MACM 101 — Discrete Mathematics I

Exercises on Functions and Induction. Due: Tuesday, November 10th (at the beginning of the class)

Reminder: the work you submit must be your own. Any collaboration and consulting outside resourses must be explicitely mentioned on your submission.
Please, use a pen. 30 points will be taken off for pencil written work.

1. Find the domain and range of the following function f. It assigns to each string of bits (that is, of 0s and 1s) the number equal to twice of the number of zeros in that string. For example $f(011001010101010) = 16$.

   Solution

   By definition f assigns a value to "each string of bits" thus the domain is the set of all bit strings.

   It's easy to see that the range is the set of all positive even numbers. An odd number can not be a value of f since the value is twice the number of zeros. Any even number $2k$ has a preimage: a string of k zeros.

2. Determine whether or not the function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto, if $f((m, n)) = m - n$.

   Solution

   Yes it is onto. For any number k we have $f((k, 0)) = k - 0 = k$.

3. Give an explicit formula for a function from the set of all integers to the set of positive integers that is onto but is not one-to-one.

   Solution

   The formula is $f(x) = |x| + 1$.

   That is $f(x)$ is the absolute value of x plus 1. The definition is correct because for any integer except for zero $|x|$ is a positive integer and $|x| + 1$ is a positive integer. For 0 we shall have $f(0) = 1$.

It is onto since any positive integer is an absolute value of itself, so any positive integer x will be equal to $f(x-1)$. It is not one-to-one because $f(5) = f(-5) = 6$.

4. Suppose that g is a function from A to B and f is a function from B to C. Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

   Solution

   Take any $x \in A, y \in A, x \neq y$. Since g is injective we have $g(x) \neq g(y)$. Thus since f is injective we have $f(g(x)) \neq f(g(y))$. By definition of $\circ$ it means that $(f \circ g)(x) \neq (f \circ g)(y)$.QED.

5. Show that $\frac{n-5}{n^2}$ is in $o(n)$.

   Solution

   By definition of o we need to show that

   $$\forall m \in \mathbb{R}\ \exists k \in \mathbb{N}\ \forall n \in \mathbb{N} \quad (n > k) \rightarrow \left(\left| \frac{n-5}{n^2} \right| < m|n| \right).$$

   For $n > 5$ we have

   $$\left| \frac{n-5}{n^3} \right| = \frac{n-5}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}.$$

   It is easy to see that for $m > 0$ and $n > \left\lceil \frac{1}{\sqrt{m}} \right\rceil$ we have $m > \frac{1}{n^2}$.

   Thus for $n > \max(5, \left\lceil \frac{1}{\sqrt{m}} \right\rceil)$ we have $m > \frac{1}{n^2} \geq \frac{n-5}{n^3}$ and consequently $m|n| > \left| \frac{n-5}{n^2} \right|$.

   Therefore for any m we should set $k = \max(5, \left\lceil \frac{1}{\sqrt{m}} \right\rceil)$ and the statement is proven.

6. Prove that for every positive integer n

   $$3 \sum_{i=1}^{n} i \cdot (i+1) = n(n+1)(n+2).$$

   Solution

Base Step: $3 \cdot 1(1+1) = 1(1+1)(1+2) \Leftrightarrow 6 = 6$. Proven.

Inductive Step:

$$\begin{aligned}
3\sum_{i=1}^{k+1} i \cdot (i+1) &= \\
3\sum_{i=1}^{k} i \cdot (i+1) + 3(k+1)(k+2) &= \text{inductive hypothesis} \\
k(k+1)(k+2) + 3(k+1)(k+2) &= \\
(k+3)(k+1)(k+2)
\end{aligned}$$

QED

7. A guest at a Computer Science Halloween party is a celebrity if this person is known by every other guest, but knows none of them. A particular party may have no celebrity. Your task is (1) to prove that there is at most one celebrity at a party; and (2) to find the celebrity if one exists, at a party, by asking only one type of question — asking a guest whether they know a second guest. Everyone must answer your question truthfully. That is, if Alice and Bob are two people at the party, you can ask Alice if she knows Bob; she must answer correctly. Use mathematical induction to show that if there are n people at the Halloween party then $3(n-1)$ questions are enough to find out the celebrity (if there is one).

   Solution

   Base Step: If we have two people A and B at the party then we just ask them if they know each other. If one of the answers is YES and the other is NO then the person who answered NO is a celebrity, otherwise there is no celebrity. We need only 2 questions for that which is less than $3(2-1) = 3$.

   Inductive Step: We assume that the statement is true for a party of k people and prove it for a party of $k+1$ people. Let A and B be two people at the party. We ask A if (s)he knows B. If the answer is YES then A is not a celebrity, if the answer is NO then B is not a celebrity. In any case we can exclude one person. Next we consider the remaining k people as a party and use the inductive hypothesis to find a celebrity among them in $3(k-1)$ steps. If there is no celebrity among them then we are done. If there is a celebrity C then we need to two more questions: we ask C if (s)he knows the excluded person and the excluded person if (s)he knows C. If C does not know the excluded person and the excluded person knows C then C is a celebrity.

8. Recall that $P(N)$ is a powerset of natural numbers, i.e. the set of all subsets of N. Let the function $f : N \to P(N)$ be defined as follows. For any number n the value $f(n)$ is the set of positions on which in the decimal writing of n we have 5. For instance $f(5) = \{1\}, f(55) = \{1, 2\}, f(2515) = \{1, 3\}, f(123523515) = \{1, 3, 6\}$. (The positions are counted from the right).

   What is image of f? Use induction to prove your answer.

   Solution

   The image (range) of f is the set of all finite subsets of N. Indeed any natural number has finite number of digits and consequently 5s can not occupy an infinite set of positions.

   We use induction to show that any finite subset has a preimage. The induction goes on the number n of elements of the set.

   Base step: We have $f(0) = \varnothing$ and have the statement proven for the size 0.

   Inductive step: Let $F \subset N$, $|F| = k+1$. Let x be the largest element in F. By the inductive hypothesis there is $y \in N$ such that $f(y) = F \backslash \{x\}$. Then we have $f(5 \cdot 10^x + y) = F$. QED.

9. Recall that for sets A and B we write $|A| \geq |B|$ for "cardinality of A is greater or equal to the cardinality of B". Are there such A, B and c that $c \notin$A, $A \cap B = \varnothing, |B| \geq |A|$ and $|A \cup \{c\}| \geq |B|$? **Prove** your answer.

   Solution

   The problem is actually simple. Let $A = B = \varnothing$ and $c = 0$. Then $|A| = |B| = 0$, and $|A| \geq |B|$. Moreover $|A \cup \{c\}| = 1 \geq 0 = |B|$.