MACM 101 — Discrete Mathematics I

Exercises on Functions and Relations. Due: Tuesday, October 27th (at the beginning of the class)

Reminder: the work you submit must be your own. Any collaboration and consulting outside resourses must be explicitely mentioned on your submission.
Please, use a pen. 30 points will be taken off for pencil written work.
Please, write your name clearly, the way it is entered in the Gradebook. Make you TA happy.

1. Using laws of set theory show that

$$(A - B) - C = (A - C) - (B - C)$$

Solution

From right to left:

$$\begin{aligned}
(A - C) - (B - C) &= \text{by definition of “$-$”} \\
(A \cap \overline{C}) \cap \overline{(B \cap \overline{C})} &= \text{by De Morgan's laws and complementation law} \\
(A \cap \overline{C}) \cap \overline{C}) \cap (\overline{B} \cup \overline{C}) &= \text{by distributivity} \\
(A \cap \overline{C} \cap \overline{B}) \cup (A \cap \overline{C} \cap C) &= \begin{cases} \text{definition of $-$, complement law,} \\ \text{domination law and commutativ-} \\ \text{ity} \end{cases} \\
(A - B) - C &
\end{aligned}$$

2. Let A, B, and C be sets. Show that

$$(A - C) \cap (C - B) = \varnothing$$

Draw Venn diagrams for the expression on the left side.

Solution

By definition of $-$, associativity of $-$ and complement law we have

$$(A - C) \cap (C - B) = A \cap \overline{C} \cap C \cap \overline{B} = \varnothing \cap A \cap \overline{B} = \varnothing.$$

3. What can you say about the sets A and B if we know that

$$A \cup B = B \cap A?$$

Solution

We can conclude that $A = B$. Consider an arbitrary $x \in A$. Then $x \in B \cap A$. Using the given equation we get $x \in B \cup A$. By definition of $\cup$ we have $x \in B$. Thus we've proven that $A \subseteq B$. It is shown analogously that $A \supseteq B$.

4. Show that for any sets A and B

$$A \Delta B = (A \cup B) - (A \cap B).$$

Solution

$$\begin{aligned} A\Delta B = \{x \mid x \in A \oplus x \in B\} &= \text{by the truth table} \\ \{x \mid (x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)\} &= \text{by definition of } \cup, \cap, - \\ (A \cup B) \cap \overline{A \cap B} &= \text{by definition of } - \\ (A \cup B) - (A \cap B) & \end{aligned}$$

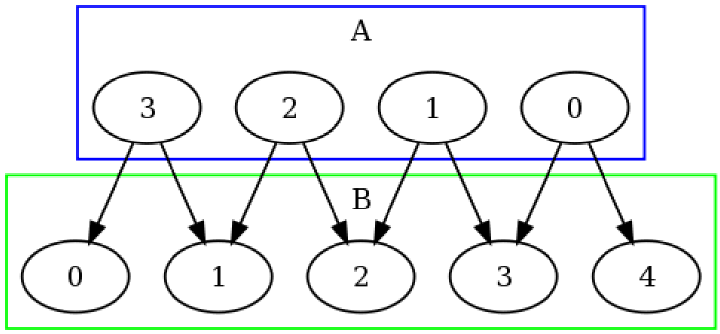
Note. *If you use this method, please draw the truth table.*

5. Make a list of pairs, construct the matrix, and draw the graph of the relation R from the set $A = \{0, 1, 2, 3\}$ to the set $B = \{0, 1, 2, 3, 4\}$ such that $(a, b) \in R$ if and only if $a + b \in \{3, 4\}$.

Solution

The list of pairs is $\{(0, 3), (0, 4), (1, 2), (1, 3), (2, 1), (2, 2), (3, 0), (3, 1)\}$. The matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$



6. Prove that

$$A \times (A \cup B) \times C = (A \times A \times C) \cup (A \times B \times C).$$

Solution

$$\begin{aligned} A \times (A \cup B) \times C = & \\ \{(x,y,z) \mid x \in A \wedge (y \in A \vee y \in B) \wedge z \in C\} &= \text{by distributivity} \\ \{(x,y,z) \mid (x \in A \wedge y \in A \wedge z \in C) \vee (x \in A \wedge y \in B \wedge z \in C)\} &= \text{by definition of } \times, \cup \\ (A \times A \times C) \cup (A \times B \times C) & \end{aligned}$$

7. Show that the relation R, consisting of all pairs (x, y) where x and y are bit strings of length three or more that agree except perhaps in their first three bits, is an equivalence relation on the set of all bit strings.

Solution

1) Reflexivity: any string agrees with itself everywhere.

2) Symmetricity: If a agrees with b on any character after 4 then b also agrees with a.

3) Transitivity: If a agrees with b and b agrees with c on some character then a agrees with c on this character, thus the property holds.

8. Relation R is given by matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Is R an order? If yes, what its minimal, maximal, least, and greatest elements are?

Solution

Let us assume that the elements of the set are $1, 2, 3$ and 4 and i-th row/column of the matrix correspond to i.

This relation is not an order, because order must be transitive, while we have $(4, 1) \in R, (1, 3) \in R, (4, 3) \notin R$.

9. Let $A = \{1, 2, 3\}$, and let R be a binary relation on $A \times A \times A$ given by: $((a, b, c), (d, e, f)) \in R$ if and only if $a \leq d$, $b \leq e$, and $c \leq f$. Show that R is an order and draw its diagram.

Solution

Indeed we have reflexivity:

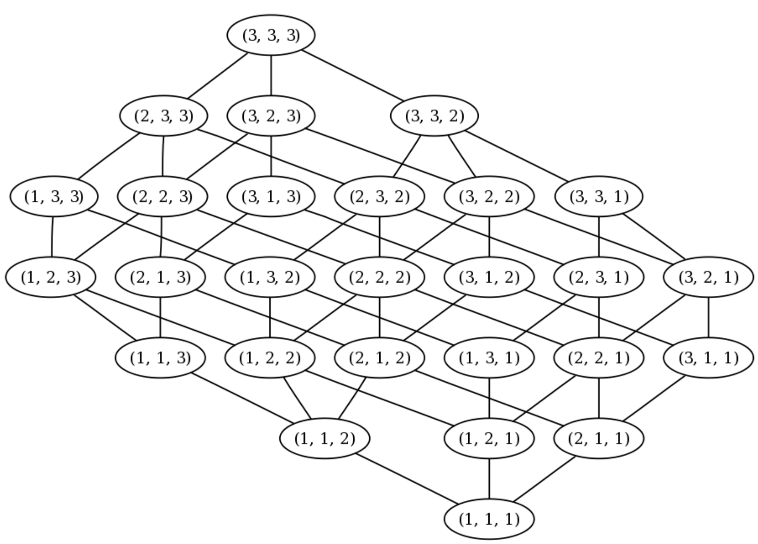
$$a \leq a, b \leq b, c \leq c \Rightarrow ((a, b, c), (a, b, c)) \in R,$$

antisymmetry:

$$a \leq d, d \leq a, b \leq e, e \leq b, c \leq f, f \leq c \Rightarrow a \leq d, b \leq e, c \leq f \Rightarrow (a, b, c) = (d, e, f),$$

and transitivity

$$a \leq d, d \leq g, b \leq e, e \leq h, c \leq f, f \leq i \Rightarrow a \leq g, b \leq h, c \leq i \Rightarrow ((a, b, c), (g, h, i)) \in R.$$



10. Let R be a relation that is symmetric, antisymmetric and reflexive. Show that R is equivalence relation.

Solution

In fact it is equality relation. That is $(x, y) \in R$ if and only if $x = y$.

Indeed if $(x, y) \in R$ then by symmetry $(y, x) \in R$ and by anty-symmetry $x = y$. It is known that equality is transitive.