q(x))$$

## Quantifiers and Compound Statements (cntd)

- Logic connectives can be put between quantified statements

“Every car is blue, or there is a red car”

$P(x)$  - “car  $x$  is blue”

$Q(x)$  - “car  $x$  is red”

$$(\forall x P(x)) \vee (\exists x Q(x))$$

“For every number there is a smaller one, or there is the least number”

We use predicate  $x < y$

$$(\forall x \exists y (y < x)) \vee (\exists x \forall y (x < y))$$

# Homework

Exercises from the Book:

No. 1, 2, 4acij, 9a(i,iv), 12(vii,viii) (page 100-102)