MACM 101 — Discrete Mathematics I

Exercises on Predicates and Quantifiers. Due: Tuesday, October 13th (at the beginning of the class)

Reminder: the work you submit must be your own. Any collaboration and consulting outside resourses must be explicitely mentioned on your submission.
Please, use a pen. 30 points will be taken off for pencil written work.

1. Determine the truth value of each of these statements if the universe of each variable consists of (i) all real numbers, (ii) all integers.

   (a) $\exists x \exists y (x + y \neq y + x)$
   (b) $\forall x \exists y (x + y = 2 \wedge 2x - y = 2)$

   Solution

   (a) Formally negating the statement we get

   $$\forall x \forall y (x + y = y + x),$$

   which is the law of commutativity of addition.

   Thus statement (a) is false in both universes, because addition is commutative and for any x, y we have $x + y = y + x$.

   (b) The statement is false in both universes. To prove it we need to prove that negation of this statement is true.

   $$\neg(\forall x \ \exists y \ (x + y = 2 \wedge 2x - y = 2)) \Leftrightarrow$$
   $$\exists x \ \forall y \ x + y \neq 2 \vee 2x - y \neq 2)$$

   Let us assign $x = 2$ and then the quantified predicate turns into

   $$2 + y \neq 2 \vee 4 - y \neq 2 \Leftrightarrow y \neq 0 \vee y \neq 2.$$

   We see that the quantified statement is true for all y and thus we have the negation of statement (b) proven.

2. Use predicates and quantifiers to express this statement

   "There is a man who has visited some park in every province of Canada"

   Solution

   Let $V(x, y)$, where x is a person and y is a park be a predicate "Person x visited park y".

   Let $L(x, y)$, where x is a park and y is a province be a predicate "Park x is located in province y".

   Then the statement under consideration can be expressed as follows

   $$\exists x \ \forall y \ \exists z \ V(x, z) \wedge L(z, y).$$

3. Find a counterexample, if possible, to this universally quantified statement, where the universe for all variables consists of all integers

   $$\forall x \exists y \ (3xy = 12)$$

   Solution

   A counterexample is in particular $x = 12$. If we assume that there is y such that

   $$3xy = 12,$$

   then we would have $3 \cdot 12 \cdot y = 12$ and thus $y = 1/3$, which is does not belong to the universe of all integers.

4. Rewrite the following statement so that negations appear only within predicates (that is, no negation is outside a quantifier or an expression involving logical connectives)

   $$\neg \forall x \ ((\exists y \forall z \ P(x, y, z)) \leftrightarrow (\exists z \exists y \ R(x, y, z))).$$

   Solution

   Method 1 If you use connective $\oplus$ the problem gets easier:

   $$\begin{aligned} \neg \forall x \ ((\exists y \forall z \ P(x, y, z)) \leftrightarrow (\exists z \exists y \ R(x, y, z))) &\Leftrightarrow \\ \exists x \ \neg((\exists y \forall z \ P(x, y, z)) \leftrightarrow (\exists z \exists y \ R(x, y, z))) &\Leftrightarrow \\ \exists x \ ((\exists y \forall z \ P(x, y, z)) \oplus (\exists z \exists y \ R(x, y, z))) & \end{aligned}$$

Method 2 If you do not want to use $\oplus$ then we have to recall that

$$(\neg(p \leftrightarrow q)) \Leftrightarrow (p \leftrightarrow \neg q).$$

And then after the second step we continue:

$$\exists x\ \neg((\exists y \forall z\ P(x,y,z)) \leftrightarrow (\exists z \exists y\ R(x,y,z))) \Leftrightarrow$$
$$\exists x\ ((\exists y \forall z\ P(x,y,z)) \leftrightarrow (\neg(\exists z \exists y\ R(x,y,z)))) \Leftrightarrow$$
$$\exists x\ ((\exists y \forall z\ P(x,y,z)) \leftrightarrow (\forall z \forall y\ \neg R(x,y,z)))$$

5. Let $Q(x,y)$ be the statement "$x+y = x-y$". If the universe of discourse for both variables is the set of integers, what are the truth values of the following?

   a) $Q(1,1)$

   b) $Q(2,0)$

   c) $\exists x\ Q(x,2)$

   d) $\exists x \forall y\ Q(x,y)$

   e) $\exists y \forall x\ Q(x,y)$

   Prove your answers.

   Solution

   a) $1+1 = 1-1$ — False.

   b) $2+0 = 2-0$ — True.

   c) $\exists x\ x+2 = x-2$ — False, because the equality is equivalent to $2 = -2$.

   d) $\exists x \forall y\ x+y = x-y$ — False. Consider negation of this statement:

$$\forall x \exists y\ x+y \neq x-y. \tag{1}$$

   For arbitrary x we have $x+2 \neq x-2$ and we prove the statement (**??**) using $x+2 \neq x-2$ as a premise:

   1) $x+2 \neq x-2$ (premise)

   2) $\exists y\ x+y \neq x-y$ (rule of existential generalization applied to 1))

   3) $\forall x\ \exists y\ x+y \neq x-y$ (rule of universal generalization app. to 2))

   e) $\exists y\ \forall x\ x+y = x-y$ — True. Indeed if we set $y=0$ then the $Q(x,y)$ turns into $x+0 = x-0$ which is true for any x.

6. Are the following statements logically equivalent?

$$\exists x P(x) \wedge Q(x) \quad \text{and} \quad (\exists x P(x)) \wedge (\exists y Q(y))$$

Solution

No they are not. In case the universe of discourse is the set of all integer numbers, $P(x)$ is "x is even", and $Q(y)$ is "y is odd" we have the first statement saying

"There exists a number which is odd and even", while the second means

"There is an even number and there is an odd number". So the second statement gets to be true and the first gets to be false.

I have received several emails asking whether its a typo that one of the expressions contain y while another does not. Note that some statements containing different variables are equivalent. For instance

$$(\forall x P(x)) \rightarrow (\forall y Q(y))$$

is equivalent to

$$(\exists z \neg P(z)) \vee (\forall y Q(y)).$$

That happens because once variable is bounded by a quantifier it does not matter what name it was.

7. Determine whether the following argument is valid or invalid and explain why.

   'No ducks are willing to waltz.'
   'No officers ever decline to waltz.'
   'All my poultry are ducks.'
   'Therefore, my poultry are not officers'.

Solution

The argument is valid. Let's prove the concluded statement by contradiction. Let x be an officer member of my poultry. All my poultry are ducks, consequently x is a duck. No ducks are willing to waltz thus x is not willing to waltz. But x is an officer and no officers ever decline to waltz, thus x never declines a waltz. We came to a contradiction x is not willing to waltz and x is willing to waltz. QED.

8. Given premises: 'All clear explanations are satisfactory'
'Some excuses are unsatisfactory'
infer 'Some excuses are not clear explanations.'

   Write the proof formally.

   Solution

   Let $S(x)$ be "x is a satisfactory", $C(x)$ be "x is a clear explanation" and $E(x)$ be "x is an excuse". Then the premises turn into:
$\forall x\ C(x) \to S(x)$
$\exists x\ E(x) \wedge \neg S(x)$.

   While conclusion is $\exists x\ E(x) \wedge \neg C(x)$.

   The proof goes as follows:

   1) $\exists x\ E(x) \wedge \neg S(x)$ (premise)

   2) $E(x_0) \wedge \neg S(x_0)$ (for some x_0, by existential specification of 1))

   3) $\forall x\ C(x) \to S(x)$ (premise)

   4) $C(x_0) \to S(x_0)$ (by universal specification of 3))

   5) $\neg S(x_0)$ (by simplification of 2))

   6) $\neg C(x_0)$ (Modus Tollens of 4) and 5))

   7) $E(x_0)$ (by simplification of 2))

   8) $E(x_0) \wedge \neg C(x_0)$ (conjunction of 6) and 7))

   9) $\exists x\ E(x) \wedge \neg C(x)$ (existential generalization of 8))

9. Find a universe for variables x, y, z for which the statement

$$\forall x \forall y((x = y) \to \exists z \quad z \neq x)$$

   is true and another universe in which it is false.

   Solution

   If the universe is $\{0, 1\}$ then the statement is true. Indeed consider arbitrary x and y and the statement

$$((x = y) \to \exists z \quad z \neq x).$$

   If $x \neq y$ then this statement is true, because premise is false. If $x = y = 0$ then there exists $z = 1$ for which $z \neq x$. Similarly with $x = y = 1$.

If the universe is $\{0\}$ then when $x = y = 0$ there is no such element in the universe that does not equal to x, because the universe contains the only element.