1. Equivalence Classes

(1) Define what is meant by an equivalence relation on a set.
(2) Describe an equivalence relation on \{1, 2, 3, 4, 5, 6\} whose equivalence classes are \{1, 2\}, \{3\}, \{4, 5, 6\}.
(3) Let \(A = \{1, 2, 3\}\). Determine all the equivalence relations \(R\) on \(A\). For each of these list all ordered pairs in the relation \(R\). (No other representation of the relation will be accepted.)
(4) If the following statement is true, prove it; if it is false, provide a counterexample: On the set \(R\) of real numbers, the relation \(R\) defined by \((x, y) \in R, x \neq y\) is an equivalence relation.
(5) Prove or disprove: Let \(k\) be a fixed integer, \(k \geq 5\), and let a relation \(\equiv\) be defined on the set \(\mathbb{N} \times \mathbb{N}\) by
\[
(x, y) \equiv (u, v) \text{ iff } (x = u \mod k) \land (y = v \mod k).
\]
Then \(\equiv\) is an equivalence relation.
(6) Prove or disprove: On the set \(\{1, 2, 3\}\) there is no equivalence relation \(R\) for which \(|R| = 6\).
(7) On a set \(B\) with \(\|B\| = 4\) there is no equivalence relation \(R\) for which \(|R| = 6\).
(8) Suppose that \(f : A \to B\) is any injective function, and \(g : B \to C\) is any surjective function. Prove or disprove each of the following statements:
The relation \(R\) defined as follows on \(B\) is an equivalence relation:
\[
\forall b_1 \in B \, \forall b_2 \in B \, [(b_1, b_2) \in R \iff g(b_1) = g(b_2)].
\]
(9) Showing all your work, determine exactly how many equivalence relations there are on the 4-element set \(S = \{a, b, c, d\}\).
(10) Let \(R\) be the relation on the set of ordered pairs of positive integers such that \(((a, b), (c, d)) \in R\) iff \(ad = bc\). Show that \(R\) is an equivalence relation. For the relation \(R\) on the set of pairs \((a, b)\) where \(1 \leq a, b \leq 3\), sketch the directed graph.
(11) Determine all equivalence relations on the 4-element set \(S = \{a, b, c, d\}\). For each of these determine the number of ordered pairs in the relation.

2. Partial Ordered Sets

(1) Give an example of each of the following, if one exists. If none exists, prove that fact.
- a partial ordering \(R_2\) on \(\mathbb{N}\) such that \(R_2 = R_2^{-1}\).
• a partial ordering $R_4$ on a set $S$ containing elements $x$, $y$ such that $x$ is minimal, $y$ is minimum, and $x \neq y$.

2. If the following statement is true, prove it; if it is false, provide counterexamples:
   On the set $Z$ of all integers, the relation $S$ defined by $(a, b) \in S \iff a|b$ is a partial ordering.

3. Prove or disprove: The number of total orders that can be defined on a set of $n$ elements ($n \geq 2$) is $2^n n!$.

4. For a binary relation $R$ on a set $A$, you are to consider the possibility that there exists a relation $R'$ such that $R \subseteq R'$, where $R'$ is to have certain specified properties. In each of the following separate cases either prove that, for any $R$, a relation $R'$ must always exist; or prove by a counterexample that it may happen that no such $R'$ exists.
   (a) $R'$ is symmetric.
   (b) $R'$ is reflexive.
   (c) $R'$ is a partial ordering.

5. Showing all your work, determine exactly how many total orderings there are on the 4-element set $S = \{a, b, c, d\}$.

6. Showing all your work, determine exactly all the partial orderings of the set $S = \{a, b, c, d\}$ that contain exactly 5 ordered pairs of elements of $S$.

7. Determine the number of binary relations $R$ on a set $A = \{a_1, a_2, ..., a_m\}$ of $n$ elements, which have each of the following combinations of properties.
   (a) $R$ is not restricted in any way.
   (b) $R$ is not antisymmetric.
   (c) $R$ is not both symmetric and reflexive.
   (d) $R$ is symmetric and is not reflexive.
   (e) $R$ is symmetric and irreflexive.
   (f) Suppose that $A$ is the union of disjoint sets $B$ and $\overline{B}$, where $|B| = r$. Let $R_1$ be the relation on $B$ defined to be $R \cap (B \times B)$. Now count the relations $R$ on $A$ such that $R_1$ is a symmetric relation on $B$. 