

Rules of Inference

Previous Lecture

- Logically equivalent statements
- Statements Φ and Ψ are equivalent iff $\Phi \leftrightarrow \Psi$ is a tautology
- Main logic equivalences
 - double negation
 - DeMorgan's laws
 - commutative, associative, and distributive laws
 - idempotent, identity, and domination laws
 - the law of contradiction and the law of excluded middle
 - absorption laws

Expressing Connectives

● Some connectives can be expressed through others

- $p \oplus q \iff \neg(p \leftrightarrow q)$

- $p \leftrightarrow q \iff (p \rightarrow q) \wedge (q \rightarrow p)$

- $p \rightarrow q \iff \neg p \vee q$



Theorem Every compound statement is logically equivalent to a statement that uses only conjunction, disjunction, and negation

Example

● Show that

$$\neg(p \leftrightarrow q) \Leftrightarrow \neg p \leftrightarrow q$$

We show that $\neg(p \leftrightarrow q) \Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$

and

$$\neg p \leftrightarrow q \Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$$

Example

“If you are a computer science major or a freshman and you are not a computer science major or you are granted access to the Internet, then you are a freshman or have access to the Internet”

p - ‘you can access the Internet from campus’

q - ‘you are a computer science major’

r - ‘you are a freshman’

First Law of Substitution

● Suppose that the compound statement Φ is a tautology. If p is a primitive statement that appears in Φ and we replace each occurrence of p by the same statement q , then the resulting compound statement Ψ is also a tautology.

● Let $\Phi = (p \rightarrow q) \vee (q \rightarrow p)$, and we substitute p by $p \vee (s \oplus r)$

Therefore $((p \vee (s \oplus r)) \rightarrow q) \vee (q \rightarrow (p \vee (s \oplus r)))$ is a tautology

Second Law of Substitution

● Let Φ be a compound statement, p an arbitrary (not necessarily primitive!) statement that appears in Φ , and let q be a statement such that $p \Leftrightarrow q$. If we replace one or more occurrences of p by q , then for the resulting compound statement Ψ we have $\Phi \Leftrightarrow \Psi$.

● Let $\Phi = (p \rightarrow q) \vee (q \rightarrow p)$, and we substitute the first occurrence of p by $p \vee (p \wedge q)$.

Recall that $p \Leftrightarrow p \vee (p \wedge q)$ by Absorption Law.

Therefore

$$(p \rightarrow q) \vee (q \rightarrow p) \Leftrightarrow ((p \vee (p \wedge q)) \rightarrow q) \vee (q \rightarrow p).$$

Logic Inference

- The goal of an argument is to **infer** the required **conclusion** from given **premises**
- Formally, an argument is a sequence of statements, each of which is either a premises, or obtained from preceding statements by means of a rule of inference

Logic Inference

- One of the main goals of logic is to distinguish valid and invalid arguments

What can we say about the following arguments:

“If you have a current password, then you can log onto the network.
You have a current password. Therefore, you can log onto the network.”

and

“If you have a current password, then you can log onto the network.
You can log onto the network. Therefore, you have a current password.”

Logic Inference (cntd)

- Write these arguments in symbolic form

p - you have a current password

q - you can log onto the network

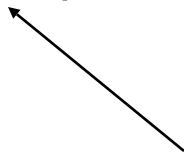
$$\frac{p \rightarrow q \quad p}{\text{---}}$$

$\vdash q$

$$\frac{p \rightarrow q \quad q}{\text{---}}$$

$\vdash p$

‘Therefore’



Inference and Tautologies

● Check that $\Phi = ((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology

If $(p \rightarrow q) \wedge p$ is false, that is if one of $p \rightarrow q$ and p is false, then Φ is true

If $(p \rightarrow q) \wedge p$ is true, then both $p \rightarrow q$ and p are true. Since the implication $p \rightarrow q$ is true and p is true, q must also be true.

Therefore Φ is true.

Therefore whatever values of p and q are, if $((p \rightarrow q) \wedge p)$ is true, then q is also true.

The first example is a **valid** argument!

Inference and Tautologies

● Let us try $\Psi = ((p \rightarrow q) \wedge q) \rightarrow p$

p	q	$p \rightarrow q$	$((p \rightarrow q) \wedge q)$	Ψ
0	0	1	0	1
0	1	1	1	0
1	0	0	0	1
1	1	1	1	1

In the case $p = 0$, $q = 1$ both conditions $(p \rightarrow q)$ and q are true, but p is false.

This is not a valid argument!

General Definition of Inference

- The general form of an argument in symbolic form is

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \vdash q$$

Diagram illustrating the structure of an argument:

- The expression $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ represents the **premises**.
- The expression q represents the **conclusion**.

- The argument is **valid** if whenever each of the premises is true the conclusion is also true
- The argument is valid if and only if the following compound statement is a tautology

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \longrightarrow q$$

Rules of Inference


- Verifying if a complicated statement is a tautology is nearly impossible, even for computer. Fortunately, general arguments can be replaced with small collection of simple ones, **rules of inference**.

- *Modus ponens*

$$\begin{array}{c} p \rightarrow q \\ p \\ \hline \vdash q \end{array}$$

“If you have a current password, then you can log onto the network.
You have a current password.
Therefore, you can log onto the network.”

Rule of Syllogism


$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \vdash p \rightarrow r \end{array}$$

 The corresponding tautology $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

“If you send me an e-mail, then I’ll finish writing the program.
If I finish writing the program, then I’ll go to sleep early.”


p - ‘you will send me an e-mail’

q - ‘I will finish writing the program’

r - ‘I will go to sleep early’

“Therefore, if you send me an e-mail, then I’ll go to sleep early”

Modus Tollens


$$\begin{array}{c} p \rightarrow q \\ \neg q \\ \hline \vdash \neg p \end{array}$$

 The corresponding tautology $((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$

“If today is Friday, then tomorrow I’ll go skiing.
I will not go skiing tomorrow.

p - ‘today is Friday’

q - ‘I will go skiing tomorrow’

“Therefore, today is not Friday”

Rule of Disjunctive Syllogism

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \vdash q \end{array}$$

● The corresponding tautology $((p \vee q) \wedge \neg p) \rightarrow q$


“I’ll go skiing this weekend.
I will not go skiing on Saturday.”


p - ‘I will go skiing on Saturday’

q - ‘I will go skiing on Sunday’

“Therefore, I will go skiing on Sunday”

Rule for Proof by Cases


$$\frac{p \rightarrow r \quad q \rightarrow r}{\vdash (p \vee q) \rightarrow r}$$

 The corresponding tautology $((p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$

“If today is Saturday, then I’ll go skiing.
If today is Sunday, then I’ll go skiing.

p - ‘today is Saturday’

q - ‘today is Sunday’

r - ‘I’ll go skiing’

“Therefore, if today is weekend, then I will go skiing”

Rules of Contradiction, Simplification, and Amplification

● Rule of Contradiction

Reductio ad Absurdum

$$\frac{\neg p \rightarrow F}{\vdash p} \quad \text{The corresponding tautology } (\neg p \rightarrow F) \rightarrow p$$

● Rule of Simplification

$$\frac{p \wedge q}{\vdash p} \quad \text{The corresponding tautology } (p \wedge q) \rightarrow p$$

● Rule of Amplification

$$\frac{p}{\vdash p \vee q} \quad \text{The corresponding tautology } p \rightarrow (p \vee q)$$

Example

● Premises:

“It is not sunny this afternoon and it is colder than yesterday.

We will go swimming only if it is sunny.

If we do not go swimming, then we will take a canoe trip.

If we take a canoe trip, then we will be home by sunset.”

● Conclusion: “We will be home by sunset.”

● Notation:	p - it is sunny this afternoon
q - it is colder than yesterday	s - we will take a canoe trip
r - we will go swimming	t - we will be home by sunset

● Premises: $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$

● Conclusion: t

Example (cntd)

● We have $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$

Step	Reason
1. $\neg p \wedge q$	premise
2. $\neg p$	simplification of (1)
3. $r \rightarrow p$	premise
4. $\neg r$	modus tollens of (2) and (3)
5. $\neg r \rightarrow s$	premise
6. s	modus ponens of (4) and (5)
7. $s \rightarrow t$	premise
8. t	modus ponens of (6) and (7)

Homework

Exercises from the Book:

No. 1aii (express implications by \neg and \vee), 3b, 13 (page 66)

- Express conjunction and disjunction through implication and negation (*)

No. 1a, 3c (page 84)