Propositional Logic

Discrete Mathematics

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Use of Logic

- In mathematics and rhetoric:

  - Give precise meaning to statements.
  - Distinguish between valid and invalid arguments.
  - Provide rules of `correct' reasoning.

Natural language can be very ambiguous

`If you do your homework, then you'll get to watch the game.'

≬

`If you don't do your homework, then you will not get to watch ...'

`You do your homework, or you'll fail the exam.'

=

`If you don't do your homework, then you'll fail the exam.'
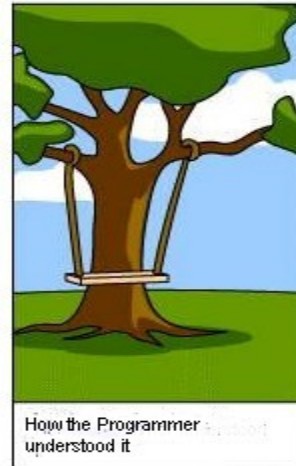
Use of Logic (cntd)

- In computing:
    - Derive new data / knowledge from existing facts
    - Design of computer circuits.
    - Construction of computer programs.
    - Verification of correctness of programs and circuit design.
    - Specification

What the customer
really needed

How the Programmer
understood it

What the customer
got

Statements (propositions)

- Propositional logic deals with statements and their truth values

- A statement is a declarative sentence that can be true or false

- Truth values are TRUTH (T or 1) and FALSE (F or 0).

- Examples:
    - $1 + 1 = 2$ (statement, T)
    - The moon is made of cheese   (statement, F)
    - Go home!   (not statement, imperative)
    - What a beautiful garden!   (not statement, exclamation)
    - Alice said, `What a beautiful garden!' (statement, depends on Alice)
    - $y + 1 = 2$ (not statement, uncertain)
    - Human descended from an ape   (statement, ?)

Compound Statements

● Simplest statement are called primitive statement

● We cannot decide the truth value of a primitive statement. This is not what logic does.

We shall use propositional variables to denote primitive statements, *p, q, r*, …

● Instead we combine primitive statements by means of logic connectives into compound statements or formulas and look how the truth value of a compound statement depends on the truth values of the primitive statements it includes.

We will denote compound statements by $\Phi, \Psi, \ldots$

Logic Connectives

- negation (not, ¬)   `It is not true that at least one politician was honest'

- conjunction (and, ∧)   `In this room there is a lady, and in the other room there is a tiger'

- disjunction (or, ∨)   `Margaret Mitchell wrote `Gone with the Wind or I am going home'

- implication (if…, then…, →)   `If there is a tiger in this room, then there is no lady there'

- exclusive or (either …, or …, ⊕)   `There is either a tiger in this room, or a lady'

- biconditional (equivalence) (if and only if, ↔)

      `There is a lady in this room if and only if there is a tiger in the other room'

Truth tables

Truth table is a way to specify the exact dependence of the truth value of a compound statement through the values of primitive statements involved

truth values of primitive statements
(propositional variables)

truth value of compound statements
(formulas)

p	q	Φ
0	0	0
0	1	1
…	…	…

Truth Tables of Connectives (negation and conjunction)

● Negation

p	¬p
F (0)	T (1)
T (1)	F (0)

unary connective

`Today is Friday' ⟹
`Today is not Friday'

● Conjunction

p	q	p ∧ q
0	0	0
0	1	0
1	0	0
1	1	1

binary
connective

`Today is Friday'
`It is raining'   ⟹

`Today is Friday and it
is raining'

Truth Tables of Connectives (disjunction)

🔴 Disjunction – inclusive `or'

p	q	p ∨ q
0	0	0
0	1	1
1	0	1
1	1	1

`Students who have taken calculus can take this course'

⇒

`Students who have taken computing can take this course'

`Students who have taken calculus or computing can take this course'

Be careful with `or' constructions in natural languages!

`You do your homework, or you'll fail the exam.'

`Today is Friday or Saturday'

Truth Tables of Connectives (exclusive or)

🔴 Exclusive `or'

One of the statements is true but not both

p	q	p $\oplus$ q
0	0	0
0	1	1
1	0	1
1	1	0

`You can follow the rules or be disqualified.'

`Natalie will arrive today or Natalie will not arrive at all.'

Truth Tables of Connectives (implication)

● Implication

p	q	p $\rightarrow$ q
0	0	1
0	1	1
1	0	0
1	1	1

Note that logical (*material*) implication does not assume any causal connection.

`**If** black is white, **then** we live in Antarctic.'

'**If** pigs fly, **then** Paris is in France.'
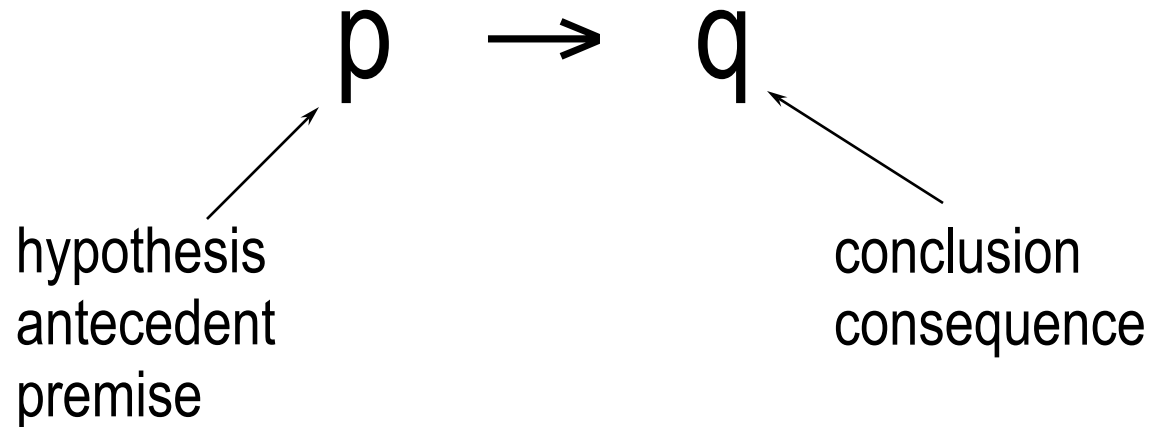
Implication as a promise

● Implication can be thought of as a promise, and it is true if the promise is kept

`**If** I am elected, **then** I will lower taxes'

- He is not elected and taxes are not lowered   promise kept!

- He is not elected and taxes are lowered   promise kept!

- He is elected, but (=and) taxes are not lowered   promise broken!

- He is elected and taxes are lowered   promise kept!

Playing with Implication

🔴 Parts of implication

$$p \rightarrow q$$

hypothesis
antecedent
premise

conclusion
consequence

`if p, then q'            `q unless ¬p'   `a sufficient con-
                                                     dition for q is p'
`if p, q'                 `p implies q'

`p is sufficient for q'   `p only if q'   `a necessary con-
                                                     dition for p is q'
`q if p'                  `q whenever p'

`q when p'                `q follows from p'

Playing with Implication (cntd)

- Converse, contrapositive, and inverse

$$p \rightarrow q$$ `The home team wins whenever it is raining'
(`If it is raining then the home team wins')

- Converse $$q \rightarrow p$$

`If the home team wins, then it is raining'

- Contrapositive $$\neg q \rightarrow \neg p$$

`If the home team does not win, then it is not raining'

- Inverse $$\neg p \rightarrow \neg q$$

`If it is not raining, then the home team does not win'

Truth Tables of Connectives (biconditional)

🔴 Biconditional or Equivalence

One of the statements is true if and only if the other is true

p	q	p ⟷ q
0	0	1
0	1	0
1	0	0
1	1	1

`You can take the flight if and only if you buy a ticket.'

Example

`You can access the Internet from campus if you are a computer science major or if you are not a freshman.'

p - `you can access the Internet from campus'

q - `you are a computer science major'

r - `you are a freshman'

Tautologies

- Tautology is a compound statement (formula) that is true for all combinations of truth values of its propositional variables

$$(p \rightarrow q) \lor (q \rightarrow p)$$

p	q	$(p \rightarrow q) \lor (q \rightarrow p)$
0	0	1
0	1	1
1	0	1
1	1	1

Contradictions

- Contradiction is a compound statement (formula) that is false for all combinations of truth values of its propositional variables

$$(p \oplus q) \wedge (p \oplus \neg q)$$

p	q	$(p \oplus q) \wedge (p \oplus \neg q)$
0	0	0
0	1	0
1	0	0
1	1	0

Homework

Exercises from the Book:

No. 1 ,3, 4, 8a, 8c (page 54)