Inference in first-order logic

Chapter 9
Outline

♦ Reducing first-order inference to propositional inference
♦ Unification
♦ Generalized Modus Ponens
♦ Forward and backward chaining
♦ Resolution
Universal instantiation (UI)

Every instantiation of a universally quantified sentence is entailed by it:

\[ \forall v \alpha \]

\[ \text{SUBST}\left(\{v/g\}, \alpha\right) \]

for any variable \(v\) and ground term \(g\)

E.g., \(\forall x \ King(x) \land Greedy(x) \Rightarrow Evil(x)\) yields

\(King(John) \land Greedy(John) \Rightarrow Evil(John)\)
\(King(Richard) \land Greedy(Richard) \Rightarrow Evil(Richard)\)
\(King(Father(John)) \land Greedy(Father(John)) \Rightarrow Evil(Father(John))\)
\[\vdots\]
**Existential instantiation (EI)**

For any sentence $\alpha$, variable $v$, and constant symbol $k$ that does not appear elsewhere in the knowledge base:

$$\exists v \quad \alpha \quad \text{Subst}(\{v/k\}, \alpha)$$

E.g., $\exists x \quad \text{Crown}(x) \land \text{OnHead}(x, \text{John})$ yields

$$\text{Crown}(C_1) \land \text{OnHead}(C_1, \text{John})$$

provided $C_1$ is a new constant symbol, called a **Skolem constant**

Another example: from $\exists x \quad d(x^y)/dy = x^y$ we obtain

$$d(e^y)/dy = e^y$$

provided $e$ is a new constant symbol
Existential instantiation contd.

UI can be applied several times to **add** new sentences; the new KB is logically equivalent to the old

EI can be applied once to **replace** the existential sentence; the new KB is **not** equivalent to the old, but is satisfiable iff the old KB was satisfiable
Reduction to propositional inference

Suppose the KB contains just the following:

\[ \forall x \ King(x) \land Greedy(x) \Rightarrow Evil(x) \]

King(John)
Greedy(John)
Brother(Richard, John)

Instantiating the universal sentence in all possible ways, we have

King(John) \land Greedy(John) \Rightarrow Evil(John)
King(Richard) \land Greedy(Richard) \Rightarrow Evil(Richard)
King(John)
Greedy(John)
Brother(Richard, John)

The new KB is propositionalized: proposition symbols are

King(John), Greedy(John), Evil(John), King(Richard) etc.
Claim: a ground sentence* is entailed by new KB iff entailed by original KB

Claim: every FOL KB can be propositionalized so as to preserve entailment

Idea: propositionalize KB and query, apply resolution, return result

Problem: with function symbols, there are infinitely many ground terms, e.g., \( \text{Father}(\text{Father}(\text{Father}(\text{John}))) \)

Theorem: Herbrand (1930). If a sentence \( \alpha \) is entailed by an FOL KB, it is entailed by a finite subset of the propositional KB

Idea: For \( n = 0 \) to \( \infty \) do

create a propositional KB by instantiating with depth-\( n \) terms
see if \( \alpha \) is entailed by this KB

Problem: works if \( \alpha \) is entailed, loops if \( \alpha \) is not entailed

Theorem: Turing (1936), Church (1936), entailment in FOL is semidecidable
Problems with propositionalization

Propositionalization seems to generate lots of irrelevant sentences. E.g., from

\[ \forall x \ (\text{King}(x) \land \text{Greedy}(x)) \Rightarrow \text{Evil}(x) \]
\[ \text{King}(\text{John}) \]
\[ \forall y \ \text{Greedy}(y) \]
\[ \text{Brother}(\text{Richard}, \text{John}) \]

it seems obvious that \( \text{Evil}(\text{John}) \), but propositionalization produces lots of facts such as \( \text{Greedy}(\text{Richard}) \) that are irrelevant.
Unification

We can get the inference immediately if we can find a substitution $\theta$ such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$.

$\theta = \{x/John, y/John\}$ works

$\text{UNIFY}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

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$\theta = \{x/\text{John}, y/\text{John}\}$ works.

**Unify**($\alpha$, $\beta$) = $\theta$ if $\alpha\theta = \beta\theta$

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\[ \theta = \{x/\text{John}, y/\text{John}\} \text{ works} \]

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We can get the inference immediately if we can find a substitution \( \theta \) such that \( \text{King}(x) \) and \( \text{Greedy}(x) \) match \( \text{King}(\text{John}) \) and \( \text{Greedy}(y) \)

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Standardizing apart eliminates overlap of variables, e.g., \(\text{Knows}(z_{17}, \text{OJ})\)
Generalized Modus Ponens (GMP)

\[
p_1', p_2', \ldots, p_n', (p_1 \land p_2 \land \ldots \land p_n \Rightarrow q) \quad q\theta
\]

where \( p_i'\theta = p_i\theta \) for all \( i \)

\( p_1' \) is \( King(John) \)  \( p_1 \) is \( King(x) \)
\( p_2' \) is \( Greedy(y) \)  \( p_2 \) is \( Greedy(x) \)
\( \theta \) is \( \{x/John, y/John\} \)  \( q \) is \( Evil(x) \)
\( q\theta \) is \( Evil(John) \)

GMP used with KB of definite clauses (\textbf{exactly} one positive literal)
All variables assumed universally quantified
The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

Prove that Col. West is a criminal
Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:
Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:

\[ \text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x) \]

Nono ... has some missiles
Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:

\[ \text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x) \]

Nono ... has some missiles, i.e., \( \exists x \ \text{Owns}(\text{Nono}, x) \land \text{Missile}(x) : \text{Owns}(\text{Nono}, M_1) \text{ and } \text{Missile}(M_1) \)

... all of its missiles were sold to it by Colonel West
... it is a crime for an American to sell weapons to hostile nations:
\[\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)\]
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\[\text{Owns}(\text{Nono}, M_1) \text{ and } \text{Missile}(M_1)\]
... all of its missiles were sold to it by Colonel West
\[\text{Missile}(x) \land \text{Owns}(\text{Nono}, x) \Rightarrow \text{Sells}(\text{West}, x, \text{Nono})\]
Missiles are weapons:
Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:

\[ \text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x) \]

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\[ \text{Owns}(\text{Nono}, M_1) \text{ and } \text{Missile}(M_1) \]

... all of its missiles were sold to it by Colonel West

\[ \text{Missile}(x) \land \text{Owns}(\text{Nono}, x) \Rightarrow \text{Sells}(\text{West}, x, \text{Nono}) \]

Missiles are weapons:

\[ \text{Missile}(x) \Rightarrow \text{Weapon}(x) \]

An enemy of America counts as “hostile”:
Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:
\[ \text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x) \]

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\[ \text{Missile}(x) \land \text{Owns}(\text{Nono}, x) \Rightarrow \text{Sells}(\text{West}, x, \text{Nono}) \]

Missiles are weapons:
\[ \text{Missile}(x) \Rightarrow \text{Weapon}(x) \]

An enemy of America counts as “hostile”:
\[ \text{Enemy}(x, \text{America}) \Rightarrow \text{Hostile}(x) \]

West, who is American ...
\[ \text{American}(\text{West}) \]

The country Nono, an enemy of America ...
\[ \text{Enemy}(\text{Nono}, \text{America}) \]
function FOL-FC-Ask(KB, α) returns a substitution or false

repeat until new is empty

new ← {}

for each sentence r in KB do

(p₁ ∧ ... ∧ pₙ ⇒ q) ← STANDARDIZE-APART(r)

for each θ such that (p₁ ∧ ... ∧ pₙ)θ = (p'₁ ∧ ... ∧ p'ₙ)θ

for some p'₁, ..., p'ₙ in KB

q' ← SUBST(θ, q)

if q' is not a renaming of a sentence already in KB or new then do

add q' to new

φ ← UNIFY(q', α)

if φ is not fail then return φ

add new to KB

return false
Forward chaining proof

\begin{align*}
\text{American(\textit{West})} & \\
\text{Missile(M1)} & \\
\text{Owns(Nono,M1)} & \\
\text{Enemy(Nono,America)} & \\
\end{align*}
Forward chaining proof

American(West)  
Missile(M1)  
Weapon(M1)  
Sells(West,M1,Nono)  
Owns(Nono,M1)  
Hostile(Nono)  
Enemy(Nono,America)
Properties of forward chaining

Sound and complete for first-order definite clauses
(proof similar to propositional proof)

**Datalog** = first-order definite clauses + **no functions** (e.g., crime KB)
FC terminates for Datalog in poly iterations: at most $p \cdot n^k$ literals

May not terminate in general (with functions) if $\alpha$ is not entailed

This is unavoidable: entailment with definite clauses is semidecidable
Backward chaining algorithm

function FOL-BC-Ask(KB, goals, θ) returns a set of substitutions
  inputs: KB, a knowledge base
           goals, a list of conjuncts forming a query (θ already applied)
           θ, the current substitution, initially the empty substitution {} 
  local variables: answers, a set of substitutions, initially empty

if goals is empty then return {θ}
q' ← Subst(θ, First(goals))
for each sentence r in KB
  where Standardize-Apart(r) = (p₁ ∧ ... ∧ pₙ ⇒ q)
  and θ' ← Unify(q, q') succeeds
  new_goals ← [p₁, ..., pₙ | Rest(goals)]
  answers ← FOL-BC-Ask(KB, new_goals, Compose(θ', θ)) ∪ answers
return answers
Backward chaining example

Criminal(West)
Backward chaining example

Criminal(West)

{x/West}

American(x)  Weapon(y)  Sells(x,y,z)  Hostile(z)
Backward chaining example

American(West)  Weapon(y)  Sells(x,y,z)  Hostile(z)

Criminal(West)  \{x/West, y/M1\}

Missile(y)  \{y/M1\}
Backward chaining example

\[ \text{Criminal(West)} \quad \{x/\text{West}, y/M1, z/\text{Nono}\} \]

- \text{American(West)}
  - \{\}\n- \text{Weapon(y)}
- \text{Sells(West,M1,z)}
  - \{z/\text{Nono}\}
- \text{Hostile(z)}
- \text{Missile(y)}
  - \{y/M1\}
- \text{Missile(M1)}
- \text{Owns(Nono,M1)}
Backward chaining example

- **Criminal(West)**
  - **American(West)**
  - **Weapon(y)**
  - **Sells(West,M1,z)**
  - **Hostile(Nono)**
- **Missile(y)**
- **Missile(M1)**
- **Owns(Nono,M1)**
- **Enemy(Nono,America)**

Inference:
- \( x/West, y/M1, z/Nono \)
Properties of backward chaining

Depth-first recursive proof search: space is linear in size of proof

Incomplete due to infinite loops
⇒ fix by checking current goal against every goal on stack

Inefficient due to repeated subgoals (both success and failure)
⇒ fix using caching of previous results (extra space!)

Widely used (without improvements!) for logic programming
## Logic programming

Sound bite: computation as inference on logical KBs

<table>
<thead>
<tr>
<th>Logic programming</th>
<th>Ordinary programming</th>
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<tbody>
<tr>
<td>1. Identify problem</td>
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<tr>
<td>2. Assemble information</td>
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</tr>
<tr>
<td>3. Tea break</td>
<td>Figure out solution</td>
</tr>
<tr>
<td>4. Encode information in KB</td>
<td>Program solution</td>
</tr>
<tr>
<td>5. Encode problem instance as facts</td>
<td>Encode problem instance as data</td>
</tr>
<tr>
<td>6. Ask queries</td>
<td>Apply program to data</td>
</tr>
<tr>
<td>7. Find false facts</td>
<td>Debug procedural errors</td>
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Should be easier to debug $\text{Capital}(\text{NewYork, US})$ than $x := x + 2$!
Prolog systems

Basis: backward chaining with Horn clauses + bells & whistles
Widely used in Europe, Japan (basis of 5th Generation project)
Compilation techniques ⇒ approaching a billion LIPS

Program = set of clauses = head :- literal₁, ..., literalₙ.

    criminal(X) :- american(X), weapon(Y), sells(X,Y,Z), hostile(Z).

Depth-first, left-to-right backward chaining
Built-in predicates for arithmetic etc., e.g., X is Y*Z+3
Closed-world assumption (“negation as failure”)
    e.g., given alive(X) :- not dead(X).
    alive(joe) succeeds if dead(joe) fails
Prolog examples

Depth-first search from a start state X:

\[
\text{dfs}(X) :\text{-} \text{goal}(X).
\]
\[
\text{dfs}(X) :\text{-} \text{successor}(X,S),\text{dfs}(S).
\]

No need to loop over S: \textbf{successor succeeds for each}

Appending two lists to produce a third:

\[
\text{append}([],Y,Y).
\]
\[
\text{append}([X|L],Y,[X|Z]) :\text{-} \text{append}(L,Y,Z).
\]

query: \text{append}(A,B,[1,2]) ?

answers: A=[] B=[1,2]
A=[1,2] B=[]
Resolution: brief summary

Full first-order version:

\[
\ell_1 \lor \cdots \lor \ell_k, \quad m_1 \lor \cdots \lor m_n
\]

\[
(\ell_1 \lor \cdots \lor \ell_{i-1} \lor \ell_{i+1} \lor \cdots \lor \ell_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \lor \cdots \lor m_n)\theta
\]

where \(\text{UNIFY}(\ell_i, \neg m_j) = \theta\).

For example,

\[
\neg \text{Rich}(x) \lor \text{Unhappy}(x).
\]

\[
\text{Rich}(\text{Ken})
\]

\[
\text{Unhappy}(\text{Ken})
\]

with \(\theta = \{x/\text{Ken}\}\)

Apply resolution steps to \(\text{CNF}(KB \land \neg \alpha)\); complete for FOL
Conversion to CNF

Everyone who loves all animals is loved by someone:
\[ \forall x \ [\forall y \ Animal(y) \Rightarrow Loves(x, y)] \Rightarrow [\exists y \ Loves(y, x)] \]

1. Eliminate biconditionals and implications

\[ \forall x \ [\neg \forall y \ \neg Animal(y) \lor Loves(x, y)] \lor [\exists y \ Loves(y, x)] \]

2. Move \( \neg \) inwards:

\[ \neg \forall x, p \equiv \exists x \ \neg p, \quad \neg \exists x, p \equiv \forall x \ \neg p: \]

\[ \forall x \ [\exists y \neg \neg Animal(y) \lor Loves(x, y))] \lor [\exists y \ Loves(y, x)] \]
\[ \forall x \ [\exists y \neg Animal(y) \land \neg Loves(x, y))] \lor [\exists y \ Loves(y, x)] \]
\[ \forall x \ [\exists y \ Animal(y) \land \neg Loves(x, y))] \lor [\exists y \ Loves(y, x)] \]
Conversion to CNF contd.

3. Standardize variables: each quantifier should use a different one

$$\forall x \ [\exists y \ Animal(y) \land \neg Loves(x, y)] \lor [\exists z \ Loves(z, x)]$$

4. Skolemize: a more general form of existential instantiation.
   Each existential variable is replaced by a Skolem function
   of the enclosing universally quantified variables:

$$\forall x \ [Animal(F(x)) \land \neg Loves(x, F(x))] \lor Loves(G(x), x)$$

5. Drop universal quantifiers:

$$[Animal(F(x)) \land \neg Loves(x, F(x))] \lor Loves(G(x), x)$$

6. Distribute $\land$ over $\lor$:

$$[Animal(F(x)) \lor Loves(G(x), x)] \land [\neg Loves(x, F(x)) \lor Loves(G(x), x)]$$
Resolution proof: definite clauses

\[ \neg \text{American}(x) \lor \neg \text{Weapon}(y) \lor \neg \text{Sells}(x,y,z) \lor \neg \text{Hostile}(z) \lor \text{Criminal}(x) \]

\[ \neg \text{Criminal}(\text{West}) \]

\[ \neg \text{American}(\text{West}) \]

\[ \neg \text{Weapon}(y) \lor \neg \text{Sells}(\text{West},y,z) \lor \neg \text{Hostile}(z) \]

\[ \neg \text{Missile}(x) \lor \text{Weapon}(x) \]

\[ \neg \text{Missile}(y) \lor \neg \text{Sells}(\text{West},y,z) \lor \neg \text{Hostile}(z) \]

\[ \neg \text{Missile}(x) \lor \neg \text{Owns}(\text{Nono},x) \lor \text{Sells}(\text{West},x,\text{Nono}) \]

\[ \neg \text{Sells}(\text{West},\text{M1},z) \lor \neg \text{Hostile}(z) \]

\[ \neg \text{Hostile}(\text{M1}) \lor \neg \text{Owns}(\text{Nono},\text{M1}) \lor \neg \text{Hostile}(\text{Nono}) \]

\[ \neg \text{Enemy}(x,\text{America}) \lor \text{Hostile}(x) \]

\[ \neg \text{Hostile}(\text{Nono}) \]

\[ \text{Enemy}(\text{Nono},\text{America}) \]

\[ \text{Enemy}(\text{Nono},\text{America}) \]