MACM 101 — Discrete Mathematics I

Exercises on Combinatorics, Probability, Languages and Integers. Due: Tuesday, November 24th (at the beginning of the class)

Reminder: the work you submit must be your own. Any collaboration and consulting outside resourses must be explicitely mentioned on your submission.
Please, use a pen. 30 points will be taken off for pencil written work.

1. How many bit strings of length 10 contain at least three 1s and at least three 0s.

   Solutions

   A string contains less than 3 ones if it contains no ones (1 string), exactly one 1 (10 strings) and exactly 2 ones ($\binom{10}{2} = 10 \cdot 9/2$ strings). There is the same number of strings that contain less than 3 zeros. There are 2^{10} strings of length 10. So the answer is

   $$2^{10} - 2(10 \cdot 9/2 + 10 + 1).$$

2. A five-card poker hand is a collection of five different cards that are selected from all 52 cards.

   1) What is the probability that a five-card poker hand contains exactly one ace?

   2) What is the probability that a five-card poker hand contains at least one ace?

   Solution

   1) We pick and ace and then 4 other cards so there are:

   $$4 \cdot \binom{48}{4}$$

poker hands like that. Thus probability to get such hand is

$$\frac{4 \cdot \binom{48}{4}}{\binom{52}{5}}.$$

2) There are

$$\binom{48}{5}$$

hands NOT like that. So the probability is

$$1 - \frac{\binom{48}{5}}{\binom{52}{5}}.$$

3. Let $L_1 = \{aaaa\}^*$, $L_2 = \{a,b\}\{a,b\}\{a,b\}\{a,b\}\{a,b\}\{a,b\}$, and $L_3 = L_2^*$. Describe strings in the language $L_1 \cap L_3$.

   Solution

   A string is in L_1 if it consists of as and its length is divisible by 4. A string is in L_2 if its length is divisible by 6. Thus a string is in $L_1 \cap L_3$ if it consists of as and its length is divisible by both 6 and 4. We know that the latter is the case if the length is divisible by $lcm(6,4) = 12$. Thus $L_1 \cap L_3 = \{aaaaaaaaaaaa\}^*$.

4. Give a regular expression that represents the language of strings over $\{a,b,c\}$, in which the number of a's is divisible by four.

   Solution

   $$(\{b,c\}^*a)(\{b,c\}^*a)(\{b,c\}^*a)(\{b,c\}^*a)\{b,c\}^*.$$

5. A palindrome is a string that reads the same backward as it does forward, that is, a string w where $w = w^R$, where w^R is reversal of the string w. Find a grammar that generates the set of all palindromes over the alphabet $\{0,1\}$.

   Solution

   The grammar is similar to the grammar generating the language of properly placed parentheses:

$S \to 0S0$
$S \to 1S1$
$S \to 0$
$S \to 1$
$S \to \lambda$.

Any string that can be obtained from S is a palindrome (possibly containing S in the middle). Proof by induction:

Base step: S is a palindrome. Done.

Inductive step: If we apply any of the rules the string remains a palindrome.

QED.

6. Build a finite automaton that accepts the language of all strings over $\{1, 2, 3\}$ such that the sum of all digits is divisible by seven.

The automaton has seven states $s_0, \ldots, s_6$. The only accepting state is s_0. The transition function works as follows:

$$v(s_k, d) = s_r,$$

where r is the remainder of division of $k + d$ by 7.

After reading string w the automaton is at state s_r, where r is the remainder of the division of sum of the digits in w by 7. Proof by induction:

Base step: Initially it's in s_0 and the sum of the digits in the empty string is 0.

Inductive step: Let $w' = wd$. Once the automaton has read w it's in the state s_r. Then after reading w' it will be in the state $s_{r' mod 7}$ where r' is the remainder of division of $r + d$ by 7.

Let x be the sum of the digits in w, the sum of the digits in w' equals

$$x + d = 7q + r + d = 7q + 7q' + r' = 7(q + q') + r',$$

and thus r' is the remainder of division of $x + d$ by 7. QED.

Therefore the automaton accepts the word w if and only if the sum of digits in w is divisible by 7 with remainder 0, i.e. just divisible by 7.

7. Prove that for every positive n, there are n consecutive composite integers.

   Solution

   Consider $(n+1)!+k$, where $k \in \{2, \ldots, n, n+1\}$. $(n+1)!$ is divisible by k and k is divisible by k, thus $(n+1)!+k$ is divisible by k. So for any n numbers $\{n!+2, \ldots, (n+1)!+n+1\}$ are composite.

8. Convert 53252 from its decimal expansion to its hexadicimal expansion.

   Solution Dividing 52352 by 16 with remainder repeatedly we get: 53252 = 4 + 16 (0 + 16 (0 + 16 * 13)). So

$$53252 = D004.$$

9. Use the Euclidean algorithm to find the greatest common divisor d of 1529 and 14038. Find integer numbers u and v such that $d = 1529u + 14038v$. (Note and use the fact that when you divide r_k by r_{k+1} with remainder you get u' and v' such that $r_{k+2} = u'r_k + v'r_{k+1}$.)

   Solution

$$\begin{aligned}
14308 &= 1529 \cdot 9 + 547 \\
1529 &= 547 \cdot 2 + 435 \\
547 &= 435 + 112 \\
435 &= 3 \cdot 112 + 99 \\
112 &= 99 + 13 \\
99 &= 13 \cdot 7 + 8 \\
13 &= 8 + 5 \\
8 &= 5 + 3 \\
5 &= 3 + 2 \\
3 &= 2 + 1
\end{aligned}$$

Thus the least common divisor is 1 and we have:

$$\begin{aligned}
1 &= \\
3 - 2 &= \\
2 \cdot 3 - 5 &= \\
2 \cdot 8 - 3 \cdot 5 &= \\
5 \cdot 8 - 3 \cdot 13 &= \\
5 \cdot 99 - 38 \cdot 13 &= \\
43 \cdot 99 - 38 \cdot 113 &= \\
43 \cdot 435 - 167 \cdot 112 &= \\
210 \cdot 435 - 167 \cdot 547 &= \\
210 \cdot 1529 - 587 \cdot 547 &= \\
5493 \cdot 1529 - 587 \cdot 14308&.
\end{aligned}$$

10. Find the prime factorization of numbers 92928 and 123552, and their least common multiple.

    Solution Trying to divide these numbers by small primes we luckily get:

$$92928 = 2^8 \cdot 3 \cdot 11^2,$$

$$123552 = 2^5 \cdot 3^3 \cdot 11 \cdot 13.$$

    Thus the least common multiple is $2^8 \cdot 3^3 \cdot 11^2 \cdot 13$.