Bounded relational width

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July 12, 2009

Abstract

We prove that $\mathrm{CSP}(\mathbb{A})$ for an idempotent algebra $\mathbb{A}$ has bounded relational width if and only if $\mathsf{var}(\mathbb{A})$ omits the unary and affine types. If it has bounded relational width it has relational width at most 3. The problem, given a relational structure $\mathcal{A}$, decide if $\mathrm{CSP}(\mathcal{A})$ has bounded relational width (equivalently, if the variety generated by the corresponding algebra omits the Boolean and affine types) is polynomial time.

1 Introduction

For introduction into relational and other widths see [6].

Theorem 1 *Let $\mathbb{A}$ be a finite idempotent algebra. Problem $\mathrm{CSP}(\mathbb{A})$ for an idempotent algebra $\mathbb{A}$ has bounded relational width if and only if $\mathsf{var}(\mathbb{A})$ omits the unary and affine types. If it has bounded relational width it has relational width at most 3.*

For a relational structure $\mathcal{A}$ let $\mathrm{Alg}(\mathcal{A})$ denote the corresponding algebra. In the RELATIONAL STRUCTURE OF TYPE 2 problem we are given a finite relational structure $\mathcal{A}$ such that all its polymorphisms are idempotent, and the question is whether $\mathsf{var}(\mathrm{Alg}(\mathcal{A}))$ omits the unary and affine types, or, equivalently, by Theorem 1, whether $\mathrm{CSP}(\mathcal{A})$ has bounded relational width.

Theorem 2 RELATIONAL STRUCTURE OF TYPE 2 *is polynomial time.*

2 Preliminaries

For introduction for multi-sorted CSP and other technicalities see e.g. [2].

2.1 Colored graphs.

Let $\mathbb{A}$ be a finite idempotent algebra. The subalgebra of $\mathbb{A}$ generated by set $B \subseteq \mathbb{A}$ will denoted by $\mathsf{Sg}(B)$. A pair ab of elements from $\mathbb{A}$ is called an *edge* if and only if there exists a congruence θ of $\mathsf{Sg}(a, b)$ and a term operation f of $\mathbb{A}$ such that either f^θ is an affine operation on $\mathsf{Sg}(a, b)/_\theta$, or f^θ is a semilattice operation on $\{a^\theta, b^\theta\}$, or f^θ is a majority operation on $\{a^\theta, b^\theta\}$. Edge ab is called *thin* if θ is the equality relation. Otherwise it is called *thick*.

The color of an edge is defined in the same way as for conservative algebras. If there exists a congruence θ and a term operation $f \in \mathsf{Term}(\mathbb{A})$ such that f^θ is a semilattice operation on $\{a^\theta, b^\theta\}$ then ab is said to be *red* or to have the *semilattice type*. An edge ab is *yellow* or of the *majority type* if it is not red and there are a congruence θ and $f \in \mathsf{Term}(\mathbb{A})$ such that f^θ is a majority operation on $\{a^\theta, b^\theta\}$. Finally, ab is *blue* or of the *affine type* if it is not red or yellow and there are a congruence θ and $f \in \mathsf{Term}(\mathbb{A})$ such that f^θ is an affine operation on $\mathsf{Sg}(a, b)/_\theta$.

In this paper we denote by $\mathrm{Gr}(\mathbb{A})$ the graph with vertex set $\mathbb{A}$, whose edge set consists of thin red edges of $\mathbb{A}$ and yellow edges (possibly thick) of $\mathbb{A}$. Algebra $\mathbb{A}$ is called *hereditarily red-yellow connected* if $\mathrm{Gr}(\mathbb{B})$ is connected for any subalgebra of $\mathbb{A}$. The results of [5] imply the following

Proposition 1 *Let $\mathbb{A}$ be an idempotent algebra. Then $\mathbb{A}$ is hereditarily red-yellow connected if and only if* $\mathsf{var}(\mathbb{A})$ *omits the unary and affine type.*

We will also need two other useful properties. An algebra $\mathbb{A}$ such that every its thin red edge and yellow edge is a subalgebra will be called *conglomerate*. Every algebra can be made conglomerate without destroying the connectivity of $\mathrm{Gr}(\mathbb{A})$ and that of subalgebras of $\mathbb{A}$.

Proposition 2 ([5]) *Let $\mathbb{A} = (A, F)$ be a hereditarily red-yellow connected idempotent algebra, ab an edge of $\mathrm{Gr}(\mathbb{A})$ of semilattice or majority type, θ a congruence witnessing this, and let $R_{ab} = (a/_\theta \cup b/_\theta)$ be a thick yellow edge or a thin edge ab of $\mathrm{Gr}(\mathbb{A})$. Let also $\mathbb{A}' = (A; F')$ be a reduct of $\mathbb{A}$ such that F' contains all operations from F preserving R_{ab}. Then $\mathbb{A}'$ is hereditarily red-yellow connected.*

The reduct of an algebra $\mathbb{A}$ constructed by applying Proposition 2 iteratively until every edge of the resulting algebra is a subalgebra will be called the *conglomerate reduct* of $\mathbb{A}$.

Operations witnessing the type of edges can be significantly uniformized.

Proposition 3 ([5]) *Let $\mathbb{A}$ be a finite idempotent algebra. For each edge ab let θ_{ab} denote the congruence witnessing that. There are term operations f, g, h of $\mathbb{A}$ such that*

$f|_{\{a/\theta_{ab}, b/\theta_{ab}\}}$ *is a semilattice operation if* ab *is a red edge (not necessarily thin), it is the first projection if* ab *is a yellow or blue edge;*

$g|_{\{a/\theta_{ab}, b/\theta_{ab}\}}$ *is a majority operation if* ab *is a yellow edge, it is the first projection if* ab *is a blue edge, and* $g|_{\{a/\theta_{ab}, b/\theta_{ab}\}}(x,y,z) = f|_{\{a/\theta_{ab}, b/\theta_{ab}\}}(x, f|_{\{a/\theta_{ab}, b/\theta_{ab}\}}(y,z))$ *if* ab *is red;*

$h|_{\mathsf{Sg}(ab)/\theta_{ab}}$ *is an affine operation operation if* ab *is a blue edge, it is the first projection if* ab *is a yellow edge, and* $h|_{\{a/\theta_{ab}, b/\theta_{ab}\}}(x,y,z) = f|_{\{a/\theta_{ab}, b/\theta_{ab}\}}(x, f|_{\{a/\theta_{ab}, b/\theta_{ab}\}}(y,z))$ *if* ab *is red.*

Fixing operation f which is semilattice on all red edges we can also choose and fix orientation of such edges: if ab is a red edge then it is directed from a to b if $f(a,b) = f(b,a) = b$ (we will only need thin edges). This will also be denoted by $a \leq b$. We will always assume that operation f is fixed. If there is a directed path in $\mathrm{Gr}(\mathbb{A})$ connecting a and b and consisting of red edges then we write $a \prec b$.

Proposition 4 ([5]) *Let* $\mathbb{A}$ *be an idempotent algebra. There is a binary term operation* f *of* $\mathbb{A}$ *such that* f *is a semilattice operation on every red edge of* $\mathrm{Gr}(\mathbb{A})$ *and, for any* $a, b \in \mathbb{A}$*, either* $a = f(a,b)$ *or the pair* $(a, f(a,b))$ *is a thin red edge.*

Let $\mathrm{Gr}'(\mathbb{A})$ denote the subgraph of $\mathrm{Gr}(\mathbb{A})$ obtained by removing all yellow edges. It will sometimes be convenient to consider the partial order $\mathrm{Scc}(\mathbb{A})$ of strongly connected components (scc) of $\mathrm{Gr}'(\mathbb{A})$. The scc containing an element a will be denoted by $\widehat{a}$. The *filter* generated by a, denoted $\mathrm{Ft}(a)$, is the union of all scc's from the filter generated by $\hat{a}$ in $\mathrm{Scc}(\mathbb{A})$. By $\max(\mathbb{A})$ we denote the set of the *maximal elements* of the graph $\mathrm{Gr}'(\mathbb{A})$, that is, elements from maximal (under the partial order) scc's of $\mathrm{Gr}(\mathbb{A})$.

An *r-path* in $\mathrm{Gr}(\mathbb{A})$ is a path in this graph that contains only red edges. An r-path is said to be *directed* if all its edges are oriented in the right direction. Analogously, An *ry-path* in $\mathrm{Gr}(\mathbb{A})$ is a path in this graph that contains both red and yellow edges. An ry-path is said to be *directed* if all its red edges are oriented in the right direction.

3 Prerequisites

In this section we prove several auxiliary statements.

Lemma 1 (Path Expansion Lemma) *Let* R *be a subdirect product of relations* R_1 *and* R_2*,* R_1 m*-ary, and* $(\mathbf{a}, \mathbf{b}) \in R$*. If* $\mathbf{a} = \mathbf{a}_1, \mathbf{a}_2, , \dots, \mathbf{a}_k \in R_1$ *is an r-path,* *then there are* $\mathbf{b} = \mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_k \in R_2$ *such that* $(\mathbf{a}_1, \mathbf{b}_1), \dots, (\mathbf{a}_k, \mathbf{b}_k)$ *is an r-path in* R*. Moreover, if* $\mathbf{b}$ *is maximal, then the* $\mathbf{b}_i$ *are also maximal and belong to the same scc.*

Proof: See [2]. □

Lemma 2 *Let* $\mathbb{B}$ *be a subalgebra of* $\mathbb{A}$*. Then, for every* $a, b \in \mathbb{B}$ *such that* $b \in \mathrm{Ft}(a)$*, there exists* $n \in \mathbb{N}$ *such that the equation*

$$f(f(\dots f(f(a, x_1), x_2) \dots, x_{n-1})x_n) = b$$

is solvable in $\mathbb{B}$*.*

Proof: The lemma follows straightforwardly from the definition of r-connectedness. □

By $\mathcal{F}(\mathbb{A})$ for an algebra $\mathbb{A}$ we denote the set of its *factors*, that is homomorphic images of subalgebras.

Lemma 3 *Let* R *be a subdirect product of* $\mathbb{D}_1, \dots, \mathbb{D}_n \in \mathcal{F}(\mathbb{A})$ *and* $I_1, \dots, I_k \subseteq \{1, \dots, n\}$*. Then for any tuple* $\mathbf{a}' = (\mathbf{a}[i])_{i \in I_1} \in \max(\mathrm{pr}_{I_1} R)$ *there exist* $\mathbf{a}[j] \in \mathbb{D}_j$*,* $j \in [n] - I_1$*, such that* $\mathbf{a} \in R$ *and* $\mathrm{pr}_{I_t}\mathbf{a} \in \max(\mathrm{pr}_{I_t} R)$ *for each* $t \in \{2, \dots, k\}$*.*

Proof: See [2]. □

Lemma 4 *Let* R *be a subdirect product of* $\mathbb{D}_1, \mathbb{D}_2$ *and* B, C *maximal scc's of* $\mathbb{D}_1, \mathbb{D}_2$*, respectively, such that* $R \cap (B \times C) \neq \varnothing$*. Then for any* $b \in B$ *there is* $c \in C$ *with* $(b, c) \in R$*.*

Proof: See [2]. □

An algebra $\mathbb{B}$ from $\mathcal{F}(\mathbb{A})$ is said to be *maximal generated* if $\mathbb{B} = \mathsf{Sg}(C)$ where for some maximal scc C. It is called *arbitrarily maximal generated* if $\mathbb{B} = \mathsf{Sg}(C)$ where for some maximal scc C. We will need three versions of the following lemma.

Lemma 5 *Let* R *be a subdirect product of* $\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{F}(\mathbb{A})$*, and* $\mathbb{D}_1, \mathbb{D}_2$ *are generated by maximal scc's* B_1, B_2*, respectively, such that there exist an element* $a \in B_1$ *with* $\{a\} \times B_2 \subseteq \mathbb{D}$*. Then* $R = \mathbb{D}_1 \times \mathbb{D}_2$*.*

Proof: We prove by induction that $\{c\} \times \mathbb{D}_2 \subseteq R$ for every $c \in \mathbb{D}_1$. The inclusion $\{a\} \times \mathbb{D}_2 = \mathsf{Sg}(\{a\} \times B_2) \subseteq R$ forms the base case of induction. Further, suppose that there is $d \in \mathbb{D}_1$ such that $d \le c$ and $\{d\} \times \mathbb{D}_2 \subseteq \mathbb{D}$. Take an arbitrary $b \in B_2$. For a certain $b_1 \in \mathbb{D}_2$, we have $\begin{pmatrix} c \\ b' \end{pmatrix} \in \mathbb{D}$. Set $\begin{pmatrix} c \\ b_1 \end{pmatrix} = f\left(\begin{pmatrix} d \\ b \end{pmatrix}, \begin{pmatrix} c \\ b' \end{pmatrix}\right) \in R$. Then $b_1 \in \widehat{b} = B_2$. By Lemma 2, there exist $b_2, \dots, b_l \in \mathbb{D}_2$ such that $f(f(\dots f(b_1, b_2) \dots), b_l) = b$. It is easy to see that

$$f\left(f\left(\dots\left(\begin{pmatrix} c \\ b_1 \end{pmatrix}, \begin{pmatrix} d \\ b_2 \end{pmatrix}\right), \dots\right), \begin{pmatrix} d \\ b_l \end{pmatrix}\right) = \begin{pmatrix} c \\ b \end{pmatrix} \in \mathbb{D}.$$

Therefore, $\{c\} \times \mathbb{D}_2 = \mathsf{Sg}(\{c\} \times b_2) \subseteq R$. Since there is a path from a to every $c \in \max(\mathbb{D}_1)$, this holds for every $c \in b_1$. Finally, as $\mathbb{D}_1 = \mathsf{Sg}(B_1)$, the result follows. $\square$

Lemma 6 *Let R be a subdirect product of an maximal generated algebras $\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{F}(\mathbb{A})$ generated by maximal scc's A_1, A_2, respectively, such that $R \cap (A_1 \times A_2) \neq \varnothing$ and there exists an element $a \in \mathbb{D}_1$ with $\{a\} \times B_2 \subseteq \mathbb{D}$. Then $R = \mathbb{D}_1 \times \mathbb{D}_2$.*

Proof: We prove that $\{c\} \times \mathbb{D}_2 \subseteq R$ for every $c \in A_1$. As in the proof of Lemma 5 we can show that if $\{b\} \times \mathbb{D}_2 \subseteq R$ then $\{c\} \times \mathbb{D}_2 \subseteq R$ for any $c \in \mathrm{Ft}(b)$. Therefore it suffices to prove the result only for one element from A_1.

First we show that there are a term operation f' of $\mathbb{A}$ and $b \in A_1$ such that $f'(b, a) = b$ and f' is a semilattice operation on every thin red edge. If $f(b, a) = b$ for some $b \in A_1$ we set $f' = f$. Otherwise take an arbitrary element $b_0 \in A_1$ and construct a sequence $b_0, b_1, b_2, \dots$ with $b_{i+1} = f(b_i, a)$. Clearly all the b_i belong to A_1. There are i, j such that $b_i = b_j$. Then choose $b = b_i$ and

$$f'(x, y) = \underbrace{f(\dots f(f}_{j-i \text{ times}}(x, y), y) \dots y).$$

As is easily seen, $f'(b, a) = b$.

Since $R \cap (A_1 \times A_2) \neq \varnothing$, by Lemma 4 this set is a subdirect product of A_1 and A_2. There is a maximal $d \in A_2$ with $(b, d) \in R$. Let C be the set of all $e \in A_2$ such that $(b, e) \in R$, note that $C \neq \varnothing$. If $C \neq A_2$ then there are $e \in C$ and $e' \in A_2 - C$ and ee' is a thin red edge. Then

$$\begin{pmatrix} b \\ e' \end{pmatrix} = f\left(\begin{pmatrix} b \\ e \end{pmatrix}, \begin{pmatrix} a \\ e' \end{pmatrix}\right) \in R,$$

a contradiction.

To complete the prove we use the fact that A_1 generates $\mathbb{D}_1$ and A_2 generates $\mathbb{D}_2$. $\square$

Lemma 7 *Let R be a subdirect product of $\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{F}(\mathbb{A})$, let A_1, A_2 be maximal scc's of $\mathbb{D}_1, \mathbb{D}_2$, respectively, such that $R \cap (A_1 \times A_2) \neq \varnothing$. If there exist an element $a \in \mathbb{D}_1$ with $\{a\} \times A_2 \subseteq R$, then $A_1 \times A_2 \subseteq R$.*

Proof: We prove that $\{c\} \times A_2 \subseteq R$ for every $c \in A_1$. As before it suffices to prove the result only for one element from A_1. Note also that element a can be chosen maximal.

As in the proof of Lemma 6, there are a term operation f' of $\mathbb{A}$ and $b \in A_1$ such that $f'(b, a) = b$ and f' is a semilattice operation on every thin red edge. Since $R \cap (A_1 \times A_2) \neq \varnothing$, we complete the proof as in Lemma 6. □

Lemma 8 *Let $\mathbb{A} = \mathsf{Sg}(a, b)$ be simple, $a, b \in \max(\mathbb{A})$, and R a subdirect square of $\mathbb{A}$. Let also S be the tolerance defined by $\{(c, d) \in \mathbb{A}^2 \mid (e, c), (e, d) \in R \text{ for some } e\}$. If S is a connected tolerance then there is a sequence $a = d_1, \dots, d_k = b'$ such that $(d_i, d_{i+1}) \in S$, d_i is maximal, $b' \in \widehat{b}$, and if a_i are such that $(a_i, d_i), (a_i, d_{i+1}) \in R$ then a_i can also be chosen maximal.*

Proof: We start with any sequence $a = d_1, \dots, d_k = b$, $(d_i, d_{i+1}) \in S$ connecting a and b. Such a sequence exists because S is a connected tolerance. We prove by induction on k. The base case of induction is obvious by the choice of a. Suppose d_i is maximal. Let $d_{i+1} = e_1 \leq \dots \leq e_s$ be a directed r-path and e_s a maximal element. Let also $(b_j, e_j) \in R$ be extensions of the e_j and $(a_q, d_q), (a_q, d_{q+1}) \in R$ for $q \in [k-1]$. Then for each q, $i \leq q \leq k-1$, we construct the sequence $a_q = a_q^1 \leq \dots \leq a_q^s$ and for each q, $i \leq q \leq k$, the sequence $d_q = d_q^1 \leq \dots \leq d_q^s$, where

$$a_q^j = f(a_q^{j-1}, b_j) \qquad \text{and} \qquad d_q^j = f(d_q^{j-1}, e_j).$$

Then observing that

$$\begin{pmatrix} a_q^1 \\ d_q^1 \end{pmatrix} = \begin{pmatrix} a_q \\ d_q \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a_q^{j+1} \\ d_q^{j+1} \end{pmatrix} = f\left(\begin{pmatrix} a_q^j \\ d_q^j \end{pmatrix}, \begin{pmatrix} b_{j+1} \\ e_{j+1} \end{pmatrix} \right), \qquad \text{and}$$

$$\begin{pmatrix} a_q^1 \\ d_{q+1}^1 \end{pmatrix} = \begin{pmatrix} a_q \\ d_{q+1} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a_q^{j+1} \\ d_{q+1}^{j+1} \end{pmatrix} = f\left(\begin{pmatrix} a_q^j \\ d_{q+1}^j \end{pmatrix}, \begin{pmatrix} b_{j+1} \\ e_{j+1} \end{pmatrix} \right)$$

we get that $\begin{pmatrix} a_q^s \\ d_q^s \end{pmatrix}, \begin{pmatrix} a_q^s \\ d_{q+1}^s \end{pmatrix} \in R$ for any $i \leq q \leq n-1$. Note also that d_{i+1}^s is a maximal element. Continuing in a similar way we also can guarantee that a_q^s is a maximal element.

Since d_i is maximal and $d_i \prec d_i^s$, these two elements belong to the same scc. Therefore, there is a directed r-path $d_i^s = e_1' \leq \dots \leq e_t' = d_i$. Let also $(b_j', e_j') \in R$

be extensions of the e'_j. Then for each q, $i \le q \le k-1$, we construct sequence $a_q^s = p_q^1 \le \ldots \le p_q^t$ and for each q, $i \le q \le k$, sequence $d_q^s = r_q^1 \le \ldots \le r_q^t$, where

$$p_q^j = f(p_q^{j-1}, b'_j) \qquad \text{and} \qquad r_q^j = f(r_q^{j-1}, e'_j).$$

Then observing that

$$\begin{pmatrix} p_q^1 \\ r_q^1 \end{pmatrix} = \begin{pmatrix} a_q^s \\ d_q^s \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} p_q^{j+1} \\ r_q^{j+1} \end{pmatrix} = f\left(\begin{pmatrix} p_q^j \\ r_q^j \end{pmatrix}, \begin{pmatrix} b'_{j-1} \\ e'_{j-1} \end{pmatrix} \right), \qquad \text{and}$$

$$\begin{pmatrix} p_q^1 \\ r_{q+1}^1 \end{pmatrix} = \begin{pmatrix} p_q \\ r_{q+1} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} p_q^{j+1} \\ r_{q+1}^{j+1} \end{pmatrix} = f\left(\begin{pmatrix} p_q^j \\ r_{q+1}^j \end{pmatrix}, \begin{pmatrix} b'_{j-1} \\ e'_{j-1} \end{pmatrix} \right)$$

we get that $\begin{pmatrix} p_q^t \\ r_q^t \end{pmatrix}, \begin{pmatrix} p_q^t \\ r_{q+1}^t \end{pmatrix} \in R$ for any $i \le q \le n-1$. Note also that $r_i^t = d_i$, r_{i+1}^t is a maximal element, and r_n^t belongs to the same scc as b. □

4 Maximal-generated algebras

First, we prove Theorem 1 in a narrow particular case. While the main result of this section will be used only in the very end, several intermediate results will be very helpful throughout the paper.

Arguments in this section follow the line of those in [3].

4.1 Binary relations

Our first purpose is to show that a subdirect product of simple arbitrarily maximal generated algebras has a very restricted form. The *graph* of a mapping $\pi\colon \mathbb{D}_1 \to \mathbb{D}_2$ is the binary relation $G_\pi = \{(a, \pi(a)) \mid a \in \mathbb{D}_1\}$ over $\mathbb{D}_1, \mathbb{D}_2$.

Lemma 9 *Let R be a subdirect product of simple maximal generated $\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{F}(\mathbb{A})$, generated by maximal scc's A_1, A_2, respectively, and let $R \cap (A_1 \times A_2) \ne \varnothing$. Then R is either the graph of a bijective mapping from $\mathbb{D}_1$ to $\mathbb{D}_2$, or $\mathbb{D}_1 \times \mathbb{D}_2$.*

Proof: Notice first, that if R is the graph of a mapping $\pi : \mathbb{D}_1 \to \mathbb{D}_2$, then the kernel of π is a congruence of $\mathbb{D}_1$ and, since $\mathbb{D}_1$ is simple, π is a bijection. The same holds if R is the graph of a mapping from $\mathbb{D}_2$ into $\mathbb{D}_1$.

Suppose that R is neither $\mathbb{D}_1 \times \mathbb{D}_2$ nor the graph of a bijective mapping, and that $|\mathbb{D}_1| + |\mathbb{D}_2|$ is the smallest number such that there exists a subdirect product of simple maximal generated with this property. We show that there is $b \in \mathbb{D}_1$ [or $b \in \mathbb{D}_2$] such that $\{b\} \times \mathbb{D}_2 \subseteq R$ [respectively, $\mathbb{D}_1 \times \{b\} \subseteq R$].

For $a \in \mathbb{D}_1, b \in \mathbb{D}_2$ by B_a, C_b we denote the sets $\{c \mid (a,c) \in R\}$, $\{c \mid (c,b) \in R\}$ respectively.

Claim 1. For any $A \subset \mathbb{D}_1$ [any $A \subset \mathbb{D}_2$], there is $a \in \mathbb{D}_2$ [respectively, $a \in \mathbb{D}_1$] and $b \in A$, $c \in \mathbb{D}_1 - A$ [respectively, $c \in \mathbb{D}_2 - A$] such that $(b,a),(c,a) \in R$ [respectively, $(a,b),(a,c) \in R$].

The claim follows from the fact that the tolerances $\varrho_1 = \{(a,b) \mid \text{ there is } c \text{ such that } (a,c),(b,c) \in R\}$ and $\varrho_2 = \{(a,b) \mid \text{ there is } c \text{ such that } (c,a),(c,b) \in R\}$ on $\mathbb{D}_1$ and $\mathbb{D}_2$, respectively, are connected.

Take $a \in \mathbb{D}_1$ such that $|B_a| > 1$ and set $E_1 = \{a\}$, and for each $i > 0$

$$E_{i+1} = \begin{cases} \bigcup_{b \in E_i} B_b & \text{if } i \text{ is odd} \\ \bigcup_{b \in E_i} C_b & \text{if } i \text{ is even.} \end{cases}$$

By Claim 1, for each $i > 0$, $E_i \subset E_{i+2}$ unless $E_i = \mathbb{D}_1$ or $E_i = \mathbb{D}_2$. Therefore, for some $l > 1$, $E_l = \mathbb{D}_1$ or $E_l = \mathbb{D}_2$. Without loss of generality, suppose $E_l = \mathbb{D}_2$, and $E_{l-1} \neq \mathbb{D}_1$, $E_{l-2} \neq \mathbb{D}_2$.

Claim 2. For each i, $1 \le i \le l$, E_i is a subalgebra of $\mathbb{D}_1$ or $\mathbb{D}_2$.

We prove the claim by induction. In the base case of induction $E_1 = \{a\}$ is a subalgebra, because $\mathbb{D}_1$ is idempotent. If E_i is a subalgebra, and $E_i \subseteq \mathbb{D}_1$, then for any $a_2, b_2 \in E_{i+1}$ there are $a_1, b_1 \in E_i$ such that $(a_1,a_2),(b_1,b_2) \in R$. Then $\begin{pmatrix} f(a_1,b_1) \\ f(a_2,b_2) \end{pmatrix} = f\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) \in R$, $f(a_1,b_1) \in E_i$, hence, $a_2 * b_2 \in E_{i+1}$. The proof in the case $E_i \subseteq \mathbb{D}_2$ is analogous.

Thus, E_{l-1} is a proper subalgebra of $\mathbb{D}_1$ such that $\bigcup_{b \in E_{l-1}} B_b = \mathbb{D}_2$.

Define a sequence $\mathbb{B}_0, \mathbb{B}_1, \ldots, \mathbb{B}_k$ of algebras and a sequence of congruences $\theta_0, \theta_1, \ldots, \theta_k$ where θ_i is a congruence of $\mathbb{B}_i$ through the following rules.

1) $\mathbb{B}_0$ is generated by a maximal scc of E_{l-1} such that it contains $a \in \mathbb{D}_1$ with $|B_a| > 1$ and $B_a \cap A_2 \neq \varnothing$ (later we show that such an scc exists).

2) Suppose that $\mathbb{B}_i$ is already defined. Let θ_i be its maximal congruence or the identity relation if $\mathbb{B}_i$ is simple.

3) If $\mathbb{B}_i$ is a singleton, then $k = i$ and the process stops. Otherwise set $\mathbb{B}_{i+1}$ to be the algebra generated by a maximal scc of a class of θ_i that contains $a \in \mathbb{D}_1$ with $|B_a| > 1$ and $B_a \cap A_2 \neq \varnothing$ (as we shall prove later, such a class exists).

Set $\mathbb{B}'_i = \mathbb{B}_i/_{\theta_i}$ and

$$\begin{aligned} R^{(i)} &= \{(a,b) \mid a = a'^{\theta_i} \text{ where } a' \in \mathbb{B}_i,\ (a',b) \in R\} \\ &\subseteq \mathbb{B}'_i \times \mathbb{D}_2. \end{aligned}$$

We prove that, for every i, (i) for any $b \in \mathbb{D}_2$ there exists $a \in \mathbb{B}_i$ such that $(a,b) \in R$, (ii) if $\mathbb{B}_i$ is generated by its maximal scc S then $(c,d) \in R$ for some $c \in S$ and $d \in A_2$, and (iii) $R^{(i)} = \mathbb{B}'_i \times \mathbb{D}_2$.

If $i = 0$, then (i) holds by the choice of E_{l-1}, and Lemma 3. Consider $R' = R \cap (E_{\ell-1} \times \mathbb{D}_2)$. Since $|B_a| > 1$ for some $a \in E_{\ell-1}$, the tolerance $\varrho'_2 = \{(a,b) \mid$ there is c such that $(c,a),(c,b) \in R'\}$ is connected. Take two maximal elements $b,c \in \mathbb{D}_2$ such that $b \in A_2$. By Lemma 8 there are maximal elements $a' \in E_{\ell-1}$ and $b' \in \mathbb{D}_2$ such that $(a',b),(a',b') \in R'$. Therefore for the maximal component generating $\mathbb{B}_0$ we can choose $\widehat{a'}$. Also (ii) holds, as $(a',b) \in R$ and $a' \in \widehat{a'}$, $b \in A_2$. Therefore, $R^{(0)}$ is not the graph of a mapping, and, since $\mathbb{B}'_0$ is simple, maximal generated and $|\mathbb{B}'_0| + |\mathbb{D}_2| < |\mathbb{D}_1| + |\mathbb{D}_2|$, we get $R^{(0)} = \mathbb{B}'_0 \times \mathbb{D}_2$.

Suppose that for $i-1$ properties (i), (ii), (iii) hold. Then, for any $a' \in \mathbb{B}'_{i-1}$ we have $\{a'\} \times \mathbb{D}_2 \subseteq R^{(i-1)}$, that is, by Lemma 3, for every $b \in \mathbb{D}_2$ there exists $a \in \mathbb{B}_i$ such that $(a,b) \in R$, that proves (i) for i. By (ii) for $i-1$ the θ_{i-1}-class A containing $\mathbb{B}_i$ contains an element b such that $|B_b| > 1$. Arguing as in the previous paragraph, $\mathbb{B}_i$ can be chosen such that $b \in \mathbb{B}_i$ and $(b,c) \in R$ for some $c \in A_2$. Finally, as $\mathbb{B}'_i$ is simple we have $R^{(i)} = \mathbb{B}'_i \times \mathbb{D}_2$.

We have proved $R^{(k)} = \mathbb{B}'_k \times \mathbb{D}_2$. Since $\mathbb{B}_k$ is a singleton, say, $\mathbb{B}_k = \{b\}$ this implies $\mathbb{B}_k = \mathbb{B}'_k$, that is $\{b\} \times \mathbb{D}_2 \subseteq R$.

To complete the proof we just have to apply Lemma 5. $\Box$

Note that the second part of the proof is valid for a subdirect product of not only simple maximal generated algebras.

Corollary 1 *Let R be a subdirect product of $\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{F}(\mathbb{A})$ where $\mathbb{D}_2$ is simple maximal generated and R is not the graph of any mapping $\pi\colon \mathbb{D}_1 \to \mathbb{D}_2$. Then there exists $a \in \mathbb{D}_1$ such that $\{a\} \times \mathbb{D}_2 \subseteq R$.*

To prove this we should put $\mathbb{B}_0$ equal to $\max(\mathbb{D}_1)$.

Corollary 2 *Let R be a subdirect product of maximal generated $\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{F}(\mathbb{A})$, say, the algebras are generated by their maximal scc's A_1, A_2, respectively, and $\mathbb{D}_2$ is simple. Let also $R \cap (A_1 \times A_2) \neq \varnothing$. Then either $R = \mathbb{D}_1 \times \mathbb{D}_2$, or there is a surjective mapping $\pi\colon \mathbb{D}_1 \to \mathbb{D}_2$ such that $R = \{(a, \pi(a)) \mid a \in \mathbb{D}_1\}$.*

Proof: If R is not the graph of a mapping then we are in the conditions of Corollary 1. Therefore there exists $a \in \mathbb{D}_1$ such that $\{a\} \times \mathbb{D}_2 \subseteq R$. Since $R \cap (A_1 \times A_2) \neq \varnothing$, by Lemma 6, we get $R = \mathbb{D}_1 \times \mathbb{D}_2$. $\Box$

Observe now that if R is a subdirect product of abitrariry maximal generated $\mathbb{D}_1, \mathbb{D}_2$, then by Lemma 3 $R \cap (A_1 \times A_2) \neq \varnothing$ for some maximal scc's A_1, A_2 such that $\mathbb{D}_1 = \mathsf{Sg}(A_1)$ and $\mathbb{D}_2 = \mathsf{Sg}(A_2)$.

Corollary 3 *Let R be a subdirect product of arbitrarily maximal generated $\mathbb{D}_1, \mathbb{D}_2 \in \mathcal{F}(\mathbb{A})$, and $\mathbb{D}_2$ is simple. Then either $R = \mathbb{D}_1 \times \mathbb{D}_2$, or there is a surjective mapping $\pi\colon \mathbb{D}_1 \to \mathbb{D}_2$ such that $R = \{(a, \pi(a)) \mid a \in \mathbb{D}_1\}$.*

4.2 Multi-ary relations

In this subsection we consider non-binary relations.

Lemma 10 *Let R be a subdirect product of simple maximal generated $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3 \in \mathcal{F}(\mathbb{A})$. Let the algebras be generated by maximal scc's A_1, A_2, A_3, respectively, and $R \cap (A_1 \times A_2 \times A_3) \neq \varnothing$. If $\mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{i,j} R$ for every $i, j \in \{1, 2, 3\}$, then $R = \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3$.*

Proof: Suppose without loss of generality that $|\mathbb{D}_1| \leq |\mathbb{D}_2| \leq |\mathbb{D}_3|$. For $a \in \mathbb{D}_1$ set

$$R_a = \{(b_2, b_3) \mid (a, b_2, b_3) \in R\}.$$

Notice that, for every $a \in \mathbb{D}_1$, R_a is a subalgebra of $\mathrm{pr}_{2,3} R$, and, since $\mathrm{pr}_{1,2} R = \mathbb{D}_1 \times \mathbb{D}_2$, $\mathrm{pr}_{1,3} R = \mathbb{D}_1 \times \mathbb{D}_3$, the algebra R_a is a subdirect product of $\mathbb{D}_2, \mathbb{D}_3$. By Lemma 9, R_a is either the graph of a bijective mapping or $\mathbb{D}_2 \times \mathbb{D}_3$.

Suppose first that R_a is not the graph of a mapping for some $a \in \mathbb{D}_1$. By Corollary 1 there is $b \in \mathbb{D}_2$ such that $(a, b) \times \mathbb{D}_3 \subseteq R$. Applying Lemma 6 treating R as a subdirect product of $\mathrm{pr}_{1,2} R = \mathbb{D}_1 \times \mathbb{D}_2$ and $\mathbb{D}_3$ we get $A_1 \times A_2 \times \mathbb{D}_3 \subseteq R$. This implies $R = \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3$.

Now suppose that, for every $a \in \mathbb{D}_1$, the set R_a is the graph of a bijective mapping $\pi_a\colon \mathbb{D}_2 \to \mathbb{D}_3$. This immediately implies $|\mathbb{D}_2| = |\mathbb{D}_3|$, let us denote this number by k, and as $\mathrm{pr}_{2,3} R = \mathbb{D}_2 \times \mathbb{D}_3$, there are at least k different relations of the form R_a. Therefore, $|\mathbb{D}_1| = k$ and $|R_a| = k$ for any $a \in \mathbb{D}_1$. Moreover, $|\mathrm{pr}_{2,3} R| = k^2$, which means $R_a \cap R_{a'} = \varnothing$ whenever $a \neq a'$, $a, a' \in \mathbb{D}_1$. The equivalence relation $\sim$ on $\mathrm{pr}_{2,3} R$ where $(a, b) \sim (c, d)$ iff $(a, b), (c, d) \in R_e$ for some $e \in \mathbb{D}_1$, is a congruence of $\mathrm{pr}_{2,3} R = \mathbb{D}_2 \times \mathbb{D}_3$.

Since $\mathbb{D}_2 \cong \mathbb{D}_3$, $\mathbb{D}_2 \times \mathbb{D}_3$ can be treated as the square of $\mathbb{D}_2$. Recall that an element a of an algebra $\mathbb{A}$ is said to be *absorbing* if whenever $t(x, y_1, \ldots, y_n)$ is an $(n+1)$-ary term operation of $\mathbb{A}$ such that t depends on x and $(b_1, \ldots, b_n) \in A^n$, then $t(a, b_1, \ldots, b_n) = a$. A congruence θ of $\mathbb{A}^2$ is said to be *skew* if it is the kernel of no projection mapping of $\mathbb{A}^2$ onto its factors. $\mathbb{D}_2$ is a simple idempotent algebra, therefore, by the results of [11] one of the following holds: (a) $\mathbb{D}_2$ is term equivalent to a module; (b) $\mathbb{D}_2$ has an absorbing element; or (c) $\mathbb{D}_2^2$ has no skew congruence. Case (a) is impossible, because $\mathbb{D}_2$ has a 2-element subalgebra term equivalent to a semilattice, but no module has such a subalgebra. If in case (b) a is an absorbing element, then $f(a, b) = a$ for any $b \in \mathbb{D}_2$ that would imply that any maximal scc is a singleton, which contradicts the fact that $\mathbb{D}_2$ is maximal generated. Finally, case (c) is also impossible, because $\sim$ is a skew congruence.

Thus, our assumption that R_a is the graph of a mapping for all $a \in \mathbb{D}_1$ cannot be the case, and the lemma is proved. $\Box$

Lemma 11 *Let R be a subdirect product of simple maximal generated $\mathbb{D}_1, \dots, \mathbb{D}_n \in \mathcal{F}(\mathbb{A})$, say, $\mathbb{D}_i$ is generated by a maximal scc A_i; and let $R \cap (A_1 \times \dots \times A_n) \neq \varnothing$. If $\mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{i,j} R$ for every $i, j \in [n]$, then $R = \mathbb{D}_1 \times \dots \times \mathbb{D}_n$.*

Proof: We prove the lemma by induction. The base case of induction $n = 2, 3$ have been proved in Lemmas 9, Lemma 10. Suppose that the lemma holds for each number less than n. Take $a \in \mathbb{D}_1$ and denote by R_a the set $\{(b_2, \dots, b_n) \mid (a, b_2, \dots, b_n) \in R\}$. By Lemma 10, $\mathbb{D}_1 \times \mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{1,i,j} R$ for any $2 \leq i, j \leq n$. Then $\mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{i,j} R_a$. Now if a is such that $R_a \cap (A_2 \times \dots \times A_n) \stackrel{\mu}{\equiv} \varnothing$ then by induction hypothesis $R_a = \mathbb{D}_2 \times \dots \times \mathbb{D}_n$. The lemma is now follows from Lemma 7. $\Box$

Definition 1 *A relation $R \subseteq \mathbb{D}_1 \times \dots \times \mathbb{D}_n$ is said to be* almost trivial *if there exists an equivalence relation θ on the set $\{1, \dots, n\}$ with classes $I_1, \dots, I_k$, such that*

$$R = \mathrm{pr}_{I_1} R \times \dots \times \mathrm{pr}_{I_k} R$$

where $\mathrm{pr}_{I_j} R = \{(a_{i_1}, \pi_{i_2}(a_{i_1}), \dots, \pi_{i_l}(a_{i_1})) \mid a_{i_1} \in \mathbb{D}_{i_1}\}$, $I_j = \{i_1, \dots, i_l\}$, for certain bijective mappings $\pi_{i_2} \colon \mathbb{D}_{i_1} \to \mathbb{D}_{i_2}, \dots, \pi_{i_l} \colon \mathbb{D}_{i_1} \to \mathbb{D}_{i_l}$.

Proposition 5 *Let R be subdirect product of simple maximal generated algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$ from $\mathcal{F}(\mathbb{A})$, say, $\mathbb{D}_i$ is generated by a maximal scc A_i; and let $R \cap (A_1 \times \dots \times A_n) \neq \varnothing$. Then R is an almost trivial relation.*

Proof: See [3]. $\Box$

As before the conditions of Proposition 5 hold if all the algebras involved are arbitrarily maximal generated.

Corollary 4 *Let R be subdirect product of simple arbitrarily maximal generated algebras $\mathbb{D}_1, \dots, \mathbb{D}_n$ from $\mathcal{F}(\mathbb{A})$. Then R is an almost trivial relation.*

Corollary 5 *Let $\mathcal{P} = (V; \mathcal{F}(\mathbb{A}); \delta; \mathcal{C})$ be a 3-minimal problem instance, where $\mathcal{A}$ is a class of simple arbitrarily maximal generated algebras from $\mathcal{F}(\mathbb{A})$, and each constraint relation is a subdirect product of its domains. Then if none of the constraint relations of $\mathcal{P}$ is empty then $\mathcal{P}$ has a solution.*

Proof: See [3]. □

We complete this section with two other auxiliary lemmas.

Lemma 12 *Let R be a subdirect product of maximal generated $\mathbb{D}_1, \dots, \mathbb{D}_n \in \mathcal{F}(\mathbb{A})$, where $\mathbb{D}_1$ is simple. Let also $\mathbb{D}_1$ be generated by a maximal scc A_1, $\mathrm{pr}_{2,\dots,n} R$ is maximal generated, say, by a maximal scc Q, $R \cap (A_1 \times Q) \neq \varnothing$, and $\mathrm{pr}_{1,i} R = \mathbb{D}_1 \times \mathbb{D}_i$ for $i \in \{2, \dots, n\}$, Then $R = \mathbb{D}_1 \times \mathrm{pr}_{2,\dots,n} R$.*

Proof: We prove the lemma by induction on n. The case $n = 2$ is obvious. Consider the case $n = 3$. We use induction on $|\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3|$. The trivial case $|\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3| = 3$ gives the base case of induction. Let Q be the maximal scc of $\mathrm{pr}_{2,3} R$ generating it. If both $\mathbb{D}_2, \mathbb{D}_3$ are simple, then the result follows from Proposition 5. Otherwise, suppose that $\mathbb{D}_3$ is not simple. Take a maximal congruence θ of $\mathbb{D}_3$, fix a θ-class C and consider $R' \subseteq \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3/_\theta$, $R'' \subseteq R$ such that

$$
\begin{aligned}
R' &= \{(a, b, c^\theta) \mid (a, b, c) \in R\}, \\
R'' &= \{(a, b, c) \mid (a, b, c) \in R, c \in C\},
\end{aligned}
$$

and $R''' \subseteq R$, the algebra generated by a maximal scc Q' of R'' such that $\mathrm{pr}_{2,3} Q' \cap Q \neq \varnothing$. Obviously, $\mathrm{pr}_{1,3} R'' = \mathbb{D}_1 \times C$. Moreover, $\mathbb{D}_1 \times C'' \subseteq \mathrm{pr}_{1,3} R'''$, where C'' is the algebra generated by a certain maximal scc C' of C. By Corollary 1, $\mathrm{pr}_{2,3} R'$ is either the graph of a bijective mapping, or $\mathbb{D}_2 \times \mathbb{D}_3/_\theta$.

CASE 1. $\mathrm{pr}_{2,3} R'$ is the graph of a bijective mapping $\pi\colon \mathbb{D}_2 \to \mathbb{D}_3/_\theta$.

In this case $B'' = \mathrm{pr}_2 R'''$ is the algebra generated by a maximal scc B' of $B = \pi^{-1}(C)$. Since for each $(a, b) \in \mathbb{D}_1 \times B \subseteq \mathrm{pr}_{1,2} R$ there is $c \in C$ with $(a, b, c) \in R$, we have $\mathbb{D}_1 \times B \subseteq \mathrm{pr}_{1,2} R''$. Furthermore, $\mathbb{D}_1 \times B''$ is the algebra generated by a maximal scc of $\mathbb{D}_1 \times B$, hence, $\mathrm{pr}_{1,2} R''' = \mathbb{D}_1 \times B''$.

Since $|\mathbb{D}_1| + |B''| + |C''| < |\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3|$, and $\mathrm{pr}_{2,3} R'''$ is maximal generated, inductive hypothesis implies $\mathbb{D}_1 \times \mathrm{pr}_{2,3} R''' \subseteq R'''$. In particular, there is $(a, b) \in \mathrm{pr}_{2,3} R''' \cap Q \subseteq \mathrm{pr}_{2,3} R$ such that $\mathbb{D}_1 \times \{(a, b)\} \subseteq R$. To finish the proof we just apply Lemma 6.

CASE 2. $\mathrm{pr}_{2,3} R' = \mathbb{D}_2 \times \mathbb{D}_3/_\theta$.

Since $|\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3/_\theta| < |\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3|$, $\mathbb{D}_3/_\theta$ is simple, and $\mathrm{pr}_{1,2} R = \mathbb{D}_1 \times \mathbb{D}_2$, by inductive hypothesis, $R' = \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3/_\theta$. Therefore, $\mathrm{pr}_{1,2} R'' = \mathbb{D}_1 \times \mathbb{D}_2$. Then $\mathrm{pr}_{1,2} R''' = \mathbb{D}_1 \times \mathbb{D}_2$. Now we argue as in Case 1.

Let us assume that the lemma is proved for $n-1$. Then $\mathbb{D}_1 \times \mathrm{pr}_{3,\dots,n}R \subseteq \mathrm{pr}_{1,3,\dots,n}R$. Denoting $\mathrm{pr}_{3,\dots,n}R$ by R' we have $R \subseteq \mathbb{D}_1 \times \mathbb{D}_2 \times R'$, and the conditions of the lemma hold for this subdirect product. Thus $R = \mathbb{D}_1 \times \mathrm{pr}_{2,\dots,n}R$ as required. $\Box$

Lemma 13 *Let R be a subdirect product of $\mathbb{D}_1, zD_2, \mathbb{D}_3$ such that it is generated by $R \cap (\widehat{a} \times \widehat{b} \times \widehat{c})$ for some maximal elements a, b, c, and such that $\widehat{a} \times \widehat{c} \subseteq \mathrm{pr}_{13}R$, $\widehat{b} \times \widehat{c} \subseteq \mathrm{pr}_{23}R$, and $Q = \max(\mathrm{pr}_{12}R \cap (\widehat{a} \times \widehat{b}))$ is strongly r-connected. Then $\mathrm{pr}_{12}R \times \widehat{c} \subseteq R$.*

Proof: Let R' be the relation generated by $R \cap (Q \times \widehat{c})$. Then $\mathrm{pr}_{1,2}R'$ is maximal generated. It is not hard to see that for any $(b', c') \in \widehat{b} \times \widehat{c}$ there is $a' \in \widehat{a}$ such that $(a', b', c') \in R$. Since (b', c') is maximal by Lemma 3 a' can be chosen such that $(a', b') \in \max(\mathrm{pr}_{1,2}R \cap (\widehat{a} \times \widehat{b}) = Q$. This implies $\widehat{b} \times \widehat{c} \subseteq \mathrm{pr}_{2,3}R$, and, therefore $\mathbb{D}_2 \times \mathbb{D}_3 \subseteq \mathrm{pr}_{2,3}R'$. Similarly, $\mathbb{D}_1 \times \mathbb{D}_3 \subseteq \mathrm{pr}_{1,3}R'$. Finally, R' is a subdirect product of $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$ and these algebras are maximal generated by $\widehat{a}, \widehat{b}, \widehat{c}$, respectively.

We show that for any subalgebra S of pr_3R such that (a) S is maximal generated by a maximal scc A, (b) $\widehat{a} \times A \subseteq \mathrm{pr}_{13}R$,$\widehat{b} \times A \subseteq \mathrm{pr}_{23}R$; (c) $Q \subseteq \mathrm{pr}_{12}(R \cap (\widehat{a} \times \widehat{b} \times S))$; (d) $R \cap (Q \times A) \neq \varnothing$, the following is true: there is $d \in S$ such that $Q \times \{d\} \subseteq R$. Observe that (b) implies (b') $\mathrm{pr}_1R \times Q \subseteq \mathrm{pr}_{1,3}R'$ and $\mathrm{pr}_2R \times Q \subseteq \mathrm{pr}_{2,3}R'$

We prove by induction on the size of S. If $|S| = 1$, then the claim is obvious. Suppose that the result holds for all subalgebras satisfying conditions (a)–(d) and smaller than S. Set $Q' = \{(c_1, c_2, c_3) \in Q \mid c_3 \in S\}$. If S is simple then the result follows from Lemma 10. Otherwise let θ be a maximal congruence of S, and let

$$R^\theta = \{(c_1, c_2, c_3^\theta) \mid (c_1, c_2, c_3) \in R'\}.$$

By Lemma 12 $R^\theta = \mathrm{pr}_{12}R' \times S/_\theta$. Take a class S' of θ, a maximal scc B of S' such that $R \cap (Q \times B) \neq \varnothing$, and let $S'' = \mathsf{Sg}(B)$. By the induction hypothesis it suffices to show that S'' satisfies conditions (a)–(d).

Condition (a) holds by the choice of S''. Condition (b) follows from (b') for Q. Let $C = Q \cap \mathrm{pr}_{12}(R \cap (\widehat{a} \times \widehat{b} \times B))$, since $R \cap (Q \times B) \neq \varnothing$, $C \neq \varnothing$. As Q is strongly r-connected there are $\mathbf{d} \in C$ and $\mathbf{d}' \in Q - C$ such that $\mathbf{d}\mathbf{d}'$ is a thin red edge. Take $e \in B$ and $e' \in S$ such that $(\mathbf{d}, e), (\mathbf{d}', e') \in R$. Such e' exists, as $Q \subseteq \mathrm{pr}_{12}(R \cap (\widehat{a} \times \widehat{b} \times S'))$. Set

$$\begin{pmatrix} \mathbf{d}' \\ e'' \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{d} \\ e \end{pmatrix}, \begin{pmatrix} \mathbf{d}' \\ e' \end{pmatrix}\right) \in R.$$

Here $e'' \in B$ therefore $\mathbf{d}' \in C$, a contradiction. Thus $C = Q$ and (c) is true for S''. Finally (d) is true by the choice of B. By inductive hypothesis $Q \times \{d\} \subseteq Q$ for $d \in S''$.

Since $\mathrm{pr}_3 R'$ satisfies (a)–(d), there is $d \in \mathrm{pr}_3 R'$ with $\mathrm{pr}_{1,2} R' \times \{d\} \subseteq R'$. Applying Lemma 5 we obtain the result. $\Box$

Corollary 6 *Let R be a subdirect product of arbitrarily maximal generated $\mathbb{D}_1, \dots, \mathbb{D}_n \in \mathcal{F}(\mathbb{A})$, where $\mathbb{D}_1$ is simple, $\mathrm{pr}_{2,\dots,n} R$ is maximal generated, and $\mathrm{pr}_{1,i} R = \mathbb{D}_1 \times \mathbb{D}_i$ for $i \in \{2, \dots, n\}$, Then $R = \mathbb{D}_1 \times \mathrm{pr}_{2,\dots,n} R$.*

5 Connectivity

Let $\mathbb{A}$ be a finite algebra such that $\mathsf{var}(\mathbb{A})$ omits the unary and affine types.

We say that an algebra $\mathbb{A}$ satisfies the *yellow connectivity property* if for any maximal scc's B, C of $\mathrm{Gr}'(\mathbb{A})$ there are $b \in B$ and $c \in C$ such that bc is a thin yellow edge. The main goal of this section is to prove

Proposition 6 *Any conglomerate algebra $\mathbb{A}$ satisfies the yellow connectivity property.*

We will prove by induction on the size of $\mathbb{A}$. The base case of induction, $|\mathbb{A}| = 2$ is trivial. Proposition 6 will follow from a series of lemmas. In all the lemmas we assume the inductive hypothesis. Thus, we assume that $\mathbb{A}$ is a conglomerate algebra such that every its proper subalgebra satisfies the yellow connectivity property.

Lemma 14 *Let R be a subalgebra of the direct product of algebras $\mathbb{A}_1, \dots, \mathbb{A}_n$ each of which satisfies the yellow connectivity property. Then R satisfies the yellow connectivity property.*

Proof: We proceed by induction on the arity of R. If R is unary, the claim is obvious that gives us the base case of induction. Suppose the result is true for all relations of smaller arity. Take $\mathbf{a}, \mathbf{b} \in \max(R)$. We distinguish two cases.

CASE 1. $\mathbf{a}[n] = \mathbf{b}[n]$

Consider the algebra $R' = R \cap (\mathbb{A}_1 \times \dots \times \mathbb{A}_{n-1} \times \{\mathbf{a}[n]\})$, and take $\mathbf{a}', \mathbf{b}' \in \max(R')$ such that $\mathbf{a} \prec \mathbf{a}'$ and $\mathbf{b} \prec \mathbf{b}'$. Observe that we might not be able to take $\mathbf{a}' = \mathbf{a}$ and $\mathbf{b}' = \mathbf{b}$, because $\mathbf{a}, \mathbf{b}$ may not be maximal elements in R'. By inductive hypothesis there are $\mathbf{a}'' \in \widehat{\mathrm{pr}_{[n-1]}\mathbf{a}'}$ and $\mathbf{b}'' \in \widehat{\mathrm{pr}_{[n-1]}\mathbf{b}'}$ such that $\mathbf{a}''\mathbf{b}''$ is a thin yellow edge. Clearly, $(\mathbf{a}'', \mathbf{a}[n])(\mathbf{b}'', \mathbf{a}[n])$ is also a thin yellow edge. Finally we note that these tuples belong to $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{b}}$, respectively.

CASE 2. $\mathbf{a}[n] \neq \mathbf{b}[n]$

Since $\mathbf{a}[n], \mathbf{b}[n] \in \max(\mathbb{A}_n)$, there is a directed ry-path $\mathbf{a}[n] = a_1, \dots, a_k = \mathbf{b}[n]$ in $\mathrm{Gr}'(\mathbb{A}_n)$ that contains exactly one yellow edge. We will show that there are $\mathbf{a}'_1, \mathbf{a}''_1, \mathbf{a}'_2, \dots, \mathbf{a}'_k, \mathbf{a}''_k$ such that

(a) there are directed r-paths from $\mathbf{a}$ to $\mathbf{a}'_1$ and from $\mathbf{a}''_k$ to $\mathbf{b}$;

(b) $\mathbf{a}'_i, \mathbf{a}''_i \in \max(R_i)$, where $R_i = R(\mathbb{A}_1 \times \dots \times \mathbb{A}_{n-1} \times \{a_i\})$;

(c) either $\mathrm{pr}_{[n-1]}\mathbf{a}''_i = \mathrm{pr}_{[n-1]}\mathbf{a}'_{i+1}$, or

- if $a_i a_{i+1}$ is a thin yellow edge then $\mathbf{a}''_i, \mathbf{a}'_{i+1} \in \max(R_{ii+1})$, where $R_{ii+1} = R(\mathbb{A}_1 \times \dots \mathbb{A}_{n-1} \times \{a_i, a_{i+1}\})$ (observe that, since $a_i a_{i+1}$ is a thin edge, $\{a_i, a_{i+1}\}$ is a subalgebra), and there is a directed ry-path containing only one yellow edge from $\mathbf{a}''_i$ to $\mathbf{a}'_{i+1}$ in R_{ii+1};
- if $a_i a_{i+1}$ is an r-edge then $\mathbf{a}''_i \mathbf{a}'_{i+1}$ is an r-edge and $\mathbf{a}''_i, \mathbf{a}'_{i+1}$ belong to the same scc of $\mathrm{Gr}'(R)$.

We proceed by induction on i. Choose $\mathbf{a}'_1 \in \max(R_1)$ such that $\mathbf{a} \prec \mathbf{a}'_1$. It is possible because $\mathbf{a} \in R_1$. This gives the base case of induction. Suppose $\mathbf{a}'_1, \mathbf{a}''_1, \dots, \mathbf{a}'_i$ are chosen.

CLAIM. If $a_i a_{i+1}$ is a thin yellow edge and there is $\mathbf{c} \in \mathrm{pr}_{[n-1]}R$ such that $(\mathbf{c}, a_i), (\mathbf{c}, a_{i+1}) \in R$, then there is $\mathbf{d} \in \mathrm{pr}_{[n-1]}R$ with the same property and such that $\mathbf{d} \in \mathrm{pr}_{[n-1]} \max(R_i)$, $\mathbf{d} \in \mathrm{pr}_{[n-1]} \max(R_{i+1})$, and $\mathbf{d} \in \mathrm{pr}_{[n-1]} \max(R_{ii+1})$.

It suffices to observe that if $\mathbf{e} \in \mathrm{pr}_{[n-1]}(R_i \cap R_{i+1})$ and $\mathbf{e}' \in \mathrm{pr}_{[n-1]}R_{ii+1}$ is such that $\mathbf{e} \leq \mathbf{e}'$ then $\mathbf{e}' \in \mathrm{pr}_{[n-1]}R_i$ and $\mathbf{e}' \in \mathrm{pr}_{[n-1]}R_{i+1}$. By the assumption $(\mathbf{e}, a_i), (\mathbf{e}, a_{i+1}) \in R$. Suppose that $(\mathbf{e}', a_i) \in R$ then

$$\begin{pmatrix} \mathbf{e}' \\ a_{i+1} \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{e} \\ a_{i+1} \end{pmatrix}, \begin{pmatrix} \mathbf{e}' \\ a_i \end{pmatrix} \right) \in R.$$

SUBCASE 2A. $a_i a_{i+1}$ is a thin yellow edge.

If there is $\mathbf{c} \in \mathrm{pr}_{[n-1]} \max(R_{ii+1})$ with $(\mathbf{c}, a_i), (\mathbf{c}, a_{i+1}) \in R$, then by Claim we can set we can set $\mathbf{a}''_i = (\mathbf{c}, a_i)$, $\mathbf{a}'_{i+1} = (\mathbf{c}, a_{i+1})$.

Otherwise take any $\mathbf{c} \in \max(R_i)$, $\mathbf{d} \in \max(R_{i+1})$. We have $\mathbf{c}, \mathbf{d} \in \max(R_{ii+1})$. Indeed, if $\mathbf{c} \prec \mathbf{c}' \in \max(R_{ii+1}) - \max(R_i)$, then there are $\mathbf{e}, \mathbf{e}'$ such that $\mathbf{e} \leq \mathbf{e}'$, $(\mathbf{e}, a_i) \in R$, and $(\mathbf{e}', a_{i+1}) \in R$. Then

$$\begin{pmatrix} \mathbf{e}' \\ a_i \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{e} \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{e}' \\ a_{i+1} \end{pmatrix} \right) \in R.$$

By induction hypothesis there is a directed ry-path, $\mathbf{c} = \mathbf{c}_1, \dots, \mathbf{c}_\ell = \mathbf{d}$, in $\mathrm{pr}_{[n-1]} R_{ii+1}$. Every $\mathbf{c}_j$ extends to a tuple from R_{ii+1} in a unique way. Thus either $(\mathbf{c}_j, a_i) \in R$ or $(\mathbf{c}_j, a_{i+1}) \in R$. Let j be such that $(\mathbf{c}_j, a_i) \in R$ and $(\mathbf{c}_{j+1}, a_{i+1}) \in R$, or $(\mathbf{c}_{j+1}, a_i) \in R$ and $(\mathbf{c}_j, a_{i+1}) \in R$. We claim that $\mathbf{c}_j\mathbf{c}_{j+1}$ is a thin yellow edge. Indeed, if $\mathbf{c}_j\mathbf{c}_{j+1}$ is a red edge then

$$\begin{pmatrix} \mathbf{c}_{j+1} \\ a_i \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{c}_j \\ a_i \end{pmatrix}, \begin{pmatrix} \mathbf{c}_{j+1} \\ a_{i+1} \end{pmatrix} \right) \in R,$$

a contradiction. Finally, for any $\mathbf{c}_j\mathbf{c}_{j+1}$, r-edge $(\mathbf{c}_j, a_i)(\mathbf{c}_{j+1}, a_i)$ (or $(\mathbf{c}_j, a_{i+1})(\mathbf{c}_{j+1}, a_{i+1})$) belongs to the same scc of R_{ii+1}, and therefore to the same scc of R. We set $\mathbf{a}''_i = \mathbf{c}$ and $\mathbf{a}'_{i+1} = \mathbf{d}$.

SUBCASE 2B. $a_i a_{i+1}$ is a red edge.

First, we construct a sequence of elements of R_{ii+1} as follows:

- Set $\mathbf{c}_0$ to be any maximal element from R_i, thus, $\mathbf{c}_0[n] = a_i$.
- If $\mathbf{c}_j$ is found, let $\mathbf{d}_j$ be a maximal element from R_{ii+1} such that $\mathbf{c}_j \le \mathbf{d}_j$ and $\mathbf{d}_j[n] = a_{i+1}$. For instance, we can choose $\mathbf{d}'_j = f(\mathbf{c}_j, \mathbf{e})$, where $\mathbf{e}$ is any tuple $\mathbf{e} \in R$ with $\mathbf{e}[n] = a_{i+1}$, and then $\mathbf{d}_j$ as any maximal element with $\mathbf{d}'_j \prec \mathbf{d}_j$.
- Since a_i, a_{i+1} belong to the same scc, there is a directed r-path from a_{i+1} to a_i in $\mathbb{A}_n$. Expanding this path to an r-path from $\mathbf{d}_j$, we obtain a tuple $\mathbf{c}'_{j+1} \in R$ such that $\mathbf{c}'_{j+1}[n] = a_i$ and $\mathbf{d}_j \prec \mathbf{c}'_{j+1}$ in R. Now set $\mathbf{c}_{j+1}$ to be any maximal element from R_i with $\mathbf{c}'_{j+1} \prec \mathbf{c}_j$.

Since R_{ii+1} is finite for some $j, \ell \in \mathbb{N}$ with $j \ne \ell$ (say, $j < \ell$), we have $\mathbf{c}_j = \mathbf{c}_\ell$. This means that R contains a directed r-path from $\mathbf{c}_j$ to $\mathbf{d}_j$, and from $\mathbf{d}_j$ to $\mathbf{c}_j$. Set $\mathbf{a}''_i = \mathbf{c}_j$ and $\mathbf{a}'_{i+1} = \mathbf{d}_j$.

Summarizing, by Case 1, there is a directed ry-path from $\mathbf{a}'_i$ to $\mathbf{a}''_i$ for each i; by Subcase 2a there is a directed ry-path from $\mathbf{a}''_i$ to $\mathbf{a}'_{i+1}$ for the i such that $a_i a_{i+1}$ is a thin yellow edge; and, by Subcase 2b, there is a directed path from $\mathbf{a}''_i$ to $\mathbf{a}'_{i+1}$ for each i such that $a_i a_{i+1}$ is a red edge. Moreover, in the latter case $\mathbf{a}''_i, \mathbf{a}'_{i+1}$ belong to the same scc of R. □

Lemma 15 *If ab is a yellow edge and θ a witnessing congruence of $\mathsf{Sg}(a, b)$ then there is an automorphism φ of $\mathsf{Sg}(a, b)/_\theta$ such that $\varphi(a^\theta) = b^\theta$ and $\varphi(b^\theta) = a^\theta$.*

Proof: Follows from [5]. □

Next we prove that a thick yellow edge can always be replaced with a thin yellow edge.

Lemma 16 *Let ab be a (thick) yellow edge in $\mathrm{Gr}(\mathbb{A})$, $\mathbb{A} = \mathsf{Sg}(a, b)$. Then there is $a' \in \max(a^\theta) \cap \mathrm{Ft}(a)$ and $b' \in \max(b^\theta) \cap \mathrm{Ft}(b)$ such that $a'b'$ is a thin yellow edge.*

Proof: We prove by induction of the size of $\mathbb{A}$. The base case of induction, $|\mathbb{A}| = 2$, is obvious. So suppose the statement holds for all algebras $\mathbb{B}$ with $|\mathbb{B}| < |\mathbb{A}|$. We also assume the induction hypothesis of Proposition 6.

CLAIM 1. a and b are not connected in $\mathsf{Sg}(a, b)$ with a r-path.

Indeed, if a, b are r-connected then there are $a'' \in a^\theta$, $b'' \in b^\theta$ such that $a''b''$ or $b''a''$ is a red edge. However f^θ is a projection on $\{a^\theta, b^\theta\}$, a contradiction.

Suppose first that there are $c \in \max(a^\theta)$ and $d \in \max(b^\theta)$ such that $a \prec c$, $b \prec d$, and $\mathsf{Sg}(c, d) \neq \mathbb{A}$. Let θ' denote the restriction of θ onto $\mathsf{Sg}(c, d)$. Then by the induction hypothesis there are $c' \in \max(c^{\theta'})$ and $d' \in \max(d^{\theta'})$ such that $c \prec c'$, $d \prec d'$, and $c'd'$ is a thin y-edge. Thus we may assume that, for any $c \in \max(a^\theta) \cap \mathrm{Ft}(a), d \in \max(b^\theta) \cap \mathrm{Ft}(b)$, we have $\mathsf{Sg}(c, d) = \mathbb{A}$. In particular, we may assume that a and b are maximal elements. Observe also that $\max(\mathbb{A}) = \max(a^\theta) \cup \max b^\theta$.

First we show that a, b can be chosen such that $(a, b), (b, a)$ are maximal in the subalgebra generated by $\{(a, b), (b, a)\}$. Indeed, choose maximal $a' \in a^\theta \cap \mathrm{Ft}(a)$, $b' \in b^\theta \cap \mathrm{Ft}(b)$ such that $\mathsf{Sg}((a', b'), (b', a'))$ is smallest possible. Let (c, d) be a maximal element in this algebra such that $(c, d) \in \mathrm{Ft}((a', b')) \cup \mathrm{Ft}((b', a'))$. By Claim 1, either $c \in a^\theta, d \in b^\theta$, or $c \in b^\theta, d \in a^\theta$. Since c, d are maximal, by the choice of a', b' we have $\mathsf{Sg}((c, d), (d, c)) = \mathsf{Sg}((a', b'), (b', a'))$. Thus c, d satisfy the required conditions.

We prove that subalgebra R of $\mathbb{A}^6$ generated by (a, b, a, b, b, a), (a, b, b, a, a, b), (b, a, a, b, a, b) contains (a, b, a, b, a, b). It will be convenient for us to denote $c_i = a$ for $i = 1, 3, 5$ and $c_i = b$ for $i = 2, 4, 6$.

CLAIM 2. (a, b, a, b) is maximal in $\mathrm{pr}_{1234}R$.

Note that $\mathrm{pr}_{12}R = \mathsf{Sg}((a, b), (b, a))$ and that (a, b) is maximal in $\mathrm{pr}_{12}R$. Since $(a, b, a, b), (b, a, a, b) \in \mathrm{pr}_{1234}R$, $\mathrm{pr}_{12}R \times \{(a, b)\} \subseteq \mathrm{pr}_{1234}R$. Let $\mathbf{c} \in \mathrm{pr}_{1234}R$ be such that $(a, b, a, b) \prec \mathbf{c}$. Then $(\mathbf{c}[1], \mathbf{c}[2])$ and $(\mathbf{c}[3], \mathbf{c}[4])$ are from the same scc as (a, b). There is a directed r-path from $(\mathbf{c}[3], \mathbf{c}[4])$ to (a, b). By Lemma 1 this path can be extended to a path from $\mathbf{c}$ to some tuple $\mathbf{d} \in \mathrm{pr}_{1234}R$ such that $(\mathbf{d}[3], \mathbf{d}[4]) = (a, b)$ and $(\mathbf{d}[1], \mathbf{d}[2])$ belongs to the same scc as $(\mathbf{c}[1], \mathbf{c}[2])$, and therefore (a, b). There is a path $(\mathbf{d}[1], \mathbf{d}[2]) = \mathbf{d}_1 \leq \ldots \leq \mathbf{d}_k = (a, b)$ in $\mathrm{pr}_{12}R$. Since $\mathrm{pr}_{12}R \times \{(a, b)\} \subseteq \mathrm{pr}_{1234}R$, this path is extendible by (a, b) to a path in $\mathrm{pr}_{1234}R$. This implies that (a, b, a, b) is maximal.

CLAIM 3. For any $i \in \{1, 3, 5\}$, and any $(d_i, d_{i+1}) \in \max(\mathrm{pr}_{ii+1} R \cap (a^\theta \times b^\theta))$ there are $a_j \in c_j^\theta$, $j \in [6] - \{i, i+1\}$, such that $\mathbf{a} \in R$, where

$$\mathbf{a}[j] = \begin{cases} d_j, & \text{if } j = i \text{ or } j = i+1, \\ a_j, & \text{otherwise} \end{cases}$$

Without loss of generality we may assume that $i = 1$. It suffices to prove that there are a_3, a_5 with the required properties, since by Lemma 15 $\mathrm{pr}_{34} R^\theta$ and $\mathrm{pr}_{56} R^\theta$ are graphs of an isomorphism, it implies $a_4, a_6 \in b^\theta$.

Observe first that, since $\begin{pmatrix} a \\ b \\ a \end{pmatrix}, \begin{pmatrix} b \\ a \\ a \end{pmatrix} \in \mathrm{pr}_{123} R, \mathrm{pr}_{125} R$, we have $\mathrm{pr}_{12} R \times \{a\} \subseteq \mathrm{pr}_{123} R, \mathrm{pr}_{125} R$.

Let $R'' = \mathrm{pr}_{12} R \cap (a^\theta \times b^\theta)$ and

$$D = \{(a', b') \in R'' \mid \text{there are } a_3, a_5 \in a^\theta \text{ such that } (a', b', a_3, a_5) \in \mathrm{pr}_{1235} R\}.$$

Since

$$g\left(\begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \\ b \\ a \end{pmatrix}, \begin{pmatrix} b \\ a \\ a \\ a \end{pmatrix}\right) \in \begin{pmatrix} a^\theta \\ b^\theta \\ a^\theta \\ a^\theta \end{pmatrix},$$

this set is non-empty. By the inductive hypothesis a^θ and in b^θ satisfy the yellow connectivity property. By Lemma 14 the same is true for R''. Therefore if $\max(R'') \not\subseteq D$, there are $(a', b') \in D$ and $(a'', b'') \in R'' - D$ such that $(a', b')(a'', b'')$ is a thin red or yellow edge.

By R' we will denote the relation $\mathrm{pr}_{1235} R \cap (a^\theta \times b^\theta \times a^\theta \times a^\theta)$. If $(a', b')(a'', b'')$ is a red edge then take any $\mathbf{a}'' = (a'', b'', a''_3, a''_5) \in \mathrm{pr}_{1235} R$ and any $\mathbf{a}' = (a', b', a'_3, a'_5) \in R'$. We have

$$\begin{pmatrix} a'' \\ b'' \\ a'''_3 \\ a'''_5 \end{pmatrix} = f\left(\begin{pmatrix} a' \\ b' \\ a'_2 \\ a'_3 \end{pmatrix}, \begin{pmatrix} a'' \\ b'' \\ a''_3 \\ a''_5 \end{pmatrix}\right) \in R'.$$

Let $(a', b')(a'', b'')$ be a yellow edge. Take $\mathbf{a}''_1 = (a'', b'', a''_3, b_5)$, $\mathbf{a}''_2 = (a'', b'', b_3, a''_5) \in \mathrm{pr}_{1235} R$ such that $a''_3, a''_5 \in a^\theta$. Such tuples exist because as we observed above $\mathrm{pr}_{12} R \times \{a\} \subseteq \mathrm{pr}_{123} R, \mathrm{pr}_{125} R$, and a''_3, a''_5 can be chosen to be a. Let also $\mathbf{a}' = (a', b', a'_3, a'_5) \in R'$. Then for the tuple $(a'', b'', a'''_3, a'''_5) = g(\mathbf{a}', \mathbf{a}''_1, \mathbf{a}''_2) \in \mathrm{pr}_{1235} R$ we have $(a'', b'', a'''_3, a'''_5) \in R$, a contradiction. Claim 3 is proved.

CLAIM 4. For any $I \in \{\{1,2,3,4\},\{1,2,5,6\},\{3,4,5,6\}\}$, there are $b_i \in \max(c_i^\theta)$, $i \in [6]-I$, such that $\mathbf{a}' \in R$, where

$$\mathbf{a}'[i] = \begin{cases} c_i, & \text{if } i \in I, \\ b_i, & \text{if } i \notin I. \end{cases}$$

Without loss of generality let $I = \{1,2,3,4\}$. It suffices to show that b_5 with the required properties exists. If it is so for sume tuple $\mathbf{a}$ we shall say that tuple $\mathbf{a}$ is a^θ*-extendible*.

We show first that if some tuple $\mathbf{a} \in \mathrm{pr}_{12345} R$ is a^θ-extendible and there is a directed r-path from $\mathbf{a}$ to $\mathbf{b}$ in $\mathrm{pr}_{1234} R$, then $\mathbf{b}$ is also a^θ-extendible. Indeed, if $(\mathbf{a}, a_5) \in \mathrm{pr}_{12345} R$ for some $a_5 \in a^\theta$, then an r-path from $\mathbf{a}$ to $\mathbf{b}$ can be extended to an r-path from $(\mathbf{a}, a_5)$ to some $(\mathbf{b}, b_5)$ in $\mathrm{pr}_{12345} R$. Since b_5 is in the same r-connected component as a_5, it belongs to a^θ, that implies the claim.

Let $S = \max(\mathrm{pr}_{12} R \cap (a^\theta \times b^\theta))$ and $S' = \mathrm{pr}_{1234} R \cap (a^\theta \times b^\theta \times a^\theta \times b^\theta)$. It will be convinient to denote the tuple (a, b) by $\mathbf{a}$. Let

$$D = \{\mathbf{b} \in a^\theta \times b^\theta \mid \text{there is } \mathbf{a}' \in a^\theta \times b^\theta \text{ such that } (\mathbf{a}', \mathbf{b}) \in \max(S') \text{ and is } a^\theta\text{-extendible}\}$$

and

$$D' = \{\mathbf{b} \in D \mid \text{there is } \mathbf{a}' \in \widehat{\mathbf{a}} \text{ such that } (\mathbf{a}', \mathbf{b}) \text{ is } a^\theta\text{-extendible}\}.$$

By Claim 3 $D = \mathrm{pr}_{34} R \cap (a^\theta \times b^\theta)$. If $\max(D) \subseteq D'$ then we are done. Indeed, $(a, b) \in \max(D)$, hence, there is $\mathbf{a}' \in \widehat{\mathbf{a}}$ such that $(\mathbf{a}', a, b)$ is a^θ-extendible, say, $(\mathbf{a}', a, b, a_5) \in \mathrm{pr}_{12345} R$. Since $\mathrm{pr}_{12} R \times \{(a, b)\} \subseteq \mathrm{pr}_{1234} R$, there is a directed path from $(\mathbf{a}', a, b)$ to $(\mathbf{a}, a, b)$. This path can be extended to a path from $(\mathbf{a}', a, b, a_5)$ to $(\mathbf{a}, a, b, a_5')$ in $\mathrm{pr}_{12345} R$ for some $a_5' \in a^\theta$.

Now assume that $\max(D) \not\subseteq D'$. Claim 3 implies that $D' \neq \varnothing$: First take $\mathbf{a}' = \mathbf{a}$ and then extend it correspondingly to Claim 3 obtaining some tiple $\mathbf{b} \in D'$. As D is a subalgebra of direct product of a^θ and b^θ, by the induction hypothesis and Lemma 14 there are $\mathbf{b}' \in D'$ and $\mathbf{b}'' \in D - D'$ such that $\mathbf{b}'\mathbf{b}''$ a thin red or yellow edge.

If $\mathbf{b}'\mathbf{b}''$ is a red edge then take $(\mathbf{a}', \mathbf{b}', a_5) \in \mathrm{pr}_{12345} R$ for some $\mathbf{a}' \in \widehat{\mathbf{a}}$ and $a_5 \in a^\theta$, and $(\mathbf{a}'', \mathbf{b}'', a_5') \in \mathrm{pr}_{12345} R$ such that $a_5' \in a^\theta$ and consider

$$\begin{pmatrix} \mathbf{a}''' \\ \mathbf{b}'' \\ a_5'' \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \\ a_5 \end{pmatrix}, \begin{pmatrix} \mathbf{a}'' \\ \mathbf{b}'' \\ a_5' \end{pmatrix} \right).$$

For this tuple we have $\mathbf{a}' \leq \mathbf{a}'''$ and therefore $\mathbf{a}''' \in \widehat{\mathbf{a}}$ and $a_5'' \in a^\theta$, a contradiction.

If $\mathbf{b}'\mathbf{b}''$ is a yellow edge then consider the relation

$$\begin{aligned} S &= \mathrm{pr}_{12}(R \cap (\mathbb{A}^2 \times \{\mathbf{b}'\} \times \mathbb{A}^2)) \cap \mathrm{pr}_{12}(R \cap (\mathbb{A}^2 \times \{\mathbf{b}''\} \times \mathbb{A}^2)) \\ & \quad \cap \mathrm{pr}_{12}(R \cap (\mathbb{A}^2 \times \{\mathbf{b}', \mathbf{b}''\} \times a^\theta \times b^\theta)). \end{aligned}$$

This relation consists of all tuples $\mathbf{a}'$ such that $(\mathbf{a}', \mathbf{b}'), (\mathbf{a}', \mathbf{b}'') \in \mathrm{pr}_{1234}R$ and at least one of them is a^θ-extendible.

We show that $\widehat{\mathbf{a}} \subseteq \mathrm{pr}_{12}(R \cap (\mathbb{A}^2 \times \{\mathbf{b}'\} \times \mathbb{A}^2)) \cap \mathrm{pr}_{12}(R \cap (\mathbb{A}^2 \times \{\mathbf{b}''\} \times \mathbb{A}^2))$. Since $(a,b,a,b), (a,b,b,a) \in \mathrm{pr}_{1234}R$, $\mathbf{a} \times \mathrm{pr}_{34}R \subseteq \mathrm{pr}_{1234}R$. Then if $\mathbf{d}$ satisfies the condition $\mathbf{d} \times \mathrm{pr}_{34}R \subseteq \mathrm{pr}_{1234}R$, and $\mathbf{d} \le \mathbf{e}$ then $\mathbf{e}$ also satisfies this condition. Let $F = \{\mathbf{c} \in \max(\mathrm{pr}_{34}R) \mid (\mathbf{e}, \mathbf{c}) \in R_{1234}R\}$. If there is $\mathbf{c} \in F$ and $\mathbf{c}' \in \max(\mathrm{pr}_{34}R)$ such that $\mathbf{c}\mathbf{c}'$ is a red edge then

$$\begin{pmatrix} \mathbf{e} \\ \mathbf{c}' \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{d} \\ \mathbf{c}' \end{pmatrix}, \begin{pmatrix} \mathbf{e} \\ \mathbf{c} \end{pmatrix} \right) \in \mathrm{pr}_{1234}R,$$

a contradiction. Otherwise if there is a maximal scc C such that $C \cap F = \varnothing$, then take $\mathbf{c}' \in C$ and $\mathbf{c} \in F$. We have

$$\begin{pmatrix} \mathbf{e} \\ \mathbf{c}'' \end{pmatrix} = f\left(\begin{pmatrix} \mathbf{d} \\ \mathbf{c}' \end{pmatrix}, \begin{pmatrix} \mathbf{e} \\ \mathbf{c} \end{pmatrix} \right) \in \mathrm{pr}_{1234}R,$$

where $\mathbf{c}' \le \mathbf{c}''$, and so $\mathbf{c}'' \in F$, a contradiction again. In the remaining case $F = \max(\mathrm{pr}_{34}R)$, in particular, $(a,b), (b,a) \in F$. This implies $\mathbf{e} \times \mathrm{pr}_{34}R \subseteq \mathrm{pr}_{1234}R$.

Therefore, as for some $\mathbf{a}' \in \widehat{\mathbf{a}}$ we have $(\mathbf{a}', \mathbf{b}')$ is a^θ-extendible, $S \ne \varnothing$. Now let $R' = R \cap (S \times \{\mathbf{b}', \mathbf{b}''\} \times \mathbb{A}^2)$ and

$$E = \{\mathbf{a}' \mid (\mathbf{a}', \mathbf{b}'') \in \mathrm{pr}_{1234}R' \text{ and } (\mathbf{a}', \mathbf{b}'') \text{ is } a^\theta\text{-extendible}\}.$$

If $\max(E) = \max(\mathrm{pr}_{12}R')$ then $\mathbf{a}' \in \max(\mathrm{pr}_{12}R')$ for some $\mathbf{a}' \in \widehat{\mathbf{a}}$, and we get a contradiction with $\mathbf{b}'' \notin D'$. Thus assume $\max(E) \ne \max(\mathrm{pr}_{12}R')$

Next we show that $E \ne \varnothing$. Suppose the contrary, $E = \varnothing$. This means, in particular, that the relation

$$S' = \mathrm{pr}_{1234}(R \cap (a^\theta \times b^\theta \times \{\mathbf{b}', \mathbf{b}''\} \times a^\theta \times b^\theta))$$

is the graph of a mapping $\pi\colon \mathrm{pr}_{12}S' \to \{\mathbf{b}', \mathbf{b}''\}$. Since $\mathbf{b}', \mathbf{b}'' \in D$, both $\pi^{-1}(\mathbf{b}')$ and $\pi^{-1}(\mathbf{b}'')$ are non-empty. Moreover, as there is $\mathbf{a}' \in \pi^{-1}(\mathbf{b}') \cap \widehat{\mathbf{a}}$ and $\{\mathbf{a}'\} \times \mathbb{A}^2 \subseteq \mathrm{pr}_{1234}R$, there is also $(\mathbf{a}', \mathbf{b}'', b_5) \in \mathrm{pr}_{12345}R$ for some $b_5 \in b^\theta$. For any $\mathbf{a}'' \in \pi^{-1}(\mathbf{b}'')$ we have

$$g\left(\begin{pmatrix} \mathbf{a}'' \\ \mathbf{b}'' \end{pmatrix}, \begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix}, \begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix} \right) \in S'.$$

Since $g(\mathbf{b}'', \mathbf{b}', \mathbf{b}') = \mathbf{b}'$ we have that $\mathbf{a}''' = g(\mathbf{a}'', \mathbf{a}', \mathbf{a}') \in \pi^{-1}(\mathbf{b}')$. Now let $a_5, a_5' \in a^\theta$ be such that $(\mathbf{a}'', \mathbf{b}'', a_5), (\mathbf{a}', \mathbf{b}', a_5') \in \mathrm{pr}_{12345} R$. Such a tuple $(\mathbf{a}'', \mathbf{b}'', a_5)$ exists, as $\mathbf{b}'' \in D$. Then

$$g\left(\begin{pmatrix}\mathbf{a}''\\ \mathbf{b}''\\ a_5\end{pmatrix}, \begin{pmatrix}\mathbf{a}'\\ \mathbf{b}''\\ b_5\end{pmatrix}, \begin{pmatrix}\mathbf{a}'\\ \mathbf{b}'\\ a_5'\end{pmatrix}\right) = \begin{pmatrix}\mathbf{a}'''\\ \mathbf{b}''\\ a_5''\end{pmatrix} \in R'.$$

However, $\mathbf{a}''' \in \pi^{-1}(\mathbf{b}')$ and $a_5'' \in a^\theta$, a contradiction with the assumption on the structure of S'.

If $E \cap \max(\mathrm{pr}_{12} R') = \varnothing$ then there are $\mathbf{a}' \in E$ and $\mathbf{a}'' \in \mathrm{pr}_{12} R' - E$ such that $\mathbf{a}' \leq \mathbf{a}''$. Therefore $(\mathbf{a}', \mathbf{b}'', a_5), (\mathbf{a}'', \mathbf{b}'', b_5') \in \mathrm{pr}_{12345} R$ for some $a_5 \in a^\theta$, and $b_5 \in \mathbb{A}$. Therefore

$$\begin{pmatrix}\mathbf{a}''\\ \mathbf{b}''\\ a_5'\end{pmatrix} = f\left(\begin{pmatrix}\mathbf{a}'\\ \mathbf{b}''\\ a_5\end{pmatrix}, \begin{pmatrix}\mathbf{a}''\\ \mathbf{b}''\\ b_5\end{pmatrix}\right),$$

belongs to $\mathrm{pr}_{12345} R$ and $a_5' \in a^\theta$. A contradiction.

Now we also need to show that $E \cap (a^\theta \times b^\theta) \neq \varnothing$. As $E \neq \varnothing$, $(\mathbf{c}, \mathbf{b}'', a_5) \in \mathrm{pr}_{12345} R$ for some $\mathbf{c} \in E$ and $a_5 \in a^\theta$. As $\{(a, b)\} \times \mathrm{pr}_{34} R \subseteq \mathrm{pr}_{1234} R$, $(\mathbf{a}, \mathbf{b}'', b_5) \in \mathrm{pr}_{12345} R$ for some $b_5 \in \mathbb{A}$. Finally, as $\mathbf{b}' \in D'$, there are $\mathbf{a}' \in \widehat{\mathbf{a}}$ and $a_5' \in a^\theta$ such that $(\mathbf{a}', \mathbf{b}', a_5') \in \mathrm{pr}_{12345} R$. Then the tuple

$$\begin{pmatrix}\mathbf{a}''\\ \mathbf{b}''\\ a_5''\end{pmatrix} = g\left(\begin{pmatrix}\mathbf{c}\\ \mathbf{b}''\\ a_5\end{pmatrix}, \begin{pmatrix}\mathbf{a}\\ \mathbf{b}''\\ b_5\end{pmatrix}, \begin{pmatrix}\mathbf{a}'\\ \mathbf{b}'\\ a_5'\end{pmatrix}\right),$$

where $\mathbf{a}'' \in a^\theta \times b^\theta$ and $a_5'' \in a^\theta$ belongs to $\mathrm{pr}_{12345} R$.

Therefore by Lemma 14 there are $\mathbf{a}' \in E$ and $\mathbf{a}'' \in \mathrm{pr}_{12} R' - E$ such that $\mathbf{a}'\mathbf{a}''$ is a thin red or yellow edge.

If $\mathbf{a}'\mathbf{a}''$ is a red edge then we argue as before. If $\mathbf{a}'\mathbf{a}''$ is a yellow edge then observe that $(\mathbf{a}', \mathbf{b}'')$ and $(\mathbf{a}'', \mathbf{b}')$ are a^θ-extendible. Moreover, $(\mathbf{a}'', \mathbf{b}'') \in \mathrm{pr}_{1234} R'$. Take $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in R'$ such that $\mathrm{pr}_{1234}\mathbf{a}_1 = (\mathbf{a}', \mathbf{b}'')$, $\mathrm{pr}_{1234}\mathbf{a}_2 = (\mathbf{a}'', \mathbf{b}')$, $\mathrm{pr}_{1234}\mathbf{a}_3 = (\mathbf{a}'', \mathbf{b}'')$, and $\mathbf{a}_1[5], \mathbf{a}_2[5] \in a^\theta$. We get

$$g(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = g\left(\begin{pmatrix}\mathbf{a}'\\ \mathbf{b}''\\ \mathbf{a}_1[5]\end{pmatrix}, \begin{pmatrix}\mathbf{a}''\\ \mathbf{b}'\\ \mathbf{a}_2[5]\end{pmatrix}, \begin{pmatrix}\mathbf{a}''\\ \mathbf{b}''\\ \mathbf{a}_3[5]\end{pmatrix}\right) = \begin{pmatrix}\mathbf{a}''\\ \mathbf{b}''\\ a_5'''\end{pmatrix} \in R,$$

for some $a_5 \in a^\theta$. A contradiction.

Observation 1 *Since $(a,b,a,b,b,a)\in R$ and $(a',b',a,b,a,b)\in R$ for some $a'\in a^\theta$, $b'\in b^\theta$ by Claim 4, $\{(a,b)\}\times \mathrm{pr}_{56}R\subseteq \mathrm{pr}_{3456}(R\cap(a^\theta\times b^\theta\times \mathrm{pr}_{3456}R)$. The same holds for any combination of pairs $\{1,2\},\{3,4\},\{5,6\}$, of coordinate positions.*

CLAIM 5. Relation $R'=R\cap(\widehat{\mathbf{a}}\times\widehat{\mathbf{a}}\times\widehat{\mathbf{a}})$ is a subdirect product of $\widehat{\mathbf{a}}$, $\widehat{\mathbf{a}}$, and $\widehat{\mathbf{a}}$. Moreover, any tuple from $\widehat{\mathbf{a}}\times\widehat{\mathbf{a}}$ is extendible by a tuple from $\widehat{\mathbf{a}}$.

Let $\overline{R}=\mathrm{pr}_{1234}R\cap(\widehat{\mathbf{a}}\times\widehat{\mathbf{a}})$. We show that every $\mathbf{b}\in\overline{R}$ can be extended by a tuple $\mathbf{a}'\in\widehat{\mathbf{a}}$. By Observation 1 we know that $(\{\mathbf{a}\}\times\widehat{\mathbf{a}})\cup(\widehat{\mathbf{a}}\times\{\mathbf{a}\})\subseteq C=\{\mathbf{b}\in\mathrm{pr}_{1234}R\mid(\mathbf{b},\mathbf{d})\in R,\mathbf{d}\in a^\theta\times b^\theta\}$. Show that $\widehat{\mathbf{a}}\times\widehat{\mathbf{a}}$ belongs to this set. Let $B=\{\mathbf{c}\in\widehat{\mathbf{a}}\mid\{\mathbf{c}\}\times\widehat{\mathbf{a}}\subseteq C\}$. If $B\neq\widehat{\mathbf{a}}$, there are $\mathbf{c}\in B$ and $\mathbf{c}'\in\widehat{\mathbf{a}}-B$ such that $\mathbf{c}\mathbf{c}'$ is a red edge. Let also $B'=\{\mathbf{c}''\in\widehat{\mathbf{a}}\mid(\mathbf{c}',\mathbf{c}'')\in C\}$. Again as $B'\neq\widehat{\mathbf{a}}$ there are $\mathbf{c}''\in B'$ and $\mathbf{c}'''\in\widehat{\mathbf{a}}-B'$ such that $\mathbf{c}''\mathbf{c}'''$ is a red edge. Then

$$\begin{pmatrix}\mathbf{c}'\\ \mathbf{c}'''\end{pmatrix}=f\left(\begin{pmatrix}\mathbf{c}\\ \mathbf{c}'''\end{pmatrix},\begin{pmatrix}\mathbf{c}'\\ \mathbf{c}''\end{pmatrix}\right)\in R,$$

a contradiction.

If, for some $\mathbf{b}\in\overline{R}$ and some $\mathbf{d}\in\widehat{\mathbf{a}}$, it holds that $(\mathbf{b},\mathbf{d})\in R$ then every $\mathbf{b}\in\overline{R}$ is extendible in $\widehat{\mathbf{a}}$. This easily follows from the fact that $\overline{R}=\widehat{\mathbf{a}}\times\widehat{\mathbf{a}}$ by the same argument as above.

Now due to directed ry-connectedness there are $\mathbf{c},\mathbf{c}'\in\mathrm{pr}_{56}R\cap(a^\theta\times b^\theta)$ such that $\mathbf{c}$ extends to $\widehat{\mathbf{a}}\times\widehat{\mathbf{a}}$, while $\mathbf{c}'$ does not, and $\mathbf{c}\mathbf{c}'$ is a thin red edge or yellow edge. If $\mathbf{c}\mathbf{c}'$ is a thin red edge then we quickly get a contradiction. Indeed, let $(\mathbf{b},\mathbf{c}),(\mathbf{b}',\mathbf{c}')\in R$ where $\mathbf{b}\in\widehat{\mathbf{a}}\times\widehat{\mathbf{a}}$. Then

$$\begin{pmatrix}\mathbf{b}''\\ \mathbf{c}'\end{pmatrix}=f\left(\begin{pmatrix}\mathbf{b}\\ \mathbf{c}\end{pmatrix},\begin{pmatrix}\mathbf{b}'\\ \mathbf{c}'\end{pmatrix}\right)R,$$

where $\mathbf{b}\le\mathbf{b}''$, and so $\mathbf{b}''\in\widehat{\mathbf{a}}\times\widehat{\mathbf{a}}$.

Suppose that $\mathbf{c}\mathbf{c}'$ is a thin yellow edge. Set $R'=R\cap(a^\theta\times b^\theta\times a^\theta\times b^\theta\times\{\mathbf{c},\mathbf{c}'\})$. It is easy to see that $\widehat{\mathbf{a}}\cap\max(\mathrm{pr}_{12}R')\neq\varnothing$ and $\widehat{\mathbf{a}}\cap\max(\mathrm{pr}_{34}R')\neq\varnothing$. Indeed, by Observation 1 $\mathbf{a}\in\mathrm{pr}_{12}R'$, and if $\mathbf{a}\not\in\max(\mathrm{pr}_{12}R')$ then there is $\mathbf{a}'$ with $\mathbf{a}\prec\mathbf{a}'$ and $\mathbf{a}'\in\max(\mathrm{pr}_{12}R')$. However, $\mathbf{a}'\in\widehat{\mathbf{a}}$.

Let

$$\begin{aligned}D' &= \{\mathbf{b}\in\mathrm{pr}_{34}R'\mid(\mathbf{d},\mathbf{b})\in\mathrm{pr}_{1234}R',\ \text{for }\mathbf{d}\in\widehat{\mathbf{a}},\ \text{and }(\mathbf{b},\mathbf{c}')\in\mathrm{pr}_{3456}R'\},\quad\text{and}\\ D &= \{\mathbf{b}\in\mathrm{pr}_{34}R'\mid(\mathbf{d},\mathbf{b},\mathbf{c}')\in R',\ \text{for }\mathbf{d}\in\widehat{\mathbf{a}}\}.\end{aligned}$$

Again by Observation 1 $D\neq\varnothing$, and by the assumption $\max(D')\not\subseteq D$ (as is easily seen $\mathbf{a}\in D'$, so $\widehat{\mathbf{a}}\cap D'\neq\varnothing$). On the other hand, if $\mathbf{b}\in D$ then $\mathrm{Ft}(\mathbf{b})\subseteq D$

(here $\mathrm{Ft}(\mathbf{b})$ is computed in $\mathrm{pr}_{34}R'$). Therefore there are $\mathbf{b}, \mathbf{b}'$ such that $\mathbf{b} \in D$ and $\mathbf{b}' \in D' - D$, and $\mathbf{b}\mathbf{b}'$ is a thin yellow edge. Note that for any $\mathbf{d} \in \widehat{\mathbf{a}} \cap \mathrm{pr}_{12}R'$ with $(\mathbf{d}, \mathbf{b}') \in \mathrm{pr}_{1234}R'$ we have $(\mathbf{d}, \mathbf{b}', \mathbf{c}) \in R'$.

Set $R'' = R' \cap (a^\theta \times b^\theta \times \{\mathbf{b}, \mathbf{b}'\} \times \{\mathbf{c}, \mathbf{c}'\})$. Observe that $\max(\mathrm{pr}_{12}R'') \cap \widehat{\mathbf{a}} \neq \varnothing$. Indeed, $(\mathbf{b}, \mathbf{c}')$ is extendible by some $\mathbf{d} \in \widehat{\mathbf{a}}$. Then we argue as before.

Set

$$\begin{aligned} E' &= \{\mathbf{d} \in \max(\mathrm{pr}_{12}R'') \mid (\mathbf{d}, \mathbf{c}') \in \mathrm{pr}_{1256}R'', (\mathbf{d}, \mathbf{b}') \in \mathrm{pr}_{1234}R''\}, \quad \text{and} \\ E &= \{\mathbf{d} \in \max(\mathrm{pr}_{12}R'') \mid (\mathbf{d}, \mathbf{b}', \mathbf{c}') \in R\}. \end{aligned}$$

Note that $E \neq \varnothing$, since $\mathbf{b}' \in D'$. Moreover, if $\mathbf{d} \in E$ then $\mathrm{Ft}(\mathbf{d}) \subseteq E$ (here $\mathrm{Ft}(\mathbf{d})$ is computed in $\mathrm{pr}_{12}R''$). Observe that $\max(E') \not\subseteq E$. Indeed, otherwise by the choice of $\mathbf{b}$ we have $(\mathbf{e}, \mathbf{b}, \mathbf{c}') \in R$ for all $\mathbf{e} \in \mathrm{Ft}(\mathbf{a})$ (computed in E'), so, $(\mathbf{e}, \mathbf{c}') \in \mathrm{pr}_{1256}R''$ for all $\mathbf{e} \in \mathrm{Ft}(\mathbf{a})$. Also, as $\mathbf{b}' \in D'$, it holds that $(\mathbf{e}, \mathbf{b}') \in \mathrm{pr}_{1234}R''$ for all $\mathbf{e} \in \mathrm{Ft}(\mathbf{a})$, an by the choice of D' we have $(\mathbf{e}, \mathbf{b}', \mathbf{c}) \in R$ for all $\mathbf{e} \in \mathrm{Ft}(\mathbf{a})$. Thus $\mathrm{Ft}(\mathbf{a}) \cap \max(E') \neq \varnothing$, and $\mathrm{Ft}(\mathbf{a}) \cap E \neq \varnothing$ would contradict the construction of E.

Therefore there are $\mathbf{d}, \mathbf{d}'$ such that $\mathbf{d} \in E$, $\mathbf{d}' \in E' - E$ and $\mathbf{d}, \mathbf{d}'$ is a thin yellow edge.

We have: $(\mathbf{d}, \mathbf{b}', \mathbf{c}') \in R$ by the choice of $\mathbf{d}$; $(\mathbf{d}', \mathbf{b}, \mathbf{c}') \in R$, because $(\mathbf{d}', \mathbf{c}') \in \mathrm{pr}_{1256}R''$ and $(\mathbf{d}', \mathbf{b}', \mathbf{c}') \not\in R$ by the choice of $\mathbf{d}'$; finally, $(\mathbf{d}', \mathbf{b}', \mathbf{c}) \in R$, because, $(\mathbf{d}', \mathbf{b}') \in \mathrm{pr}_{1234}R''$ and $(\mathbf{d}', \mathbf{b}', \mathbf{c}') \not\in R$. Now

$$\begin{pmatrix} \mathbf{d}' \\ \mathbf{b}' \\ \mathbf{c}' \end{pmatrix} = g\left(\begin{pmatrix} \mathbf{d} \\ \mathbf{b}' \\ \mathbf{c}' \end{pmatrix}, \begin{pmatrix} \mathbf{d}' \\ \mathbf{b} \\ \mathbf{c}' \end{pmatrix}, \begin{pmatrix} \mathbf{d}' \\ \mathbf{b}' \\ \mathbf{c} \end{pmatrix} \right) \in R,$$

a contradiction.

Finally, since $\widehat{\mathbf{a}} \times \widehat{\mathbf{a}} \subseteq \mathrm{pr}_{1234}R$ by the standard argument we can show that any tuple from $\widehat{\mathbf{a}} \times \widehat{\mathbf{a}}$ is extendible by a tuple from $\widehat{\mathbf{a}}$.

CLAIM 6. If $\widehat{\mathbf{a}} \times \widehat{\mathbf{a}} \times \widehat{\mathbf{a}} \subseteq R$.

Let Q be the subalgebra of R generated by $R \cap (\widehat{\mathbf{a}} \times \widehat{\mathbf{a}} \times \widehat{\mathbf{a}})$. By Claim 5 $\widehat{\mathbf{a}} \times \widehat{\mathbf{a}} \subseteq \mathrm{pr}_{1234}Q, \mathrm{pr}_{3456}Q, \mathrm{pr}_{1256}Q$. Since $\widehat{\mathbf{a}} \times \widehat{\mathbf{a}}$ is strongly r-connected we can apply Lemma 13. □

In the rest of this section we show that there is a thin yellow edge between some elements of any two maximal scc's of $\mathbb{A}$. We call such scc's *yellow connected*. We first need an auxiliary lemma.

Lemma 17 *Let θ be a congruence of $\mathbb{A}$ and A, B elements of $\mathbb{A}/_\theta$. Then*
(1) if $A \leq B$ then for any $a \in A$ there is $b \in B$ such that ab is a red thin edge;
(2) any maximal element A of $\mathbb{A}/_\theta$ contains a maximal element of $\mathbb{A}$.

Proof: (1) Take $a \in A$ and any $c \in B$. Then $b = f(a,c) \in B$ and ab is a red edge.

(2) Take $a \in A$ and any maximal $b \in \mathbb{A}$ with $a \prec b$. Clearly, $A = a^\theta \prec b^\theta$ in $\mathbb{A}/_\theta$. Since A is maximal, we also have $b^\theta \prec A$ in $\mathbb{A}/_\theta$. By part (1) of this lemma, it means that there is a directed r-path from b to a certain element $d \in A$. As b is maximal, d is also maximal. $\square$

Next we show that the relation of yellow connectivity is transitive.

Lemma 18 *Suppose elements a, b are connected with a directed ry-path containing only maximal elements. Then there are $a' \in \widehat{a}$ and $b' \in \widehat{b}$ such that $a'b'$ is a thin yellow edge.*

Proof: We prove by induction on (1) the size of $\mathbb{A}$, and (2) the number of yellow edges in the directed y-path connecting a and b. Thus, we may assume that $\mathbb{A} = \mathsf{Sg}(a,b)$. Moreover, we may assume that $\mathbb{A}$ is simple. Indeed, if θ is a proper congruence of $\mathbb{A}$, then there are a', b' such that $a'^\theta \in \widehat{a^\theta}$, $b'^\theta \in \widehat{b^\theta}$, and $a'^\theta b'^\theta$ is a thin yellow edge in $\mathbb{A}/_\theta$. This means $a'b'$ is a yellow edge (not necessarily thin) of $\mathrm{Gr}(\mathbb{A})$. By Lemma 16 there are $a'' \in \widehat{a'}$, $b'' \in \widehat{b'}$ such that $a''b''$ is a thin yellow edge.

We also assume that $\mathbb{A}$ is minimal among algebras $\mathsf{Sg}(a', b')$ for $a' \in \widehat{a}$, $b' \in \widehat{b}$. Thus, $\mathbb{A} = \mathsf{Sg}(a', b')$ for any $a' \in \widehat{a}$, $b' \in \widehat{b}$. Consider relation R generated by (a,b) and (b,a). Again we assume that this relation is such that for any $a' \in \widehat{a}$, $b' \in \widehat{b}$ with $(a',b'), (b',a') \in R$ the relation is also generated by $(a',b'), (b',a')$. As usual this relation defines a tolerance on $\mathbb{A}$: $\varrho = \{(c,d) \mid$ there is e such that $(e,c), (e,d) \in R\}$. We consider two cases.

CASE 1. R is a graph of an automorphism.

Let $a = a_1$, and let $a'_1 a_2, \dots, a'_{k-1} a_k = b$ be the yellow edges from the ry-path connecting a and b, and also $a'_i \in \widehat{a_i}$ for all i. Now we proceed by induction on k. If $\mathsf{Sg}(a, a_{k-1}) \neq \mathbb{A}$ then by induction hypothesis there are $a' \in \widehat{a}$ and $a'' \in \widehat{a_{k-1}}$ such that $a'a''$ is a thin yellow edge. If $\mathsf{Sg}(a, a_{k-1}) = \mathbb{A}$ then a and a_{k-1} are connected with an ry-path with fewer yellow edges and again by the induction hypothesis there are $a' \in \widehat{a}$ and $a'' \in \widehat{a_{k-1}}$ such that $a'a''$ is a thin yellow edge. In either cases we can assume $k = 3$.

So suppose $a'c$ and $c'b'$ are y-edges and $c' \in \widehat{c}$ (and both, c and c' are maximal). If the relation generated by $(a',b'), (b',a')$ does not satisfy conditions of Case 1, we replace a with a', b with b' and prove the result under Case 2. Otherwise we assume $a' = a$, $b' = b$. There is a term operations $p(x,y), r(x,y)$ such that $p(a,b) = c$ and $r(a,b) = c'$. Let $d = p(b,a)$ and $d' = r(b,a)$. Since R is an automorphism mapping a to b and b to a, bd and ad' are yellow edges. Set

$$g'(x,y,z) = g(x, p(x,y), p(x,z)).$$

It is not hard to see that

$$g'(a,a,b) = g'(a,b,a) = a, \quad g'(b,a,a) = d.$$

Let Q be the relation generated by $(a,a,b),(a,b,a),(b,a,a)$. It is easy to see that $(\{a\}\times\mathbb{A})\cup(\mathbb{A}\times\{a\}) \subseteq \mathrm{pr}_{1,2}Q = \mathrm{pr}_{1,3}Q = \mathrm{pr}_{2,3}Q$. Let Q' be the relation generated by $R\cap(\widehat{a}\times\widehat{a}\times\widehat{d})$. For this relation we have: (a) $\mathrm{pr}_{1,3}Q' = \mathrm{pr}_1 Q'\times\mathrm{pr}_3 Q'$ and $\mathrm{pr}_{2,3}Q' = \mathrm{pr}_2 Q'\times\mathrm{pr}_3 Q'$ (it follows from Lemma 5 and the observation above), and (b) $\widehat{a}\times\widehat{a} \subseteq \mathrm{pr}_{12}Q$. Therefore $\max(\mathrm{pr}_{12}Q\cap(\widehat{a}\times\widehat{a})) = \widehat{a}\times\widehat{a}$, and so is strongly r-connected. By Lemma 13 $\mathrm{pr}_{12}Q\times\widehat{d} \subseteq Q'$. In particular, $(a,a,d')\in Q'$.

Similarly, $(d',a,a),(a,d',a)\in Q$. Finally applying g to these three tuples we get $(a,a,a)\in Q$, which implies that there is a majority operation on $\{a,b\}$.

CASE 2. R is not the graph of an automorphism.

Since $\mathbb{A}$ is simple this means that ϱ is a connected tolerance. There are classes of this tolerance $D_1,\dots,D_k$ such that $a\in D_1$, $b\in D_k$, and for each i there is e_i such that $\{e_i\}\times D_i\subseteq R$, and $D_i\cap D_{i+1}\neq\varnothing$. Again we have two cases to consider.

SUBCASE 2A. For some i, $D_i=\mathbb{A}$, or, equivalently, $k=1$.

Let $\{e\}\times\mathbb{A}\subseteq R$. If $\mathsf{Sg}(a,e)=\mathbb{A}$, then, as $(a,b),(e,b)\in R$, pair (b,b) is also in R. This means that there is a term operation $p(x,y)$ such that $p(a,b) = p(b,a) = b$, and so $a\le b$, a contradiction.

Suppose that $\mathsf{Sg}(a,e)\neq\mathbb{A}$ and $\mathsf{Sg}(b,e)\neq\mathbb{A}$. By induction hypothesis there are $a'\in\widehat{a}$, $e',e''\in\widehat{e}$, and $b'\in\widehat{b}$ such that $a'e'$ and $e''b'$ are thin yellow edges. By Lemma 7 $\{e'\}\times\mathbb{A}\subseteq R$, in particular, $(e'e'')\in R$. By symmetricity, also $(e'',e')\in R$. Since $\widehat{e}\times\widehat{e}\subseteq R$, this pair is a maximal element in R. Again by Lemma 7 we have $(\widehat{a}\times\widehat{b})\cup(\widehat{b}\times\widehat{a})\subseteq R$. In particular $(a',b'),(b',a')\in R$, by the assumption made, these two pairs generate R. There is a term operation $p(x,y)$ such that $p(a,b)=e'$, $p(b,a)=e''$.

Consider 6-ary relation Q generated by $(a',b',a',b',b',a'),(a',b',b',a',a',b')$, (b',a',a',b',a',b'). As in Case 1 let $g'(x,y,z) = g(x,p(x,y),p(x,z))$. Applying g' to those three tuples we get $(a',b',a',b',e',e'')\in Q$. Let $\mathbf{a}=(a',b')$ and $\mathbf{e}=(e',e'')$. As before $(\{\mathbf{a}\}\times\mathbb{A}^2)\cup(\mathbb{A}^2\times\{\mathbf{a}\})\subseteq\mathrm{pr}_{1,2,3,4}Q = \mathrm{pr}_{3,4,5,6}Q = \mathrm{pr}_{1,2,5,6}Q$. Let also Q' be the relation generated by $Q\cap(\widehat{\mathbf{a}}\times\widehat{\mathbf{a}}\times\widehat{\mathbf{e}})$. Observe that (a) $\mathrm{pr}_{1,2,5,6}Q' = \mathrm{pr}_{1,2}Q'\times\mathrm{pr}_{5,6}Q'$ and $\mathrm{pr}_{3,4,5,6}Q' = \mathrm{pr}_{3,4}Q'\times\mathrm{pr}_{5,6}Q'$ (it follows from Lemma 5 and the observation above), and (b) $\widehat{\mathbf{a}}\times\widehat{\mathbf{a}}\subseteq\mathrm{pr}_{1,2,3,4}Q$. Therefore $\max(\mathrm{pr}_{1,2,3,4}Q\cap(\widehat{\mathbf{a}}\times\widehat{\mathbf{a}})) = \widehat{\mathbf{a}}\times\widehat{\mathbf{a}}$, and so is strongly r-connected. By Lemma 13 $\mathrm{pr}_{1,2,3,4}Q'\times\widehat{\mathbf{e}}\subseteq Q'$. In particular, $(\mathbf{a},\mathbf{a},e'',e')\in Q'$.

Similarly, $(e'', e', \mathbf{a}, \mathbf{a}), (\mathbf{a}, e'', e', \mathbf{a}) \in Q$. Finally applying g to these three tuples we get $(\mathbf{a}, \mathbf{a}, \mathbf{a}) \in Q$, which implies that there is a majority operation on $\{a', b'\}$.

SUBCASE 2B. For every i, $D_i \neq \mathbb{A}$.

In this case use yet another induction parameter, the number k. Then Subcase 2a gives the base case for induction. Choose $a_i \in D_{i-1} \cap D_i$ and $a_1 = a$. By Lemma 8 a_i can be chosen maximal. If $\mathsf{Sg}(a, a_k) \neq \mathbb{A}$ then by induction hypothesis there are $a' \in \widehat{a}$ and $a'_k \in \widehat{a_k}$ such that $a'a'_k$ is a thin yellow edge. If $\mathsf{Sg}(a, a_k) = \mathbb{A}$ then again the same true by induction hypothesis, as a and a_k are connected with shorter chain of ϱ-classes.

If $\mathsf{Sg}(b, e_k) \neq \mathbb{A}$ then there is $b' \in \widehat{b}$ and $e' \in \widehat{e_k}$ such that $b'e'$ is a thin yellow edge and there are $a'' \in \widehat{a}$, $b'' \in \widehat{b}$ with $(b', a''), (e', b'') \in R$. By the assumption made $\mathsf{Sg}(a'', b'') = \mathbb{A}$ therefore for any $d \in \mathbb{A}$ either $(b', d) \in R$ or $(e', d) \in R$. Since $(a, b) \in R$, argueing as in the proof of Lemma 6 we obtain $\widehat{a} \times \widehat{b} \subseteq R$ and $\widehat{b} \times \widehat{a} \subseteq R$.

Suppose first that $(e', e') \in R$. Then as before it is not hard to show that $\{e'\} \times \widehat{e'} \subseteq R$. If $\mathsf{Sg}(a'', e') = \mathbb{A}$ then, since $(b', a''), (b', e') \in R$, we have Subcase 2a. So, suppose $\mathsf{Sg}(a'', e') \neq \mathbb{A}$. Then there are $a''' \in \widehat{a''}$ and $e'' \in \widehat{e'}$ such that $a'''e''$ is a thin yellow edge. As we showed before, $(\widehat{a} \times \widehat{b}) \cup (\widehat{b} \times \widehat{a}) \subseteq R$, in particular, $(a''', b''), (b'', a''') \in R$. By the assumption made these two pairs generate R, and we can assume $a''' = a$, $b'' = b$.

There is a term operation $p(x, y)$ such that $p(a, b) = e''$ and $p(b, a) = e'$. Consider 6-ary relation Q generated by $(a, b, a, b, b, a), (a, b, b, a, a, b), (b, a, a, b, a, b)$. As in Case 1 let $g'(x, y, z) = g(x, p(x, y), p(x, z))$. Applying g' to those three tuples we get $(a, b, a, b, e', e'') \in Q$. Let $\mathbf{a} = (a, b)$ and $\mathbf{e} = (e', e'')$. As before $(\{\mathbf{a}\} \times \mathbb{A}^2) \cup (\mathbb{A}^2 \times \{\mathbf{a}\}) \subseteq \mathrm{pr}_{1,2,3,4}Q = \mathrm{pr}_{3,4,5,6}Q = \mathrm{pr}_{1,2,5,6}Q$. Let also Q' be the relation generated by $Q \cap (\widehat{\mathbf{a}} \times \widehat{\mathbf{a}} \times \widehat{\mathbf{e}})$. Observe that (a) $\mathrm{pr}_{1,2,5,6}Q' = \mathrm{pr}_{1,2}Q' \times \mathrm{pr}_{5,6}Q'$, $\mathrm{pr}_{3,4,5,6}Q' = \mathrm{pr}_{3,4}Q' \times \mathrm{pr}_{5,6}Q'$ (it follows from Lemma 5 and the observation above), and (b) $\widehat{\mathbf{a}} \times \widehat{\mathbf{a}} \subseteq \mathrm{pr}_{1,2,3,4}Q$. Therefore $\max(\mathrm{pr}_{1,2,3,4}Q \cap (\widehat{\mathbf{a}} \times \widehat{\mathbf{a}})) = \widehat{\mathbf{a}} \times \widehat{\mathbf{a}}$, and so is strongly r-connected. By Lemma 13 $\mathrm{pr}_{1,2,3,4}Q' \times \widehat{c} \subseteq Q'$. In particular, $(\mathbf{a}, \mathbf{a}, e'', e') \in Q'$. Observe that (e', e'') and (e'', e') are both maximal in $R = \mathrm{pr}_{5,6}Q$, since $\widehat{e'} \times \widehat{e'} \subseteq R$.

Similarly, $(e'', e', \mathbf{a}, \mathbf{a}), (\mathbf{a}, e'', e', \mathbf{a}) \in Q$. Finally applying g to these three tuples we get $(\mathbf{a}, \mathbf{a}, \mathbf{a}) \in Q$, which implies that there is a majority operation on $\{a, b\}$. $\square$

Let $a \in \mathbb{A}$. The *depth* of a, denoted $\mathsf{dep}(a)$, is the maximal number k such that there is a directed r-path from a to a maximal element that goes through exactly k scc of $\mathrm{Gr}'(\mathbb{A})$. In particular, a is maximal if and only if $\mathsf{dep}(a) = 1$.

An ry-path is said to be *1-stage* if it has the form $a_1, \ldots, a_k, a_{k+1}, \ldots, a_m$, where $a_1 \geq a_2 \geq \ldots \geq a_k$, pair $a_k a_{k+1}$ is a thin yellow edge; and $a_{k+1} \leq \ldots \leq a_m$. We also allow *0-stage* paths defined in a similar way except that such a path does not contain a yellow edge and $a_k \leq \ldots \leq a_m$. The depth of 1-stage ry-path $\mathcal{A} = a_1, \ldots, a_k, \ldots, a_\ell, \ldots, a_m$, denoted $\mathsf{dep}(\mathcal{A})$, is the maximum of $\mathsf{dep}(a_k), \mathsf{dep}(a_{k+1})$, and that of a 0-stage path is $\mathsf{dep}(a_k)$. Elements a_k and a_{k+1} of a 1-stage path we will sometimes call *distinguished*.

Lemma 19 *It suffices to prove the result for $a, b \in \max(\mathbb{A})$ under the following assumptions:*

(1) a, b are connected with a 1-stage or 0-stage ry-path;
(2) if $a = a_1, \ldots, a_k, a_{k+1}, \ldots, a_m = b$ is a 1-stage path connecting a and b, then $\mathsf{dep}(a_k) + \mathsf{dep}(a_{k+1}) > 2$;
(3) $\mathbb{A} = \mathsf{Sg}(a', b')$ for any $a' \in \widehat{a}$ and $b' \in \widehat{b}$;
(4) $\mathsf{Sg}(a, b)$ is simple.

Proof: (1) Clearly, every ry-path can be split into 1-stage or 0-stage fragments (with a possibility that $k = 1$ or $k + 1 = m$). However, we also need to guarantee that the ends of those fragments are maximal elements. Let $a_1, \ldots, a_k, a_{k+1}, \ldots, a_m = a'_1, \ldots, a'_{k'}, a'_{k'+1}, \ldots, a'_{m'}$ be an ry-path and $a_1, \ldots, a_k, a_{k+1}, \ldots, a_m$ and $a'_1, \ldots, a'_{k'}, a'_{k'+1}, \ldots, a'_{m'}$ are 1-stage ry-paths. Choose a maximal element c such that there is a directed r-path $a_m = c_1 \leq c_2 \leq \ldots \leq c_t = c$. Then $a_1, \ldots, a_k, a_{k+1}, \ldots, a_m, c_2, \ldots, c_t = c$ and $c = c_t, \ldots, c_2, c_1 = a'_1, \ldots, a'_{k'}, a'_{k'+1}, \ldots, a'_{m'}$ are 1-stage ry-paths whose ends are maximal elements. It suffices to prove that a, c and c, b are connected with a directed ry-path. Finally we apply Lemma 18. For 0-stage paths or combination of 1-stage and 0-stage paths the argument is the same.

(2) If $\mathsf{dep}(a_k) + \mathsf{dep}(a_\ell) = 2$, then a_k and a_ℓ belong to the same scc's as a_1 and a_m, respectively. Thus, a_1 is connected to a_m with a directed ry-path.

(3) If $\mathsf{Sg}(a', b') \subset \mathbb{A}$ then by the induction hypothesis $\mathsf{Sg}(a', b')$ has the yellow connectivity property. Let a'', b'' be maximal elements of $\mathsf{Sg}(a', b')$ such that $a'' \in \mathrm{Ft}(a')$ and $b'' \in \mathrm{Ft}(b')$. Then $\widehat{a''}$ and $\widehat{b''}$ are yellow connected in $\mathsf{Sg}(a', b')$. Since a, b are maximal and $a \prec a''$, $b \prec b''$, elements a, a' and b, b' are strongly r-connected in $\mathrm{Gr}(\mathbb{A})$. Thus $\widehat{a}$, $\widehat{b}$ are yellow connected.

(4) Let θ be a non-trivial congruence of $\mathbb{A} = \mathsf{Sg}(a, b)$. Then $|\mathbb{A}/_\theta| < |\mathbb{A}|$, therefore $\widehat{a^\theta}$ and $\widehat{b^\theta}$ are yellow connected. In other words there is a directed ry-path $a^\theta = A_1, \ldots, A_k = b^\theta$ in $\mathrm{Gr}(\mathbb{A}/_\theta)$ from a^θ to b^θ that goes through maximal elements and contains at most one yellow edge. We show that $\widehat{a}$, $\widehat{b}$ are yellow connected by constructing a directed ry-path in $\mathbb{A}$ as follows.

(a) Let a' be an element maximal in A_1 and such that $a \prec a'$. Elements a and a' belong to the same scc of $\mathrm{Gr}'(\mathbb{A})$, so we choose a directed r-path from a to a' as the initial part of our path.

Suppose that a path is constructed from a to a certain element $c \in A_i$ such that c is maximal in A_i.

(b) Let A_iA_{i+1} be a yellow edge. Observe that $\max(A_i \cup A_{i+1}) = \max(A_i) \cup \max(A_{i+1})$. For any $c' \in A_{i+1}$, maximal in A_{i+1}, the pair cc' is a yellow edge, but not necessarily thin. By Lemma 16, there are $d \in A_i$, $d' \in A_{i+1}$ such that dd' is a thin yellow edge, and d, d' are maximal in A_i and A_{i+1}, respectively. Moreover, $d \in \mathrm{Ft}(c)$ and a directed r-path from c to d in A_i, and therefore in $\mathbb{A}$.

(c) Let $A_i \le A_{i+1}$ $\mathrm{Gr}(\mathbb{A}/_\theta)$. Since the two classes belong to the same scc of $\mathrm{Gr}'(\mathbb{A})$ there is a directed r-path fron A_{i+1} to A_i. Let $A_i = B_0, A_{i+1} = B_1, B_2, \dots, B_{n-1}, B_0 = A_i$ be a directed r-path connecting A_i with A_{i+1}, and then A_{i+1} to A_i. By Lemma 17(1) for any $j \in \{0, \dots, n-1\}$ and any $d \in B_j$ there is $d' \in B_{j+1}$ such that dd' is a thin red edge. We construct a directed r-path $d_0, d_1, \dots$ as follows. Set $d_0 = c$. Then if $d_r \in \max(B_j)$ then set d_{r+1} to be an element in $B_{j+1 \pmod n}$ such that d_rd_{r+1} is a thin red edge. Otherwise set $d_{r+1}, \dots, d_{r+s}$ to be a directed r-path in B_j from d_r to a maximal element d_{r+s} in B_j.

Since A_i is finite there are p, q, $p < q$, such that $d_p = d_q$ is a maximal element in A_i. Also let d_u be such that $p < u < q$ and d_u is a maximal element of A_{i+1}. Since both c and d_p are maximal elements of A_i, by induction hypothesis scc's $\widehat{c}$ and $\widehat{d_p}$ are yellow connected. By Lemma 18 $\widehat{a}$ and $\widehat{d_p}$ are yellow connected. The same holds for $\widehat{a}$ and $\widehat{d_q}$. □

Let a, b are connected with a 1-stage or 0-stage path $\mathcal{A} = a_1, \dots, a_k, a_{k+1}, \dots, a_m$ with $a = a_1, b = a_m$ and a_k, a_{k+1} being the distingished elements if $\mathcal{A}$ is 1-stage. We distinguish several cases:

(a) $\mathcal{A}$ is 1-stage, and $\mathsf{dep}(a_k) > 1$, $\mathsf{dep}(a_{k+1}) > 1$;

(b) $\mathcal{A}$ is 1-stage, and $\mathsf{dep}(a_k) > 1$, $\mathsf{dep}(a_{k+1}) = 1$, or $\mathsf{dep}(a_k) = 1$, $\mathsf{dep}(a_{k+1}) > 1$,

(c) $\mathcal{A}$ is 1-stage, and $\mathsf{dep}(a_k) = 1$, $\mathsf{dep}(a_{k+1}) > 2$, or $\mathsf{dep}(a_k) > 2$, $\mathsf{dep}(a_{k+1}) = 1$,

(d) $\mathcal{A}$ is 0-stage, $u \neq v$, where u denotes the number of scc's on the path $a_1, \dots, a_k$, and v denotes the number of scc's on the path $a_k, \dots, a_m$; observe that in this case $u, v > 1$ and either $u > 2$ or $v > 2$,

(e) $\mathcal{A}$ is 0-stage, $u = v > 2$,

(f) $\mathcal{A}$ is 0-stage, $u = v = 2$.

We prove by induction on the depth of the path. If $\mathsf{dep}(\mathcal{A}) = 1$ then the result follows from Lemma 19(2) that gives the base case. Then we rank 1- and 0-stage paths so that paths of lower depth precede paths of higher depth, and 0-stage paths precede 1-stage paths of the same depth.

An element c is called *left sub-bottom* (with respect to ry-path $a_1, \ldots, a_m$) if $c \in \widehat{a_{k'}}$, where $a_{k'}$ is the last element with $k' < k$ and such that $\mathsf{dep}(a_{k'}) < \mathsf{dep}(a_k)$. Similarly, d is called *right sub-bottom* (with respect to ry-path $a_1, \ldots, a_m$) if $d \in \widehat{a_{\ell'}}$, where $a_{\ell'}$ is the first element with $\ell < \ell'$ and such that $\mathsf{dep}(a_{\ell'}) < \mathsf{dep}(a_\ell)$.

Lemma 20 *Let c, d be chosen as follows (according to the cases above):*

(a) if $\mathsf{dep}(a_k) \geq \mathsf{dep}(a_{k+1})$ then $c \in \widehat{a_1}$ and $d \in \widehat{a_{k+1}}$; if $\mathsf{dep}(a_k) < \mathsf{dep}(a_{k+1})$ then $c \in \widehat{a_k}$ and $d \in \widehat{a_m}$;

(b) $c \in \widehat{a_1}$ and $d \in \widehat{a_m}$;

(c) $c \in \widehat{a_1}$ and d is a right sub-bottom element;

(d) if $u > v$ then c is a left sub-bottom element and $d \in \widehat{a_m}$, otherwise $c \in \widehat{a_1}$ and d is a right sub-bottom element;

(e) $c \in \widehat{a_1}$ and d is a right sub-bottom element;

(f) $c \in \widehat{a_1}$ and $d \in \widehat{a_m}$.

Then if $\mathsf{Sg}(c, d) \neq \mathbb{A}$ then a, b are connected with a directed ry-path.

Proof:

(a) We assume $\mathsf{dep}(a_k) \geq \mathsf{dep}(a_{k+1})$, the second case is similar. Let c', d' be elements maximal in $\mathsf{Sg}(c, d)$ such that $c \prec c'$ and $d \prec d'$. By inductive hypothesis, $\widehat{c'}$ and $\widehat{d'}$ are yellow connected in$\mathsf{Sg}(c, d)$. Let c'' and d'' be maximal elements of $\mathbb{A}$ such that $c' \prec c''$ and $d' \prec d''$. Since $c \in \widehat{a_1}$, element a_1 is connected with d'' with a directed r-path. Let $d'' = c_0 \geq \ldots \geq c_{t_2} = d' \geq c_{t_2+1} \geq \ldots \geq c_{s_2} = a_{k+1}$ be an r-path connecting d'' with a_{k+1}. This is a 0-stage path of depth less than $\mathsf{dep}(\mathcal{A})$.

• As was mentioned, a_1 and a'' are connected by a directed r-path.

• c'' and d'' are connected with the path $\mathcal{A}'$, namely, $c'' = b_{s_1}, \ldots, b_{t_1}, d_1, \ldots, d_u, c_{t_2}, \ldots, c_0 = d''$, where $d_1, \ldots, d_u$ is a 1- or 0-stage path in $\mathsf{Sg}(()c, d)$ connecting c' and d'. Then $\mathsf{dep}(\mathcal{A}') < \mathsf{dep}(\mathcal{A})$, since $\mathsf{dep}(c') < \mathsf{dep}(a_k)$ and $\mathsf{dep}(d') \leq \mathsf{dep}(a_{k+1})$. Therefore $\widehat{c''}$ and $\widehat{d''}$ are yellow connected.

• d'' and a_m are connected by the 0-stage path $d'' = c_0, \ldots, c_{s_2} = a_\ell, a_{\ell+1}, \ldots, a_m$. By inductive hypothesis, $\widehat{d''}$ and $\widehat{b}$ are yellow connected.

Finally Lemma 18 completes the proof.

(b) Similarly to the previous case we choose c', d', c'', d''. Since $c \in \widehat{a_1}$, $d \in \widehat{a_m}$, and so $c', c'' \in \widehat{a_1}$ and $d', d'' \in \widehat{a_m}$, there is a directed r-path connecting a_1 with c'', and a directed r-path connecting d'' with a_m. Moreover, by the inductive hypothesis $\widehat{c''}$ and $\widehat{d''}$ are yellow connected.

(c) We consider case $\mathsf{dep}(a_{k+1}) > 2$. Again we choose c', d', c'', d''. Then a_1 is strongly r-connected with c''. Let $c'' = b_1, \dots, b_s = c'$ be an r-path connecting c'' with c'. By induction hypothesis $\widehat{c'}$ and $\widehat{d'}$ are yellow connected in $\mathsf{Sg}(c, d)$, i.e. there is a 1- or 0-stage path, $c' = d_1, \dots, d_u = d'$ from c' to d' in $\mathsf{Sg}(c, d)$. Let also $d' = b_{s+1}, \dots, b_t = d''$, be a directed r-path from d' to d''. Path $\mathcal{A}$: $b_1, \dots, b_s = d_1, \dots, d_u = b_{s+1}, \dots, b_t$, is a 1- or 0-stage path. Since $\mathsf{dep}(d_u) < \mathsf{dep}(a_{k+1})$, the $\mathsf{dep}(\mathcal{A}') < \mathsf{dep}(\mathcal{A})$, and by the inductive hypothesis $\widehat{c''}$ and $\widehat{d''}$ are yellow connected. Finally, $d'' = b_t, \dots, b_{s+1} = d', c_1, \dots, c_v = d, c_{v+1}, \dots, c_w = a_\ell, a_{\ell+1}, \dots, a_m$, where c_ℓ is the chosen sub-bottom element, is a 0-stage path of depth less than that of $\mathcal{A}$. Therefore $\widehat{d''}$ and $\widehat{b}$ are yellow connected.

(d) We assume that $u > v$, the case $v > u$ is very similar. Choose c', d', c'' as before. Since $d' \in \widehat{a_m}$ we do not need to choose d''. Path $a = a_1, \dots, a_{k'} = b_1, \dots, b_q = c', b_{q+1}, \dots, b_r = c''$, where $a_{k'}$ is the sub-bottom element chosen, $a_{k'} = b_1, \dots, b_q = c', b_{q+1}, \dots, b_r = c''$ is an r-path from $a_{k'}$ to c', to c'', is a 0-stage ry-path whose depth $\mathsf{dep}(a_{k'})$ is less then that of $\mathcal{A}$, which equals $\mathsf{dep}(a_k)$. Therefore $\widehat{a}$ and $\widehat{c''}$ are yellow connected. Then, there is a 1- or 0-stage ry-path $c'' = b_r, \dots, b_q = c', c_1, \dots, c_v = d'$, whose depth $\max(\mathsf{dep}(c'), \mathsf{dep}(d')) < \mathsf{dep}(a_k)$. Therefore $\widehat{c''}$ and $\widehat{d'}$ are yellow connected. Finally, d' and b are strongly r-connected.

(e) Choose c', d', d'' as usual. Elements a and c' are strongly r-connected. Elements c' and d'', as well as, elements d'' and b are connected with 1- or 0-stage ry-paths of depth smaller than $\mathsf{dep}(\mathcal{A}) = \mathsf{dep}(a_k)$. Thus, $\widehat{a}$ and $\widehat{b}$ are yellow connected.

(f) This case is very similar to case (b). □

Proof:[of Proposition 6] By Lemma 19(1) we may assume that $\mathsf{dep}(a_k) + \mathsf{dep}(a_\ell) \geq 3$. We also assume that $\mathcal{A}$ is a path of least depth.

Choose elements c, d in each of the cases (a)–(f) as described in Lemma 20. Let R be the relation generated by $(c, d), (d, c)$. The relation $S_{c,d} = \{(c', d') \in \mathbb{A}^2 \mid$ there is e such that $(e, c'), (e, d') \in R\}$ is a tolerance of $\mathbb{A}$, because, by Lemma 20 $\mathsf{Sg}(c, d) = \mathbb{A}$. Since $\mathbb{A}$ is simple we have two cases.

CASE 1. For some $c' \in \widehat{c}$, $d' \in \widehat{d}$ relation $S_{c',d'}$ is non-trivial, and therefore is a connected tolerance.

For convenience we rename $c = c'$ and $d = d'$. In this case there is a sequence

$a = d_1, \dots, d_n = b$ such that $(d_i, d_{i+1}) \in S$ for any $i \in [n-1]$. By Lemma 8 the d_i can be chosen to be maximal, if we allow the last element of this sequence to belong to the same scc as b rather than b itself.

There are two possibilities.

SUBCASE 1A. For any $i \in [n-1]$ the algebra $\mathsf{Sg}(d_i, d_{i+1})$ does not equal $\mathbb{A}$.

By the inductive hypothesis, for any i, any two maximal scc's in $\mathsf{Sg}(d_i, d_{i+1})$ are yellow connected. Since d_i and d_{i+1} are maximal in $\mathbb{A}$, scc's $\widehat{d_i}, \widehat{d_{i+1}}$ are also yellow connected.

SUBCASE 1B. For some i, $\mathsf{Sg}(d_i, d_{i+1}) = \mathbb{A}$.

In this case there is $e \in \mathbb{A}$ such that $(e, a), (e, b) \in R$. Since R is symmetric, this means that $(\{e\} \times \mathbb{A}) \cup (\mathbb{A} \times \{e\}) \subseteq R$. Using the same arguments as before we may assume that e is maximal. Indeed, if e' is such that $e \le e'$ and $(e', a') \in R$ then set

$$\begin{pmatrix} e' \\ a'' \end{pmatrix} = f\left(\begin{pmatrix} e \\ a \end{pmatrix}, \begin{pmatrix} e' \\ a' \end{pmatrix}\right), \quad \text{and} \quad \begin{pmatrix} e' \\ b'' \end{pmatrix} = f\left(\begin{pmatrix} e \\ b \end{pmatrix}, \begin{pmatrix} e' \\ a' \end{pmatrix}\right),$$

where $a'' \in \widehat{a}$, $b'' \in \widehat{b}$. There is a directed r-path $a'' = a_1, \dots, a_s = a$. We set

$$\begin{pmatrix} e' \\ a_1 \end{pmatrix} = \begin{pmatrix} e' \\ a'' \end{pmatrix}, \quad \begin{pmatrix} e' \\ a_i \end{pmatrix} = f\left(\begin{pmatrix} e \\ a_i \end{pmatrix}, \begin{pmatrix} e' \\ a_{i-1} \end{pmatrix}\right).$$

Thus $(e', a) \in R$. Similarly, $(e', b) \in R$.

Suppose first that both $\mathsf{Sg}(c, e)$ and $\mathsf{Sg}(e, d)$ are smaller than $\mathbb{A}$. Take c', d', maximal elements in $\mathsf{Sg}(c, e)$ with $c \prec c'$ and $e \prec d'$, and c'', d'', maximal elements in $\mathbb{A}$ with $c' \prec c''$, $d' \prec d''$. By induction hypothesis $\widehat{c'}$ and $\widehat{d'}$ are yellow connected. Therefore, as before $\widehat{a}$ is yellow connected to $\widehat{e}$. Similar arguments are valid for $\mathsf{Sg}(e, d)$.

Suppose that $\mathsf{Sg}(c, e) = \mathbb{A}$. Then $(c, d), (e, d) \in R$ and hence $(d, d) \in R$. Thus cd is a thin red edge, a contradiction with the choice of c, d. In the case $\mathsf{Sg}(d, e) = \mathbb{A}$ the argument is similar.

CASE 2. $S_{c',d'}$ is trivial for all $c' \in \widehat{c}$, $d' \in \widehat{d}$, and therefore R is the graph of an automorphism π of $\mathbb{A}$ that maps c to d and c to d.

CLAIM. For any $e \in \mathbb{A}$, $\mathsf{dep}(\pi(e)) = \mathsf{dep}(e)$.

The claim follows from easy observation that the image (an the preimage) of a directed r-path is a directed r-path.

We consider the seven cases corresponding the cases (a)–(g) from Lemma 20.

SUBCASES 2A, 2C, 2D, 2E. In these cases by Claim $\mathsf{dep}(c) = \mathsf{dep}(d)$. However, $\mathsf{dep}(c) = 1$ while $\mathsf{dep}(d) > 1$ (or in cases 2a, 2b, 2c it can be that $\mathsf{dep}(c) > 1$ and $\mathsf{dep}(d) = 1$).

SUBCASE 2B. $\mathsf{dep}(a_k) = 1$, $\mathsf{dep}(a_{k+1}) = 2$ or $\mathsf{dep}(a_k) = 2$, $\mathsf{dep}(a_{k+1}) = 1$.

Suppose first that $\mathsf{dep}(a_k) = 1$, $\mathsf{dep}(a_{k+1}) = 2$. We choose $a' = a_k$ (so, $a' \in \widehat{a_1}$), and $u \in \widehat{a_{k+1}}$ so that $a'u$ is a thin yellow edge. By Lemma 20 $\mathsf{Sg}(a', b) = \mathbb{A}$, by the conditions of Case 2 $S_{a',b}$ is the graph of an automorphism, and we may replace a with a'. There is a term operation $p(x, y)$ such that $p(a, b) = u$; let $w = p(b, a)$. Since π is an automorphism taking a to b and b to a, we have that wb is a thin yellow edge. Set $g'(x, y, z) = g(x, p(x, y), p(x, z))$. Then $g'(a, a, b) = a$, $g'(a, b, a) = a$, $g'(b, a, a) = w$.

Consider ternary relation Q generated by triples $(a, a, b), (a, b, a), (b, a, a)$. As usual, $(\widehat{a} \times \mathbb{A}) \cup (\mathbb{A} \times \widehat{a}) \subseteq \mathrm{pr}_{1,2}Q, \mathrm{pr}_{1,3}Q, \mathrm{pr}_{2,3}Q$. Let also Q' be the relation generated by $Q \cap (\widehat{a} \times \widehat{a} \times \widehat{a})$. Observe that Q' is nonempty. Indeed, $d \prec a$; let $d = d_1, \dots, d_m \in \widehat{a}$ be an r-path, and let $(a_i, b_i, d_i) \in Q$ be triples extending the d_i such that $a_1 = b_1 = a$. Then setting (a'_1, b'_1, d_1) to be (a, a, d), and

$$\begin{pmatrix} a'_{i+1} \\ b'_{i+1} \\ d_{i+1} \end{pmatrix} = f\left(\begin{pmatrix} a'_i \\ b'_i \\ d_i \end{pmatrix}, \begin{pmatrix} a_{i+1} \\ b_{i+1} \\ d_{i+1} \end{pmatrix}\right)$$

we obtain a tuple $(a'_m, b'_m, d_m) \in Q \cap (\widehat{a} \times \widehat{a} \times \widehat{a})$. By the standard argument $(\widehat{a} \times \mathbb{A}) \cup (\mathbb{A} \times \widehat{a}) \subseteq \mathrm{pr}_{1,2}Q', \mathrm{pr}_{1,3}Q', \mathrm{pr}_{2,3}Q'$. Now Lemma 13 implies $\widehat{a} \times \widehat{a} \times \widehat{a} \subseteq Q$. Therefore there is a ternary operation g'' such that $g''(a, a, b) = g''(a, b, a) = g''(b, a, a) = a$. Applying the automorphism π we get $g''(b, b, a) = g''(b, a, b) = g''(a, b, b) = b$. Thus a, b is a thin yellow edge.

The case $\mathsf{dep}(a_k) = 2$, $\mathsf{dep}(a_{k+1}) = 1$ is similar.

SUBCASE 2F. By Lemma 20 we can assume that there is $u \in \widehat{a_k}$ such that $u \le b$ (and changing b if necessary). Take a term operation $p(x, y)$ such that $p(a, b) = u$ and set $w = p(b, a)$. Due to automorphism π we have $w \le a$. Let $g'(x, y, z) = f(p(y, x), p(z, x))$. As is easily seen, $g(a, a, b) = g'(a, b, a) = a$ and $g'(b, a, a) = u$.

Consider ternary relation Q generated by triples $(a, a, b), (a, b, a), (b, a, a)$. As usual, $(\widehat{a} \times \mathbb{A}) \cup (\mathbb{A} \times \widehat{a}) \subseteq \mathrm{pr}_{1,2}Q, \mathrm{pr}_{1,3}Q, \mathrm{pr}_{2,3}Q$. Let also Q' be the relation generated by $Q \cap (\widehat{a} \times \widehat{a} \times \widehat{a})$. Similar to the previous case Q' is nonempty. By the standard argument $(\widehat{a} \times \mathbb{A}) \cup (\mathbb{A} \times \widehat{a}) \subseteq \mathrm{pr}_{1,2}Q', \mathrm{pr}_{1,3}Q', \mathrm{pr}_{2,3}Q'$. Now Lemma 13 implies $\widehat{a} \times \widehat{a} \times \widehat{a} \subseteq Q$. Therefore there is a ternary operation g' such that $g'(a, a, b) = g'(a, b, a) = g'(b, a, a) = a$. $\square$

6 Quasi-2-Decomposability

Recall that an (n-ary) relation over a set A is called *2-decomposable* if, for any tuple $\mathbf{a} \in A^n$, $\mathbf{a} \in R$ if and only if, for any $i, j \in [n]$, $\mathrm{pr}_{ij}\mathbf{a} \in \mathrm{pr}_{ij}R$. 2-decomposability is closely related to the existence of majority polymorphisms of the relation. In our case relations in general do not have a majority polymorphism, but they still have a property close to 2-decomposability. We say that a relation R, a subdirect product of $\mathbb{A}_1, \dots, \mathbb{A}_n$, is *quasi-2-decomposable*, if for any elements $a_1, \dots, a_n$, $a_i \in \max(\mathbb{A}_i)$, such that $(a_i, a_j) \in \max(\mathrm{pr}_{i,j}R)$ for any i, j, there is a tuple $\mathbf{b} \in R$ with $(\mathbf{b}[i], \mathbf{b}[j]) \in \widehat{(a_i, a_j)}$ for any $i, j \in [n]$.

Proposition 7 *Any relation invariant under $\mathbb{A}$ is quasi-2-decomposable.*

Moreover, if R is an n-ary relation, $X \subseteq [n]$, tuple $\mathbf{a}$ is such that $(\mathbf{a}[i], \mathbf{a}[j]) \in \max(\mathrm{pr}_{i,j}R)$ for any i, j, and $\mathrm{pr}_X\mathbf{a} \in \max(\mathrm{pr}_X R)$, there is a tuple $\mathbf{b} \in R$ with $(\mathbf{b}[i], \mathbf{b}[j]) \in (\widehat{\mathbf{a}[i], \mathbf{a}[j]})$ for any $i, j \in [n]$, and $\mathrm{pr}_X\mathbf{b} = \mathrm{pr}_X\mathbf{a}$.

Proof: Let $\mathbf{a}$ be a tuple satisfying the conditions of quasi-2-decomposability. By induction on ideals of the power set of $[n]$ we prove that for any ideal I there is $\mathbf{a}'$ such that $(\mathbf{a}'[i], \mathbf{a}'[j]) \in (\widehat{\mathbf{a}[i], \mathbf{a}[j]})$, and for any $U \in I$ $\mathrm{pr}_U\mathbf{a}' \in \max(\mathrm{pr}_U R)$ and $\mathrm{pr}_U\mathbf{a}' \in \widehat{\mathrm{pr}_U\mathbf{a}}$. The base case, the ideal that consists of all at most 2-elements sets, set X, and its subsets, is given by the tuple $\mathbf{a}$.

Suppose that the claim is true for an ideal I, set W does not belong to I, but all its proper subsets do. Let $\mathcal{D}$ be the set of all tuples $\mathbf{c}$ such that $\mathrm{pr}_U\mathbf{c} \in \max(\mathrm{pr}_U R)$ and $\mathrm{pr}_U\mathbf{c} \in \widehat{\mathrm{pr}_U\mathbf{a}}$ for every $U \in I$. If a tuple belongs to $\mathcal{D}$ it is said to *support* $\mathcal{D}$. We show that $\mathcal{D}$ contains a tuple $\mathbf{b}$ with $\mathrm{pr}_W\mathbf{b} \in \max(\mathrm{pr}_W R)$.

CLAIM 1. If $\mathbf{b} \in \mathcal{D}$ and $\mathbf{c} \in R$ then $\mathbf{b}' = f(\mathbf{b}, \mathbf{c}) \in \mathcal{D}$.

Clearly for any $U \in I$ $\mathrm{pr}_U\mathbf{b}' \in \mathrm{pr}_U R$. Then $\mathbf{b} \le \mathbf{b}'$ that implies $\mathrm{pr}_U\mathbf{b}' \in \widehat{\mathrm{pr}_U\mathbf{b}}$ for any $U \in I$.

Assume that $W = \{1, \dots, \ell\}$ and fix $\mathbf{b} \in \mathcal{D}$. We prove the following statement:

Let $\mathbf{c} \in \mathcal{D}$ be such that $\mathrm{pr}_U\mathbf{c} \in \widehat{\mathrm{pr}_U\mathbf{b}}$ for all $U \in I$ and $Q \subseteq \max(\mathrm{pr}_W R)$ such that for any $U \subset W$ there is $\mathbf{c}_U \in R$ with $\mathrm{pr}_U\mathbf{c}_U = \mathrm{pr}_U\mathbf{c}$ and $\mathrm{pr}_W\mathbf{c}_U \in Q$. Then there is $\mathbf{d}$ supporting $I \cup \{W\}$ such that $\mathrm{pr}_W\mathbf{d} \in Q$ and $\mathrm{pr}_U\mathbf{d} \in \widehat{\mathrm{pr}_U\mathbf{b}}$ for $U \in I$.

We prove the statement by induction on the sum of sizes of unary projections of Q. If one of these projections is 1-element then the statement trivially follows from the assumption $\mathrm{pr}_U\mathbf{c} \in Q$ for U including all coordinate positions whose projections contain more than 1 element. So suppose that the statement is proved for all relations with unary projections smaller than Q.

By assumption there are $\mathbf{c}_1,\dots,\mathbf{c}_\ell \in Q$ with $\mathrm{pr}_{W-\{i\}}\mathbf{c}_i = \mathrm{pr}_{W-\{i\}}\mathbf{c}$. Clearly these tuples can be chosen such that $\mathbf{c}_i[i]$ is maximal.

Suppose that for some i the unary projection $\mathrm{pr}_i Q \neq \mathsf{Sg}(\mathbf{c}[i],\mathbf{c}_i[i])$. Assume $i=1$. Then set

$$Q' = Q \cap \left(\mathsf{Sg}(\mathbf{c}[1],\mathbf{c}_1[1]) \times \prod_{i\in W-\{1\}} \mathsf{Sg}(\widehat{\mathbf{c}[i]}) \right).$$

We show that $\mathbf{c}$ can be changed so that Q' satisfies the conditions of the statement. If $\mathrm{pr}_{W-\{1\}}\mathbf{c}$ is not maximal in $\mathrm{pr}_{W-\{1\}}Q'$ then take an r-path $\mathrm{pr}_{W-\{1\}}\mathbf{c} = \mathbf{b}_1,\dots,\mathbf{b}_k$ in $\mathrm{pr}_{W-\{1\}}Q'$ so that $\mathbf{b}_k$ is maximal. Then let $\mathbf{b}'_i$ form an r-path in R such that $\mathrm{pr}_W\mathbf{b}'_i \in Q'$ and $\mathrm{pr}_{W-\{1\}}\mathbf{b}'_i = \mathbf{b}_i$. Then we set $\mathbf{d}_1 = \mathbf{c}$, and $\mathbf{d}_{i+1} = f(\mathbf{d}_i,\mathbf{b}'_{i+1})$. By Claim 1 $\mathbf{d}_k \in \mathcal{D}$; moreover, for any $U \subset W$, $\mathrm{pr}_U\mathbf{d} \subseteq \mathrm{pr}_U Q'$, and for any $U \in I$, $\mathrm{pr}_U\mathbf{d} \in \widehat{\mathrm{pr}_U\mathbf{c}} = \widehat{\mathrm{pr}_U\mathbf{b}}$. Continuing the path if necessary we ensure that $\mathbf{d}[1]$ is maximal in $\mathrm{pr}_1 Q'$. Then just apply the induction hypothesis.

Let $\mathbf{c}_i$ be chosen such that $\mathsf{Sg}(\mathbf{c}[i],\mathbf{c}_i[i])$ are minimal possible. We will prove that $\mathbf{c} \in Q$. Replacing Q with relation

$$Q'(x,y,z) = \exists x_3,\dots,x_n(Q(x,y,z,x_3,\dots,x_n) \wedge (x_3 = \mathbf{c}[3]) \wedge \dots \wedge (x_n = \mathbf{c}[n]))$$

Q can be assumed ternary. Let $Y = \{i \in [3] \mid \mathbf{c}_i[i] \notin \widehat{\mathbf{c}[i]}\}$ and $Z = [n] - Y = \{i \in [n] \mid \mathbf{c}_i[i] \in \widehat{\mathbf{c}[i]}\}$. Without loss of generality assume $Z = \{1,\dots,\ell\}$ and $Y = \{\ell+1,\dots,n\}$.

CLAIM 2. $(\widehat{\mathbf{c}_1[1]} \times \widehat{\mathbf{c}[2]} \times \widehat{\mathbf{c}[3]}) \cup (\widehat{\mathbf{c}[1]} \times \widehat{\mathbf{c}_2[2]} \times \widehat{\mathbf{c}[3]}) \cup (\widehat{\mathbf{c}[1]} \times \widehat{\mathbf{c}[2]} \times \widehat{\mathbf{c}_3[3]}) \subseteq Q$.

Observe that for any $i,j \in [3]$, $(\widehat{\mathbf{c}[i]} \times \widehat{\mathbf{c}[j]}) \cup (\widehat{\mathbf{c}[i]} \times \widehat{\mathbf{c}_j[j]}) \subseteq \mathrm{pr}_{i,j}Q$. Indeed, $(\mathbf{c}[i],\mathbf{c}[j]),(\mathbf{c}[i],\mathbf{c}_j[j]) \in \mathrm{pr}_{i,j}Q$ implying $\{\mathbf{c}[i]\} \times \mathrm{pr}_j Q \subseteq \mathrm{pr}_{i,j}Q$ and then applying Lemma 7. Then the conditions of Lemma 13 are satisfied for the relation Q' generated by $R \cap (\widehat{\mathbf{c}_1[1]} \times \widehat{\mathbf{c}[2]} \times \widehat{\mathbf{c}[3]})$: $\widehat{\mathbf{c}_1[1]} \times \widehat{\mathbf{c}[2]} \subseteq \mathrm{pr}_{1,2}Q'$, $\widehat{\mathbf{c}_1[1]} \times \widehat{\mathbf{c}[3]} \subseteq \mathrm{pr}_{1,3}Q'$; $\mathrm{pr}_{2,3}Q'$ is generated by $\widehat{\mathbf{c}[2]} \times \widehat{\mathbf{c}[3]}$, which is strongly r-connected; finally $(\mathbf{c}_1[1],\mathbf{c}[2],\mathbf{c}[3]) \in Q'$. Hence by Lemma 13 we have $\widehat{\mathbf{c}_1[1]} \times \widehat{\mathbf{c}[2]} \times \widehat{\mathbf{c}[3]} \subseteq Q$.

For $\widehat{\mathbf{c}[1]} \times \widehat{\mathbf{c}_2[2]} \times \widehat{\mathbf{c}[3]})$ and $\widehat{\mathbf{c}[1]} \times \widehat{\mathbf{c}[2]} \times \widehat{\mathbf{c}_3[3]}$ proof is similar.

Note that if $Z \neq \varnothing$ then we are done. Suppose that $Z = \varnothing$. By Proposition 6 for each $i \in [3]$ there are $b_i \in \widehat{\mathbf{c}[i]}$ and $c_i \in \widehat{\mathbf{c}_i[i]}$ such that $b_i c_i$ is a thin yellow edge. By Claim 2 $(c_1, b_2, b_3), (b_1, c_2, b_3), (b_1, b_2, c_3) \in Q$, therefore

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = g\left(\begin{pmatrix} c_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ c_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ c_3 \end{pmatrix}\right) \in Q.$$

Again the conditions of Lemma 13 are satisfied for the relation Q' generated by $R \cap (\widehat{\mathbf{c}[1]} \times \widehat{\mathbf{c}[2]} \times \widehat{\mathbf{c}[3]})$: $\widehat{\mathbf{c}[1]} \times \widehat{\mathbf{c}[2]} \subseteq \mathrm{pr}_{1,2} Q'$, $\widehat{\mathbf{c}[1]} \times \widehat{\mathbf{c}[3]} \subseteq \mathrm{pr}_{1,3} Q'$; $\mathrm{pr}_{2,3} Q'$ is generated by $\widehat{\mathbf{c}[2]} \times \widehat{\mathbf{c}[3]}$, which is strongly r-connected; finally $(b_1, b_2, b_3) \in Q'$. Hence $\widehat{\mathbf{c}[1]} \times \widehat{\mathbf{c}[2]} \times \widehat{\mathbf{c}[3]} \subseteq Q$, in particular, $\mathbf{c} \in Q$.

To finish the prove it suffices to use the fact that the resulting tuple $\mathbf{b}$ is such that $\mathrm{pr}_X \mathbf{b} \in \widehat{\mathrm{pr}_X \mathbf{a}}$. There is an r-path in R that starts at $\mathbf{b}$ and ends at $\mathbf{b}'$ such that $\mathrm{pr}_X \mathbf{b}' = \mathrm{pr}_X \mathbf{a}$. As is easily seen, tuple $\mathbf{b}'$ satisfies all the remaining conditions. $\Box$

7 Proof of Theorem 1

In this section we prove Theorem 1 in the case of multi-sorted problem instances over arbitrary algebras from $\mathcal{F}(\mathbb{A})$. Let $\mathcal{P} = (V; \mathcal{F}(\mathbb{A}); \delta; \mathcal{C})$ be a 3-minimal problem instance. For $u, v, w \in V$ by $\mathcal{S}_u, \mathcal{S}_{u,v}, \mathcal{S}_{u,v,w}$ we denote sets of partial solutions to $\mathcal{P}$ on $\{u\}, \{u, v\}, \{u, v, w\}$, respectively. We show that $\mathcal{P}$ can be transformed to another 3-minimal problem instance which satisfies some additional conditions.

Proposition 8 *Let $\mathcal{P} = (V; \mathcal{F}(\mathbb{A}); \delta; \mathcal{C})$ be a 3-minimal problem instance without empty constraint relations. Let $v \in V$ and B be a maximal scc of $\mathrm{Gr}'(\mathbb{A}_{\delta(v)})$. Then the problem instance $\mathcal{P}_{v,B} = (V; \mathcal{F}(\mathbb{A}); \delta; \mathcal{C}')$, where*

- *for each $C = \langle \mathbf{s}, R \rangle \in \mathcal{C}$ there is $C' = \langle \mathbf{s}, R' \rangle \in \mathcal{C}'$ where R' contains all tuples $\mathbf{a}$ from R such that for any $u, w \in \mathbf{s}$ there is $c \in \mathsf{Sg}(B)$ with $(c, \mathbf{a}[u], \mathbf{a}[w]) \in \mathcal{S}_{v,u,w}$.*

satisfies the following conditions

- *$\mathcal{P}'$ is 3-minimal, and has no empty constraint relation;*
- *if $\mathcal{P}'$ has a solution, then $\mathcal{P}$ does.*

Proof: The second claim of the proposition is straightforward from the construction.

Take any variables $x, y, z \in V$. The set of partial solutions of $\mathcal{P}_{v,B}$ on $\{x\}$, $\{x, y\}$, and $\{x, y, z\}$ will be denoted by $\mathcal{S}'_x, \mathcal{S}'_{x,y}$, and $\mathcal{S}'_{x,y,z}$, respectively. Let also $\mathcal{S}''_{x,y}$ denote the set of maximal elements from

$$\{(a, b) \mid \text{ such that } (d, a, b) \in \mathcal{S}_{v,x,y} \text{ for some } d \in B\}.$$

CLAIM 1. For any $x, y, z \in V - \{v\}$ and any $(a, b) \in \mathcal{S}''_{x,y}$ there is c such that $(a, c) \in \mathcal{S}''_{x,z}$ and $(b, c) \in \mathcal{S}''_{y,z}$.

Consider the following relation

$$R(x_1, x_2, x_3, x_4) = \exists u(\mathcal{S}_{v,x,z}(x_1, x_2, u) \wedge \mathcal{S}_{v,y,z}(x_3, x_4, u).$$

Let d be the element of B_v such that $(d, a, b) \in \mathcal{S}_{v,x,y}$, and let $\mathbf{a} = (d, a, d, b)$. We show that $\mathrm{pr}_{i,j}\mathbf{a} \in \mathrm{pr}_{i,j}R$ for any $i, j \in [4]$. If $i = 2$, $j = 4$ or the other way round then we set y to be an extention e of (a, b) in $\mathcal{S}_{x,y,z}$, and x_1, x_3 to extentions of (a, e) and (b, e) in $\mathcal{S}_{v,x,z}$ and $\mathcal{S}_{v,y,z}$, respectively. If $i = 1$, $j = 2$ or $i = 3$, $j = 4$ then set y to be an extention e of (d, a) or (d, b) in $\mathcal{S}_{v,x,z}$ and $\mathcal{S}_{v,y,z}$, respectively. Then extend e to a tuple from $\mathcal{S}_{v,y,z}$ or $\mathcal{S}_{v,x,z}$, respectively. If $i = 1$, $j = 4$ or $i = 3$, $j = 2$ then extend (d, b) or (d, a) by an element e to a tuple in $\mathcal{S}_{v,y,z}$ or $\mathcal{S}_{v,x,z}$, respectively. Then set x_2 (resp., x_4) to be a value extending (d, e) in $\mathcal{S}_{v,z,x}$ (resp., $\mathcal{S}_{v,z,y}$), and x_3 (resp., x_1) to be a value extending (d, b) (resp., (d, a)) to a tuple in $\mathcal{S}_{v,z,y}$ (resp., $\mathcal{S}_{v,z,x}$). Finally, if $i = 1$, $j = 3$ then choose e so that $(d, e) \in \mathcal{S}_{v,z}$ and extend this pair to tuples from $\mathcal{S}_{v,z,x}$ and $\mathcal{S}_{v,y,z}$.

By Proposition 7 there is $\mathbf{b} \in R$ such that $\mathbf{b}[2] = \mathbf{a}[2] = a$, $\mathbf{b}[4] = \mathbf{a}[4] = b$, $\mathbf{b}[1], \mathbf{b}[3] \in \widehat{d} = B$, and $(\mathbf{b}[i], \mathbf{b}[j]) \in (\widehat{\mathbf{a}[i], \mathbf{a}[j]})$. Therefore there is c such that $(\mathbf{b}[1], a, c) \in \mathcal{S}_{v,x,z}$ and $(\mathbf{b}[3], b, c) \in \mathcal{S}_{v,y,z}$, which implies $(a, c) \in \mathcal{S}''_{x,z}$ and $(b, c) \in \mathcal{S}''_{y,z}$. The claim is proved.

CLAIM 2. (1) For any $x, y \in V - \{v\}$ and any $(a, b) \in \mathcal{S}''_{x,y}$, there is mapping $\varphi : V \to \mathbb{A}$ such that $\varphi(x) = a$, $\varphi(y) = b$, $\varphi(v) \in B$, and $(\varphi(u), \varphi(w)) \in \mathcal{S}''_{u,w}$ for any $u, w \in V$.

(2) Moreover, if ψ is a mapping from $U \subseteq V$ satisfying the conditions above, then φ can be chosen such that $\varphi_{|U} = \psi$.

Let $V = \{v_1, \dots, v_n\}$ and $v = v_1$, $x = v_2$, $y = v_3$. By induction on i we prove that a required φ_I can be found on $I = \{v_1, \dots, v_i\}$. For $i = 3$ the claim follows from the assumptions. So, suppose it is proved for i. Take φ_I satisfying the conditions on I and consider the relation given by

$$R(x_1, \dots, x_i) = \exists y \bigwedge_{j=1}^{i} \mathcal{S}''_{v_j,v_{i+1}}(x_j, y).$$

By the inductive hypothesis and Claim 1, for any $j, k \in [i]$ we have $(\varphi(v_j), \varphi(v_k)) \in \mathrm{pr}_{j,k} R$. By Proposition 7 there is $\mathbf{a} \in R$ such that $\mathbf{a}[2] = a$, $\mathbf{a}[3] = b$, and $(\mathbf{a}[j], \mathbf{a}[k]) \in \widehat{(\varphi(v_j), \varphi(v_k))}$ for any $j, k \in [i]$. This means that there is c such that $(\mathbf{a}[j], c) \in \mathcal{S}''_{v_j, v_{i+1}}$ for all $j \in [i]$. Observe that c is a maximal element of $\mathcal{S}_{v_{i+1}}$.

For any $j, k \in [i]$ there is $c' \in \widehat{c}$ such that $(\varphi(v_j), c') \in \mathcal{S}''_{v_j, v_{i+1}}$ and $(\varphi(v_k), c') \in \mathcal{S}''_{v_k, v_{i+1}}$. Indeed, since $(\mathbf{a}[j], \mathbf{a}[k]) \in \widehat{(\varphi(v_j), \varphi(v_k))}$, there is a directed r-path from $(\mathbf{a}[j], \mathbf{a}[k])$ to $(\varphi(v_j), \varphi(v_k))$. This r-path can be expandede to an r-path from $(\mathbf{a}[j], \mathbf{a}[k], c)$ to $(\varphi(v_j), \varphi(v_k), c')$ for some c' with $c \prec c'$.

Let $\mathbb{B}$ be a minimal subalgebra of $\mathsf{Sg}(\widehat{c})$ such that for some maximal scc C of $\mathbb{B}$ and for any $j, k \in [i]$ there is $c' \in C$ such that $(\varphi(v_j), c') \in \mathcal{S}''_{v_j, v_{i+1}}$ and $(\varphi(v_k), c') \in \mathcal{S}''_{v_k, v_{i+1}}$. Clearly $\mathbb{B}$ can be chosen maximal generated by C. Let R_j denote the binary relation $\mathsf{Sg}(\mathcal{S}''_{v_j, v_{i+1}} \cap (\widehat{\varphi(v_j)} \times C))$. Take a maximal congruence θ of $\mathbb{B}$, and let $R_j^\theta = \{(d, e^\theta) \mid (d, e) \in R_j\}$ for $j \in [i]$. By Lemma 9 R_j^θ is either the graph of a mapping $\psi_j : \mathrm{pr}_1 R_j \to \mathbb{B}/_\theta$, or $\widehat{\varphi(v_j)} \times \mathrm{pr}_2 R_j^\theta \subseteq R_j^\theta$.

Let $U \subseteq W$ be the set of those variables v_j for which R_j^θ is the graph of a mapping $\psi_j : \mathrm{pr}_1 R_j \to \mathbb{B}/_\theta$. Then for any $j, k \in U$ we have $\psi_j(\varphi(v_j)) = \psi_k(\varphi(v_k))$. Therefore there is a θ-block $\mathbb{B}'$ such that for any $j, k \in [i]$ there is $c' \in \mathbb{B}'$ such that $(\varphi(v_j), c') \in \mathcal{S}''_{v_j, v_{i+1}}$ and $(\varphi(v_k), c') \in \mathcal{S}''_{v_k, v_{i+1}}$.

Consider the relation given by

$$R'(x_1, \ldots, x_i) = \exists y \mathbb{B}'(y) \wedge \bigwedge_{j=1}^{i} \mathcal{S}''_{v_j, v_{i+1}}(x_j, y).$$

By the what is shown above, for any $j, k \in [i]$ we have $(\varphi(v_j), \varphi(v_k)) \in \mathrm{pr}_{j,k} R'$. By Proposition 7 there is $\mathbf{a}' \in R$ such that $\mathbf{a}'[2] = a$, $\mathbf{a}'[3] = b$, and $(\mathbf{a}'[j], \mathbf{a}'[k]) \in \widehat{(\varphi(v_j), \varphi(v_k))}$ for any $j, k \in [i]$. This means that there is $c'' \in \mathbb{B}'$ such that $(\mathbf{a}'[j], c'') \in \mathcal{S}''_{v_j, v_{i+1}}$ for all $j \in [i]$. Clearly c'' is a maximal element. As before, for any $j, k \in [i]$ there is $c''' \in \widehat{c''}$ such that $(\varphi(v_j), c''') \in \mathcal{S}'_{v_j, v_{i+1}}$ and $(\varphi(v_k), c''') \in \mathcal{S}'_{v_k, v_{i+1}}$. A contradiction with minimality of $\mathbb{B}$.

Finally, if $\mathbb{B}$ is simple then let $d \in C$ denote the element $\psi(\varphi(v_\ell))$, which is common for all $\ell \in U$. For any $j, k \in [i]$ consider the relation generated by $\mathcal{S}_{v_j, v_k, v_{i+1}} \cap (\widehat{\varphi(v_j)} \times \widehat{\varphi(v_k)} \times C)$. If $j, k \notin U$ then by Lemma 12 $\mathrm{pr}_{v_j, v_k}(\mathcal{S}_{v_j, v_k, v_{i+1}} \cap (\widehat{\varphi(v_j)} \times \widehat{\varphi(v_k)} \times C)) \times C \subseteq \mathcal{S}_{v_j, v_k, v_{i+1}}$, in particular, $(\varphi(v_j), \varphi(v_k), d) \in \mathcal{S}_{v_j, v_k, v_{i+1}}$. If, say, $j \in U$, then for any triple $(\varphi(v_j), \varphi(v_k), d') \in \mathcal{S}_{v_j, v_k, v_{i+1}}$ it must be $d' = d$. However, by the choice of C there is $d' \in C$ such that $(\varphi(v_j), \varphi(v_k), d') \in \mathcal{S}_{v_j, v_k, v_{i+1}}$. Thus $(\varphi(v_j), d) \in \mathcal{S}''_{v_j, v_{i+1}}$ for all $j \in [i]$.

Claim 2 is proved.

To complete the proof it suffices to show that for any $x, y, z \in V$ the set $\max(Q_{x,y,z})$, where $Q_{x,y,z}$ is the set of all tuples $\mathbf{a}$ from $\mathcal{S}_{x,y,z}$ such that $\mathrm{pr}_{x,y}\mathbf{a} \in \mathcal{S}''_{x,y}$, $\mathrm{pr}_{y,z}\mathbf{a} \in \mathcal{S}''_{y,z}$, $\mathrm{pr}_{y,z}\mathbf{a} \in \mathcal{S}''_{y,z}$, is a subset of $\mathcal{S}'_{x,y,z}$. If $v \in \{x, y, z\}$ this claim is obvious. So suppose $v \notin \{x, y, z\}$.

We show first that $Q_{x,y,z} \neq \varnothing$. Take any triple (a, b, c) such that $(a, b) \in \mathcal{S}''_{x,y} \subseteq \mathcal{S}_{x,y}$, $(b, c) \in \mathcal{S}''_{y,z} \subseteq \mathcal{S}_{y,z}$, and $(a, c) \in \mathcal{S}''_{x,z} \subseteq \mathcal{S}_{x,z}$. By Proposition 7 there is $(a', b', c') \in \mathcal{S}_{x,y,z}$ such that $(a', b') \in \widehat{(a, b)} \subseteq \mathcal{S}''_{x,y}$, $(b', c') \in \widehat{(b, c)} \subseteq \mathcal{S}''_{y,z}$, and $(a', c') \in \widehat{(a, c)} \subseteq \mathcal{S}''_{x,z}$.

Take a tuple $\mathbf{a} = (a, b, c) \in \max(Q_{x,y,z})$ and a constraint $C' = \langle \mathbf{s}, R' \rangle \in \mathcal{C}'$. We need to show that $\mathrm{pr}_{\mathbf{s}\cap\{x,y,z\}}\mathbf{a}$ can be extended by a tuple $\mathbf{a}' \in R'$. If $|\mathbf{s} \cap \{x, y, z\}| < 3$ the result follows from Claim 2 and Proposition 7. So suppose $\{x, y, z\} \subseteq \mathbf{s}$. We assume $\mathbf{s} = (v_1, \ldots, v_k)$ and $x = v_1$, $y = v_2$, $z = v_3$.

Constraint C' and relation R' are obtained from a certain constraint $C \in \mathcal{C}$ and relation R, respectively. Since $(a, b, c) \in \mathcal{S}_{x,y,z}$, we have $(a, b, c) \in \mathrm{pr}_{x,y,z}R$. By Claim 2 there is a mapping $\varphi : V \to \mathbb{A}$ such that $\varphi(x) = a, \varphi(y) = b, \varphi(z) = c$, and for any $u, w \in V$ it holds that $(\varphi(u), \varphi(w)) \in \mathcal{S}''_{u,w}$. In particular, for any $u, w \in \mathbf{s}$ we have $(\varphi(u), \varphi(w)) \in \mathrm{pr}_{u,w}R$. By Proposition 7 there is $\mathbf{b} \in R$ such that $(\mathbf{b}[i], \mathbf{b}[j]) \in \mathcal{S}''_{v_i,v_j} \subseteq \mathrm{pr}_{v_i,v_j}R$ for any $i, j \in [k]$ and $(\mathbf{b}[1], \mathbf{b}[2], \mathbf{b}[3]) = (a, b, c)$. Clearly this tuple also belongs to R' that implies the result. $\square$

Now we are in a position to prove Theorem 1.

Proof:[of Theorem 1] Let $\mathcal{P} = (V; \mathcal{F}(\mathbb{A}); \delta; \mathcal{C})$ be a 3-minimal problem instance without empty constraint relations. We prove by induction on the number of elements in $\mathbb{A}_{\delta(v)}$, $v \in V$, that $\mathcal{P}$ has a solution.

THE BASE CASE OF INDUCTION. If all $\mathbb{A}_{\delta(v)}$, $v \in V$, are arbitrarily maximal generated, and simple or 1-element, then the required result follows from Corollary 5.

INDUCTION STEP. Suppose that the theorem holds for all problem instances $\mathcal{P}' = (V; \mathcal{F}(\mathbb{A}); \delta'; \mathcal{C}')$ where $|\mathbb{A}_{\delta'(v)}| \leq |\mathbb{A}_{\delta(v)}|$ for $v \in V$ (here $\mathbb{A}_{\delta'(v)}$ denotes the set of partial solutions to $\mathcal{P}'$ on $\{v\}$) and at least one inequality is strict.

We have two cases.

CASE 1. For some $v \in V$, there is a maximal scc B of $\mathrm{Gr}'(\mathbb{A}_{\delta(v)})$ such that $\mathsf{Sg}(B) \neq \mathbb{A}_{\delta(v)}$.

In this case take any maximal scc B of $\mathrm{Gr}'(\mathbb{A}_{\delta(v)})$ with $\mathsf{Sg}(B) \neq \mathbb{A}_{\delta(v)}$ and consider the problem $\mathcal{P}_{v,B}$. By Proposition 8 this problem is 3-minimal. We get the result by the inductive hypothesis.

CASE 2. For all $v \in V$ algebra $\mathbb{A}_{\delta(v)}$ is arbitrarily maximal generated.

Let us assume that, for a certain $u \in V$, $\mathbb{A}_{\delta(u)}$ is not simple and θ is a maximal congruence of $\mathbb{A}_{\delta(u)}$. By Corollary 2, for any $v \in V - \{u\}$, $\mathcal{S}^{\theta}_{u,v} = \{(a^{\theta}, b) \mid (a, b) \in \mathcal{S}_{u,v}\}$ is either the direct product $\mathbb{A}_{\delta(u)}/_{\theta} \times \mathbb{A}_{\delta(v)}$, or the graph of a surjective mapping $\pi_v \colon \mathbb{A}_{\delta(v)} \to \mathbb{A}_{\delta(u)}/_{\theta}$. Let W denote the set consisting of u and all $v \in V$ such that $\mathcal{S}^{\theta}_{u,v}$ is the graph of π_v, and

$$\theta_v = \begin{cases} \theta, & \text{if } v = u, \\ \ker \pi_v & \text{if } v \in W, \\ =_v & \text{otherwise,} \end{cases}$$

for $v \in V$ where $=_v$ denotes the equality relation on $\mathcal{S}_v$. Consider the *factor problem* $\overline{\mathcal{P}} = (V; \mathcal{F}(\mathbb{A}); \tilde{\delta}; \widetilde{\mathcal{C}})$ where $\mathbb{A}_{\tilde{\delta}(v)} = \mathbb{A}_{\delta(v)}/_{\theta_v}$, $v \in V$, and for each $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$, $\mathbf{s} = (v_1, \dots, v_k)$, there is $\widetilde{C} = \langle \mathbf{s}, \tilde{R}\rangle \in \widetilde{\mathcal{C}}$ such that

$$\tilde{R} = \{(a_{v_1}^{\theta_{v_1}}, \dots, a_{v_k}^{\theta_{v_k}}) \mid (a_{v_1}, \dots, a_{v_k}) \in R\}.$$

As is easily seen, the factor problem is 3-minimal, therefore, by the induction hypothesis it has a solution. Let $(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ be a solution. Notice that if $v \in W$, then $\mathfrak{a}_v$ is a congruence block of $\mathbb{A}_{\delta(v)}$, that is, $\mathfrak{a}_v$ is a subset of $\mathbb{A}_{\delta(v)}$ in this case. The 3-minimality of $\mathcal{P}$ implies that, for any constraint $\langle \mathbf{s}, R\rangle \in \mathcal{C}$, any $v, w \in \mathbf{s} \cap W$, and any $\mathbf{a} \in R$, if $a_v \in \mathfrak{a}_v$ then $a_w \in \mathfrak{a}_w$. Set $\mathcal{P}' = (V; \mathcal{F}(\mathbb{A}); \delta'; \mathcal{C}')$ where

$$\mathbb{A}_{\delta'(v)} = \begin{cases} \mathfrak{a}_v & \text{if } v \in W, \\ \mathbb{A}_{\delta(v)} & \text{otherwise,} \end{cases}$$

and for each $C = \langle \mathbf{s}, R\rangle \in \mathcal{C}$ there is $C' = \langle \mathbf{s}, R'\rangle \in \mathcal{C}'$ with

$$\mathbf{a} \in R' \quad \text{if and only if} \quad \mathbf{a} \in R \text{ and } a_v \in \mathfrak{a}_v \text{ for all } v \in W \cap \mathbf{s}.$$

Since $|\mathfrak{a}_u| < |\mathbb{A}_{\delta(u)}|$, to complete the proof we just have to show that $\mathcal{P}'$ is 3-minimal. For $U = \{u_1, u_2, u_3\} \subseteq V$ set $S_U = \mathcal{S}_{u_1,u_2,u_3} \cap (S_1 \times S_2 \times S_3)$ where $S_i = \mathbb{A}_{\delta'(u_i)}$. Clearly, for any $C' = \langle \mathbf{s}, R'\rangle \in \mathcal{C}'$, we have $\mathrm{pr}_{U\cap \mathbf{s}} R' \subseteq \mathrm{pr}_{U\cap\mathbf{s}} S_U$. Therefore, if we prove the reverse inclusion then we get the equality $\mathrm{pr}_{U\cap\mathbf{s}} R' = \mathrm{pr}_{U\cap\mathbf{s}} S_U$ which implies the 3-minimality of $\mathcal{P}'$.

Take $\mathbf{b} = (a_{u_1}, a_{u_2}, a_{u_3}) \in \max(S_U)$, $\langle \mathbf{s}, R\rangle \in \mathcal{C}$, and $\mathbf{a} \in \max(R)$ such that $\mathrm{pr}_{U\cap\mathbf{s}} \mathbf{a} = \mathrm{pr}_{U\cap\mathbf{s}} \mathbf{b}$. If $U \cap W \cap \mathbf{s} \neq \varnothing$ then, for any $v \in s \cap W$, $a_v \in \mathfrak{a}_v$, and therefore $\mathbf{a} \in R'$. If $\mathbf{s} \cap W = \varnothing$ then $R' = R$, and again $\mathbf{a} \in R'$. Otherwise, consider the relation $\tilde{R}$. Choose $v \in \mathbf{s} \cap W$ and set $Q = \mathrm{pr}_{(\mathbf{s}-W)\cup\{v\}} \tilde{R}$. Since every $\mathbb{A}_{\delta(v)}$ is arbitrarily maximal generated, by Corollary 6, for any scc T of $\mathrm{pr}_{\mathbf{s}-W} R$ we have $\mathrm{pr}_v Q \times \mathsf{Sg}(T) \subseteq Q$. This means that there is $\mathbf{c} \in R$ such that $\mathrm{pr}_{\mathbf{s}-W} \mathbf{c} = \mathrm{pr}_{\mathbf{s}-W} \mathbf{a}$ and $c_v \in \mathfrak{a}_v$. Therefore, $c_w \in \mathfrak{a}_w$ for any $w \in s \cap W$, and hence $\mathbf{c} \in R'$. Since $\mathbf{s} \cap U \subseteq \mathbf{s} - W$, we have $\mathrm{pr}_{\mathbf{s}\cap U} \mathbf{c} = \mathrm{pr}_{\mathbf{s}\cap U} \mathbf{b}$, as required. $\square$

8 Testing omitting types

Finally, we consider the question whether or not the variety generated by a given algebra or by the algebra of a given relational structure omits the unary and affine types. More precisely, we consider three decision problems. In ALGEBRA OF TYPE 2 we are given a finite set A and operation tables of idempotent operations $f_1, \dots, f_n$ on A, and the question is whether $\mathsf{var}(\mathbb{A})$, where $\mathbb{A} = (A; \{f_1, \dots, f_n\})$, omits the unary and affine types. In RELATIONAL STRUCTURE OF TYPE 2 we are given a finite relational structure $\mathcal{A}$ such that all its polymorphisms are idempotent, and the question is whether $\mathsf{var}(\mathrm{Alg}(\mathcal{A}))$ omits the unary and affine types. In RELATIONAL STRUCTURE OF TYPE $2(k)$ we are given a finite relational structure $\mathcal{A}$, $|A| \le k$, again such that all its polymorphisms are idempotent, and the question is whether $\mathsf{var}(\mathrm{Alg}(\mathcal{A}))$ omits the unary and affine types. The problem ALGEBRA OF TYPE 2 was shown solvable in polynomial time in [8], and it easily follows from this result (see [4]) that RELATIONAL STRUCTURE OF TYPE $2(k)$ is also solvable in polynomial time. In this section we prove that having the Strong Bounded Width Conjecture proved the third problem, RELATIONAL STRUCTURE OF TYPE 2, is also solvable in polynomial time. More precisely we use the fact that there is an algorithm $\mathbb{A}$ that solves $\mathrm{CSP}(\mathbb{A})$ for any finite idempotent algebra $\mathbb{A}$ provided $\mathsf{var}(\mathbb{A})$ omits the unary and affine types.

Theorem 3 *The* RELATIONAL STRUCTURE OF TYPE 2 *problem is polynomial time solvable.*

We prove the theorem in two steps. First, we show how $\mathbb{A}$ can be used to approximate relations *generated* by a set of tuples, and then we use such approximations to determine wether or not the variety corresponding to a relational structure omits the unary and affine types.

Let $\mathcal{B}$ be a structure and R a (n-ary) relation definable by a pp-formula in $\mathcal{B}$. Relation R is *generated* by tuples $\mathbf{a}_1, \dots, \mathbf{a}_k \in R$ if R is generated by those tuple in the direct power $(\mathrm{Alg}(\mathcal{B}))^n$.

It will be convenient for us to represent the problem of checking whether a structure $\mathcal{A}$ belongs to $\mathrm{CSP}(\mathcal{B})$ as a constraint satisfaction problem. An instance of the CSP over $\mathcal{B}$ consists of a set of variables, V, and a set of constraints of the form $\langle s, R\rangle$ where R is a relation of $\mathcal{B}$ (say, of arity ℓ) and s is an ℓ-tuple of variables from V. A solution to such instance is a mapping $\varphi\colon V \to B$ such that $\varphi(s) \in R$ for every constraint $\langle s, R\rangle$. The correspondence between this form of the CSP and homomorphisms of structures is as follows, see, [7]. The existence of a solution to a CSP instance is equivalent to the existence of a homomorphism from σ-structure $\mathcal{A}$ to $\mathcal{B}$, where the underlying set of $\mathcal{A}$ is the set of variables V, and every tuple s of every relation $R^{\mathcal{A}}$ corresponds to constraint $\langle s, R\rangle$.

Given a σ-structure $\mathcal{B}$ with $\sigma = (R_1, \ldots, R_k)$, r_i the arity of R_i, and a relation R over B with $|R| = m$ of arity d, the *indicator problem* $IP(\mathcal{B}, R)$ is defined as follows:

- the set V of variables is the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_{|B|^m}\}$ of all m-tuples of elements from B;
- let $T = \{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ be the set of $|B|^m$-tuples with $\mathbf{b}_i[j] = \mathbf{a}_j[i]$; for each $i \in [k]$ and each r_i-element tuple s of elements from V such that $\mathrm{pr}_s T \subseteq R_i$, we introduce a constraint $\langle s, R_i \rangle$.

The next statement follows from results of [10].

Lemma 21 *Let s be a d-element subset of V such that $\mathrm{pr}_s T = R$ (up to permutations of tuples in R there is only one such set). Then R is pp-definable in $\mathcal{B}$ if and only if, for any solution φ of $IP(\mathcal{B}, R)$, $\mathrm{pr}_s \varphi \in R$.*

Lemma 22 *Assuming m and $|R|$ are bounded, $IP(\mathcal{B}, R)$ has polynomial size in $||\mathcal{B}||$.*

Proof: Since $|B|^{|R|}$ is the number of variables in $IP(\mathcal{B}, R)$, if $|R|$ is bounded, the number of variables is polynomial in $|B|$. For each R_i, containing n_i tuples, there are $O(n_i^m)$ of at most m-element sequences of tuples from R_i. For each such sequence there is exactly one tuple $s \subseteq V$ such that $(\mathrm{pr}_s \mathbf{b}_1, \ldots, \mathrm{pr}_s \mathbf{b}_m)$ is equal to the sequence. Therefore there are $k \cdot O(\max(n_i)^m)$ constraints in $IP(\mathcal{B}, R)$. $\Box$

We consider the following algorithm for the indicator problem. Suppose that the time complexity of $\mathbb{A}$ is bounded by a polynomial $p(n)$.

Lemma 23 *If $\mathsf{var}(\mathrm{Alg}(\mathcal{B}))$ omits the unary and affine types then* REL-GEN *returns the relation $\mathsf{Sg}(R)$ generated by R in $\mathcal{B}$. Otherwise it returns a relation Q such that $R \subseteq Q \subseteq \mathsf{Sg}(R)$. In both cases if m and d are bounded then* REL-GEN *runs in polynomial time.*

Proof: If $\mathsf{var}(\mathrm{Alg}(\mathcal{B}))$ omits the unary and affine types then the result follows from Lemma 21 and the assumptions on $\mathbb{A}$. Otherwise $R \subseteq Q$ since Q is assigned to be R in Step 2 and never decreases. Because of the check in Step 3.2, Q is never added an element that is not a member of $\mathsf{Sg}(R)$, which implies $Q \subseteq \mathsf{Sg}(R)$. Finally, if m is bounded then by Lemma 22 Step 1 can be performed in polynomial time. If d is bounded then the number of iterations in Step 3 is polynomial and each iteration takes time not exceeding $p(n)$. $\Box$

To continue we need more definitions and results from algebra. Let $\mathbb{A} = (A; C)$ be an algebra. A *term operation* of $\mathbb{A}$ is an operation that can be obtained

INPUT: A σ-structure $\mathcal{B}$, whose polymorphisms are idempotent, with $\sigma = (R_1, \dots, R_k)$ where R_i is of arity r_i, and a relation $R = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ of arity d.

OUTPUT: A d-ary relation R'.

Step 1. **construct** $IP(\mathcal{B}, R)$, let $s = (s_1, \dots, s_d) \subseteq V$ be the set of variables, such that $\mathrm{pr}_s \mathbf{b}_i = \mathbf{a}_i$
Step 2 **set** $Q := R$
Step 3. **for each** $\mathbf{c} \in B^d - R$ **do**
Step 3.1 **call** $\mathbb{A}$ on $\mathcal{P}_{\mathbf{c}} = IP(\mathcal{B}, R) \cup \{\langle (s_i), \{\mathbf{c}[i]\}\rangle \mid 1 \leq i \leq d\}$ along with a clock that halts the execution upon reaching $p(n)$.
Step 3.2 **if** $\mathbb{A}$ returns an assignment φ **and** φ is a solution of $\mathcal{P}_{\mathbf{c}}$ **then do**
Step 3.2.1 **set** $Q := Q \cup \{\mathbf{c}\}$
endif
endfor
Step 4 **output** Q

Figure 1: Algorithm REL-GEN

from operations in C and projections by means of substitution. If $\mathbb{A} = \mathrm{Alg}(\mathcal{B})$ for a certain structure $\mathcal{B}$ then the term operations of $\mathbb{A}$ are exactly the polymorphisms of $\mathcal{B}$. *Polynomial operations* of $\mathbb{A}$, or simply *polynomials*, are obtained from its term operations by substituting constants instead of some of the variables. The set of all unary polynomials of $\mathbb{A}$ is denoted by $\mathsf{Pol}_1\ \mathbb{A}$. Let B and C be non-void subsets of A. The sets B, C are called *polynomially isomorphic*, if there exist $f, g \in \mathsf{Pol}_1\ \mathbb{A}$ such that (i) $f(B) = C$, (ii) $g(C) = B$, (iii) $gf|_B = \mathrm{id}_B$, and (iv) $fg|_C = \mathrm{id}_C$. If B and C are polynomially isomorphic, then this is denoted by $B \cong C$.

For elements a, b of algebra $\mathbb{A}$, let $\sigma_{\mathbb{A}}(a, b)$ (or simply $\sigma(a, b)$ if $\mathbb{A}$ is clear from the context) denote the transitive closure of $\{\{f(a), f(b)\}^2 \mid f \in \mathsf{Pol}_1\ \mathbb{A}, \{f(a), f(b)\} \not\cong \{a, b\}\}$. Thus, $(c, d) \in \sigma(a, b)$ if and only if there exist $n \geq 1$ and $z_0, z_1, \dots, z_n \in A$ with $c = z_0$, $d = z_n$ and there are $f_1, \dots, f_n \in \mathsf{Pol}_1\ \mathbb{A}$ such that for each i, $1 \leq i \leq n$, $\{z_i, z_{i-1}\} = \{f_i(a), f_i(b)\}$ and $\{a, b\} \not\cong \{f_i(a), f_i(b)\}$. A 2-element set $\{a, b\} \subseteq A$ is called a *subtrace* if and only if $(a, b) \notin \sigma(a, b)$.

Let $\mathbb{A}$ be an algebra, and $(a, b) \in A^2$ with $a \neq b$. The ordered pair (a, b) is called a *2-snag* if there is a binary polynomial g of $\mathbb{A}$ such that $g(b, b) = b$ and $g(a, b) = g(b, a) = g(a, a) = a$. Theorem 7.2 from [9] implies that algebra $\mathbb{A}$ omits the unary and affine types if and only if every its subtrace is a 2-snag.

Lemma 24 *If $\{a, b\}$ and $\{c, d\}$ are polynomially isomorphic and $\{a, b\}$ is a sub-*

trace [(a,b) is a 2-snag] then $\{c,d\}$ is a subtrace [(c,d) or (d,c) is a 2-snag].

A subalgebra generated by a set S of elements is the subset generated by S as a unary relation along the restrictions of the basic operations onto this set. Let $|A| = k$ and let $G(\mathbb{A})$ denote the directed graph whose vertices are all $\binom{k}{2}$ two-element subsets of $\mathbb{A}$ and $(\{a,b\},\{c,d\})$ is an edge in $G(\mathbb{A})$ if and only if there exists a $f \in \mathsf{Pol}_1\ \mathbb{A}$ such that $\{c,d\} = \{f(a), f(b)\}$. As is easily seen, graph $G(\mathbb{A})$ is reflexive and transitive. Since $\{a,b\}$ and $\{c,d\}$ are polynomially isomorphic if and only if $(\{a,b\},\{c,d\})$, $(\{c,d\},\{a,b\})$ are both edges of $G(\mathbb{A})$, the strongly connected components of $G(\mathbb{A})$ are exactly the classes of the equivalence relation $\cong$ on the set of two-element subsets of A. Let $CG(\mathbb{A})$ denote the *component graph* $G(\mathbb{A})$. Let also for a vertex v of $G(\mathbb{A})$ v^c denote the strongly connected component containing v. The *height* of vertex v is the length of a longest directed path in $CG(\mathbb{A})$ originating at v^c.

Lemma 25 *If $\mathsf{var}(\mathbb{A})$, where $\mathbb{A}$ is idempotent, does not omit the unary or affine type, then there are $a,b \in A$ such that $\{a,b\}$ is a subtrace in the subalgebra $\mathbb{C}$ generated by a,b, but neither (a,b) nor (b,a) is a 2-snag in $\mathbb{C}$.*

Proof: By Corollary 2.2 of [8] if $\mathsf{var}(\mathbb{A})$ for an idempotent algebra $\mathbb{A}$ admits the unary or affine type then there is a subalgebra of $\mathbb{A}$ that admits one of these types. Let $\mathbb{B}$ be such a subalgebra. Theorem 7.2 from [9] implies that there is a subtrace $\{a,b\} \subseteq B$ such that neither (a,b), nor (b,a) is a 2-snag in $\mathbb{B}$. Denote by $\mathbb{C}$ the subalgebra generated by $\{a,b\}$. We choose a,b such that (i) $\mathbb{C}$ is as small as possible, and (ii) there is no pair $\{c,d\} \subseteq \mathbb{C}$ which is a subtrace in $\mathbb{B}$, but none of $(c,d),(d,c)$ is a 2-snag in $\mathbb{B}$, and its height in $G(\mathbb{C})$ is less than that of (a,b). Such a,b exist. Indeed, choose any subtrace that is not a 2-snag satisfying condition (i). Then in $\mathbb{C}$ take a pair of the lowest height possible such that $\{c,d\}$ is a subtrace in $\mathbb{B}$, but none of $(c,d),(d,c)$ is a 2-snag. By condition (i) c,d generates the same subalgebra as a,b, but also satisfy condition (ii).

Observe that every (unary) polynomial of $\mathbb{C}$ is a restriction of a (unary) polynomial of $\mathbb{B}$ onto the base set C of $\mathbb{C}$. Indeed, such a polynomial operation can be obtained by plugging in the constants (from C) into the term operation produced from the basic operations of $\mathbb{B}$ by the same chain of substitutions. Therefore $G(\mathbb{C})$ is a subgraph of $G(\mathbb{B})$. Also if there is a binary polynomial g of $\mathbb{C}$ satisfying the conditions $g(b,b) = b$ and $g(a,b) = g(b,a) = g(a,a) = a$, then $\mathbb{B}$ has a polynomial with the same properties.

If $\{a,b\}$ is a subtrace in $\mathbb{C}$ then we are done, because neither (a,b) nor (b,a) is a 2-snag in $\mathbb{C}$. Suppose that $\{a,b\}$ is not a subtrace in $\mathbb{C}$. This means $(a,b) \in \sigma_{\mathbb{C}}(a,b)$, that is, there are $n \geq 1$ and $z_0, z_1, \ldots, z_n \in A$ with $c = z_0$, $d = z_n$

and there are $f_1, \ldots, f_n \in \mathsf{Pol}_1\ \mathbb{C}$ such that for each i, $1 \leq i \leq n$, $\{z_i, z_{i-1}\} = \{f_i(a), f_i(b)\}$ and $\{a, b\} \not\cong \{f_i(a), f_i(b)\}$. Since $(a, b) \notin \sigma_{\mathbb{B}}(a, b)$, one of the pairs $\{z_i, z_{i+1}\}$ is polynomially isomorphic to $\{a, b\}$ in $\mathbb{B}$. By Lemma 24 $\{z_i, z_{i+1}\}$ is a subtrace in $\mathbb{B}$ and neither (z_i, z_{i+1}) nor (z_{i+1}, z_i) is a 2-snag. Moreover, as $\{a, b\} \not\cong \{z_i, z_{i+1}\}$, the height of (z_i, z_{i+1}) is less than that of (a, b). A contradiction with the choice of a, b. $\Box$

Now let $\mathbb{A} = \mathrm{Alg}(\mathcal{B})$. To check whether $\mathsf{var}(\mathbb{A})$ admits the unary or affine types we just should go over all pairs $a, b \in \mathbb{A}$ using REL-GEN to verify (i) if $\{a, b\}$ is a subtrace in the subalgebra generated by a, b, and (ii) if (a, b) or (b, a) is a 2-snag in the same subalgebra. To find subtraces we use the following construction from [1].

For $\{a, b\} \subseteq A$, let $G_{a,b}(\mathbb{A})$ denote the graph with vertex set A, and such that (c, d) is an edge of $G_{a,b}$ if and only if $(\{a, b\}, \{c, d\})$ is an edge of $G(\mathbb{A})$ and $\{a, b\} \not\cong \{c, d\}$. Then $\{a, b\}$ is a subtrace in $\mathbb{A}$ if and only if there is no path from a to b in the graph $G_{a,b}$.

Note that in order to determine if there exists a unary polynomial f of the subalgebra $\mathbb{C}$ generated by a, b and such that $\{c', d'\} = \{f(c), f(d)\}$ for some $c, d, c', d' \in \mathbb{C}$, it suffices to construct $\mathsf{Sg}(R)$ for the binary relation $R = \{(a, a), (b, b), (c, d)\}$ and check whether or not (c', d') or (d', c') belongs to it. As for this relation $m = 3$ and $d = 2$ this can be done using algorithm REL-GEN. Analogously, by the definition of 2-snag, a subset $\{a, b\}$ is a 2-snag in the subalgebra generated by a, b if and only if the 4-ary relation generated by $(a, a, b, b), (a, b, a, b), (a, a, a, a), (b, b, b, b)$ contains (a, a, a, b).

This method works provided algorithm REL-GEN returns the right value of $\mathsf{Sg}(R)$. However, it is not always the case. As we shall see later, we still can use the results of REL-GEN even they are not correct, but we need some sort of monotonicity. Let the subalgebra generated by a set S according to our algorithm be denoted by $\mathsf{Sg}(S)'$ and the graph of polynomial mappings of pairs by $G'(\mathsf{Sg}(S)')$. Clearly, $\mathsf{Sg}(S)' \subseteq \mathsf{Sg}(S)$ and $G'(\mathsf{Sg}(S)')$ is a subgraph of $G(\mathsf{Sg}(S))$. We need to make sure that, for any pair a, b and any pair $c, d \in \mathsf{Sg}(a, b)'$, $\mathsf{Sg}(c, d)' \subseteq \mathsf{Sg}(a, b)'$, and that $G'(\mathsf{Sg}(c, d)')$ is a subgraph of $G'(\mathsf{Sg}(a, b)')$. Both conditions can be achieved by using algorithm ALG-GEN and GRAPH-GEN for generating subalgebras and constructing graphs, respectively.

Lemma 26 *For any structure $\mathcal{A}$ with idempotent polymorphisms, any $a, b \in A$, and any $c, d \in \mathsf{Sg}(a, b)'$, (i) $\mathsf{Sg}(c, d)' \subseteq \mathsf{Sg}(a, b)'$, and (ii) graph $G'(\mathsf{Sg}(c, d)')$ is a subgraph of $G'(\mathsf{Sg}(a, b)')$.*

Lemma 26 follows straightforwardly from the description of algorithms ALG-GEN and REL-GEN.

INPUT: A relational structure $\mathcal{A}$ and $a, b \in A$

OUTPUT: A set $\mathsf{Sg}(a,b)'$, $\{a,b\} \subseteq \mathsf{Sg}(a,b)' \subseteq \mathsf{Sg}(a,b)$, 'generated' by a, b

Step 1 **set** $S := \{a,b\}$, $\text{changed} := \mathbf{true}$
Step 2 **while** changed **do**
Step 2.1 **set** $\text{changed} := \mathbf{false}$
Step 2.2 **for all** $c, d \in S$ **do**
Step 2.2.1 **call** REL-GEN on $\mathcal{A}$ and unary relation $\{c,d\}$; the result denote by R
Step 2.2.2 **if** $R \not\subseteq S$ **set** $\text{changed} := \mathbf{true}$
Step 2.2.3 **set** $S := S \cup R$
endfor
endwhile
Step 3 **output** S

Figure 2: Algorithm ALG-GEN

INPUT: A relational structure $\mathcal{A}$ and $a, b \in A$

OUTPUT: A subgraph $G'(\mathsf{Sg}(a,b)')$, of $G(\mathsf{Sg}(a,b))$

Step 1 **call** ALG-GEN on $\mathcal{A}$ and a, b; denote the result by C
Step 2 **set** $V := \{\{c,d\} | c, d \in C, c \neq d\}$, $E := \emptyset$
Step 3 **for all** $(a', b') \in C^2$, $a' \neq b'$, **do**
Step 3.1 **call** ALG-GEN on $\mathcal{A}$ and a', b'; denote the result by C'
Step 3.2 **for all** $c, d \in C'$, $c \neq d$, **do**
Step 2.2.1 **call** REL-GEN on $\mathcal{A}$ and binary relation $\{(a',a'),(b',b'),(c,d),(d,c)\}$; denote the result by R
Step 2.2.2 **for all** $(c', d') \in R$, $c' \neq d'$ **set** $E := E \cup (\{c,d\},\{c',d'\})$
endfor
endwhile
Step 3 **output** (V, E)

Figure 3: Algorithm GRAPH-GEN

INPUT: A relational structure $\mathcal{A}$

OUTPUT: YES if $\mathsf{var}(\mathrm{Alg}(\mathcal{A}))$ admits the unary or affine type, NO otherwise

Step 1 **for each** pair $a, b \in A$ **do**
Step 1.1 **construct** graph $G(\mathbb{B})$ where $\mathbb{B}$ is the subalgebra generated by $\{a, b\}$
Step 1.2 **if** $\{a, b\}$ is a subtrace in $\mathbb{B}$ **then do**
Step 1.2.1 **if** neither (a, b) nor (b, a) is a 2-snag in $\mathbb{B}$ **then output** YES and **stop**
endif
endfor
Step 2. **output** NO

Figure 4: Algorithm UNARY-AFFINE-TYPE-CHECK

Now we are in a position to introduce an algorithm checking omitting the unary and affine types.

Lemma 27 *Algorithm* UNARY-AFFINE-TYPE-CHECK *correctly decides if* $\mathsf{var}(\mathrm{Alg}(\mathcal{A}))$ *omits the unary and affine types and is polynomial time.*

Proof: Since all the algorithms involved are polynomial time and are called only polynomially many times, the algorithm is polynomial time.

If $\mathsf{var}(\mathbb{A})$ omits the unary and affine types then REL-GEN generates the relation generated by certain tuples correctly, and, as there is no subtrace which is not a 2-snag, the algorithm outputs NO. Thus UNARY-AFFINE-TYPE-CHECK gives no false negatives. If $\mathsf{var}(\mathbb{A})$ admits one of the types, algorithm REL-GEN can miss some of the polynomials. This may lead to three possible mistakes. (1) Identifying a subtrace as a not 2-snag while it is. (2) Deciding that some pairs of elements are not isomorphic while they are, which may lead to a conclusion that some pair is not a subtrace while it is a subtrace. (3) Deciding that there is no $f \in \mathsf{Pol}_1\ \mathbb{A}$ with $f(\{a, b\}) = \{c, d\}$ while such polynomial exists, which may lead to identifying some pair as a subtrace while it is not.

Cases (1) and (3) do not cause any difficulties, because in these cases we only may mistakenly conclude that some pair witnesses admitting the unary or affine type. However, this can happen only if $\mathsf{var}(\mathbb{A})$ admits one of those types indeed, and therefore, although we are wrong about a particular pair, our overall decision is correct. To cope with case (2) we need to elaborate.

What we need to prove is that if $\mathbb{A} = \mathrm{Alg}(\mathcal{A})$ contains subtraces that are not 2-snags, at least one of them will be identified by the algorithm. Suppose that $\{a, b\}$ is a subtrace that is not a 2-snag. By Lemma 25, we may assume that it is also a subtrace in the subalgebra $\mathbb{C}$ generated by $\{a, b\}$, and that neither (a, b) nor (a, b) is a 2-snag in $\mathbb{C}$. Note that if the algorithm identifies it as a subtrace, it also identifies it as a non-2-snag.

We proceed by induction on the size of the set C' found by REL-GEN as the set generated by a, b and the height of $\{a, b\}$ in $G'(\mathbb{C})$, the graph constructed instead of $C(\mathbb{C})$ by the algorithm. If $|C'| = 2$ then $\{a, b\}$ is the only pair in this subalgebra that is isomorphic to itself and therefore is identified as a subtrace. If the height of $\{a, b\}$ is 0 then every pair $(f(a), f(b))$, f is a unary polynomial of $\mathbb{C}$ found by the algorithm, belongs to the same strongly connected component, and therefore the set $\sigma'_{\mathbb{C}}(a, b)$ found by the algorithm is empty. Again, the algorithm identifies $\{a, b\}$ as a subtrace.

Now suppose that $\{a, b\}$ is a subtrace and the algorithm (mistakenly) finds a sequence $a = z_0, \ldots, z_\ell = b$ such that $\{z_i, z_{i+1}\} \in \sigma'_{\mathbb{C}}(a, b)$. Since $(a, b) \notin \sigma_{\mathbb{C}}(a, b)$, there is j such that $\{z_j, z_{j+1}\} \cong \{a, b\}$ in algebra $\mathbb{C}$, but this isomorphism is overlooked by the algorithm. By Lemma 24 $\{z_j, z_{j+1}\}$ is an subtrace and neither (z_j, z_{j+1}) nor (z_{j+1}, z_j) is a 2-snag. If the subalgebras generated by $\{z_j, z_{j+1}\}$ and by $\{a, b\}$ (as found by the algorithm) are different, then we are done by inductive hypothesis. Otherwise $G'(\mathsf{Sg}(z_j, z_{j+1})') = G'(\mathsf{Sg}(a, b)')$, but the height of $\{z_j, z_{j+1}\}$ is strictly less than that of $\{a, b\}$. We again use the inductive hypothesis. $\square$

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