

Laws of Logic

Previous Lecture

- Truth tables
- Tautologies and contradictions
- Logic equivalences

Logic Equivalences

- Compound statements Φ and Ψ are said to be **logically equivalent** if the statement Φ is true (false) if and only if Ψ is true (respectively, false)

or

- The truth tables of Φ and Ψ are equal

or

- For any choice of truth values of the primitive statements (propositional variables) of Φ and Ψ , formulas Φ and Ψ have the same truth value
- If Φ and Ψ are logically equivalent, we write

$$\Phi \Leftrightarrow \Psi$$

Example Equivalences

- Implication and its contrapositive

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

- All tautologies are equivalent to T
- All contradictions are equivalent to F

Laws of Logic

Double negation

$$\neg \neg p \Leftrightarrow p$$

p	$\neg p$	$\neg \neg p$
0	1	0
1	0	1

Laws of Logic (cntd)

DeMorgan's laws

$$\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

$$\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg (p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

Why do we need laws of logic?

- Anything can be checked by the truth table. So why study laws of logic?
- Hmmmm....
- End of the lecture?
- By the way if we have 3 variables how many rows are there in the truth table?
- 8
- 10 variables - 1024 rows
- 20 variables ~ million rows
- 30 variable ~ billion
- 100 variables ~ **gazillion**
- Thus, truth tables are too large. Lecture continues...


Example

● Construct the negation of


‘Miguel has a cell phone and he has a laptop’

‘Heather will go to the concert or Steve will go to the concert’


'Algebraic' Laws of Logic


$$\begin{array}{l} p \wedge q \Leftrightarrow q \wedge p \\ p \vee q \Leftrightarrow q \vee p \end{array}$$

Commutative laws


$$\begin{array}{l} p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r \\ p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r \end{array}$$

Associative laws


$$\begin{array}{l} p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \\ p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r) \end{array}$$

Distributive laws



$$\begin{array}{l} p \wedge p \Leftrightarrow p \\ p \vee p \Leftrightarrow p \end{array}$$

Idempotent laws

'Logic' Laws of Logic


$$\begin{aligned} p \wedge T &\Leftrightarrow p \\ p \vee F &\Leftrightarrow p \end{aligned}$$

Identity laws



$$\begin{aligned} p \wedge \neg p &\Leftrightarrow F \\ p \vee \neg p &\Leftrightarrow T \end{aligned}$$

Inverse laws

the law of contradiction
the law of excluded middle


$$\begin{aligned} p \wedge F &\Leftrightarrow F \\ p \vee T &\Leftrightarrow T \end{aligned}$$

Domination laws


$$\begin{aligned} p \wedge (p \vee q) &\Leftrightarrow p \\ p \vee (p \wedge q) &\Leftrightarrow p \end{aligned}$$

Absorption laws

Example

- Simplify the statement

$$\neg(q \vee r) \vee \neg(\neg q \vee p) \vee r \vee p$$

Expressing Connectives

● Some connectives can be expressed through others

- $p \oplus q \iff \neg(p \leftrightarrow q)$
- $p \leftrightarrow q \iff (p \rightarrow q) \wedge (q \rightarrow p)$
- $p \rightarrow q \iff \neg p \vee q$



Theorem Every compound statement is logically equivalent to a statement that uses only conjunction, disjunction, and negation

Example

“If you are a computer science major or a freshman and you are not a computer science major or you are granted access to the Internet, then you are a freshman or have access to the Internet”

p - ‘you can access the Internet from campus’

q - ‘you are a computer science major’

r - ‘you are a freshman’

Example

- Simplify the statement

$$(p \vee q) \leftrightarrow (p \rightarrow q)$$

First Law of Substitution

● Suppose that the compound statement Φ is a tautology. If p is a primitive statement that appears in Φ and we replace each occurrence of p by the same statement q , then the resulting compound statement Ψ is also a tautology.

● Let $\Phi = (p \rightarrow q) \vee (q \rightarrow p)$, and we substitute p by $p \vee (s \oplus r)$

Therefore $((p \vee (s \oplus r)) \rightarrow q) \vee (q \rightarrow (p \vee (s \oplus r)))$ is a tautology

Second Law of Substitution

- Let Φ be a compound statement, p an arbitrary (not necessarily primitive!) statement that appears in Φ , and let q be a statement such that $p \Leftrightarrow q$. If we replace one or more occurrences of p by q , then for the resulting compound statement Ψ we have $\Phi \Leftrightarrow \Psi$.
- Let $\Phi = (p \rightarrow q) \vee (q \rightarrow p)$, and we substitute the first occurrence of p by $p \vee (p \wedge q)$.
Recall that $p \Leftrightarrow p \vee (p \wedge q)$ by Absorption Law.

Therefore

$$(p \rightarrow q) \vee (q \rightarrow p) \Leftrightarrow ((p \vee (p \wedge q)) \rightarrow q) \vee (q \rightarrow p).$$

Homework

Exercises from the Book:

No. 1ai, 2, 6a, 6b, 14a (page 66)

- Express conjunction and disjunction through implication and negation
(*)