Inference in first-order logic

Chapter 9

Outline

♦ Reducing first-order inference to propositional inference
♦ Unification
♦ Generalized Modus Ponens
♦ Forward and backward chaining
♦ Resolution

Universal instantiation (UI)

Every instantiation of a universally quantified sentence is entailed by it:

\[
\forall v \alpha \quad \text{Subst}(\{v/g\}, \alpha)
\]

for any variable \(v\) and ground term \(g\)

E.g., \(\forall x \text{ King}(x) \land \text{ Greedy}(x) \Rightarrow \text{ Evil}(x)\) yields

\[
\text{King}(\text{Richard}) \land \text{ Greedy}(\text{Richard}) \Rightarrow \text{ Evil}(\text{Richard})
\]

Existential instantiation (EI)

For any sentence \(\alpha\), variable \(v\), and constant symbol \(k\) that does not appear elsewhere in the knowledge base:

\[
\exists v \alpha \quad \text{Subst}(\{v/k\}, \alpha)
\]

E.g., \(\exists x \text{ Crown}(x) \land \text{ OnHead}(x, \text{John})\) yields

\[
\text{Crown}(C_1) \land \text{ OnHead}(C_1, \text{John})
\]

provided \(C_1\) is a new constant symbol, called a Skolem constant

Another example: from \(\exists x \ d(x^y)/dy = x^y\) we obtain

\[
d(e^y)/dy = e^y
\]

provided \(e\) is a new constant symbol

Existential instantiation contd.

UI can be applied several times to add new sentences; the new KB is logically equivalent to the old

EI can be applied once to replace the existential sentence; the new KB is not equivalent to the old, but is satisfiable iff the old KB was satisfiable

Reduction to propositional inference

Suppose the KB contains just the following:

\[
\forall x \text{ King}(x) \land \text{ Greedy}(x) \Rightarrow \text{ Evil}(x)
\]

\[
\text{King}(\text{John})
\]

\[
\text{Greedy}(\text{John})
\]

\[
\text{Brother} (\text{Richard}, \text{John})
\]

Instantiating the universal sentence in all possible ways, we have

\[
\text{King}(\text{John}) \land \text{ Greedy}(\text{John}) \Rightarrow \text{ Evil}(\text{John})
\]

\[
\text{King}(\text{Richard}) \land \text{ Greedy}(\text{Richard}) \Rightarrow \text{ Evil}(\text{Richard})
\]

\[
\text{King}(\text{John}) \land \text{ Greedy}(\text{John}) \Rightarrow \text{ Evil}(\text{John})
\]

\[
\text{King}(\text{Richard}) \land \text{ Greedy}(\text{Richard}) \Rightarrow \text{ Evil}(\text{Richard})
\]

\[
\text{King}(\text{John}) \land \text{ Greedy}(\text{John}) \Rightarrow \text{ Evil}(\text{John})
\]

\[
\text{Brother} (\text{Richard}, \text{John})
\]

\[
\text{Brother} (\text{Richard}, \text{John})
\]

The new KB is propositionalized: proposition symbols are

\[
\text{King}(\text{John}), \text{ Greedy}(\text{John}), \text{ Evil}(\text{John}), \text{ King}(\text{Richard})\] etc.
Problems with propositionalization

Propositionalization seems to generate lots of irrelevant sentences. E.g., from

\[ \forall x \, \text{King}(x) \land \text{Greedy}(x) \Rightarrow \text{Evil}(x) \]
\[ \text{King}(\text{John}) \]
\[ \forall y \, \text{Greedy}(y) \]
\[ \text{Brother}(\text{Richard}, \text{John}) \]

it seems obvious that \( \text{Evil}(\text{John}) \), but propositionalization produces lots of facts such as \( \text{Greedy}(\text{Richard}) \) that are irrelevant

Unification

We can get the inference immediately if we can find a substitution \( \theta \) such that \( \text{King}(x) \) and \( \text{Greedy}(x) \) match \( \text{King}(\text{John}) \) and \( \text{Greedy}(y) \)

\[ \theta = \{ x/\text{John}, y/\text{John} \} \text{ works} \]

\[ \text{UNIFY}(\alpha, \beta) = \theta \text{ if } \alpha \theta = \beta \theta \]

\[ \begin{array}{ccc}
    p & q & \theta \\
    \text{Knows}(\text{John}, x) & \text{Knows}(\text{John}, \text{Jane}) & \{ x/\text{Jane} \} \\
    \text{Knows}(\text{John}, x) & \text{Knows}(y, \text{OJ}) & \{ x/\text{OJ}, y/\text{John} \} \\
    \text{Knows}(\text{John}, x) & \text{Knows}(y, \text{Mother}(y)) & \{ y/\text{John}, x/\text{Mother}(\text{John}) \} \\
    \text{Knows}(\text{John}, x) & \text{Knows}(x, \text{OJ}) & \\
\end{array} \]
**Unification**

We can get the inference immediately if we can find a substitution $\theta$ such that $\text{King}(x)$ and $\text{Greedy}(x)$ match $\text{King}(\text{John})$ and $\text{Greedy}(y)$.

$\theta = \{x/\text{John}, y/\text{John}\}$ works

$\text{UNIFY}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

$p_1 \land p_2 \land \ldots \land p_n \Rightarrow q$

$p_i' \quad q'$

$p_i \quad \text{Knows}(\text{John}, x)$
$p_i \quad \text{Knows}(\text{John}, x)$
$p_i \quad \text{Knows}(\text{John}, y)$
$p_i \quad \text{Knows}(\text{John}, x)$

Standardizing apart eliminates overlap of variables, e.g., $\text{Knows}(z, \text{OJ})$

**Example knowledge base**

The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American. Prove that Col. West is a criminal.

**Generalized Modus Ponens (GMP)**

$p_1' \land p_2' \land \ldots \land p_n' \Rightarrow q'$

$p_i' \quad q'$

$p_i \quad \text{Knows}(\text{John}, x)$
$p_i \quad \text{Knows}(\text{John}, x)$
$p_i \quad \text{Knows}(\text{John}, y)$
$p_i \quad \text{Knows}(\text{John}, x)$

$\theta$ is $\{x/\text{John}, y/\text{John}\}$
$q\theta$ is $\text{Evil}(\text{John})$

GMP used with KB of definite clauses (exactly one positive literal)

All variables assumed universally quantified

**Example knowledge base contd.**

... it is a crime for an American to sell weapons to hostile nations:

$\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)$

Nono... has some missiles

... all of its missiles were sold to it by Colonel West

$\exists x \quad \text{Owens}(\text{Nono}, x) \land \text{Missile}(x)$

... it is a crime for an American to sell weapons to hostile nations:

$\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)$

Nono... has some missiles, i.e., $\exists x \quad \text{Owens}(\text{Nono}, x) \land \text{Missile}(x)$

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Generalized Modus Ponens (GMP)

$p_1 ' \land p_2 ' \land \ldots \land p_n ' \Rightarrow q '$

$p_i ' \quad q '$

$p_i \quad \text{Knows}(\text{John}, x)$
$p_i \quad \text{Knows}(\text{John}, x)$
$p_i \quad \text{Knows}(\text{John}, y)$
$p_i \quad \text{Knows}(\text{John}, x)$

$\theta$ is $\{x/\text{John}, y/\text{John}\}$
$q\theta$ is $\text{Evil}(\text{John})$

GMP used with KB of definite clauses (exactly one positive literal)

All variables assumed universally quantified

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Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:

$\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)$

Nono... has some missiles

... all of its missiles were sold to it by Colonel West

$\exists x \quad \text{Owens}(\text{Nono}, x) \land \text{Missile}(x)$

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Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:

$\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)$

Nono... has some missiles

... all of its missiles were sold to it by Colonel West

$\exists x \quad \text{Owens}(\text{Nono}, x) \land \text{Missile}(x)$

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Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:

$\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)$

Nono... has some missiles

... all of its missiles were sold to it by Colonel West

$\exists x \quad \text{Owens}(\text{Nono}, x) \land \text{Missile}(x)$

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Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:

$\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)$

Nono... has some missiles

... all of its missiles were sold to it by Colonel West

$\exists x \quad \text{Owens}(\text{Nono}, x) \land \text{Missile}(x)$

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Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:

$\text{American}(x) \land \text{Weapon}(y) \land \text{Sells}(x, y, z) \land \text{Hostile}(z) \Rightarrow \text{Criminal}(x)$

Nono... has some missiles

... all of its missiles were sold to it by Colonel West

$\exists x \quad \text{Owens}(\text{Nono}, x) \land \text{Missile}(x)$
Example knowledge base contd.

... it is a crime for an American to sell weapons to hostile nations:
American(x) ∧ Weapon(y) ∧ Sells(x, y, z) ∧ Hostile(z) ⇒ Criminal(x)
Nono . . . has some missiles, i.e., ∃ x Owns(Nono, x) ∧ Missile(x):
Owns(Nono, M₁) and Missile(M₁)
... all of its missiles were sold to it by Colonel West
Missile(x) ∧ Owns(Nono, x) ⇒ Sells(West, x, Nono)
Missiles are weapons:

Example knowledge base contd.

American(West)
Missile(M₁)
Owns(Nono, M₁)
Enemy(Nono, America)

Example knowledge base contd.

An enemy of America counts as “hostile”:

\[ \text{Enemy}(x, \text{America}) \Rightarrow \text{Hostile}(x) \]

West, who is American . . .
American(West)
The country Nono, an enemy of America . . .
Enemy(Nono, America)
Properties of forward chaining

Sound and complete for first-order definite clauses
(proof similar to propositional proof)

Datalog = first-order definite clauses + no functions (e.g., crime KB)
FC terminates for Datalog in poly iterations: at most \( p \cdot n^{k_l} \) literals
May not terminate in general (with functions) if \( \alpha \) is not entailed
This is unavoidable: entailment with definite clauses is semidecidable

Backward chaining algorithm

```plaintext
function FOL-BC-Ask(KB, goals, \( \theta \)) returns a set of substitutions
inputs: KB, a knowledge base
goals, a list of conjuncts forming a query (\( \theta \) already applied)
\( \theta \), the current substitution, initially the empty substitution (\{\})
local variables: answers, a set of substitutions, initially empty
if goals is empty then return (\( \theta \))
\( q' \leftarrow \text{SUBST}(\theta, \text{FIRST}(goals)) \)
for each sentence \( r \) in KB
where STANDARDIZE-Apart(\( r \)) = \( (p_1 \land \ldots \land p_n \Rightarrow q) \)
and \( \theta' \leftarrow \text{UNIFY}(\theta, q') \) succeeds
new_goals \( \leftarrow \{p_1, \ldots, p_n|\text{REST}(goals)\} \)
answers \( \leftarrow \text{FOL-BC-Ask}(KB, \text{newgoals, COMPOSE}(\theta', \theta)) \cup \text{answers} \)
return answers
```

Backward chaining example

```
Criminal(West)
[s/West]

American(s)
Weapon(s)
Sells(s,W,y)
Hostile(z)

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```
Backward chaining example

Properties of backward chaining

Logic programming

Sound bite: computation as inference on logical KBs
Unify

CN F
θ
∨
¬∃
¬∀

Skolem function

¬

Enemy(Nono,America)

For example,

query: append(A,B,[1,2]) ?

append([X|L],Y,[X|Z]) :- append(L,Y,Z).

append([],Y,Y).

Appending two lists to produce a third:

Apply resolution steps to

where

Full first-order version:

\[ \ell_1 \lor \cdots \lor \ell_k, \quad m_1 \lor \cdots \lor m_n \]

\[ \ell_1 \lor \cdots \lor \ell_{k-1} \lor \ell_{k+1} \lor \cdots \lor \ell_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \lor \cdots \lor m_n \theta \]

where \( \text{UNIFY}(\ell_i, m_j) = \theta \).

For example,

\[ \neg \text{Rich}(x) \lor \text{Unhappy}(x) \]

\[ \text{Rich}(Ken) \]

with \( \theta = \{ x/Ken \} \)

Apply resolution steps to \( \text{CNF}(KB \land \neg \alpha) \); complete for FOL

### Conversion to CNF

Everyone who loves all animals is loved by someone:

\[ \forall x \left[ \forall y \left( \text{Animal}(y) \Rightarrow \text{Loves}(x,y) \right) \Rightarrow [\exists y \text{Loves}(y,x)] \right] \]

1. Eliminate biconditionals and implications

\[ \forall x \left[ \neg \forall y \left( \neg \text{Animal}(y) \lor \text{Loves}(x,y) \right) \lor [\exists y \text{Loves}(y,x)] \right] \]

2. Move \( \neg \) inwards:

\[ \forall x \left[ \exists y \left( \neg \text{Animal}(y) \lor \text{Loves}(x,y) \right) \lor [\exists y \text{Loves}(y,x)] \right] \]

3. Standardize variables: each quantifier should use a different one

\[ \forall x \left[ \exists y \left( \text{Animal}(y) \land \neg \text{Loves}(x,y) \right) \lor [\exists y \text{Loves}(y,x)] \right] \]

4. Skolemize: a more general form of existential instantiation.

Each existential variable is replaced by a Skolem function of the enclosing universally quantified variables:

\[ \forall x \left[ \text{Animal}(F(x)) \land \neg \text{Loves}(x,F(x)) \lor \text{Loves}(G(x),x) \right] \]

5. Drop universal quantifiers:

\[ [\text{Animal}(F(x)) \lor \neg \text{Loves}(x,F(x))] \lor \text{Loves}(G(x),x) \]

6. Distribute \( \land \) over \( \lor \):

\[ [\text{Animal}(F(x)) \lor \text{Loves}(G(x),x)] \land [\neg \text{Loves}(x,F(x)) \lor \text{Loves}(G(x),x)] \]

### Resolution: brief summary

Full first-order version:

\[ \ell_1 \lor \cdots \lor \ell_k, \quad m_1 \lor \cdots \lor m_n \]

\[ \ell_1 \lor \cdots \lor \ell_{k-1} \lor \ell_{k+1} \lor \cdots \lor \ell_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \lor \cdots \lor m_n \theta \]

Resolution proof: definite clauses

\[ \neg \text{Animal}(x) \lor \neg \text{Weapon}(x) \lor \neg \text{Sells}(x,y,z) \lor \neg \text{Hostile}(z) \lor \text{Criminal}(x) \]

\[ \neg \text{American}(West) \lor \neg \text{Weapon}(x) \lor \neg \text{Sells}(x,y,z) \lor \neg \text{Hostile}(z) \]

\[ \text{Missing}(x) \lor \text{Weapon}(y) \lor \text{Sells}(x,y,z) \lor \text{Hostile}(z) \]

\[ \text{Missile}(M1) \lor \text{Weapon}(y) \lor \text{Sells}(x,y,z) \lor \text{Hostile}(z) \]

\[ \text{Criminal}(x) \lor \text{Weapon}(y) \lor \text{Sells}(x,y,z) \lor \text{Hostile}(z) \]

\[ \text{Criminal}(x) \lor \text{Weapon}(y) \lor \text{Sells}(x,y,z) \lor \text{Hostile}(z) \]

\[ \text{Hostile}(x) \lor \text{Weapon}(y) \lor \text{Sells}(x,y,z) \lor \text{Hostile}(z) \]

\[ \text{Missile}(M1) \lor \text{Weapon}(y) \lor \text{Sells}(x,y,z) \lor \text{Hostile}(z) \]

\[ \text{Hostile}(x) \lor \text{Weapon}(y) \lor \text{Sells}(x,y,z) \lor \text{Hostile}(z) \]

\[ \text{Hostile}(x) \lor \text{Weapon}(y) \lor \text{Sells}(x,y,z) \lor \text{Hostile}(z) \]

\[ \text{Hostile}(x) \lor \text{Weapon}(y) \lor \text{Sells}(x,y,z) \lor \text{Hostile}(z) \]

\[ \text{Hostile}(x) \lor \text{Weapon}(y) \lor \text{Sells}(x,y,z) \lor \text{Hostile}(z) \]