MACM 101 — Discrete Mathematics I

Exercises on Propositional Logic. Due: Tuesday, September 29th (at the beginning of the class)

SOLUTIONS

1. Construct a truth table for the following compound proposition: $(p \leftrightarrow q) \rightarrow (p \vee q)$

   Solution

p	q	$(p \leftrightarrow q) \rightarrow (p \vee q)$
0	0	0
0	1	1
1	0	1
1	1	1

2. Which of the following statements are tautologies?

   $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

   $(p \rightarrow q) \leftrightarrow (\neg(p \wedge \neg q))$

   $(p \vee q) \rightarrow (p \wedge q)$

   Solution

   Method 1: Use truth tables.

   Method 2: Reason about it:

   1) Yes, each of $(p \rightarrow q)$ and $(\neg p \vee q)$ is false if and only if p is true and q is false.

   2) Yes, it follows from 1), De Morgan's law and double negation law:

$$
\begin{gathered}
(p \rightarrow q) \leftrightarrow (\neg p \vee q) \Leftrightarrow \\
(p \rightarrow q) \leftrightarrow (\neg\neg(\neg p \vee q)) \Leftrightarrow \\
(p \rightarrow q) \leftrightarrow (\neg(\neg\neg p \wedge \neg q)) \Leftrightarrow \\
(p \rightarrow q) \leftrightarrow (\neg(p \wedge \neg q))
\end{gathered}
$$

   3) No, if we set $p = 0, q = 1$ then the formula turns into 0.

3. Show that $(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalent.

   Method 1: Use truth table.

Method 2: Use the laws of logic.

$$\begin{aligned}
(p \to r) \wedge (q \to r) &\Leftrightarrow \text{(definition of } \to) \\
(\neg p \vee r) \wedge (\neg q \vee r) &\Leftrightarrow \text{(distributivity)} \\
(\neg p \wedge \neg q) \vee r &\Leftrightarrow \text{(De Morgan)} \\
(\neg(p \vee q)) \vee r &\Leftrightarrow \text{(definition of } \to) \\
(p \vee q) \to r &
\end{aligned}$$

4. Show that $(a \wedge b \wedge c \wedge d \wedge e) \to f$ and $(a \vee b \vee c \vee d \vee e) \to f$ are not logically equivalent

   Solution

   If we set $a = b = c = d = 0, e = 1, f = 0$ then we have the first statement to be 1 and the second to be 0.

5. Simplify the compound statement

$$\neg(p \wedge (q \vee r) \wedge ((p \wedge q) \to r)).$$

   Solution

   Method 1: Use laws of logic.

   Method 2:

p	q	r	$p \wedge (q \vee r) \wedge ((p \wedge q) \to r)$	$\neg(p \wedge (q \vee r) \wedge ((p \wedge q) \to r))$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	1	0
1	1	0	0	1
1	1	1	1	0

   We see that the third column is 1 if and only if $p \wedge r$. Thus our expression is equivalent to $\neg p \vee \neg r$.

6. Prove that you can infer p from $p \oplus q$ and $\neg q$.

   Solution

   Build a truth table of $(p \wedge (p \oplus q)) \to \neg q$ and observe that this statement is a tautology.

7. There are two tribes living on the island of Knights and Knaves: knights and knaves. Knights always tell truth and knaves always lie. You encounter two people A and B. What are A and B if A says "If B is a knight then I am a knave!".

   Solution

1) Let p be "A is a knight" and q be "B is a knight".

Then what we know is

$$p \leftrightarrow (q \rightarrow \neg p).$$

Method 1: Truth table.

Method 2:

Let us assume that $p = 0$. Then statement $q \rightarrow \neg p$ is true and consequently $p \leftrightarrow (q \rightarrow \neg p)$ is false. We came to contradiction, thus our assumption was false and we have $p = 1$. Thus $q \rightarrow \neg p$ is true and by Modus Tollens $q = 0$.

8. Use inference rules to find out what relevant conclusion or conclusions can be drawn from this set of premises? Explain the rules of inference used to obtain each conclusion from the premises.
"I am going to hike this weekend."
"I will not go hiking on Sunday."
"If I go hiking on Sunday I will be tired on Monday."

Solution

Let
$p =$ "I will hike on Saturday",
$q =$ "I will hike on Sunday",
$r =$ "I will be tired on Monday".

Then conditions can be written as $p \vee q, \neg q, q \rightarrow r$.

By the rule of disjunctive syllogism we can infer p from $p \vee q$ and $\neg q$. No other interesting facts can be inferred. In particular we can not say anything about the value of r.

9. Write the following argument in symbolic form. Then establish the validity of the argument or give a counterexample to show that it is invalid.

> If Dominic goes to the racetrack, then Helen will be mad. If Ralph plays cards all night, then Carmela will be mad. If either Helen or Carmela gets mad, then Veronica (their attorney) will be notified. Veronica has not heard from either of these two clients. Consequently, Dominic didn't make it to racetracks and Ralph didn't play cards all night.

Solution

Let
$p =$ "Dominic goes to the racetrack",
$q =$ "Helen will be mad",

$r =$ "Ralph plays cards all night",
$s =$ "Carmela will be mad",
$t =$ "Veronica is notified".

Then the argument can be written as

$$p \to q, r \to s, (q \vee s) \to t, \neg t \vdash \neg p \wedge \neg q.$$

The proof goes as follows.

1) $\neg t$ (premise)
2) $(q \vee s) \to t$ (premise)
3) $\neg(q \vee s)$ (modus tollens)
4) $\neg q \wedge \neg s$ (De Morgan's law)
5) $\neg q$ (simplification)
6) $p \to q$ (premise)
7) $\neg p$ (modus tollens)

and analogously you show $\neg q$.

10. Using the Rule of Conjunction

$$\begin{array}{c} p \\ q \\ \hline p \wedge q \end{array}$$

and other rules of inference and logic equivalences give the reasons for the steps verifying the following argument.
Premises: $(\neg p \vee q) \to r$, $r \to (s \vee t)$, $\neg s \wedge \neg u$, $\neg u \to \neg t$.
Conclusion: p.

Steps	Reasons
1) $\neg s \wedge \neg u$	
2) $\neg u$	
3) $\neg u \to \neg t$	
4) $\neg t$	
5) $\neg s$	
6) $\neg s \wedge \neg t$	
7) $r \to (s \vee t)$	
8) $\neg(s \vee t) \to \neg r$	
9) $(\neg s \wedge \neg t) \to \neg r$	
10) $\neg r$	
11) $(\neg p \vee q) \to r$	
12) $\neg r \to \neg(\neg p \vee q)$	
13) $\neg r \to (p \wedge \neg q)$	

14) $p \wedge \neg q$
15) p

Solution

Steps	Reasons
1) $\neg s \wedge \neg u$	premise
2) $\neg u$	conjunctive simplification to Step 1
3) $\neg u \rightarrow \neg t$	premise
4) $\neg t$	modus ponens to Steps 2 and 3
5) $\neg s$	conjunctive simplification to Step 1
6) $\neg s \wedge \neg t$	rule of conjunction to Steps 4 and 5
7) $r \rightarrow (s \vee t)$	premise
8) $\neg(s \vee t) \rightarrow \neg r$	contrapositive to Step 7
9) $(\neg s \wedge \neg t) \rightarrow \neg r$	DeMorgan's law to Step 8
10) $\neg r$	modus ponens to Steps 6 and 9
11) $(\neg p \vee q) \rightarrow r$	premise
12) $\neg r \rightarrow \neg(\neg p \vee q)$	contrapositive to Step 11
13) $\neg r \rightarrow (p \wedge \neg q)$	DeMorgan's law to Step 12
14) $p \wedge \neg q$	modus ponens to Steps 10 and 13
15) p	conjunctive simplification to Step 14.